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On Markovian Short Rates in Term Structure Models Driven by Jump-Diffusion Processes

P. V. Gapeev and U. Küchler*

We study a bond market model and related term structure of interest rates where prices of zero coupon bonds are driven by a jump-diffusion process. We present a criterion on the deterministic forward rate volatilities under which the short rate process is Markovian and give sufficient conditions on the bond price volatility structure depending on the short rate for existing a finite-dimensional Markovian realization of the term structure model.

1 Introduction

The question of the short rate having the Markov property has attracted a considerable attention in the literature on term structure of interest rates. The reason for that is the simplification of pricing formulas for bonds and derivatives in the underlying bond market in this case. If the bond market is driven by Wiener processes the mentioned problem was studied by Carverhill [6], who proved that the short rate process is Markovian within the Heath-Jarrow-Morton framework with a deterministic volatility function if and only if this volatility factorizes into a product of two functions only depending on the actual time and maturity time, respectively. A more general question of interest is the problem under which conditions there exists a multidimensional Markov process (a so-called state process) having the short rate process as one of its components such that the mentioned prices depend on this Markov process only. In this case the Markov process is said to be a *finite-dimensional Markovian realization* of the considered term structure model. Finite-dimensional realizations for bond markets driven by Wiener processes have been studied e.g. in Jeffrey [15], Ritchken and Sankarasubramanian [20], Bhar and Chiarella [1], Inui and Kijima [13], Chiarella and Kwon [7], Björk and Gombani [2], Björk and Svenson [5].

During the nineties term structure models driven by processes with jumps have been investigated in detail. Shirakawa [21] studied a model driven by both a Wiener and Poisson process with stationary forward rate volatilities. A general approach was proposed by Björk, Kabanov

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and Runggaldier [3], who supposed the driving stochastic process to be a sum of a diffusion and a marked point process having finitely many jumps during every finite time interval. Björk et al. [4] introduced a general jump-diffusion bond market model. A model of another type was introduced by Eberlein and Raible [10] (see also Eberlein [9]). They started with a Lévy process and generalized Carverhill's result to that case under an additional assumption on the characteristic function of the marginal distribution. Küchler and Naumann [17] extended their result to all Lévy processes possible in the Eberlein-Raible model and included new examples like the class of bilateral gamma processes which contains variance gamma processes playing a role in recent discussions of stochastic models in financial markets (see e.g. Madan and Seneta [19] and Madan [18]). Björk and Gombani [2] found necessary and sufficient conditions for existing a finite-dimensional realization for the jump-diffusion forward rate having a deterministic volatility structure.

In the present paper we investigate a jump-diffusion bond market model and the corresponding term structure of interest rates. We study the questions of the short rate being Markovian in the case of deterministic bond volatility structure and the existence of a finite-dimensional Markovian realization for the term structure in the case where the bond price volatilities are random and depend on the short rate.

The paper is organized as follows. In Section 2 we introduce a jump-diffusion bond market model and the corresponding term structure of interest rates which is a slight modification of models in [3] and [4]. We present the relations between the bond prices, the instantaneous forward rates, as well as the short rate process under a martingale measure and define the notions of state variables and finite-dimensional Markovian realizations.

In Section 3 we prove that under deterministic volatility structure the short rate is Markovian if and only if the forward rate volatilities of both the continuous and jump part decompose into products of two factors and have the common factor depending on the time to maturity (the latter was not remarked in Proposition 2.2 and the formula (2.4) in [6]). The proof of this fact is based on the extensions of arguments in [6], [10], [16] and [17] to our jump-diffusion model. We also mention the possible generalizations of the result to the case where the bond price is driven by both multidimensional Wiener process and Poisson random measure and describe the analogues of Vasiček and Ho-Lee model in the jump-diffusion case.

In Section 4 we consider the case where the bond price volatilities depend on the short rate. We prove that if the volatility of the continuous part satisfies the multiplicative type condition (2.11) in [20] and the volatility of the jump part satisfies an additive type condition, then the short rate is Markovian with finite-dimensional state space and the bond price admits a finite-dimensional Markovian realization. For proof of this fact we use the Markov property of unique (strong) solutions of multidimensional stochastic differential equations. We also show that if we replace the additive type condition for the volatility of the jump part by the multiplicative type condition (2.11) in [20], then we obtain an infinite-dimensional state space for the short rate process.

2 The jump-diffusion bond market model

In this section, following [3], we define the basic objects of the bond market model driven by both a Wiener process and a Poisson random measure. For the definition of notions from

stochastic analysis we refer to [14].

2.1. Suppose that on some complete stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T^*]}, Q)$ with a fixed time horizon $T^* > 0$ there exist a standard Wiener process $W = (W_t)_{t\in[0,T^*]}$ and a homogeneous Poisson random measure $\mu(dt, dx)$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ with the intensity measure (compensator) $\nu(dt, dx) = dt \otimes F(dx)$ where $F(dx)$ is a positive σ-finite measure on $\mathcal{B}(\mathbb{R})$. Let W and μ be independent and $(\mathcal{F}_t)_{t\in[0,T^*]}$ be their natural filtration, i.e. $\mathcal{F}_t = \sigma\{W_s, \mu([0,s] \times A) \,|\, s \in [0,t], A \in \mathcal{B}(\mathbb{R})\}$ for all $t \in [0,T^*]$.

