

Zank, Horst; Schmidt, Ulrich; Diecidue, Enrico

Working Paper

Parametric Weighting Functions

Economics Working Paper, No. 2007-01

Provided in Cooperation with:

Christian-Albrechts-University of Kiel, Department of Economics

Suggested Citation: Zank, Horst; Schmidt, Ulrich; Diecidue, Enrico (2007) : Parametric Weighting Functions, Economics Working Paper, No. 2007-01, Kiel University, Department of Economics, Kiel

This Version is available at:

<https://hdl.handle.net/10419/22019>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

Parametric Weighting Functions

by Enrico Diecidue, Ulrich Schmidt and Horst Zank



Christian-Albrechts-Universität Kiel

Department of Economics

Economics Working Paper

No 2007-01



Parametric Weighting Functions

by Enrico Diecidue, Ulrich Schmidt, and Horst Zank*

28 January 2007

Abstract. This paper provides behavioral foundations for parametric weighting functions under rank-dependent utility. This is achieved by decomposing the independence axiom of expected utility into separate meaningful properties. These conditions allow us to characterize rank-dependent utility with power and exponential weighting functions. Moreover, by restricting the conditions to subsets of the probability interval, foundations of rank-dependent utility with parametric inverse- S shaped weighting functions are obtained.

Keywords: Comonotonic independence, probability weighting function, preference foundation, rank-dependent utility.

Journal of Economic Literature Classification Numbers: D81.

*We thank Duncan R. Luce for detailed and helpful comments. Also, we acknowledge financial support from the Manchester School Visiting Fellowship Scheme.

Correspondence: Horst Zank, Economics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom; Telephone: ++44 161 275 4872, Fax: ++44 161 275 4812, E-mail: horst.zank@manchester.ac.uk

1 Introduction

Many empirical studies have shown that expected utility theory (EU), in particular its crucial independence axiom, does not provide an accurate description of people's actual choice behavior. This evidence has motivated researchers to develop alternative more flexible models. One prominent class of these alternatives is rank-dependent utility (RDU), which was introduced by Quiggin (1981, 1982), and which is the basis of prospect theory (Tversky and Kahneman 1992, Luce and Fishburn 1991).²

Most derivations of RDU require some structural richness on the set of consequences because the proposed preference conditions focus on the derivation of continuous cardinal utility. In those approaches the weighting functions are obtained as a bonus. In this paper we follow the traditional approach put forward by von Neumann and Morgenstern (1944) by focusing on the structure naturally offered by the probability interval, and we provide preference conditions that focus on the derivation of the probability weighting function. Typical for this approach is that cardinal utility is obtained as a bonus.

Axiomatizations of general RDU, without invoking any structural assumptions on the set of consequences, have been provided by Nakamura (1995) and more recently by Abdellaoui (2002) and Zank (2004). In these approaches the weighting function is unrestricted. Empirical evidence, however, suggests a particular pattern for probability weighting: small probabilities are overweighted while large ones are underweighted. Specific parametric forms have been proposed in the literature to accommodate these features. Some involve a single parameter (Karmarkar 1978, 1979, R  ell 1987, Currim and Sarin 1989, Tversky and Kahneman 1992, Luce, Mellers and Chang 1993, Hey and Orme 1994, Safra and Segal 1998) while others use

²Because prospect theory comes down to RDU if consequences are of the same sign (that is either all gains or all losses), the arguments presented in this paper apply to prospect theory as well.

two or more parameters (Bell 1985, Goldstein and Einhorn 1987, Currim and Sarin 1989, Lattimore Baker and Witte 1992, Prelec 1998). A recent experimental investigation of various parametric weighting functions by Stott (2006) favors the variant of Prelec (1998).

Despite the large interest in parametric specifications for the weighting function under RDU, little research has been invested in the axiomatic foundation of testable preference conditions in the RDU framework with general lotteries. There exist, however, several recent foundations restricted to binary lotteries where one consequence is the zero payoff (see e.g. Luce 2000, Narens 1996, Luce 2001, Aczél and Luce 2006, al-Nowaihi and Sanjit Dhami 2006). The motivation for restricting the analysis to binary lotteries stems from the descriptive shortcomings of the independence conditions required to derive RDU for general lotteries (for a review see Marley and Luce 2005). Additionally, once cardinal utility is derived, RDU for binary lotteries can be reduced to a simple and tractable multiplicative form. However, this approach rests on additional technical assumptions. Indeed, all preference foundations we are aware of require a rich topological structure for the set of consequences (Safra and Segal 1998, Prelec 1998, Gonzalez and Wu 1999). This means that those models cannot immediately be adopted to many real world applications because the set of consequences may lack such additional structure (e.g., health). As a consequence, it is unclear how to extend the existing preference foundations and, therefore, it is unclear whether these models remain valid for general outcomes.

Our goal is to derive parametric weighting functions by employing behavioral conditions. The preference foundations presented in this paper apply to general sets of consequences, which makes the resulting models generally applicable. Except for weak ordering and continuity, the properties that we propose are all implied by the independence axiom. For instance, we retain stochastic dominance and, in line with all rank-dependent theories, we assume comonotonic independence. These two implications ensure additive separability. But further assumptions

on preferences are required to derive the separation of probability weights and utility. In fact, by focusing on specific functional forms for the weighting functions, the preference conditions that characterize these forms deliver this latter separation free of charge. This is an important difference compared to the afore mentioned derivations.

Specific implications of the independence axiom have been analyzed before and, although the focus has not been on the weighting function under RDU, there are some common aspects underlying those preference conditions and the ones proposed in this paper. Machina (1989) distinguished two properties, termed mixture separability and replacement separability, respectively. Mixture separability demands that the preference between two lotteries is invariant to mixing them with a common degenerate lottery. Replacement separability holds if the preference between two lotteries remains unaffected when in both lotteries a common consequence with identical probability is replaced by any different consequence. We explore the implications of these separability conditions within our rank-dependent framework, where we have to restrict these conditions. It turns out that our restricted conditions can be employed to characterize RDU with a power weighting function and RDU with a linear or an exponential weighting function.

However, the separability conditions are descriptively problematic. For example, they are violated by the two famous paradoxes of Allais (1953). More precisely, the common ratio effect constitutes a direct violation of our version of mixture separability that generates the power weighting function, while the common consequence effect provides a violation of our version of replacement separability. More generally, because the afore mentioned weighting functions each involve a single parameter, they cannot accommodate at the same time probabilistic risk seeking and probabilistic risk aversion within the probability interval. That is, they are incompatible with the inverse- S shaped form, concave for small probabilities and convex for

large probabilities, that received extensive empirical support (e.g., Camerer and Ho 1994, Wu and Gonzalez 1996, Tversky and Fox 1995, Gonzalez and Wu 1999, Abdellaoui 2000, Bleichrodt and Pinto 2000, Kilka and Weber 2001, Abdellaoui, Vossmann and Weber 2005).

To accommodate mixed probabilistic risk attitudes, we need to relax the previous preference conditions further, namely to hold only on specific subsets of the probability interval. This way, we can provide foundations for inverse- S shaped weighting functions under RDU, which are entirely based on behavioral preference conditions that do not require additional structural assumptions on the set of consequences.

Our analysis of inverse- S shaped weighting functions focuses on functional forms that may involve three parameters. One parameter describes the probabilistic risk attitudes for small probabilities while a second one describes such attitudes for large probabilities. The role of the third parameter is to separate the region of probabilistic risk aversion from the region of probabilistic risk seeking. As it turns out, these parametric forms are in agreement with the interpretation of modeling sensitivity towards changes from impossibility and certainty, as proposed by Tversky and Kahneman (1992). In particular, the parameters can be used to measure the degrees of sensitivity, and to quantify the relative sensitivity between certainty and impossibility.

The organization of the paper is as follows. In Section 2 general notation and preliminary results are presented. We indicate how the results of Wakker (1993) and Chateauneuf and Wakker (1993) can be used to derive additive separability, the latter property being a common point of departure for all our models. Next, we proceed with a separation of the independence axiom of EU into specific variants of the separability conditions proposed by Machina (1989). In Section 3 we analyze mixture separability restricted to worst consequences, and in Section 4 we analyze replacement separability restricted to best and worst consequences. Section 5 analyses

the implications of mixture separability now restricted to the best consequence. Finally, in Section 6 we provide results for parametric inverse- S shaped probability weighting functions and Section 7 concludes. All proofs are deferred to the Appendix.

2 Preliminaries

Let X denote the set of consequences. For simplicity of exposition, we assume a finite set of consequences, such that $X = \{x_0, \dots, x_n\}$ for $n \geq 3$. A *lottery* is a finite probability distribution over the set X . It is represented by $P = (\tilde{p}_0, x_0; \dots; \tilde{p}_n, x_n)$ meaning that probability \tilde{p}_j is assigned to consequence $x_j \in X$, for $j = 0, \dots, n$. Let L denote the set of all lotteries. A preference relation \succsim is assumed over L , and its restriction to subsets of L (e.g., all degenerate lotteries) is also denoted by \succsim . The symbol \succ denotes strict preference while \sim denotes indifference. We assume that no two consequences in X are indifferent, and further, that consequences are ordered from worst to best, i.e., $x_0 \prec \dots \prec x_n$.

