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On oscillations of the geometric Brownian motion with time delayed drift*

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Abstract

The geometric Brownian motion is the solution of a linear stochastic differential equation in the Itô-sense. If one adds to the drift term a possible nonlinear time delayed term and starts with a nonnegative initial process then the process generated in this way, may hit zero and may oscillate around zero infinitely often depending on properties of both drift terms and the diffusion constant.

Key words. geometric Brownian motion, stochastic delay, differential equations, oscillations.
AMS-Classifications. 60H10, 34K50, 93E03 .

1 Introduction

Assume $W = (W(t), t \geq 0)$ is a real valued standard Wiener process and U_0 is a real random variable both defined on some probability space $(\Omega, \mathcal{A}, P)$. Let U_0 be independent of W. Denote by a and σ two real numbers with $\sigma > 0$.
The geometric Brownian motion $U = (U(t), t \geq 0)$ with drift coefficient a and diffusion coefficient σ is defined to be the solution of the stochastic differential equation

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$$dU(t) = aU(t)dt + \sigma U(t)dW(t) \tag{1}$$

with $U(0) = U_0$ as initial condition. The solution is given by

$$U(t) = U_o \exp[\sigma W(t) + (a - \frac{\sigma^2}{2})t] \tag{2}$$

what can be easily affirmed by Itô's formula.
Many properties of U can be derived from this formula, we mention here only its (obvious) positiveness if U_0 is positive.
A complete other picture can appear if we add to the drift term a time delayed drift:

$$dX(t) = \{aX(t) + f(X(t-r))\}dt + \sigma X(t)dW(t) \tag{3}$$

where F is a measurable real function on R, and r is a positive constant.

To ensure the uniqueness of the solution we establish the initial condition

$$X(t) = X_0(t), \quad t \in [-r, 0], \tag{4}$$

where X_0 is a continuous process on $(\Omega, \mathcal{A}, P)$ being independent of W.

Principally the solution can be obtained step by step by solving the (ordinary) stochastic differential equation:

$$X(t) = \\ X(nr) + a\int_{nr}^{t} X(s)ds + \int_{nr}^{t} f(X(s-r))ds + \sigma\int_{nr}^{t} X(s)dW(s) \quad t \in [nr, (n+1)r], \tag{5}$$

$n \geq 0$. By using the process U from (2) the equation (5) yields

$$X(t) = U(t)\left[\frac{X(nr)}{U(nr)} + \int_{nr}^{t} \frac{f(X(s-r))}{U(s)}ds\right], \; t \in [nr, (n+1)r], \; n \geq 0. \tag{6}$$

Once again use Itô's formula to show that $X(.)$ given by (6) and the initial condition (4) solves equation (3). The uniquenes of the solution of (3) and (4) provided by (6) follows from a Gronwall-type argument, compare for example Mao(1997), Chapter 5.
From (6) it is clear that $X(t)$ remains positive for every $t \geq 0$ as long as the initial condition $X_0(.)$ is nonnegative but not identical zero and f is positive on $(0, \infty)$.

For other functions f this property does not keep in force.
In this note we will show that under the condition (A) on a, σ and f formulated below for any positive r and any nonnegative and not identical zero initial condition $X_0(s), s \in [-r, 0]$, the solution $X(.)$ of (3) P-a.s. oscillates around zero infinitely often if the time t increases unboundedly. Moreover, the set of zeros of P-almost every trajectory $X(.)$ turns out to be a countable infinite set, say $\{\sigma_k, k \geq 1\}$, with $\sigma_{k+1} - \sigma_k \geq r, k \geq 1$, see the Theorem and the Corollary in Section 2.

Condition (A): We say Condition (A) is satisfied, if at least one of the following two conditions hold.

(A1) $a \leq \frac{\sigma^2}{2}$, f is nonincreasing on R and there exist reals δ, b with $\delta > 0, b < 0$ such that

$$x^{-1} f(x) \leq b \tag{7}$$

for all $x \neq 0$ satisfying $|x| < \delta$,

(A2) $a \in R$, f is nonincreasing on R and there exists a negative number b such that (7) holds for all $x \neq 0$.
Note that (7) is equivalent to $f(x) \leq bx < 0$ for $x > 0$ and to $f(x) \geq bx > 0$ for $x < 0$.