Let us consider a *term structure of bond prices* $\{P(t,T) \,|\, 0 \le t \le T \le T^*\}$ where the (positive) process $P = (P(t,T))_{t\in[0,T]}$ satisfying the normalization condition:

$$P(T,T) = 1 \tag{2.1}$$

denotes the price of a zero coupon bond at time t maturing at time T for each $T \in [0,T^*]$. Let us suppose that for all but fixed $T \in [0,T^*]$ the logarithm of the bond price process $P = (P(t,T))_{t\in[0,T]}$ is given by the expression:

$$\log P(t,T) = \log P(0,T) + \int_0^t \alpha(s,T)\, ds + \int_0^t \sigma(s,T)\, dW_s + \int_0^t \int \delta(s,T,x)\, \mu(ds,dx) \tag{2.2}$$

where

$$\int_0^T \left(|\alpha(t,T)| + \sigma^2(t,T) + \int |\delta(t,T,x)|\, F(dx) \right) dt < \infty \quad (Q - \text{a.s.}), \tag{2.3}$$

$\sigma(t,T)$ is measurable with respect to the predictable σ-algebra $\mathcal{P}$ on the initial stochastic base, and $\delta(t,T,x)$ is measurable with respect to $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ (we denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$). It is assumed that for fixed $\omega \in \Omega$ and $x \in \mathbb{R}$ the functions $\sigma(t,T)$ and $\delta(t,T,x)$ defined on the triangle $\{(t,T) \,|\, 0 \le t \le T \le T^*\}$ are twice continuously differentiable in the second variable and satisfy the condition:

$$\sigma(T,T) = \delta(T,T,x) = 0 \tag{2.4}$$

for all $x \in \mathbb{R}$ and $T \in [0,T^*]$, and the function $\alpha(t,T)$ will be specified below. We will also suppose that we are allowed (by the regularity of the functions) to differentiate under the integral sign, to interchange limits and integrals, as well as to interchange the order of integration and differentiation.

Assuming that for fixed $t \in [0,T]$ the bond price $P(t,T)$ is (Q-a.s.) continuously differentiable in the variable T on $[0,T^*]$, let us introduce the corresponding *term structure of interest rates* $\{f(t,T) \,|\, 0 \le t \le T \le T^*\}$ where:

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T} \tag{2.5}$$

is the *instantaneous forward rate* contracted at time t for maturity T. On the other hand, integrating equation (2.5) and using condition (2.1), we get:

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\, du \right) \tag{2.6}$$

for all $0 \leq t \leq T \leq T^*$, and hence, we see the one-to-one correspondence between the bond prices and the forward rates. Let us also define the *short rate* process $r = (r(t))_{t\in[0,T^*]}$ by:

$$r(t) = f(t,t) \tag{2.7}$$

being the forward rate at time t for maturity t, and the associated with it *money account* process $B = (B(t))_{t\in[0,T^*]}$ by:

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right) \tag{2.8}$$

playing the role of *numéraire* in the model. Then taking:

$$\alpha(t,T) = r(t) - \frac{1}{2}\sigma^2(t,T) - \int \left(e^{\delta(t,T,x)} - 1\right) F(dx) \tag{2.9}$$

for all $t \in [0,T]$ and assuming that the condition:

$$E\left[\exp\left(\int_0^T \left(\frac{1}{2}\sigma^2(t,T) + \int \left(e^{\delta(t,T,x)} - 1\right) F(dx)\right) dt\right)\right] < \infty \tag{2.10}$$

is satisfied for every $T \in [0,T^*]$, by means of the arguments in [14; Chapter II, Section 2], we conclude that the *discounted bond price* process $(P(t,T)/B(t))_{t\in[0,T]}$ forms an $(\mathcal{F}_t, Q)$ -martingale (see [4; Section 5]).

Therefore, using the expression (2.9) for $\alpha(t,T)$, we get that under the measure Q and for each $T \in [0,T^*]$ the logarithm of the bond price (2.2) admits the representation:

$$\begin{aligned}\log P(t,T) =& \log P(0,T) + \int_0^t r(s)\,ds - \frac{1}{2}\int_0^t \sigma^2(s,T)\,ds + \int_0^t \sigma(s,T)\,dW_s \\ &- \int_0^t \int \left(e^{\delta(s,T,x)} - 1\right) F(dx)\,ds + \int_0^t \int \delta(s,T,x)\,\mu(ds,dx),\end{aligned} \tag{2.11}$$

the forward rate process (2.5) takes the expression:

$$\begin{aligned}f(t,T) =& f(0,T) + \int_0^t \sigma_T'(s,T)\sigma(s,T)\,ds - \int_0^t \sigma_T'(s,T)\,dW_s \\ &+ \int_0^t \int e^{\delta(s,T,x)}\delta_T'(s,T,x)\,F(dx)\,ds - \int_0^t \int \delta_T'(s,T,x)\,\mu(ds,dx)\end{aligned} \tag{2.12}$$

(the subscript T by a derivative denotes partial differentiation with respect to the second variable), as well as the short rate process (2.7) is given by:

$$\begin{aligned}r(t) =& f(0,t) + \int_0^t \sigma_T'(s,t)\sigma(s,t)\,ds - \int_0^t \sigma_T'(s,t)\,dW_s \\ &+ \int_0^t \int e^{\delta(s,t,x)}\delta_T'(s,t,x)\,F(dx)\,ds - \int_0^t \int \delta_T'(s,t,x)\,\mu(ds,dx)\end{aligned} \tag{2.13}$$

and, by means of (2.4), satisfies the equation:

$$dr(t) = f_T'(t,t)\,dt - \sigma_T'(t,t)\,dW_t - \int \delta_T'(t,t,x)\,\mu(dt,dx) \tag{2.14}$$

where

$$f'_T(t,t) = f'_T(0,t) + \int_0^t [\sigma''_{TT}(s,t)\sigma(s,t) + (\sigma'_T(s,t))^2]\, ds - \int_0^t \sigma''_{TT}(s,t)\, dW_s \tag{2.15}$$
$$+ \int_0^t \int e^{\delta(s,t,x)} [\delta''_{TT}(s,t,x) + (\delta'_T(s,t,x))^2]\, F(dx)\, ds - \int_0^t \int \delta''_{TT}(s,t,x)\, \mu(ds,dx)$$

and all the integrals in (2.12) - (2.15) are well-defined by virtue of the assumption (2.10) and the allowance to differentiate under the integral sign.