In this paper we present several preference conditions which become more transparent if formulated for decumulative distributions instead of lotteries. With this in mind we can identify lotteries with their corresponding decumulative probability distribution through the mapping

$$P \mapsto (p_1, \dots, p_n),$$

where $p_j = \sum_{i=j}^n \tilde{p}_i$ denotes the likelihood of getting at least x_j , $j = 1, \dots, n$. As the set of consequences is fixed we have simplified the notation above by suppressing the consequences and by noting that the worst consequence x_0 always has decumulative probability equal to 1. Therefore, the set of lotteries L is identified with the set $\{(p_1, \dots, p_n) : 1 \geq p_1 \geq \dots \geq p_n \geq 0\}$, which consists of probability tuples that are rank-ordered from highest to lowest.

In what follows we provide preference conditions for \succsim in order to *represent* the preference

relation over L by a function V . That is, V is a mapping from L into the set of real numbers, \mathbb{R} , such that for all $P, Q \in L$,

$$P \succsim Q \Leftrightarrow V(P) \geq V(Q).$$

This necessarily implies that \succsim must be a *weak order*, i.e. \succsim is *complete* ($P \succsim Q$ or $P \precsim Q$ for all $P, Q \in L$) and *transitive* ($P \succsim Q$ and $Q \succsim R$ implies $P \succsim R$ for all $P, Q, R \in L$).

The preference relation satisfies (probability-wise) *monotonicity* if $P \succ Q$ whenever $p_j \geq q_j$ for all $j = 1, \dots, n$ and $P \neq Q$. The preference relation \succsim satisfies *Jensen-continuity* on the set of lotteries L if for all lotteries $P \succ Q$ and R there exist $\rho, \mu \in (0, 1)$ such that

$$\rho P + (1 - \rho)R \succ Q \text{ and } P \succ \mu R + (1 - \mu)Q.$$

A monotonic weak order that satisfies Jensen-continuity on L also satisfies the stronger Euclidean-continuity on L (see Abdellaoui 2002, Lemma 18). We can then invoke a classical result of Debreu (1954) to derive the following statement:

THEOREM 1 *Assume that the preference relation \succsim on the set of decumulative distributions L is a Jensen-continuous monotonic weak order. Then there exists a continuous function $V : L \rightarrow \mathbb{R}$, strictly increasing in each decumulative probability, that represents \succsim . The function V is unique up to strictly increasing continuous transformations.* \square

Next, we focus on two forms of separability properties. The first property will ensure that the function in Theorem 1 is additively separable. The second family of properties is concerned with the separation of utility and probability weighting, and is discussed in the subsequent sections.

To derive additive separability we require independence of common decumulative probabilities. To define this property we introduce some useful notation. For $i \in \{1, \dots, n\}$, $P \in L$ and

$\alpha \in [0, 1]$, we denote by $\alpha_i P$ the distribution that agrees with P except that p_i is replaced by α . Whenever this notation is used it is implicitly assumed that $p_{i-1} \geq \alpha \geq p_{i+1}$ (respectively, $\alpha \geq p_{i+1}$ if $i = 1$ and $p_{i-1} \geq \alpha$ if $i = n$) to ensure that $\alpha_i P \in L$. Similarly, for $I \subset \{1, \dots, n\}$ we write $\alpha_I P$ for the distribution that agrees with P except that p_i is replaced by α for $i \in I$, whenever the probabilities in $\alpha_I P$ are ranked from highest to lowest.

The preference relation \succsim satisfies *comonotonic independence* if $\alpha_i P \succsim \alpha_i Q \Leftrightarrow \beta_i P \succsim \beta_i Q$ for all $\alpha_i P, \alpha_i Q, \beta_i P, \beta_i Q \in L$.

Comonotonic independence is a weak form of replacement separability as analyzed in Machina (1989). Recall that replacement separability demands that the preference between two lotteries is invariant when common consequences with equal probability are replaced by other common consequences. Comonotonic independence restricts replacement separability such that when comparing two lotteries, common consequences can be replaced by other common consequences only if they have a common decumulative likelihood. On reflection, one observes that this restriction implies that only common consequences of adjacent rank can be replaced.

Without the comonotonicity restriction on decumulative distributions in L we could adopt well-known results of Debreu (1960) to derive additive separability of the representing function in Theorem 1. Deriving additive separability on rank-ordered sets is not trivially extended from Debreu's classical result, but invokes more complex mathematical tools. The next theorem follows by using results of Wakker (1993) and Chateauneuf and Wakker (1993).

THEOREM 2 *The following two statements are equivalent for a preference relation \succsim on L :*

(i) *The preference relation \succsim on L is represented by an additive function*

$$V(P) = \sum_{j=1}^n V_j(p_j),$$

with continuous strictly monotonic functions $V_1, \dots, V_n : [0, 1] \rightarrow \mathbb{R}$ which are bounded except maybe V_1 and V_n which could be infinite at extreme probabilities (i.e., at 0, or 1).

(ii) The preference relation \succsim is a Jensen-continuous monotonic weak order that satisfies comonotonic independence.

The functions V_1, \dots, V_n are jointly cardinal, that is, they are unique up to location and common scale. \square

In the next sections we provide preference foundations for specific rank-dependent utility models using as common point of departure the results obtained above. Before proceeding we recall the general form of rank-dependent utility.

Rank-dependent utility (RDU) holds if the preference relation is represented by the function

$$V(P) = u(x_0) + \sum_{j=1}^n w(p_j)[u(x_j) - u(x_{j-1})], \quad (1)$$

where the utility function $u : X \rightarrow \mathbb{R}$ agrees with \succsim on X , and the *weighting function* $w : [0, 1] \rightarrow [0, 1]$ is strictly increasing and continuous with $w(0) = 0$ and $w(1) = 1$. Under RDU utility is cardinal and the weighting function is uniquely determined. If the weighting function is linear then RDU reduces to EU.

For completeness we recall the classical preference condition leading to EU. The preference relation \succsim satisfies *vNM-independence* (short for *von Neumann-Morgenstern independence*) if for all $P, Q, R \in L$ and all $\alpha \in (0, 1)$ it holds that

$$P \succsim Q \Leftrightarrow \alpha P + (1 - \alpha)R \succsim \alpha Q + (1 - \alpha)R.$$

In concluding this section, note without proof, two immediate implications of vNM-independence and weak ordering: monotonicity and comonotonic independence, which were introduced before.

3 Common Ratio Invariant Preferences

One of the difficulties of EU is to accommodate preferences that exhibit the common ratio effect. Allais (1953) compared the choice behavior for the following two decision problems. In problem 1 there is the choice between the following lotteries:

$$A_1 = (1, 1M) \text{ and } B_1 = (0.2, 0M; 0.8, 5M),$$

where M denotes \$-millions. In problem 2 the choice is between

$$A_2 = (0.95, 0M; 0.05, 1M) \text{ and } B_2 = (0.96, 0M; 0.04, 5M).$$

The literature has reported (e.g. Allais 1953, MacCrimmon and Larsson 1979, Chew and Waller 1986, Wu 1994) that a significant majority of people exhibited a preference for A_1 in the first choice problem and a preference for B_2 in the second choice problem. Substituting EU immediately reveals that this leads to a conflicting relationship.

Looking at the implications of vNM-independence we can observe that common ratio type behavior is not in conflict with monotonicity and neither with comonotonic independence. It is a different aspect of vNM-independence that is violated by such preferences, which gives rise to the following property. The preference relation \succsim satisfies *common ratio invariance for decumulative distributions* if

$$(p_1, \dots, p_n) \sim (q_1, \dots, q_n) \Leftrightarrow (\alpha p_1, \dots, \alpha p_n) \sim (\alpha q_1, \dots, \alpha q_n),$$

whenever $(p_1, \dots, p_n), (q_1, \dots, q_n), (\alpha p_1, \dots, \alpha p_n), (\alpha q_1, \dots, \alpha q_n) \in L$.

Common ratio invariance for decumulative distributions says that shifting proportionally probability mass from good consequences to the worst consequence (or doing the opposite) leaves preferences unaffected.

Common ratio invariance for decumulative distributions is a weak form of mixture separability (Machina 1989). The latter demands that a preference between two lotteries is maintained if each of the lotteries is mixed with any common consequence. In contrast, common ratio invariance for decumulative distributions demands that such mixtures are only permitted if the common consequence is the worst.

The condition has also appeared in Safra and Segal (1998), called zero-independence, where it has been used in the derivation of a specific version of Yaari (1987)'s dual theory, namely RDU with linear utility and power weighting function. The next result shows that the condition is powerful enough to yield RDU-preferences with power weighting without restricting the generality of the utility function.

THEOREM 3 *The following two statements are equivalent for a preference relation \succsim on L :*

- (i) *The preference relation \succsim on L is represented by rank-dependent utility with a power weighting function, i.e.,*

$$V(P) = u(x_0) + \sum_{j=1}^n p_j^b [u(x_j) - u(x_{j-1})],$$

with $b > 0$, and monotonic utility function $u : X \rightarrow \mathbb{R}$.

- (ii) *The preference relation \succsim is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance for decumulative distributions.*

The function u is cardinal.

□

It has previously been documented that preferences exhibiting the paradoxical common ratio effect exclude RDU preferences with power weighting. Our result above demonstrates that it is precisely this class of RDU-preferences with power weighting, including EU-preferences, that

cannot accommodate common ratio effect preferences. That the result is very general can also be inferred from the fact that, except for monotonicity, no further restrictions apply to utility.