Example: If $c < 0$ then both $f(x) = c \text{ sgn } (x)$ and $f(x) = c \tan h(x)$ satisfy (A1) but not (A2).

Our proof is a probabilistic one. For $f(x) = b\,x$ the result was recently shown by Appleby, Buckwar [1] who used functional theoretic arguments.

2 Results

We use the notations and assumptions of Chapter 1.

Let $U = (U(t), t \geq 0)$ be the geometric Brownian motion defined by

$$\left.\begin{aligned} dU(t) &= aX(t)dt + \sigma U(t)dW(t)\,, \quad t \geq 0 \\ U(0) &= 1. \end{aligned}\right\} \tag{8}$$

Put $Y(t) = \frac{X(t)}{U(t)}$, $t \geq 0$. By Itô's formula it follows

$$\left.\begin{aligned} dY(t) &= \frac{f(X(t-r))}{U(t))}\, dt \quad , \quad t \geq 0, \\ Y(0) &= X(0). \end{aligned}\right\} \tag{9}$$

If we define $U(t) = 1, Y(t) = X_0(t)$ for $-r \le t < 0$, then the process $Y = (Y(t), \quad t \ge -r)$ satisfies

$$\left.\begin{aligned} dY(t) &= \frac{f(Y(t-r)U(t-r))}{U(t)}\,dt, \quad t \ge 0, \\ Y(t) &= X_0(t), \qquad\qquad\qquad t \in [-r, 0]. \end{aligned}\right\} \quad (10)$$

It is clear that the equation (10) has a unique solution (for every ω) and can be solved step by step on the intervals $[nr, (n+1)r], n = 0, 1, 2, ...$ This solution is a continuously differentiable on $(0, \infty)$ function.
Conversely, if $Y = (Y(t), t \ge -r)$ is a solution of (10), then the process $X(t) := Y(t)U(t), t \ge -r$ (where $U(t) = 1$ for $t < 0$) is a solution of (3) due to Itô's formula.

Theorem *Assume that $(X(t), t \ge -r)$ is a solution of (3), (4) with the nonnegative initial condition $X_0(t) \ge 0, t \in [-r, 0]$ as well as a, σ, f satisfy Condition (A). Put $\tau := \inf\{t \ge 0 | X(t) = 0\}$. Then $P(\tau < \infty) = 1$.*

Proof. Observe that $\tau = \inf\{t \ge 0 : Y(t) = 0\}$, where $Y(.)$ denotes the solution of (10).
We have $\tau(\omega) = 0$ if and only if $X_0(0, \omega) = 0$. If $\tau(\omega) > 0$ then $X_0(0, \omega) > 0$ and the function $Y(t, \omega)$ is nonincreasing and strictly positive on $[0, \tau(\omega)[$ due to (10). In particular, on the set $\{\tau = \infty\}$ we have $X(t) > 0$ for all $t \ge 0$ and, by assuming *Condition A(i)*,

$$\limsup_{t\to\infty} X(t) \le X_0(0) \limsup_{t\to\infty} U(t) = 0 \qquad \text{a.s.} \quad (11)$$

Let $\tau_0 = 0$ and define recursively

$$\tau_{n+1} = \begin{cases} \infty \quad , \quad \text{if } \tau_n = \infty \text{ or } \tau \le \tau_n + r, \\ \inf\{t > \tau_n + r : X(t) = \inf\limits_{\tau_n \le s \le \tau_n + r} X(s)\}, \text{ otherwise.} \end{cases}$$

It is clear that $\tau_1, \tau_2, ..., \tau_n, ...$ are stopping times, $\tau_{n+1} \ge \tau_n + r$ on the set $\{\tau_n < \infty\}, n = 0, 1, 2, ...$ Moreover $\tau_{n+1} < \infty$ a.s. on the set $\{\tau_n < \infty, \tau > \tau_n + r\}$; $n = 0, 1, 2, ...$ Indeed, on this set $\inf\limits_{\tau_n \le s \le \tau_n + r} X(s) > 0$, and the finiteness of τ_{n+1} follows from (7). Therefore,

$$\{\tau = \infty\} = \bigcap_{n=1}^{\infty} \{\tau_n < \infty\}. \quad a.s. \quad (12)$$