If in the Heath-Jarrow-Morton approach (see [12]) one starts with the specification of the forward rates (2.12), the discounted bond prices turn out to be $(\mathcal{F}_t, Q)$ - martingales, or in other words, Q is a *martingale measure*. In this case, integrating expression (2.12), we easily get the following representation for the bond price (2.6):

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\Bigg(- \int_t^T \int_0^t \sigma'_T(s,u)\sigma(s,u)\, ds\, du + \int_t^T \int_0^t \sigma'_T(s,u)\, dW_s\, du \tag{2.16}$$
$$- \int_t^T \int_0^t \int e^{\delta(s,u,x)} \delta'_T(s,u,x)\, F(dx)\, ds\, du + \int_t^T \int_0^t \int \delta'_T(s,u,x)\, \mu(ds,dx)\, du \Bigg)$$

where by means of assumptions (2.4), we have:

$$\sigma(t,T) = \int_t^T \sigma'_T(t,v)\, dv \quad \text{and} \quad \delta(t,T,x) = \int_t^T \delta'_T(t,v,x)\, dv \tag{2.17}$$

for all $0 \le t \le T \le T^*$.

2.2. Now let us describe the place of our model defined in (2.11) - (2.17) in the wide spectrum of bond market models considered in the literature. If $\delta(t,T,x) = 0$ for all $0 \le t \le T \le T^*$ and $x \in \mathbb{R}$, then we receive the term structure model introduced by Heath, Jarrow and Morton [12]. In the case when $F(dx) = \lambda I\{x = 1\}dx$ for some $\lambda > 0$ we get the model of Shirakawa [21]. If $F(\mathbb{R}) < \infty$ then we have a particular case of the model considered by Björk, Kabanov and Runggaldier [3] where a more general driving marked point process was considered. If $\sigma(t,T)$ and $\delta(t,T,x)$ admit the representations:

$$\sigma(t,T) = \int_t^T \beta(t,s)\, ds \quad \text{and} \quad \delta(t,T,x) = \int_t^T \gamma(t,s)\, ds\, x \tag{2.18}$$

with deterministic continuously differentiable functions $\beta(t,T)$ and $\gamma(t,T)$ such that $\beta(t,T)$ or $\gamma(t,T)$ is identical zero or $\beta(t,T) = c\,\gamma(t,T)$ for some constant $c \in \mathbb{R}$ and all $0 \le t \le T \le T^*$, then we obtain the model proposed by Eberlein and Raible [10]. A more general jump-diffusion model than that defined in (2.11) - (2.17) was considered in [4]. In our case we impose conditions (2.3) and (2.10) in order to avoid some technical difficulties.

2.3. Since under conditions (2.9) and (2.10) the process $(P(t,T)/B(t))_{t\in[0,T]}$ turns out to be an $(\mathcal{F}_t, Q)$ - martingale, by means of (2.8), it is easily shown that the bond price $P = (P(t,T))_{t\in[0,T]}$ can be computed as:

$$P(t,T) = E\left[\exp\left(- \int_t^T r(s)\, ds \right) \Big| \mathcal{F}_t \right]. \tag{2.19}$$

Observe that if the short rate $r = (r(t))_{t\in[0,T^*]}$ is an $(\mathcal{F}_t, Q)$ - Markov process, then (2.19) admits the representation:

$$P(t,T) = E\left[\exp\left(-\int_t^T r(s)\,ds\right)\Big|\, r(t)\right], \tag{2.20}$$

hence $P(t,T) = H(t, r(t), T)$, $0 \le t \le T \le T^*$, for some measurable function H defined on $[0,T]\times\mathbb{R}\times[0,T^*]$, and we see that the bond price can be evaluated by means of the Markovian short rate.

In general, the process $r = (r(t))_{t\in[0,T^*]}$ is possibly non-Markovian, but in some cases there exists a finite number of processes $(r_i(t))_{t\in[0,T^*]}$, $i = 1,\dots,n$, $n \in \mathbb{N}$, such that the process $(r(t), r_1(t),\dots,r_n(t))_{t\in[0,T^*]}$ is $(\mathcal{F}_t, Q)$ - Markovian. It can be e.g. if the process $(r(t), r_1(t),\dots,r_n(t))_{t\in[0,T^*]}$ satisfy a multidimensional stochastic differential equation, and then the processes $(r_i(t))_{t\in[0,T^*]}$, $i = 1,\dots,n$, are called *state variables* of the short rate $r = (r(t))_{t\in[0,T^*]}$. In this case the bond price has the following representation:

$$P(t,T) = E\left[\exp\left(-\int_t^T r(s)\,ds\right)\Big|\, r(t), r_1(t),\dots,r_n(t)\right], \tag{2.21}$$

hence $P(t,T) = H_n(t, r(t), r_1(t),\dots,r_n(t), T)$, $0 \le t \le T \le T^*$, for some measurable function H_n defined on $[0,T]\times\mathbb{R}^{n+1}\times[0,T^*]$, $n \in \mathbb{N}$, and we see that the a priori infinite-dimensional bond price process can be realized by means of a finite-dimensional Markovian system. The process $(r(t), r_1(t),\dots,r_n(t))_{t\in[0,T^*]}$ is called a *finite-dimensional Markovian realization* of the term structure model.

3 Markovian short rates

In this section we consider a generalization of the model proposed by Eberlein and Raible [10].

3.1. In addition to the previous assumptions, let us suppose that $\sigma(t,T)$ and $\delta(t,T,x)$ are given by (2.18) where $t \mapsto \beta(t,T)$ and $t \mapsto \gamma(t,T)$ are deterministic functions continuously differentiable on $[0,T]$ for $T \in [0,T^*]$. (Note that unlike [10] we do not assume the continuous differentiability of $\beta(t,T)$ and $\gamma(t,T)$ with respect to the second variable.) In this case the short rate process (2.13) admits the representation:

$$r(t) = f(0,t) + \int_0^t \beta(s,t)\int_s^t \beta(s,u)\,du\,ds + \int_0^t\int e^{\int_s^t \gamma(s,u)du\,x}\gamma(s,t)\,x\,F(dx)\,ds - Z(t) \tag{3.1}$$

where the process $Z = (Z(t))_{t\in[0,T^*]}$ is defined by:

$$Z(t) = \int_0^t \beta(s,t)\,dW_s + \int_0^t \gamma(s,t)\,dJ_s \tag{3.2}$$

and the process $J = (J_t)_{t\in[0,T^*]}$ is given by:

$$J_t = \int_0^t\int x\,\mu(ds,dx). \tag{3.3}$$

The main result of this section will be the criterion for the short rate $r = (r(t))_{t\in[0,T^*]}$ to be Markovian. We start with a simple extension of an assertion from [10] to the case of jump-diffusion processes.