4 Extreme Replacement Separability

We start in this section by reconsidering the common consequence paradox of Allais (1953), and relate this to a new preference condition concerning the replacement of common consequences.

The common consequence paradox originates from observing behavior among the following pairs of choice problems. In problem 3 the choice is between

$$A_3 = (1, 1M) \text{ and } B_3 = (0.01, 0M; 0.89, 1M; 0.1, 5M),$$

and in problem 4 the choice is between

$$A_4 = (0.89, 0M; 0.11, 1M) \text{ and } B_4 = (0.9, 0M; 0.1, 5M).$$

It has been observed in experiments that a significant majority of people exhibit a preference for A_3 in the former choice problem and a preference for B_4 in the latter choice problem (e.g. Allais 1953, MacCrimmon and Larsson 1979, Chew and Waller 1986, Wu 1994, but see also related evidence in Wakker, Erev and Weber 1994, Birnbaum and Navarette 1998, Birnbaum 2004). If one writes the previous lotteries as decumulative distributions over consequences 0, 1M, and 5M, then one can immediately see that $A_4 = (0.11, 0)$ and $A_3 = A_4 + (0.89, 0)$, and that $B_4 = (0.1, 0.1)$ and $B_3 = B_4 + (0.89, 0)$. Clearly, exhibiting initially $A_3 \succ B_3$ together with a second preference $A_4 \prec B_4$ directly violates vNM-independence but not monotonicity and neither comonotonic independence.

In the common consequence paradox the interpretation is that people are sensitive to replacing the good common consequence of getting “1 Million with probability 0.89” with a bad

common consequence of getting “0 with probability 0.89.” Therefore, also replacement separability (Machina 1989) is violated. Although empirically it has to be verified, we think that such sensitivity would also be exhibited when the best consequence is replaced by the worst consequence, which leads to the following property. The preference relation \succsim satisfies *extreme replacement separability* if

$$(p_1, \dots, p_n) \sim (q_1, \dots, q_n) \Leftrightarrow (p_1 + \alpha, \dots, p_n + \alpha) \sim (q_1 + \alpha, \dots, q_n + \alpha),$$

whenever $(p_1, \dots, p_n), (q_1, \dots, q_n), (p_1 + \alpha, \dots, p_n + \alpha), (q_1 + \alpha, \dots, q_n + \alpha) \in L$.

The following theorem shows that for RDU-preferences the only weighting functions that are able to accommodate extreme replacement separability are linear or exponential ones.

THEOREM 4 *The following two statements are equivalent for a preference relation \succsim on L :*

- (i) *The preference relation \succsim on L is either represented by expected utility, or it is represented by rank-dependent utility with an exponential weighting function, i.e.,*

$$V(P) = u(x_0) + \sum_{j=1}^n \frac{e^{cp_j} - 1}{e^c - 1} [u(x_j) - u(x_{j-1})],$$

with $c \neq 0$, and monotonic utility function $u : X \rightarrow \mathbb{R}$.

- (ii) *The preference relation \succsim is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and extreme replacement separability.*

The function u is cardinal.

□

Note that RDU-preferences satisfying both common ratio invariance for decumulative distributions and extreme replacement separability can only be represented by EU. This follows immediately by observing that the only possible weighting function that is common in Theorems 3 and 4 is the linear weighting function $w(p) = p$.

5 A Dual Analysis

The properties considered in the previous sections can easily be formulated for cumulative distributions. Jensen-continuity, monotonicity, comonotonic independence, and also extreme replacement separability have mathematically equivalent counterparts which are obtained by simply replacing the decumulative distributions by the corresponding cumulative ones. However, doing the same for the afore mentioned common ratio invariance property leads to a different but closely related property. This can be inferred from the corresponding RDU-representation with a weighting function that is the dual of a power function (see Theorem 5 below).

Before we formulate this new property we note that if a lottery is written as a decumulative distribution $P = (p_1, \dots, p_n)$ then writing the same lottery as a cumulative distribution results in $\tilde{P} = (1 - p_1, \dots, 1 - p_n)$. The difference in the latter notation lies in the interpretation of the cumulative probability $1 - p_i$, which now refers the likelihood of getting at most x_{i-1} , $i = 1, \dots, n$, whereas the decumulative probability p_i was associated with the consequences x_i , $i = 1, \dots, n$. We denote by \tilde{L} the set of cumulative distributions.

The preference relation \succsim satisfies *common ratio invariance for cumulative distributions* if

$$(1 - p_1, \dots, 1 - p_n) \sim (1 - q_1, \dots, 1 - q_n)$$

$$\Leftrightarrow$$

$$(\alpha(1 - p_1), \dots, \alpha(1 - p_n)) \sim (\alpha(1 - q_1), \dots, \alpha(1 - q_n)),$$

whenever $(1 - p_1, \dots, 1 - p_n), (1 - q_1, \dots, 1 - q_n), (\alpha(1 - p_1), \dots, \alpha(1 - p_n)), (\alpha(1 - q_1), \dots, \alpha(1 - q_n)) \in \tilde{L}$.

This variant of common ratio invariance, which says that shifting probability mass proportionally from all consequences to the best consequence leaves preferences unaffected, is also

a weak form of mixture separability (Machina 1989). We get the following analog result to Theorem 3.

THEOREM 5 *The following two statements are equivalent for a preference relation \succsim on L :*

- (i) *The preference relation \succsim on L is represented by rank-dependent utility with a dual power weighting function, i.e.,*

$$V(P) = u(x_0) + \sum_{j=1}^n [1 - (1 - p_j)^d][u(x_j) - u(x_{j-1})],$$

with $d > 0$, and monotonic utility function $u : X \rightarrow \mathbb{R}$.

- (ii) *The preference relation \succsim is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance for cumulative distributions.*

The function u is cardinal.

□

Note that, similarly to the arguments presented at the end of the previous section, RDU-preferences satisfying both common ratio invariance for cumulative distributions and extreme replacement separability can only be represented by EU. Also, RDU-preferences satisfying both common ratio invariance properties must be EU-preferences.

6 Inverse- S shaped Weighting Functions

The parametric forms derived in the previous sections are somewhat inflexible in modeling probabilistic risk attitudes. Such risk attitudes are reflected in the curvature of the probability weighting function (see Chew, Karni and Safra 1987, Yaari 1987, Chateauneuf and Cohen 1994, Wakker 1994, Abdellaoui 2002, Chateauneuf, Cohen and Meilijson 2004). The afore mentioned RDU-preferences either exhibit exclusively probabilistic risk aversion (i.e., a convex weighting

function) or exclusively probabilistic risk seeking (i.e., a concave weighting function) throughout the probability interval. That is, in Theorem 3 either the parameter $b > 1$ (w convex) or $b < 1$ (w concave); in Theorem 4 either $c > 0$ (w convex) or $c < 0$ (w concave); and in Theorem 5 either the parameter $d < 1$ (w convex) or $d > 1$ (w concave). While there is theoretical interest in overall convex/concave probability weighting, empirical findings suggest that a combination of probabilistic risk seeking for small probabilities and probabilistic risk aversion for large probabilities is an appropriate way of modeling sensitivity towards probabilities. Because the concave region for small probabilities is followed smoothly by a convex region for larger probabilities (Tversky and Kahneman 1992, Tversky and Fox 1995, Wu and Gonzalez 1996, Abdellaoui 2000), such weighting functions are referred to as inverse- S shaped.

A few parametric forms have been proposed for inverse- S shaped weighting functions (Karmarkar 1978, 1979, Goldstein and Einhorn 1987, Currim and Sarin 1989, Lattimore, Baker and Witte 1992, Tversky and Kahneman 1992, Prelec 1998), and their parameters have been estimated in many empirical studies (Camerer and Ho 1994, Tversky and Fox 1995, Wu and Gonzalez 1996, Gonzalez and Wu 1999, Abdellaoui 2000, Bleichrodt and Pinto 2000, Kilka and Weber 2001, Etchart-Vincent 2004, Abdellaoui, Vossman and Weber 2005). Most of these parametric forms lack an appropriate axiomatic underpinning. This is problematic because it is unclear what kind of preference condition must be assumed to generate such weighting functions, and therefore, it is unclear what kind of behavioral properties are captured within a specific parametric family of weighting functions.

Axiomatizations have been proposed for the class of weighting functions introduced by Prelec (1998) (see also Luce 2001, Aczél and Luce 2006). The class introduced by Goldstein and Einhorn (1987) has been discussed in Gonzalez and Wu (1999), where necessary preference conditions have been proposed. In the axiomatic derivation of these families of weighting functions

it is necessary to assume a rich set of consequences, and further, the representing functional must also be continuous with respect to consequences. From an empirical point of view, this dependence on consequences is a demanding restriction. A further restrictive point in these axiomatizations is that a representing functional, where the continuous utility is already separated from probability weighting, must be assumed prior to invoking the additional invariance property that generates the required parametric form. An open and from an empirical point of view important question is whether, on their own, those characterizing properties are powerful enough to induce such a separation once additive separability as given in Theorem 2 has been derived.

Recall that the results presented in the previous sections are free of restrictions on the richness of the set of consequences, and also free of additional separability conditions that ensure RDU to hold prior to invoking the invariance properties. But note at the same time that these preference conditions are too rigid to permit inverse- S shaped probability weighting functions under RDU. We would like to have both preference conditions for general consequences and also axiomatizations that allow for inverse- S shaped weighting functions under RDU. In what follows we propose such preference conditions, and show that these lead to new families of parametric weighting functions.