Let n, m be integers with $1 \le m \le n$ and choose any ε from $(0, \delta)$. It follows from the definition of τ_n that $\tau_n \ge r$. On the set $\{\tau_n < \infty\}$ we have $\tau > \tau_n$ and $0 < X(\tau_n) \le X(s)$ for all $s \in [\tau_n - r, \tau_n]$. Therefore on $\{\tau_n < \infty, X(\tau_m) < \varepsilon\}$ due to (8) as well as assumption A(1)we get $X(\tau_n) \le X(\tau_m)$ and

$$
\begin{aligned}
Y(\tau_n + r) &= Y(\tau_n) + \int_{\tau_n}^{\tau_n + r} \frac{f(X(s-r))}{U(s)} ds \leq \\
&\leq Y(\tau_n) + \int_{\tau_n}^{\tau_n + r} \frac{f(X(\tau_n))}{U(\tau_n)} \frac{U(\tau_n)}{U(s)} ds \leq Y(\tau_n)(1 + b\, U_n) \\
\text{with} \quad U_n &= \int_{\tau_n}^{\tau_n + r} \mathrm{e}^{(\frac{\sigma^2}{2} - a)(s - \tau_n)} \exp\{\sigma(W(\tau_n) - W(s))\}\, ds.
\end{aligned}
$$

Consequently, we obtain on $\{\tau_n < \infty, X(\tau_m) < \varepsilon\}$
$Y(\tau_n + r) \leq 0$ if $U_n \geq |b|^{-1}$.

In other words

$$
P(\tau_n < \infty, X(\tau_m) < \varepsilon, \tau \leq \tau_n + r) \geq P(\tau_n < \infty, X(\tau_m) < \varepsilon, U_n \geq |b|^{-1})
$$

Due to the strong Markov property of W, the events $\{\tau_n < \infty, X(\tau_m) < \varepsilon\}$ and $\{U_n \geq |b|^{-1}\}$ are independent. We obtain

$$
\begin{aligned}
&P(\tau_{n+1} < \infty, X(\tau_m) < \varepsilon) = P(\tau_n < \infty, \tau > \tau_n + r, X(\tau_m) < \varepsilon) = \\
&= P(\tau_n < \infty, X(\tau_m) < \varepsilon) - P(\tau_n < \infty, X(\tau_m) < \varepsilon, \tau \leq \tau_n + r) \\
&\leq P(\tau_n < \infty, X(\tau_m) < \varepsilon)(1 - p)
\end{aligned}
$$

with

$$
p = P\left(\int_0^r \exp\left((\frac{\sigma^2}{2} - a)s - \sigma W(s) \right) ds \geq |b|^{-1} \right) > 0.
$$

Hence
$\lim_{n \to \infty} P(\tau_n < \infty, X(\tau_m) < \varepsilon) = 0$ and therefore $P(\tau = \infty, X(\tau_m) < \varepsilon) = 0$ for every $m \geq 1$.
Consequently, because of $\lim_{m \to \infty} X(\tau_m) = 0$ on $\{\tau = \infty\}$, we obtain $P(\tau = \infty) = 0$.

Now suppose that Condition A(2) holds.
Then instead of (11) we get

$$
\limsup_{t \to \infty} \mathrm{e}^{-at} X(t) \leq X_0(0) \limsup_{t \to \infty} V(t) = 0
$$

with $V(t) = \exp[\sigma W(t) \frac{\sigma^2}{2} t]$. Now we define $\tilde{X}(t) = \mathrm{e}^{-at} X(t), t \geq 0$ and $(\tilde{\tau}_n, n \geq 0)$ related to $\tilde{X}(.)$ as above $(\tau_n, n \geq 0)$ related to $X(.)$.