Lemma 3.1. *Let $Z = (Z(t))_{t\in[0,T^*]}$ from (3.2) be an $(\mathcal{F}_t, Q)$ - Markov process. Then for all $0 \le T \le S \le T^*$ there is a Borel function G such that:*

$$\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t] = G\left(\int_0^T [\beta(t,T)\,dW_t + \gamma(t,T)\,dJ_t]\right) \quad (Q\ -\ \text{a.s.}). \tag{3.4}$$

Proof. If the process $Z = (Z(t))_{t\in[0,T^*]}$ is Markovian, then for all $0 \le T \le S \le T^*$ we have:

$$E[Z(S)\,|\,\mathcal{F}_T] = E[Z(S)\,|\,Z(T)] \quad (Q\ -\ \text{a.s.}). \tag{3.5}$$

Since the integrands $\beta(t,T)$ and $\gamma(t,T)$ for $0 \le t \le T \le T^*$ are deterministic, from independence of increments of the processes $W = (W_t)_{t\in[0,T^*]}$ and $J = (J_t)_{t\in[0,T^*]}$ it follows that:

$$\begin{aligned} &E[Z(S)\,|\,\mathcal{F}_T] \\ &\quad = E\left[\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\,\Big|\,\mathcal{F}_T\right] + E\left[\int_T^S [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\right] \quad (Q\ -\ \text{a.s.}) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &E[Z(S)\,|\,Z(T)] \\ &\quad = E\left[\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\,\Big|\,Z(T)\right] + E\left[\int_T^S [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\right] \quad (Q\ -\ \text{a.s.}) \end{aligned} \tag{3.7}$$

and hence, combining (3.5) - (3.7), we get:

$$\begin{aligned} &\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t] \\ &\qquad = E\left[\int_0^T [\beta(t,S)\,dW_t + \gamma(t,S)\,dJ_t]\,\Big|\,\int_0^T [\beta(t,T)\,dW_t + \gamma(t,T)\,dJ_t]\right] \quad (Q\ -\ \text{a.s.}) \end{aligned}$$

that immediately implies the desired assertion. □

Let us formulate a useful technical proposition.

Lemma 3.2. *Let $L = (L_t)_{t\in[0,T]}$ be a nonidentical zero (real-valued) Lévy process and $f(t)$, $g(t)$ are continuously differentiable, nonconstant functions on $[0,T]$ for some $T > 0$. Suppose that $f(t)$ and $g(t)$ are affine independent on $t \in [0,T]$ (i.e. there are no constants $c, d \in \mathbb{R}$ such that $f(t) = c\,g(t) + d$ for all $t \in [0,T]$). Then the distribution of the vector:*

$$\left(\int_0^T f(t)\,dL_t, \int_0^T g(t)\,dL_t\right) \tag{3.8}$$

has a nonzero absolutely continuous part (with respect to the Lebesgue measure λ_2 on $\mathbb{R}^2$).

The proof of this assertion is given in [17; Lemma 3.1] and uses the following lemma.

Lemma 3.3. *Let $L = (L_t)_{t\in[0,T]}$ be a nonidentical zero (real-valued) Lévy process and $f(t)$ is a continuously differentiable function on $[0,T]$ for some $T > 0$. Suppose that for a Borel function H we have:*

$$H\left(\int_0^T f(t)\,dL_t\right) = L_T \quad (Q\ -\ a.s.). \tag{3.9}$$

Then $f(t)$ is necessarily a constant on $[0,T]$.

This assertion is proved in [16; Theorem 3.1].

We now prove a lemma being an extension of corresponding assertions from [10], [16] and [17] to the case of multidimensional Lévy processes and playing a crucial role in the results presented below.

Lemma 3.4. *Let $L = (L_t^1,\ldots,L_t^n)_{t\in[0,T]}$, $n\in\mathbb{N}$, be an n-dimensional nonidentical zero Lévy process (i.e. $L^i = (L_t^i)_{t\in[0,T]}$, $i = 1,\ldots,n$, are independent Lévy processes) and $f_i(t)$, $g_i(t)$, $i = 1,\ldots,n$, are continuously differentiable functions on $[0,T]$ for some $T > 0$. Suppose that for a Borel function G we have:*

$$\sum_{i=1}^n \int_0^T f_i(t)\,dL_t^i = G\left(\sum_{i=1}^n \int_0^T g_i(t)\,dL_t^i\right) \quad (Q\ -\ a.s.). \tag{3.10}$$

Then there exists a constant c such that $f_i(t) = c\,g_i(t)$ for $i = 1,\ldots,n$ and all $t\in[0,T]$.

Proof. (i) Assume that for an arbitrary $k = 1,\ldots,n$ the functions $f_k(t)$ and $g_k(t)$ are affine independent on $[0,T]$. Then by Lemma 3.2 we get that the distribution of the vector:

$$\left(\int_0^T f_k(t)\,dL_t^k, \int_0^T g_k(t)\,dL_t^k\right) \tag{3.11}$$

has a nonzero absolutely continuous part, and thus, the distribution of the vector:

$$\left(\int_0^T f_k(t)\,dL_t^k + \sum_{i=1,i\neq k}^n \int_0^T f_i(t)\,dL_t^i, \int_0^T g_k(t)\,dL_t^k + \sum_{i=1,i\neq k}^n \int_0^T g_i(t)\,dL_t^i\right) \tag{3.12}$$

also has an absolutely continuous part, but the latter is impossible because of condition (3.10). Therefore, we conclude that there exists $c_k, d_k\in\mathbb{R}$ such that:

$$f_k(t) = c_k\,g_k(t) + d_k \tag{3.13}$$

for each $k = 1,\ldots,n$ and all $t\in[0,T]$, and hence, we have:

$$\sum_{i=1}^n \int_0^T f_i(t)\,dL_t^i = \sum_{i=1}^n c_i \int_0^T g_i(t)\,dL_t^i + \sum_{i=1}^n d_i\,L_T^i. \tag{3.14}$$