To derive RDU with inverse- S shaped weighting functions we restrict the preference conditions presented in Sections 3–5 to hold only on specific intervals of probabilities. An analogous approach for general, non-parametric weighting functions and capacities was pursued by Tversky and Wakker (1995) and Wakker (2001). This seems to be a reasonable compromise because, as we show below, these conditions are still powerful enough to separate utility from probability weighting if additive separability holds, that is, if they are added in statement (ii) of Theorem 2. The idea, in line with the empirical evidence, is to impose a first invariance condition for

distributions involving small probabilities and a second invariance property for distributions involving large probabilities. This will then give sufficient flexibility in deriving the required weighting functions. However, as we indicate in the next subsection, some unwarranted features relating to the utility functions may occur.

6.1 Switch-power Weighting Functions

The results presented in this subsection focus on the class of weighting functions which are power functions for probabilities below some $\hat{p} \in (0, 1)$, and dual power functions above \hat{p} , i.e.,

$$w(p) = \begin{cases} cp^a, & \text{if } p \leq \hat{p}, \\ 1 - d(1 - p)^b, & \text{if } p > \hat{p}, \end{cases}$$

with the parameters involved as discussed below. We call these functions *switch-power weighting functions*.

We presented the function above with five parameters a, b, c, d and \hat{p} . However, these reduce to four because of continuity of w on $[0, 1]$, and if differentiability at \hat{p} is assumed, which seems plausible in this context, a reduction to three parameters is obtained. Let us elaborate on these reductions. Continuity at 0 implies that $a > 0$, and monotonicity implies that $c > 0$. Continuity at 1 implies that $b > 0$, and monotonicity implies that $d > 0$. Continuity and differentiability at \hat{p} relates a, c to b, d and \hat{p} through

$$c = \frac{1}{\hat{p}^a} - \frac{d(1 - \hat{p})^b}{\hat{p}^a},$$

and

$$c = \frac{db(1 - \hat{p})^{b-1}}{a\hat{p}^{a-1}},$$

respectively. Combining the two gives

$$\begin{aligned} c &= \hat{p}^{-a} \left[\frac{b\hat{p}}{b\hat{p} + a(1 - \hat{p})} \right], \\ d &= (1 - \hat{p})^{-b} \left[\frac{a(1 - \hat{p})}{b\hat{p} + a(1 - \hat{p})} \right]. \end{aligned}$$

If $0 < a \leq 1$ the probability weighting function is concave on $(0, \hat{p})$, and if $0 < b \leq 1$ it is convex on $(\hat{p}, 1)$, hence has an inverse- S shape. For $a, b \geq 1$ we have a S -shaped probability weighting function, which is convex on $(0, \hat{p})$ and concave on $(\hat{p}, 1)$. When \hat{p} approaches 1 or 0, the weighting function reduces to a power weighting function or a dual power weighting function, respectively. Moreover, substitution of \hat{p} into w gives

$$\begin{aligned} w(\hat{p}) &= \frac{b\hat{p}}{b\hat{p} + a(1 - \hat{p})} \\ &= 1 - \frac{a(1 - \hat{p})}{b\hat{p} + a(1 - \hat{p})}, \end{aligned}$$

from which one can easily derive the relationship

$$w(\hat{p}) \leq \hat{p} \Leftrightarrow b \leq a.$$

In particular, this shows that whenever $a = b$ the weighting function intersects the 45° line precisely at \hat{p} (see Figure 1). One should also note that in this case the derivative of w at \hat{p} equals a , and therefore this parameter controls for the curvature of the weighting function. The parameter \hat{p} , however, indicates whether the interval for overweighting of probabilities is larger than the interval for underweighting, and therefore controls for the elevation of the weighting function (see also Gonzalez and Wu (1999) for a similar interpretation of the parameters in the “linear in log-odds” weighting function of Goldstein and Einhorn (1987)).

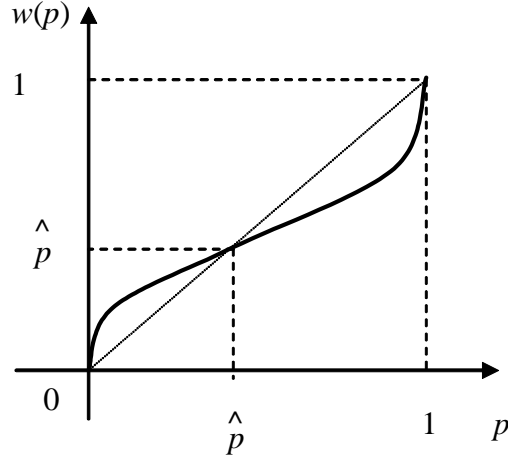


Figure 1: A two parameter switch-power weighting function.

In general, when $a \neq b$, two parameters control for curvature. In that case \hat{p} need not demarcate the regions of over and underweighting because it may not lie on the 45° line. Nevertheless, \hat{p} will still influence elevation, however, whether there is more overweighting relative to underweighting now also depends on the relationship between the magnitudes of the parameters a and b . The following figure depicts, for the case of an inverse-S shaped weighting function, the two scenarios of underweighting ($0 < b < a < 1$), respectively, overweighting ($0 < a < b < 1$) at \hat{p} .

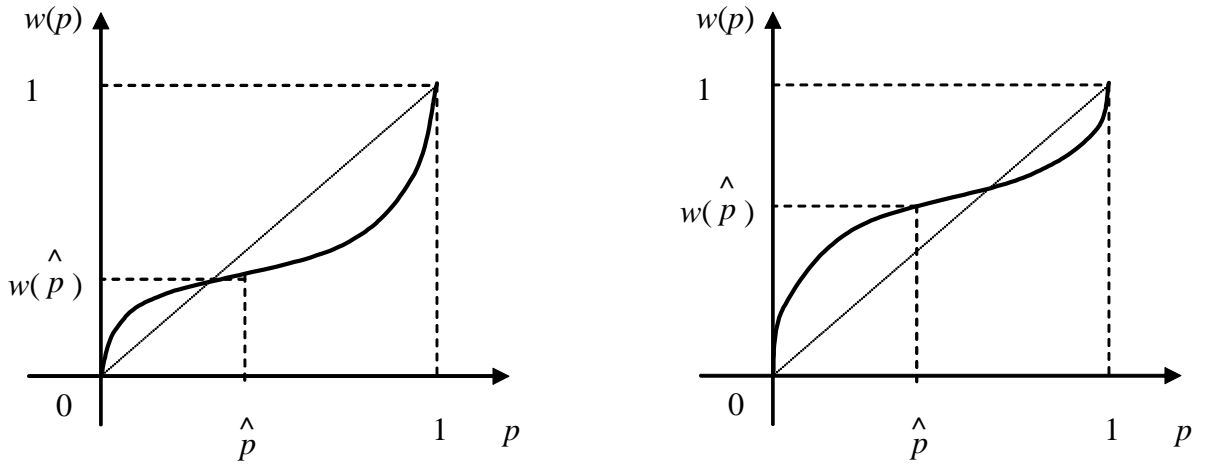


Figure 2: A 3-parameter function with underweighting respectively overweighting at \hat{p} .

As it turns out, it is more appropriate to interpret these parameters as was initially proposed by Tversky and Kahneman (1992). All parameters may influence elevation, however, the main role of \hat{p} is to demarcate the interval of probabilistic risk aversion from the interval of probabilistic risk seeking. The magnitude of the parameter a indicates diminishing (or increasing) sensitivity to changes from impossibility to possibility. This can be inferred by inspecting the derivative of w for probabilities in the range $(0, \min\{\hat{p}, 1 - \hat{p}\})$. Observe, that for $q \in (0, \min\{\hat{p}, 1 - \hat{p}\})$ we get

$$\begin{aligned} w'(p)|_{p=q} &= (cap^{a-1})|_{p=q} \\ &= \hat{p}^{-a} \left[\frac{ab\hat{p}}{b\hat{p} + a(1 - \hat{p})} \right] q^{a-1}. \end{aligned}$$

Therefore, sensitivity increases if $a > 1$ and decreases if $a < 1$. Note that for $a = 1$ sensitivity is constant. Note also that the right-derivative at 0, $w'(0_+) = 0$ if $a > 1$ and is unbounded if $a < 1$, the latter indicating extreme sensitivity for changes from possible to impossible. Similarly, as one moves away from certainty, sensitivity increases if $b > 1$ and decreases if $b < 1$, while for $b = 1$ sensitivity is constant. There is extreme sensitivity for changes from certainty to possibility if $b < 1$.