Then, analogously, it holds

$$
\{\tau = \infty\} = \bigcap_{n=1}^{\infty} \{\tilde{\tau}_n < \infty\}.
$$

We obtain on the set $\{\tilde{\tau}_n < \infty\}$ the inequalities $0 < \tilde{X}(\tau_n) \leq \tilde{X}(s),\ s \in [\tilde{\tau}_n - r, \tilde{\tau}_n]$ and therefore by assumption A(ii)

$$
\begin{aligned}
&Y(\tilde{\tau}_n + r) = Y(\tilde{\tau}_n) + \int_{\tilde{\tau}_n}^{\tilde{\tau}_n + r} \frac{f(X)(s-r))}{U(s)}\, ds \leq \\
&Y(\tilde{\tau}_n) + b\,\mathrm{e}^{-ar} \frac{\tilde{X}(\tilde{\tau}_n)}{V(\tilde{\tau}_n)} \int_{\tilde{\tau}_n}^{\tilde{\tau}_n + r} \mathrm{e}^{as} \frac{V(\tau_n)}{U(s)}\, ds \\
&= Y(\tilde{\tau}_n) + b\,\mathrm{e}^{-ar} \frac{X(\tilde{\tau}_n)}{U(\tilde{\tau}_n)} \int_{\tilde{\tau}_n}^{\tilde{\tau}_n + r} \frac{V(\tilde{\tau}_n)}{V(s)}\, ds \\
&= Y(\tilde{\tau}_n)(1 + b\,\mathrm{e}^{-ar} V_n) \\
&\text{with} \\
&V_n = \int_{\tilde{\tau}_n}^{\tilde{\tau}_n + r} \exp\Big(\sigma(W(\tilde{\tau}_n) - W(s) - \frac{\sigma^2}{2}(s - \tilde{\tau}_n)\Big) ds
\end{aligned}
$$

Now the proof can be finished for this case as under Condition A(1) above.

Remarks: (i) Note that we have used (7) for positive x only.
(ii) If $a > \frac{\sigma^2}{2}$ and (7) does not hold for every $x > 0$ then the Theorem fails. Indeed, if $f(x) = b\ \mathrm{sgn}\ (x)$ ($b < 0$ fixed, $x > 0$) then $P(\tau < \infty) < 1$.

Now assume that $X_0(t) \geq 0,\ t \in (-r, 0]$, and $X_0(t) > 0, t \in (-r', 0]$, for some $r' \in (0, r]$ and $X_0(t) = 0$ on $[-r, -r']$ if $r' < r$. Then $\tau > 0$, and $Y'(t) < 0$ on $[\tau, \tau + r[$ by (6). Hence $Y(t)$ and $X(t)$ are strictly negative on $]\tau, \tau + r[$. Now let $\tilde{X}(t) = -X(\tau + r + t)$.Then $\tilde{X}$ satisfies the equation

$$d\tilde{X}(t) = \{a\tilde{X}(t) + h(\tilde{X}(t-r))\}dt + \sigma\tilde{X}(t)d\tilde{W}(t)$$

with the initial condition $\tilde{X}_0(t) = -X(\tau + r + t), -r \leq t \leq 0$, the Wiener process $\tilde{W}(t) = W(\tau + r + t) - W(\tau + r)$ which is independent of $\tilde{X}_0$, as well as $h(x) = -f(-x), x \in R$.
If f satisfies Assumption (A) then h does it also and thus by the Theorem proved above $\tilde{X}$ hits zero with probability one, etc.
Thus we have proved the

Corollary: *Let the assumptions of the Theorem be valid and $X_0(t) > 0$ on $(-r', 0]$ for some $r' \in (0, r]$ as well as $X_0(t) = 0$ on $[-r, -r']$ if $r' < r$. . Then the stopping times $\sigma_n, n \geq 1$, defined by*

$$\sigma_{n+1} = \inf\{t > \sigma_n : X(t) = 0\},\ n \geq 0$$

with $\sigma_0 \equiv 0$, are P-a.s. finite. Moreover, it holds

$$\{t > 0 \,|\, X(t) = 0\} = \{\sigma_n, n \geq 1\}$$

$\sigma_{n+1} \geq \sigma_n + r, \quad n \geq 1$, *as well as* $X(t) > 0$ *on* $(\sigma_{2n}, \sigma_{2n+1})$ *and* $X(t) < 0$ *on* $(\sigma_{2n+1}, \sigma_{2n+2}), n \geq 0$.

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