(ii) Let us now show that in (3.13) one can take $d_k = 0$ for each $k = 1,\ldots,n$. For this, we first note that if $g_k(t)$ were a constant on $[0,T]$ then $f_k(t)$ would be a constant too, and hence, we could take $d_k = 0$ for each $k = 1,\ldots,n$. Thus, assuming for an arbitrary $k = 1,\ldots,n$ that

$g_k(t)$ is not constant and $d_k \neq 0$, we observe that, by virtue of (3.10), the expression (3.14) implies:

$$\frac{1}{d_k} G\left(\sum_{i=1}^{n} \int_0^T g_i(t)\, dL_t^i\right) - \sum_{i=1}^{n} \frac{c_i}{d_k} \int_0^T g_i(t)\, dL_t^i - \sum_{i=1, i\neq k}^{n} \frac{d_i}{d_k} L_T^i = L_T^k \quad (Q \text{ – a.s.}). \tag{3.15}$$

Then using the independence of the processes $L^1, \ldots, L^n$ and applying the result of Lemma 3.3 under $L^1, \ldots, L^{k-1}, L^{k+1}, \ldots, L^n$ fixed, from (3.15) we get that $g_k(t)$ must be constant on $[0, T]$ that contradicts the assumption. Hence we see that d_k has to be zero for each $k = 1, \ldots, n$.

(iii) Finally, we show that $c_1 = \ldots = c_n$ in (3.13). For this, we first observe that if all $g_k(t)$, $k = 1, \ldots, n$, are identical zero on $[0, T]$ then the assertion of lemma obviously holds. Thus, without loss of generality, we can assume that $g_1(t)$ is nonidentical zero on $[0, T]$. (Note that if the other $g_k(t)$, $k = 2, \ldots, n$, are identical zero on $[0, T]$ then the assertion of the lemma coincides with the result of Lemma 3.2.) Hence, denoting $c := c_1$, using condition (3.10) and the fact that in (3.13) we have $d_k = 0$ for each $k = 1, \ldots, n$, from (3.14) we get:

$$H\left(\sum_{i=1}^{n} \int_0^T g_i(t)\, dL_t^i\right) = \sum_{i=2}^{n} (c_i - c) \int_0^T g_i(t)\, dL_t^i \quad (Q \text{ – a.s.}) \tag{3.16}$$

where $H(x) = G(x) - c\,x$ for all $x \in \mathbb{R}$. Since the function $g_1(t)$ is assumed to be nonidentical zero on $[0, T]$, it is easily seen that the equality in (3.16) can be satisfied if and only if $H(x)$ is constant for all $x \in \mathbb{R}$ which immediately implies that $c_k = c$ for each $k = 2, \ldots, n$, and therefore, concludes the proof of the lemma. □

3.2. We now formulate and prove the criterion for the short rate to be Markovian. Actually, we extend the results of [6], [10] and [17] to the case where the forward rate is driven by a jump-diffusion process.

Theorem 3.5. *Suppose that both the functions $t \mapsto \beta(t, T)$ or $t \mapsto \gamma(t, T)$ are not identical zero on $[0, T]$ for all $T \in [0, T^*]$. Then the short rate process $r = (r(t))_{t\in[0,T^*]}$ is Markovian if and only if there are continuously differentiable functions $\eta(t)$ and $\kappa(t)$, $t \in [0, T]$, and the function $\zeta(T) > 0$, $T \in [0, T^*]$, such that:*

$$\beta(t, T) = \eta(t)\, \zeta(T) \quad \textit{and} \quad \gamma(t, T) = \kappa(t)\, \zeta(T) \tag{3.17}$$

for all $0 \leq t \leq T \leq T^*$.

Proof. (i) Let us first assumme that $r = (r(t))_{t\in[0,T^*]}$ is a Markov process. Then by virtue of (3.1), so is the process $Z = (Z(t))_{t\in[0,T^*]}$, and Lemma 3.1 shows that for each $0 \leq T \leq S \leq T^*$ there is a measurable function G such that (3.4) holds.

Let us fix some $T \in [0, T^*]$. Then applying Lemma 3.4 with $n = 2$, $L_t^1 = W_t$ and $L_t^2 = J_t$, $t \in [0, T^*]$, to the functions $t \mapsto \beta(t, T^*)$ and $t \mapsto \beta(t, T)$ as well as $t \mapsto \gamma(t, T^*)$ and $t \mapsto \gamma(t, T)$, we get that there exists a function $\xi(T, T^*)$, not depending on t, such that:

$$\beta(t, T^*) = \xi(T, T^*)\, \beta(t, T) \quad \text{and} \quad \gamma(t, T^*) = \xi(T, T^*)\, \gamma(t, T) \tag{3.18}$$

for all $0 \leq t \leq T \leq T^*$. Since for each $T \in [0, T^*]$ one of the functions $t \mapsto \beta(t, T)$ or $t \mapsto \gamma(t, T)$ is not identical zero on $[0, T]$, there exists some $t \in [0, T]$ such that $\beta(t, T) \neq 0$ or

$\gamma(t,T) \neq 0$. Hence, the expressions (3.18) imply that $\beta(t,T^*) \neq 0$ or $\gamma(t,T^*) \neq 0$, respectively, and thus $\xi(T,T^*) \neq 0$ for all $T \in [t,T^*]$. (Otherwise, if $\beta(t,T^*) = 0$ and $\gamma(t,T^*) = 0$, then $\xi(T,T^*) = 0$ and from (3.18) it follows that both the functions $t \mapsto \beta(t,T^*)$ and $t \mapsto \gamma(t,T^*)$ are identical zero which is a contradiction.) Since these arguments can be realized for each $T \in [0,T^*]$, we may conclude that $\xi(T,T^*) \neq 0$ for all $T \in [0,T^*]$.