The switch-power weighting function also allows for a comparison of the sensitivity to changes from 0 relative to the sensitivity to changes from 1. Considering the ratio of derivatives at q and $1 - q$ for $q \in (0, \min\{\hat{p}, 1 - \hat{p}\})$ we observe

$$\frac{w'(p)|_{p=q}}{w'(p)|_{p=1-q}} = \left[\frac{(1 - \hat{p})^{b-1}}{\hat{p}^{a-1}} \right] q^{a-b}.$$

Therefore, this relative sensitivity is constant when $a = b$, but otherwise there is more (less) sensitivity for changes from 0 than for changes from 1 if $a < b$ ($a > b$). As q approaches

$\min\{\hat{p}, 1 - \hat{p}\}$, the ratio $w'(q)/w'(1 - q)$ is decreasing (increasing) towards

$$\frac{w'(\hat{p})}{w'(1 - \hat{p})} = \begin{cases} [(1 - \hat{p})/\hat{p}]^{b-1}, & \text{if } \hat{p} \leq 1/2, \\ [\hat{p}/(1 - \hat{p})]^{a-1}, & \text{if } \hat{p} > 1/2. \end{cases}$$

There are some extreme cases that should be mentioned here. Taking limits when only a approaches 0 gives a weighting function that equals 0 at 0 and is constant equal to 1 on $(0, 1]$. Taking limits when only b approaches 0 we get a weighting function that equals 1 at 1 and is constant equal to 0 on $[0, 1)$. These latter weighting functions do not exhibit continuity or monotonicity, and therefore fall outside the RDU-functionals considered in this paper. Similarly, this holds for the classes of weighting functions where $a = b$ and a approaches 0, or when $a \neq b$ and either a or b approach infinity.

The preference condition that is necessary for RDU with (inverse) S -shaped switch-power weighting function is defined next. *Common ratio invariance* (for extreme probabilities) holds if there exists a probability $\hat{p} \in (0, 1)$ such that

$$(p_1, \dots, p_n) \sim (q_1, \dots, q_n) \Leftrightarrow (\alpha p_1, \dots, \alpha p_n) \sim (\alpha q_1, \dots, \alpha q_n),$$

whenever all $(p_1, \dots, p_n), (q_1, \dots, q_n), (\alpha p_1, \dots, \alpha p_n), (\alpha q_1, \dots, \alpha q_n) \in L_{\hat{p}} := \{R \in L : r_1 \leq \hat{p}\}$ and

$$(1 - p_1, \dots, 1 - p_n) \sim (1 - q_1, \dots, 1 - q_n)$$

$$\Leftrightarrow$$

$$(\beta(1 - p_1), \dots, \beta(1 - p_n)) \sim (\beta(1 - q_1), \dots, \beta(1 - q_n)),$$

whenever $(1 - p_1, \dots, 1 - p_n), (1 - q_1, \dots, 1 - q_n), (\beta(1 - p_1), \dots, \beta(1 - p_n))$, and $(\beta(1 - q_1), \dots, \beta(1 - q_n)) \in \tilde{L}_{\hat{p}} := \{R \in \tilde{L} : 1 - r_n \leq 1 - \hat{p}\}$.

Clearly common ratio invariance requires preferences to be immune to common proportional changes in decumulative probabilities whenever these are all smaller than some $\hat{p} \in (0, 1)$ and

it does also require immunity of preferences to common proportional changes in cumulative probabilities if these are all smaller than $1 - \hat{p}$. As the result below shows, replacing common ratio invariance for (de)cumulative distributions in (Theorem 3) Theorem 5 with the weaker common ratio invariance does not necessarily give RDU. As it turns out this property leads to a more general class of preferences represented by a RDU-like functional that combines a unique switch-power weighting function with two cardinal utility functions depending on the evaluated distribution. We state this result before we analyze this aspect further.

THEOREM 6 *The following two statements are equivalent for a preference relation \succsim on L :*

- (i) *The preference relation \succsim on L is represented by an additive representation as in Theorem 2 with functions V_j as follows:*

$$V_j(p) = \begin{cases} s_j[cp^a], & \text{if } p \leq \hat{p}, \\ \hat{s}_j[1 - d(1 - p)^b], & \text{if } p > \hat{p}, \end{cases}$$

for some $\hat{p} \in (0, 1)$ with $a, b, c, d > 0$, and positive s_j, \hat{s}_j for all $j = 1, \dots, n$.

- (ii) *The preference relation \succsim is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance.*

The parameters \hat{p}, a, b, d are uniquely determined, and $c = 1/\hat{p}^a - d(1 - \hat{p})^b/\hat{p}^a$. Further the s_j 's and the \hat{s}_j 's can be replaced by corresponding ts_j 's and $t\hat{s}_j$'s for any positive t . \square

This theorem shows that, by making the sensitivity towards small probabilities independent from that for large probabilities, a more general functional than RDU is obtained. However, when we restrict to specific sets of distributions the derived representing functional still gives RDU. We elaborate on this point next.

Take $k \in \{0, \dots, n\}$ and define $L(k) := \{P \in L : p_k \leq \hat{p} < p_{k+1}\}$. Then, on $L(k)$ the functions derived in Theorem 6 take the form

$$V_j(p) = \begin{cases} s_j[cp^a], & \text{if } p \leq \hat{p}, \\ \hat{s}_j[1 - d(1 - p)^b], & \text{if } p > \hat{p}, \end{cases}$$

for some $\hat{p} \in (0, 1)$ with $a, b, d > 0, c = 1/\hat{p}^a - d(1 - \hat{p})^b/\hat{p}^a$ and positive s_j, \hat{s}_j for all $j = 1, \dots, n$.

In this case we define $u(x_0) = 0$ and iteratively $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \dots, k$ and $u(x_j) = u(x_{j-1}) + \hat{s}_j$ for $j = k + 1, \dots, n$. This means that on $L(k)$ the preference relation is represented by

$$RDU_k(P) = u(x_0) + \sum_{j=1}^n w(p_j)[u(x_j) - u(x_{j-1})],$$

with switch-power weighting function

$$w(p) = \begin{cases} cp^a, & \text{if } p \leq \hat{p}, \\ 1 - d(1 - p)^b, & \text{if } p > \hat{p}, \end{cases}$$

and strictly monotonic cardinal utility u . Hence, RDU has been obtained for \succsim on $L(k)$.

In general, for different values of k , the RDU-functionals (or RDU-restrictions) may not agree. This shows the price that we pay for further relaxing the common ratio invariance properties of the previous sections so that they apply only on restricted sets of distributions.

An additional preference condition is now required to derive RDU for \succsim on L . Such a condition has been proposed in Zank (2004). There it was shown that, in the presence of Theorem 2, the probabilistic consistency condition is necessary and sufficient to give general RDU, hence cardinal utility, without requiring any structural assumptions on the set of consequences. In this paper we present a version of that condition that is much weaker, and on its own not sufficient to give RDU, but when added to statement (ii) of Theorem 6 above, the property implies RDU with switch-power weighting function.

The preference relation \succsim satisfies *consistency* if

$$p_I(\gamma, \dots, \gamma) \sim \hat{p}_I(\delta, \dots, \delta)$$

and

$$\hat{p}_I(\gamma, \dots, \gamma) \sim q_I(\delta, \dots, \delta)$$

imply

$$p_{I \setminus \{i\}} \hat{p}_i(\gamma, \dots, \gamma) \sim \hat{p}_{I \setminus \{i\}} q_i(\delta, \dots, \delta),$$

whenever $I = \{1, \dots, i\}$ or $I = \{i, \dots, n\}$, $i \in \{1, \dots, n\}$, and $q < \hat{p} < p$ are such that the above distributions are in L .

Note that, given monotonicity and continuity, the first two indifferences can always be derived locally. Consistency then requires that the measured indifferences for consequence x_i remains valid when measured for consequence x_{i-1} (respectively x_{i+1}). Under the assumptions of Theorem 6 the condition will preclude the possibility of having two utility functions that determine choice behavior. The next result summarizes this point.

THEOREM 7 *The following two statements are equivalent for a preference relation \succsim on L :*

(i) *The preference relation \succsim on L is represented by RDU with a switch-power utility,*

$$w(p) = \begin{cases} cp^a, & \text{if } p \leq \hat{p}, \\ 1 - d(1 - p)^b, & \text{if } p > \hat{p}, \end{cases}$$

for some $\hat{p} \in (0, 1)$ with $a, b, c, d > 0$.

(ii) *The preference relation \succsim is a Jensen-continuous monotonic weak order that satisfies comonotonic independence, common ratio invariance and consistency.*

The parameters \hat{p}, a, b, d are uniquely determined and $c = 1/\hat{p}^a - d(1 - \hat{p})^b/\hat{p}^a$. Further, the utility function u is cardinal. □

Following the line of argument presented above, one can also provide axiomatic characterizations for RDU with an analog to the switch-power weighting function that first is a dual power weighting function followed by a power weighting function.

6.2 The Switch-exponential Weighting Function

Let us now consider the switch-exponential weighting function.³ Exploiting continuity at 0 and 1, the general form of this class of weighting functions is

$$w(p) = \begin{cases} c(e^{ap} - 1), & \text{if } p \leq \hat{p}, \\ 1 - d(e^b - e^{bp}), & \text{if } p > \hat{p}, \end{cases}$$

with $ac > 0, db > 0$ by monotonicity, and due to continuity at \hat{p} it holds that

$$c = \frac{1}{e^{a\hat{p}} - 1} - \frac{d(e^b - e^{b\hat{p}})}{e^{a\hat{p}} - 1}.$$

Requiring differentiability at \hat{p} implies

$$c = \frac{db}{a} \frac{e^{b\hat{p}}}{e^{a\hat{p}}},$$

which, combined with the previous expression for c , allows us to determine both c, d in terms of a, b , and \hat{p} :

$$\begin{aligned} c &= \frac{be^{b\hat{p}}}{ae^{a\hat{p}}(e^b - e^{b\hat{p}}) + be^{b\hat{p}}(e^{a\hat{p}} - 1)}, \\ d &= \frac{ae^{a\hat{p}}}{ae^{a\hat{p}}(e^b - e^{b\hat{p}}) + be^{b\hat{p}}(e^{a\hat{p}} - 1)}. \end{aligned}$$

One can immediately derive the conditions for which there is diminishing (increasing) sensitivity at 0 and 1. An inverse- S shaped weighting function is obtained if $a < 0$ and $b > 0$, while an S -shaped weighting functions must have $a > 0$ and $b < 0$.