Therefore, defining $\eta(t) := \beta(t,T^*)$, $\kappa(t) := \gamma(t,T^*)$ and $\zeta(T) := 1/\xi(T,T^*)$, we obtain the decompositions (3.17). Furthermore, we observe that in this case without loss of generality we may assume that $\zeta(T) > 0$ for all $T \in [0,T^*]$. The continuous differentiability of the functions $\eta(t)$ and $\kappa(t)$, $t \in [0,T]$, directly follows from the assumption of $t \mapsto \beta(t,T)$ and $t \mapsto \gamma(t,T)$ to be continuously differentiable on $[0,T]$ for all $T \in [0,T^*]$, respectively.

(ii) Suppose now that the functions $\beta(t,T)$ and $\gamma(t,T)$, $0 \leq t \leq T \leq T^*$, satisfy conditions (3.17). In this case the process $Z = (Z(t))_{t\in[0,T^*]}$ from (3.2) can be represented in the form:

$$Z(t) = \zeta(t) \left(\int_0^t \eta(s)\, dW_s + \int_0^t \kappa(s)\, dJ_s \right) \tag{3.19}$$

and hence, it is Markovian and so is the process $r = (r(t))_{t\in[0,T^*]}$. $\square$

Remark 3.6. Note that, by means of Lemma 3.4, the result of Theorem 3.5 can be easily extended to the case where the bond price is driven by both multidimensional Wiener process and Poisson random measure, and in that case the volatilities have the same multiplicative structure with a common factor depending on the time to maturity.

Example 3.7. Suppose that $J = (J_t)_{t\in[0,T^*]}$ is a bilateral variance gamma process, i.e. it is a Lévy process with the triplet $(0,0,F(dx))$ where:

$$F(dx) = \left(\frac{\alpha_+}{x} e^{-\lambda_+ x} I\{x > 0\} + \frac{\alpha_-}{-x} e^{\lambda_- x} I\{x < 0\} \right) dx \tag{3.20}$$

and $\lambda_+, \lambda_-, \alpha_+, \alpha_-$ are some positive parameters (see [17; Section 5]). In this case if $|\delta(t,T,x)| \leq \min\{\lambda_+, \lambda_-\}$ for all $0 \leq t \leq T \leq T^*$ and $x \in \mathbb{R}$ then conditions (2.3) and (2.10) are satisfied, so that the assertion of Theorem 3.4 holds.

Finally, let us describe the case of stationary volatility structure for completeness.

Corollary 3.8. *Suppose that in conditions of Theorem 3.4 the volatility functions $\beta(t,T)$ and $\gamma(t,T)$ depends only on the time to maturity $T-t$, i.e. there exist functions $\widetilde{\beta}(\cdot)$ and $\widetilde{\gamma}(\cdot)$ such that $\beta(t,T) = \widetilde{\beta}(T-t)$ and $\gamma(t,T) = \widetilde{\gamma}(T-t)$ for all $0 \leq t \leq T \leq T^*$. Then the short rate $r = (r(t))_{t\in[0,T^*]}$ is a Markov process if and only if:*

$$\beta(t,T) = \widehat{\beta} e^{-\widehat{\alpha}(T-t)} \quad \textit{and} \quad \gamma(t,T) = \widehat{\gamma} e^{-\widehat{\alpha}(T-t)} \tag{3.21}$$

or

$$\beta(t,T) = \widehat{\beta}(T-t) \quad \textit{and} \quad \gamma(t,T) = \widehat{\gamma}(T-t) \tag{3.22}$$

for some constants $\widehat{\alpha} \neq 0$ and $\widehat{\beta}, \widehat{\gamma} \in \mathbb{R}$ such that $\widehat{\beta} \neq 0$ or $\widehat{\gamma} \neq 0$.

The proof of this assertion can be realized by means of a simple extension of the arguments in [10; Theorem 4.4]. If the volatility structure satisfy conditions (3.21) or (3.22), then it is called to be of Vasiček or Ho-Lee type, respectively.

Remark 3.9. If the volatilities $\beta(t,T)$ and $\gamma(t,T)$ are given by expressions (3.21), then the short rate $r = (r(t))_{t\in[0,T^*]}$ is a *mean-reverting* process satisfying the equation:

$$dr(t) = \widehat{\alpha}(\rho(t) - r(t))\,dt - \widehat{\beta}\,dW_t - \widehat{\gamma}\,dJ_t \tag{3.23}$$

where

$$\begin{aligned}\rho(t) =& f(0,t) + \frac{f_T'(0,t)}{\widehat{\alpha}} + \frac{\widehat{\beta}^2}{2\widehat{\alpha}^2}\Big(1 - e^{-2\widehat{\alpha}t}\Big) \\ &+ \frac{\widehat{\gamma}^2}{\widehat{\alpha}}\int_0^t e^{-2\widehat{\alpha}(t-s)}\int \exp\left(\frac{\widehat{\gamma}}{\widehat{\alpha}}\Big(1 - e^{-\widehat{\alpha}(t-s)}\Big)\,x\right)\,x^2\,F(dx)\,ds.\end{aligned} \tag{3.24}$$

We observe that when $\widehat{\alpha} > 0$ the process $r = (r(t))_{t\in[0,T^*]}$ fluctuates near the level $(\rho(t))_{t\in[0,T^*]}$, because from (3.23) - (3.24) it is easily seen that if $r(t) < \rho(t)$ then $dr(t)$ has a positive drift, and if $r(t) > \rho(t)$ then the drift of $dr(t)$ is negative.

4 Finite-dimensional Markovian realizations

In this section we extend the result of [20] to the jump-diffusion case.

4.1. In the sequel we will use the notation of Section 2 without assuming conditions (2.18). We immediately observe that by virtue of the conditions (2.3) and (2.10) and the assumptions of twice continuous differentiability of $\sigma(t,T)$ and $\delta(t,T,x)$ in the second variable for $\omega \in \Omega$ and $x \in \mathbb{R}$ fixed, as well as by the allowance to differentiate under the integral sign, all the integrals occuring in the sequel are well-defined.

Let us suppose that the condition:

$$\sigma_T'(t,T) = \eta(t,r(t))\,\zeta(T) \tag{4.1}$$

holds for some continuous function $\eta(t,r)$ and twice continuously differentiable function $\zeta(T)$. (Note that (4.1) coincides with the condition (2.11) in [20].) In this case applying (2.4) we get:

$$\int_t^T \sigma_T'(s,u)\,du = \theta(t,T)\sigma_T'(s,t) \tag{4.2}$$

where

$$\theta(t,T) = \int_t^T \frac{\zeta(u)}{\zeta(t)}\,du \tag{4.3}$$

and thus, using the obvious fact that:

$$\int_t^T \frac{\zeta(u)}{\zeta(t)}\int_t^u \frac{\zeta(v)}{\zeta(t)}\,dv\,du = \frac{\theta^2(t,T)}{2} \tag{4.4}$$

we see that:

$$\int_t^T \sigma_T'(s,u)\sigma(s,u)\,du = \theta(t,T)\sigma_T'(s,t)\sigma(s,t) + \frac{\theta^2(t,T)}{2}(\sigma_T'(s,t))^2. \tag{4.5}$$

Then changing the order of integration we obtain:

$$\int_t^T \int_0^t \sigma_T'(s,u)\, dW_s\, du - \int_t^T \int_0^t \sigma_T'(s,u)\sigma(s,u)\, ds\, du = \theta(t,T)\chi(t) - \frac{\theta^2(t,T)}{2}\phi(t) \tag{4.6}$$

where the processes $(\phi(t))_{t\in[0,T^*]}$ and $(\chi(t))_{t\in[0,T^*]}$ are defined by:

$$\phi(t) = \int_0^t (\sigma_T'(s,t))^2\, ds \tag{4.7}$$

and

$$\chi(t) = \int_0^t \sigma_T'(s,t)\, dW_s - \int_0^t \sigma_T'(s,t)\sigma(s,t)\, ds. \tag{4.8}$$

Let us also assume that the condition:

$$\delta(t,T,x) = \log[\kappa(t,r(t),x)\, \lambda(T)] \tag{4.9}$$

holds for some continuous function $\kappa(t,r,x)$ and twice continuously differentiable function $\lambda(T)$. In this case (under the condition $F(\mathbb{R}) < \infty$) it is easily seen that:

$$\int_t^T \int_0^t \int \delta_T'(s,t,x)\, \mu(ds,dx)\, du = \log\left(\frac{\lambda(T)}{\lambda(t)}\right) \mu(t) \tag{4.10}$$

and

$$\int_t^T \int_0^t \int e^{\delta(s,u,x)} \delta_T'(s,u,x)\, F(dx)\, ds\, du = (\lambda(T) - \lambda(t))\upsilon(t) \tag{4.11}$$

where the processes $(\mu(t))_{t\in[0,T^*]}$ and $(\upsilon(t))_{t\in[0,T^*]}$ are defined by:

$$\mu(t) = \int_0^t \int \mu(ds,dx) \tag{4.12}$$

and

$$\upsilon(t) = \int_0^t \int \kappa(s,r(s),x)\, F(dx)\, ds. \tag{4.13}$$

Summarizing the facts shown above let us formulate the main assertion of the section.

Theorem 4.1. *Assume that we have $F(\mathbb{R}) < \infty$ and conditions (4.1) and (4.9) are satisfied. Then the bond price (2.16) admits the representation:*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(\theta(t,T)\chi(t) - \frac{\theta^2(t,T)}{2}\phi(t) - (\lambda(T) - \lambda(t))\upsilon(t) + \log\frac{\lambda(T)}{\lambda(t)}\mu(t)\right) \tag{4.14}$$

where $\theta(t,T)$, $\phi(t)$, $\chi(t)$, $\mu(t)$ and $\upsilon(t)$ are defined in (4.3), (4.7), (4.8), (4.12) and (4.13), respectively. If, in addition, the function $\eta(t,r)$ satisfies the Lipschitz condition:

$$|\eta(t,r) - \eta(t,r')| \le C|r - r'| \tag{4.15}$$

for some $C > 0$ fixed, all $t \in [0,T^*]$ and $r, r' \in \mathbb{R}$, then $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{[0,T^*]}$ is a finite-dimensional Markovian realization.

Proof. Since the representation (4.14) for the bond price (2.16) easily follows from (4.6), (4.10) and (4.11) above, it remains us to prove the Markov property of the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{[0,T^*]}$ under condition (4.15). For this, let us first observe that from (4.1) - (4.8) it follows that:

$$\int_0^t [\sigma''_{TT}(s,t)\sigma(s,t) + (\sigma'_T(s,t))^2]\, ds - \int_0^t \sigma''_{TT}(s,t)\, dW_s = -\frac{\zeta'(t)}{\zeta(t)}\chi(t) + \phi(t) \tag{4.16}$$

as well as from (4.9) - (4.13) it is seen that:

$$\int_0^t \int \delta''_{TT}(s,t,x)\, \mu(ds,dx) = \frac{\lambda''(t)\lambda(t) - (\lambda'(t))^2}{\lambda^2(t)}\mu(t) \tag{4.17}$$

and

$$\int_0^t \int e^{\delta(s,t,x)}[\delta''_{TT}(s,t,x) + (\delta'_T(s,t,x))^2]\, F(dx)\, ds = \lambda''(t)\upsilon(t) \tag{4.18}$$

where, by virtue of (2.4), the processes (4.7) - (4.8) and (4.12) - (4.13) admit the representations:

$$d\phi(t) = \left[2\frac{\zeta'(t)}{\zeta(t)}\phi(t) + \eta^2(t,r(t))\zeta^2(t)\right] dt, \tag{4.19}$$