³We restrict our analysis to the cases that the weighting function is exponential below some parameter \hat{p} and exponential above it. As can be inferred from Theorem 4, the characterizing preference condition will allow also for linearity below or above the parameter \hat{p} .

In the case of an inverse- S weighting function (i.e., $a < 0, b > 0$), the condition for over-weighting at \hat{p} comes down to

$$w(\hat{p}) > \hat{p} \Leftrightarrow \frac{-a}{b} < \frac{e^{-a\hat{p}} - 1}{e^{b(1-\hat{p})} - 1} \frac{1 - \hat{p}}{\hat{p}}.$$

Observe that the sensitivity to changes from impossibility is given by $w'(0) = ca$ and the sensitivity to changes from certainty is $w'(1) = db e^b$, and that both expressions must exceed 1 in order to have overweighting for small probabilities and underweighting for large ones. By substituting for c and d we can determine if there is more sensitivity at 0 compared to sensitivity at 1 through

$$\frac{w'(0)}{w'(1)} = \frac{e^{(b-a)\hat{p}}}{e^b}.$$

Note that in the case of an inverse- S shaped weighting function one obtains

$$\frac{w'(0)}{w'(1)} > 1 \Leftrightarrow \hat{p} > \frac{b}{b-a},$$

hence, whether there is greater sensitivity at 0 compared to 1 will depend on all three parameters a, b , and \hat{p} . We compare how this relative sensitivity evolves as one moves away from the extreme probabilities. For $q \in (0, \min\{\hat{p}, 1 - \hat{p}\})$ it holds that

$$\frac{w'(p)|_{p=q}}{w'(p)|_{p=1-q}} = \frac{e^{(b-a)\hat{p}}}{e^b} e^{(a+b)q},$$

hence relative sensitivity increases if $b > -a$ (decreases if $b < -a$), reaching its maximum (minimum) at $\min\{\hat{p}, 1 - \hat{p}\}$ as follows:

$$e^{b(2\hat{p}-1)}, \text{ if } \hat{p} \leq 1/2, \text{ or } e^{a(2\hat{p}-1)}, \text{ if } \hat{p} > 1/2.$$

An analog statement can be concluded for the case of an S -shaped weighting function. Note that there is constant relative sensitivity if $b = -a$.

7 Summary

Our main objective in this paper has been to provide preference foundations for parametric weighting functions in a general RDU framework where the set of consequences is arbitrary. Inevitably, these preference foundations have to employ conditions that exploit the mathematical structure offered by the probability interval. Initially, we have derived three classes of such RDU-forms with a single parameter for probability weighting. In all these derivations cardinal utility is obtained as a bonus in addition to the specific parametric form (power, exponential, or dual power) of the weighting functions.

Building on mixture separability and replacement separability, as introduced by Machina (1989), we characterized RDU with power, linear, and exponential weighting function. This shows, once more, the relevance of the vNM-independence axiom and its implications in decision theory and in particular for the weighting functions under RDU (and prospect theory). The power weighting function is directly related to the common ratio pattern of preferences. It has also been pointed out that the exponential weighting function is directly related to the common consequence pattern of preferences (Allais 1953), a somewhat surprising connection that has not been mentioned before in the literature. The dual power weighting function has no documented EU-paradox to be linked to, but we think that a dual analog of the common ratio paradox of Allais can easily be constructed. However, viewed from a different perspective, the preference conditions that give rise to these weighting functions will hopefully lead to a better understanding of how demanding EU is, and in particular how demanding the vNM-independence axiom actually is.

The one-parameter classes of weighting functions have shortcomings for descriptive applications. In particular it is not possible to separate sensitivity to changes in small probabilities

from sensitivity to changes in large probabilities because there is a single parameter that has to govern both. Empirical studies suggest that there is extreme sensitivity to changes from certainty or impossibility to possibility, and also that this sensitivity diminishes as one approaches moderate probabilities. Taking account of this evidence, we have proposed to separate the probability interval into two exhaustive regions on which the preference conditions that implied the one-parameter weighting functions still hold. Therefore, we had to specify in advance where the boundary is that separates the intervals of distinct sensitivity to changes in probabilities, and this boundary probability appears as one of the parameters in our weighting functions. This is different to the axiomatization offered by Prelec (1998) and the one suggested in Gonzalez and Wu (1999) because there the probability value that separates the regions of distinct sensitivity is implicit in the corresponding preference conditions. It should be noted, however, that those axiomatizations do not apply to our framework, in particular, because the preference conditions characterizing those weighting functions may not be well-defined here. Also, Prelec (1998) and Gonzalez and Wu (1999), in fact, model sensitivity to changes in the logarithm of probabilities instead of probabilities as we do. From a technical point of view this is an important difference as the interval of transformed probabilities (by taking the negative of the logarithm of probabilities) is large enough (i.e., equals all positive numbers) to generate endogenously, with the appropriate axiom assumed, two regions in which changes in log-odds point in opposite direction.⁴ We think that modeling sensitivity to probability changes is more natural under RDU, certainly this is the case if one works in the general framework that we have adopted in

⁴The argument used here is best exemplified for the case of, e.g., positive power functions that apply to positive numbers. Assume that the power exceeds 1. For log-odds smaller than 1 applying the power function leads to decreases of the original number, while application to log-odds larger than 1 results in increases. So, 1 naturally demarcates the regions of opposite sensitivity in log-odds.

this paper.

By specifying exogenously the parameter separating sensitivity regions within the probability interval, we have also induced additional flexibility for the representing functions. By simply restricting some preference conditions to hold on particular subsets of the probability interval, the resulting representing functionals belong to a much larger class than that of RDU-preferences. That is, although we can obtain unique parametric weighting functions, in general there may be two cardinal utility functions that govern choice behavior. Further, the number of parameters that we get for the weighting functions —four— seems too large. To resolve these issues we have employed additional conditions. To retain RDU with a parametric inverse- S shaped weighting function we have introduced an axiom that explicitly requires consistency of measured preferences irrespective of consequences. This then gives a single cardinal utility, hence RDU. To reduce the number of parameters we assume differentiability of the weighting function, which, although it seems a reasonable constraint, is enforced exogenously.

However, except for the parametrizations presented in this paper there are no other foundations of RDU in the literature that combine parametric weighting functions and general utility. The previous parametrizations either lack preference foundations or their preference foundations are meaningful only in the special case of continuous utility. Neither is satisfactory. To some extent we have been able to resolve these shortcomings. For example, we did this for the one-parameter classes that we obtained. But, although progress has been made, our attempt to add more empirical realism and still obtain simple classes of parametric weighting functions compromises on other aspects. In particular, the problem of endogenizing the separation of the probability interval into regions of distinct probabilistic risk attitudes or distinct sensitivity, and thereby also reducing the number of parameters in the weighting functions (instead of employing differentiability), remains an open question.

8 Appendix: Proofs

PROOF OF THEOREM 3: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Common ratio invariance for decumulative distributions follows from substitution of the RDU-functional with power weighting function.

Next we prove that statement (ii) implies statement (i). Obviously statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference \succsim . We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n . To show that our additive functional in fact is a RDU form with power weighting function we use results presented in Wakker and Zank (2002). Wakker and Zank did not have the restrictions that $p_1 < 1$ and $p_n > 0$ but permitted any non-negative rank-ordered real numbers $x_i, i = 1, \dots, n$ because they worked in a setup with monetary outcomes instead of decumulative probabilities as we do here. But their results apply to our framework with minor modifications, in particular the restriction $p_1 \leq 1$ is not posing any difficulty. In their Lemma A2 they derived a similar additive representation as we have in Theorem 2, and then in their Lemma A3, using the analog of common ratio invariance for decumulative distributions, they showed that their additive representation in fact is a RDU form with common positive power function as “utility” and increasing “weighting function”. To apply their results we just need to interchange the roles of utility and weighting function. Further, because the functions $V_j, j = 1, \dots, n$ are proportional they can continuously be extended to 0 and 1 (this follows from Wakker 1993, Proposition 3.5). Hence, we can conclude that there exist positive numbers s_j such that

$$V_j(p_j) = s_j w(p_j),$$

with $w(p) = a + c(p)^b$, for some real a, b, c . Monotonicity and continuity imply that b, c are positive, and requiring further that $w(0) = 0$ and $w(1) = 1$ shows that $a = 0$ and $c = 1$. Hence, $w(p) = p^b$ is established. We define utility iteratively as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \dots, n$. Therefore, $V_j(p_j) = w(p_j)s_j = p_j^b[u(x_j) - u(x_{j-1})]$ for $j = 1, \dots, n$ with strictly monotonic utility u . We can conclude that the additive representation in Theorem 2 is RDU with a power weighting function and monotonic utility. Therefore statement (i) has been derived.

Uniqueness results follow from the joint cardinality of the functions V_j in Theorem 2, and the fact that they are proportional. These properties translate into the weighting function being unique because it assigns 0 to impossibility and 1 to certainty, and the utility being cardinal. This concludes the proof of Theorem 3. \square

PROOF OF THEOREM 4: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Extreme replacement separability follows from substitution of the RDU-functional with linear/exponential weighting function.

Next we prove that statement (ii) implies statement (i). As in the proof of Theorem 3, statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference \succsim . Attention is initially restricted to the case that $p_1 < 1$ and $p_n > 0$ to exclude unbounded V_1 and V_n . To show that this additive functional is RDU with an exponential weighting function we use results presented in Zank (2001). Zank did allow for non-negative vectors with rank-ordered monetary outcomes in his Lemma 7 instead of probabilities as we have here. However, those results apply to the case considered here if we interchange the roles of utility and decision weights. Hence, we can conclude that in the representation of

Theorem 2 the functions V_j are increasing exponential functions, i.e.,

$$V_j(p) = s_j[a \exp(cp) + b],$$

with $ac > 0$ and $s_j > 0$, and real b (or they are linear $V_j(p) = s_j[ap + b]$ with $a > 0$). As the functions are proportional, we can extend them continuously to all of $[0, 1]$ by Proposition 3.5 of Wakker (1993). We fix scale and location of the otherwise jointly cardinal V_j , i.e., $V_j(0) = 0, V_j(1) = 1$. Hence,

$$V_j(p) = s_j \left[\frac{e^{cp_j} - 1}{e^c - 1} \right],$$

with $c \neq 0$ (or $V_j(p) = s_j p$). We use the positive s_j 's to define utility as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \dots, n$. Therefore, the V_j 's are exponential or linear for $j = 1, \dots, n$ and u is strictly monotonic. Hence, statement (i) has been derived.

Uniqueness results follow by similar arguments as in the proof of Theorem 3. This concludes the proof of Theorem 4. □

PROOF OF THEOREM 5: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Common ratio invariance for cumulative distributions follows from substitution of the RDU-functional with dual power weighting function.

Next we prove that statement (ii) implies statement (i). Obviously statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference \succsim . We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n . To show that our additive functional is RDU with a dual power weighting function we use, similarly to the proof of Theorem 3, results of Wakker and Zank (2002). We define $W_j(1 - p_j) = V_j(1 - (1 - p_j)) (= V_j(p_j))$ for $j = 1, \dots, n$. These

functions are decreasing in $(1 - p_j)$ and they give an additive representation as we have in Theorem 2 but now on the set of cumulative distributions \tilde{L} . Using Lemma A3 of Wakker and Zank (2002) and common ratio invariance for cumulative distributions shows that this latter additive representation is in fact a RDU form with common positive power weighting function that is decreasing in cumulative probabilities. Further, because the functions $W_j, j = 1, \dots, n$ are proportional they can continuously be extended to 0 and 1 (this follows from Wakker 1993, Proposition 3.5). Hence, there exist positive numbers s_j such that

$$\begin{aligned} W_j(1 - p_j) &= s_j \tilde{w}(1 - p_j) \\ &= V_j(p_j), \end{aligned}$$

with $\tilde{w}(1 - p) = a - c(1 - p)^d$, for some real a, c, d . Monotonicity and continuity imply that c, d are positive, and requiring further that $\tilde{w}(0) = 0$ and $\tilde{w}(1) = 1$ shows that $a = 0$ and $c = 1$. Hence, $\tilde{w}(1 - p) = 1 - (1 - p)^d$ is established, and we define utility iteratively as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \dots, n$. Therefore, $V_j(p_j) = \tilde{w}(1 - p_j)s_j = \tilde{w}(1 - p_j)[u(x_j) - u(x_{j-1})]$ for $j = 1, \dots, n$ with strictly monotonic utility u . We can conclude that the additive representation in Theorem 2 is RDU with dual a power weighting function and monotonic utility. Therefore statement (i) has been derived.

Uniqueness results follow by similar arguments as in the proof of Theorem 3. This concludes the proof of Theorem 5. □

PROOF OF THEOREM 6: That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. For \succsim restricted to $L_{\tilde{p}}(\tilde{L}_{\tilde{p}})$, common ratio invariance comes down to common ratio invariance for decumulative (cumulative) distributions and can easily be derived by substitution of the specific RDU-like functional as discussed

following Theorem 6.

Next we prove that statement (ii) implies statement (i). Obviously statement (ii) in Theorem 2 is satisfied, hence, there exists an additively separable functional representing the preference \succsim . We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n . Similarly to the proof of Theorems 3 and 5, we use the results of Wakker and Zank (2002). The arguments used in the proof of Theorem 3 remain valid if we restrict the analysis to probability distributions in $L_{\hat{p}}$. We can conclude that the V_j 's obtained in Theorem 2 are proportional power functions for decumulative probabilities not exceeding \hat{p} . That is, there exist positive numbers s_j such that

$$V_j(p_j) = s_j w(p_j),$$

with $w(p) = cp^a$, for some positive a and c .

Similarly, the arguments used in the proof of Theorem 5 remain valid if we restrict the analysis to probability distributions in $\tilde{L}_{\hat{p}}$. We can conclude that the V_j 's obtained in Theorem 2 are proportional dual power functions for cumulative probabilities not exceeding $1 - \hat{p}$. That is, there exist positive numbers \hat{s}_j such that

$$V_j(p_j) = \hat{s}_j w(p_j),$$

with $w(p) = 1 - d(1 - p)^b$, for some positive d and b . Hence, statement (i) has been obtained.

Continuity at \hat{p} implies that the parameters are related through $c = 1/\hat{p}^a - d(1 - \hat{p})^b/\hat{p}^a$. Uniqueness results follow from the joint cardinality of the functions V_j in Theorem 2, and the fact that they are proportional. These properties translate into the weighting function being unique because it assigns 0 to impossibility and 1 to certainty, and that the s_j 's and \hat{s}_j 's can be replaced only if re-scaled by a common positive number t . This concludes the proof of Theorem 6. □

PROOF OF THEOREM 7: The proof follows from Theorem 6 and the following arguments.

Suppose that $I = \{1, \dots, i\}$ for some $1 < i < n$, and for given \hat{p} take $p > q$, and $\gamma < \delta$ such that $p_I(\gamma, \dots, \gamma) \sim \hat{p}_I(\delta, \dots, \delta)$, $\hat{p}_I(\gamma, \dots, \gamma) \sim q_I(\delta, \dots, \delta)$, and by consistency $p_{I \setminus \{i\}} \hat{p}_i(\gamma, \dots, \gamma) \sim \hat{p}_{I \setminus \{i\}} q_i(\delta, \dots, \delta)$. Then, taking the first and third indifference, substituting the functional form described in statement (i) of Theorem 6, and subtracting the two equations, we get

$$\hat{s}_i w(p) + s_i w(q) = \hat{s}_i w(\hat{p}) + s_i w(\hat{p}),$$

after cancelling common terms.

Similarly, taking the first and second indifference we get

$$\sum_{j=1}^i [\hat{s}_j w(p) + s_j w(q)] = \sum_{j=1}^i [\hat{s}_j w(\hat{p}) + s_j w(\hat{p})].$$

Therefore, for $i = 2$, we observe

$$\hat{s}_2 w(p) + s_2 w(q) = \hat{s}_2 w(\hat{p}) + s_2 w(\hat{p})$$

and

$$\hat{s}_1 w(p) + s_1 w(q) + \hat{s}_2 w(p) + s_2 w(q) = \hat{s}_1 w(\hat{p}) + s_1 w(\hat{p}) + \hat{s}_2 w(\hat{p}) + s_2 w(\hat{p}),$$

where, after substituting the first equation in the latter and cancellation of common terms, we get the equivalent equations

$$\hat{s}_2 w(p) + s_2 w(q) = \hat{s}_2 w(\hat{p}) + s_2 w(\hat{p}),$$

$$\hat{s}_1 w(p) + s_1 w(q) = \hat{s}_1 w(\hat{p}) + s_1 w(\hat{p}),$$

from which

$$\frac{\hat{s}_1}{s_1} = \frac{\hat{s}_2}{s_2}$$

follows. More generally it follows by induction on i that

$$\frac{\hat{s}_{i-1}}{s_{i-1}} = \frac{\hat{s}_i}{s_i}$$

holds for all $i = 2, \dots, n$.⁵ If one normalizes the positive s_i 's and \hat{s}_i 's such that they each sum to one, which can always be done, one observes that $s_i = \hat{s}_i$ must hold. Therefore RDU with a switch-power weighting function has been obtained. This shows that statement (ii) of the theorem implies statement (i). The proof that statement (i) implies statement (ii) follows immediate by substitution of the RDU-form with switch-powr weighting function, which completes the proof of the theorem. \square

References

- Abdellaoui, Mohammed (2000), "Parameter-free Elicitation of Utility and Probability Weighting Functions," *Management Science* 46, 1497–1512.
- Abdellaoui, Mohammed (2002), "A Genuine Rank-Dependent Generalization of the von Neumann-Morgenstern Expected Utility Theorem," *Econometrica* 70, 717–736.
- Abdellaoui, Mohammed, Frank Vossman and Martin Weber (2005), "Choice-Based Elicitation and Decomposition of Decision Weights for Gains and Losses Under Uncertainty," *Management Science* 51, 1384–1399.
- Aczél, János and R. Duncan Luce (2006), "A Behavioral Condition For Prelec's Weighting Function On The Positive Line Without Assuming $W(1) = 1$," forthcoming in *Journal of Mathematical Psychology*.