$$d\chi(t) = \left[\frac{\zeta'(t)}{\zeta(t)}\chi(t) - \phi(t)\right] dt + \eta(t,r(t))\zeta(t)\, dW_t, \tag{4.20}$$

$$d\mu(t) = \int \mu(dt,dx) \tag{4.21}$$

and

$$d\upsilon(t) = \int \kappa(t,r(t),x)\, F(dx)\, dt. \tag{4.22}$$

It follows from (4.16) - (4.18) that the process (2.14) takes the expression:

$$dr(t) = f'_T(t,t)\, dt - \eta(t,r(t))\zeta(t)\, dW_t - \frac{\lambda'(t)}{\lambda(t)}\int \mu(dt,dx) \tag{4.23}$$

where

$$f'_T(t,t) = f'_T(0,t) - \frac{\zeta'(t)}{\zeta(t)}\chi(t) + \phi(t) + \lambda''(t)\upsilon(t) - \frac{\lambda''(t)\lambda(t) - (\lambda'(t))^2}{\lambda^2(t)}\mu(t). \tag{4.24}$$

Observe that the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{t\in[0,T^*]}$ satisfies the multidimensional stochastic differential equation (4.19) - (4.23) with (4.24) which admits a solution for all possible initial conditions $(r_0, \phi_0, \chi_0, \mu_0, \upsilon_0)$ with nonnegative ϕ_0 and μ_0. We also note that, by virtue of the assumption $F(\mathbb{R}) < \infty$, the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{t\in[0,T^*]}$ has a finite number of jumps on every finite time interval. Since the function $\eta(t,r)$ is assumed to satisfy the condition (4.15), by means of the strong uniqueness argument for solutions of stochastic differential equations (see [14; Section III, Theorem 2.32] and [11; Chapter IV, Theorem 1]), we may conclude that the process $(r(t), \phi(t), \chi(t), \mu(t), \upsilon(t))_{t\in[0,T^*]}$ is a unique strong solution of (4.19) - (4.23) with (4.24), and therefore, by virtue of the arguments in [11; Chapter VI, Section 1], it is Markovian. □

Remark 4.2. Note that if in the assumptions of Theorem 4.1 we have $\delta(t,T,x)=0$ for all $0\le t\le T\le T^*$ and $x\in\mathbb{R}$, then we get the result of [20], and by means of the arguments in [13], the result of Theorem 4.1 can be also naturally extended to the case where the bond price is driven by both multidimensional Wiener process and Poisson random measure.

4.2. Finally, we show that if instead of the additive type condition (4.9) we consider one of multiplicative type, then the short rate process does not need to be a component of a finite-dimensional Markov process, but it can be a component of an infinite-dimensional one. For this, let us suppose that in the assumptions of Theorem 4.1 condition (4.9) is replaced by:

$$\delta'_T(t,T,x)=\kappa(t,r(t),x)\,\lambda(T). \tag{4.25}$$

In this case it follows that:

$$\int_0^t\int \delta''_{TT}(s,t,x)\,\mu(ds,dx)=\frac{\lambda'(t)}{\lambda(t)}\int_0^t\int \delta'_T(s,t,x)\,\mu(ds,dx) \tag{4.26}$$

and

$$\int_0^t\int e^{\delta(s,t,x)}\delta''_{TT}(s,t,x)\,F(dx)\,ds=\frac{\lambda'(t)}{\lambda(t)}\int_0^t\int e^{\delta(s,t,x)}\delta'_T(s,t,x)\,F(dx)\,ds \tag{4.27}$$

and thus, by means of the arguments of the proof of Theorem 4.1 (i), we obtain that the short rate process (2.13) - (2.14) satisfies the stochastic differential equation:

$$dr(t)=f'_T(t,t)\,dt-\eta(t,r(t))\zeta(t)\,dW_t-\int\kappa(t,r(t),x)\lambda(t)\,\mu(dt,dx) \tag{4.28}$$

where

$$f'_T(t,t)=f'_T(0,t)-\frac{\zeta'(t)}{\zeta(t)}\chi(t)+\phi(t)+\frac{\lambda'(t)}{\lambda(t)}[\xi_1(t)-\psi(t)]+\xi_2(t), \tag{4.29}$$

the processes $(\phi(t))_{t\in[0,T^*]}$ and $(\chi(t))_{t\in[0,T^*]}$ are given by (4.7) - (4.8) as well as $(\psi(t))_{t\in[0,T^*]}$ and $(\xi_n(t))_{t\in[0,T^*]}$, $n\in\mathbb{N}$, are defined by:

$$\psi(t)=\int_0^t\int\kappa(s,r(s),x)\,\mu(ds,dx) \tag{4.30}$$

and

$$\xi_n(t)=\int_0^t\int e^{\delta(s,t,x)}\left(\delta'_T(s,t,x)\right)^n F(dx)\,ds. \tag{4.31}$$

Differentiating the identities (4.30) and (4.31), we get:

$$d\psi(t)=\int\kappa(t,r(t),x)\,\mu(dt,dx) \tag{4.32}$$

and

$$d\xi_n(t)=\left(\xi_{n+1}(t)+n\frac{\lambda'(t)}{\lambda(t)}\xi_n(t)\right)dt+\int\left(\kappa(t,r(t),x)\lambda(t)\right)^n F(dx)\,dt. \tag{4.33}$$

Remark 4.3. From (4.19) - (4.20), (4.28) - (4.29), and (4.32) - (4.33) we see that the process $r=(r(t))_{t\in[0,T^*]}$ has an infinite number of state variables $(\phi(t))_{t\in[0,T^*]}$, $(\chi(t))_{t\in[0,T^*]}$,

$(\psi(t))_{t\in[0,T^*]}$, $(\xi_n(t))_{t\in[0,T^*]}$, $n \in \mathbb{N}$, so that the process $(r(t), \phi(t), \chi(t), \psi(t), \xi_1(t), \xi_2(t))_{t\in[0,T^*]}$ does not have to be Markovian.

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