⁵To get $\hat{s}_{n-1}/s_{n-1} = \hat{s}_n/s_n$ one must use consistency with $I = \{n-1, n\}$.

- Allais, Maurice (1953), “Le Comportement de l’Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l’Ecole Américaine,” *Econometrica* 21, 503–546.
- al-Nowaihi, Ali and Sanjit Dhami (2006), “A Simple Derivation of Prelec’s Probability Weighting Function,” *Journal of Mathematical Psychology* 50, 521–524.
- Bell, David E. (1985), “Disappointment in Decision Making under Uncertainty,” *Operations Research* 33, 1–27.
- Bleichrodt, Han and Jose Luis Pinto (2000), “A Parameter-Free Elicitation of the Probability weighting Function in Medical Decision Analysis,” *Management Science* 46, 1485–1496.
- Birnbaum, Michael H. (2004), “Causes of Allais Common Consequence Paradoxes: An Experimental Dissection,” *Journal of Mathematical Psychology* 48, 87–106.
- Birnbaum, Michael H. and Juan B. Navarrete (1998), “Testing Descriptive Utility Theories: Violations of Stochastic Dominance and Cumulative Independence,” *Journal of Risk and Uncertainty* 17, 49–78.
- Camerer, Colin F. and Teck-Hua Ho (1994), “Violations of the Betweenness Axiom and Non-linearity in Probability,” *Journal of Risk and Uncertainty* 8, 167–196.
- Chateauneuf, Alain and Michèle Cohen (1994), “Risk Seeking with Diminishing Marginal Utility in a Non-Expected Utility Model,” *Journal of Risk and Uncertainty* 9, 77–91.
- Chateauneuf, Alain, Michèle Cohen and Isaac Meilijson (2004), “Four Notions of Mean-Preserving Increase in Risk, Risk Attitudes and Applications to the Rank-Dependent Expected Utility Model,” *Journal of Mathematical Economics* 40, 547–571.

- Chateauneuf, Alain and Peter P. Wakker (1993), “From Local to Global Additive Representation,” *Journal of Mathematical Economics* 22, 523–545.
- Chew, Soo Hong, Edi Karni, and Zvi Safra (1987), “Risk Aversion in the Theory of Expected Utility with Rank Dependent Probabilities,” *Journal of Economic Theory* 42, 370–381
- Chew, Soo Hong and William S. Waller (1986), “Empirical Tests of Weighted Utility Theory,” *Journal of Mathematical Psychology* 30, 55–72.
- Currim, Imran S. and Rakesh K. Sarin (1989), “Prospect Versus Utility,” *Management Science* 35, 22–41.
- Debreu, Gérard (1954), “Representation of a Preference Ordering by a Numerical Function.” In Robert M. Thrall, Clyde H. Coombs, & R.L. Davis (Eds), *Decision Processes* 159–165, Wiley, New York.
- Debreu, Gérard (1960), “Topological Methods in Cardinal Utility Theory.” In Kenneth J. Arrow, Samuel Karlin, & Patrick Suppes (1960, Eds), *Mathematical Methods in the Social Sciences*, 16–26, Stanford University Press, Stanford, CA.
- Etchart-Vincent, Nathalie (2004), “Is Probability Weighting Sensitive to the Magnitude of Consequences? An Experimental Investigation on Losses,” *Journal of Risk and Uncertainty* 28, 217–235.
- Goldstein, William M. and Hillel J. Einhorn (1987), “Expression Theory and the Preference Reversal Phenomena,” *Psychological Review* 94, 236–254.
- Gonzalez, Richard and George Wu (1999), “On the Shape of the Probability Weighting Function,” *Cognitive Psychology* 38, 129–166.

- Hey, John D. and Chris Orme (1994), “Investigating Generalizations of Expected Utility Theory Using Experimental Data,” *Econometrica* 62, 1291–1326.
- Karmarkar, Uday S. (1978), “Subjectively Weighted Utility: A Descriptive Extension of the Expected Utility Model,” *Organizational Behavior and Human Performance* 21, 61–72.
- Karmarkar, Uday S. (1979), “Subjectively Weighted Utility and the Allais Paradox,” *Organizational Behavior and Human Performance* 24, 67–72.
- Kilka, Michael and Martin Weber (2001), “What Determines the Shape of the Probability Weighting Function under Uncertainty?” *Management Science* 47, 1712–1726.
- Lattimore, Pamela M., Joanna R. Baker, and Ann D. Witte (1992), “The Influence of Probability on Risky Choice,” *Journal of Economic Behavior and Organization* 17, 377–400.
- Luce, R. Duncan (2000), “*Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches*.” Lawrence Erlbaum Publishers, London.
- Luce, Duncan R. (2001) “Reduction Invariance and Prelec’s Weighting Functions,” *Journal of Mathematical Psychology* 45, 167–179.
- Luce, R. Duncan, Barbara A. Mellers, and Shi-Jie Chang (1993), “Is Choice the Correct Primitive? On Using Certainty Equivalents and Reference Levels to Predict Choices among Gambles,” *Journal of Risk and Uncertainty* 6, 115–143.
- Luce, R. Duncan & Peter C. Fishburn (1991), “Rank- and Sign-Dependent Linear Utility Models for Finite First-Order Gambles,” *Journal of Risk and Uncertainty* 4, 29–59.
- MacCrimmon, Kenneth R. and Stig Larsson (1979), “Utility Theory: Axioms versus “Paradoxes”.” In Maurice Allais and Ole Hagen (Eds), *Expected Utility Hypotheses and the*

- Allais Paradox*, 333–409, Reidel, Dordrecht, The Netherlands.
- Machina, Mark J. (1989), “Dynamic Consistency and Non-Expected Utility Models of Choice Under Uncertainty,” *Journal of Economic Literature* 27, 1622–1688.
- Marley, Anthony A.J. & R. Duncan Luce (2005), “Independence Properties vis-à-vis Several Utility Representations,” *Theory and Decision* 58, 77–143.
- Nakamura, Yutaka (1995) “Rank Dependent Utility for Arbitrary Consequence Spaces,” *Mathematical Social Science* 29, 103–129.
- Narens, Louis (1996), “A Theory of Ratio Magnitude Estimation,” *Journal of Mathematical Psychology* 40, 109–129.
- Prelec, Drazen (1998), “The Probability Weighting Function,” *Econometrica* 66, 497–527.
- Quiggin, John (1981), “Risk Perception and Risk Aversion among Australian Farmers,” *Australian Journal of Agricultural Economics* 25, 160–169.
- Quiggin, John (1982), “A Theory of Anticipated Utility,” *Journal of Economic Behaviour and Organization* 3, 323–343.
- Röell, Ailsa (1987), “Risk Aversion in Quiggin and Yaari’s Rank-Order Model of Choice under Uncertainty,” *Economic Journal* 97 (supplement), 143–160.
- Safra, Zvi and Uzi Segal (1998), “Constant Risk Aversion,” *Journal of Economic Theory* 83, 19–42.
- Stott, Henry P. (2006), “Choosing From Cumulative Prospect Theory’s Functional Menagerie.” *Journal of Risk and Uncertainty* 32, 101–130.

- Tversky, Amos and Craig R. Fox (1995), “Weighing Risk and Uncertainty,” *Psychological Review* 102, 269–283.
- Tversky, Amos and Daniel Kahneman (1992), “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty* 5, 297–323.
- Tversky, Amos and Peter P. Wakker, (1995), “Risk Attitudes and Decision Weights,” *Econometrica* 63, 1255–1280.
- von Neumann, John and Oskar Morgenstern (1944, 1947, 1953), “*Theory of Games and Economic Behavior*.” Princeton University Press, Princeton NJ.
- Wakker, Peter P. (1993), “Additive Representations on Rank-Ordered Sets II. The Topological Approach,” *Journal of Mathematical Economics* 22, 1–26.
- Wakker, Peter P. (1994) “Separating Marginal Utility and Probabilistic Risk Aversion,” *Theory and Decision* 36, 1–44.
- Wakker, Peter P. (2001), “Testing and Characterizing Properties of Nonadditive Measures through Violations of the Sure-Thing Principle,” *Econometrica* 69, 1039–1059.
- Wakker, Peter P., Ido Erev and Elke U. Weber (1994), “Comonotonic Independence: The Critical Test between Classical and Rank-Dependent Utility Theories,” *Journal of Risk and Uncertainty* 9, 195–230.
- Wakker Peter P. and Horst Zank (2002), “A Simple Preference-Foundation of Cumulative Prospect Theory with Power Utility,” *European Economic Review* 46, 1253–1271.
- Wu, George (1994), “An Empirical Test of Ordinal Independence,” *Journal of Risk and Uncertainty* 9, 39–60.

- Wu, George and Richard Gonzalez (1996), “Curvature of the Probability Weighting Function,” *Management Science* 42, 1676–1690.
- Yaari, Menahem E. (1987) “The Dual Theory of Choice under Risk,” *Econometrica* 55, 95–115.
- Zank, Horst (2001), “Cumulative Prospect Theory for Parametric and Multiattribute Utilities,” *Mathematics of Operations Research* 26, No. 1, 67–81.
- Zank, Horst (2004), “Deriving Rank-Dependent Utility Through Probabilistic Consistency,” *Economics Working Paper*, University of Manchester.