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Time-variation of higher moments in a financial market with heterogeneous agents: An analytical approach

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Time-Variation of Higher Moments in a Financial Market with Heterogeneous Agents: An Analytical Approach

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Department of Economics

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Abstract

A growing body of recent literature allows for heterogenous trading strategies and limited rationality of agents in behavioral models of financial markets. More and more, this literature has been concerned with the explanation of some of the stylized facts of financial markets. It now seems that some previously mysterious time-series characteristics like fat tails of returns and temporal dependence of volatility can be observed in many of these models as macroscopic patterns resulting from the assumed interaction of speculative traders. However, most of the available evidence stems from simulation studies of relatively complicated models which do not allow for analytical solutions. In this paper, this line of research is supplemented by analytical solutions of a simple variant of the seminal herding model introduced by Kirman (1993). Embedding the herding framework into a simple equilibrium asset pricing framework, we are able to derive closed-form solutions for the time-variation of higher moments as well as related quantities of interest enabling us to spell out under what circumstances the model gives rise to realistic behavior of the resulting time series.

1 Introduction

Until very recently, theoretical research in finance has largely ignored some of the really universal stylized facts of practically all available financial data. In fact, a glance at frequently used textbooks like the ones by O'Hara (1995) and Barucci (2003) shows that even their glossaries lack entries for some of the prevalent technical terms of the empirical finance literature. For example, while many developments in empirical finance are essentially motivated by the observation of non-Gaussian returns distributions with their "fat tails" and temporal dependence of second moments leading to "volatility clustering", these notions have been almost entirely absent from the theoretical literature.[1] While these phenomena have spurred the development of such seminal innovations like GARCH-type and stochastic volatility models in empirical finance, their behavioral origins remained an unaccessible puzzle. Two reasons might be responsible for this neglect: first, the above features characterize the behavior of financial time series as a whole, while the interest in economic theory has typically been to spell out the effect of a change of one (endogenous) economic variable on other, exogenous variables. Even when allowing for an ensemble of traders, such a comparative statics approach is not appropriate for explaining universal conditional and unconditional stochastic properties. Second,

[1] Perfectly the same holds for theoretical work on exchange rate determination where the same stylized facts also remained unnoticed for a long time.

the prevalent efficient market paradigm did, in fact, provide a very simple implicit answer to the question of the origin of *all* stylized facts of returns: since, in this framework, prices would reflect forthcoming news in an unbiased and immediate way, any property of the returns distribution would simply reflect a similar feature of the distribution of new information items. As a corollary of the efficient market hypothesis, the "news arrival process" would have to come along with fat tails and clustering of important news. Unfortunately, this corollary can hardly be subjected to econometric scrutiny. On the other hand, enough evidence had been collected against the universal validity of the efficient market paradigm to motivate alternative, behavioral approaches which then mushroomed over the nineties.

First analyses of complex data generating mechanisms based on interacting agents can be found in Kirman (1991, 1993) and De Grauwe et al. (1993). While they did not focus then on the above stylized facts (not broadly acknowledged still among theoretical researchers), they both already showed that their models could mimic the random walk nature of asset prices and exchange rates although their data-generating processes were clearly different from a true random walk. Notably, both studies also investigated what we might describe as secondary stylized facts: they repeated certain frequent econometric analyses on their simulated data and found similar behavior as with empirical records providing a possible explanation of, e.g., the forward premium puzzle of foreign exchange markets.

Evidence for volatility clustering as an emergent phenomenon of a multi-agent model appeared first in Grannan and Swindle (1994). While a large body of subsequent models studied artificial markets with heterogenous autonomous agents often endowed with some sort of artificial intelligence (classifier systems, genetic algorithms), any consideration of empirical stylized facts is curiously absent in the first wave of pertinent papers (Levy, Levy and Solomon, 1994; Arthur et al., 1997; Chen and Yeh, 1997; Arifovic, 1996). In fact, much of this early literature had been preoccupied with the question of convergence or not of their learning algorithms to the benchmark of rational expectations rather than considering empirical applications. However, subsequent research has shown that relatively simple models of interacting traders could produce realistic time series sharing the "stylized facts" of fat tails and clustering volatility even up to numerical agreement with key quantities of empirical data (Lux and Marchesi, 1999, 2000). Similar investigations of the dynamic properties of alternative models revealed that many agent-based approaches share a certain tendency of generating fat tails and volatility clustering although their quantitative manifestations are not always identical to the very robust numbers obtained with empirical data (cf. LeBaron et al., 1999; Chen and Yeh, 2002; Kirman and Teyssière, 2002; Lux and Schornstein, 2004). While often the general appearance of simulated data seemed to be quite robust with respect to most of the underlying parameters, it also turned out that the potential of generating stylized facts depends crucially on the system size (i.e., number of agents). While, realistic dynamic patterns are typically observed with the (probably natural) initial choice of a few hundred or thousand agents, increasing the number of market participants mostly leads to vanishing fat tails and volatility clustering beyond a certain threshold (cf. Egenter et al., 1999: Challet and Marsilli, 2002; Lux and Schornstein, 2004).

The present paper attempts to shed light on both of the findings detailed above. Within a relatively simple type of herding model (broadly along the lines of Kirman, 1993) we derive closed-form solutions for autocovariances of returns and their higher moments together with other statistics such as mean-passage times. Inspection of the results allows to infer in how far and under what conditions the model could mimic the empirical findings of fat tails and clustering of volatility. Investigation of different specifications of the model also allows us to point out why - in certain scenarios - increasing system size would lead to vanishing stylized facts. Our approach

is broadly complementary to recent attempts at studying asymptotic properties of related agent-based models (Horst, 2004; Föllmer, Horst and Kirman, 2004). These authors provide conditions under which the limit distribution of the price process exists in models with both global and local interactions of agents (Horst, 2004) and models with feedback from the price process on the group dynamics (Föllmer, Horst and Kirman, 2004). Since our model can be viewed as a special case of the class of models studied in Horst (2004) his result on convergence of equulibirum prices to a unique equilibrium distribution also applies in our case. However, instead of focusing on the properties of the price process alone we extend the analysis to returns to which the famous empirical regularities of fat tails and clustered volatility apply. We also go beyond asymptotic convergence results by working out various properties of the stationary distribution which are of interest in the light of empirical findings. The paper is organized as follows...

2 Kirman's Ant Model and Its Financial Interpretation

In his seminal 1993 paper, Kirman provides a simple stochastic formalization of information transmission inspired by macroscopic patterns emerging from information transmission within ant colonies. The underlying scenario is one of foraging ants who have two identical sources of food at their disposal in the vicinity of their nest. Experimental settings show that at any point in time, a majority of the ant population concentrates on exploiting one particular food source, but over time switches may occur of the preferred source. Thus, averaging over time, a bimodal distribution of the frequency of ants visiting one or the other manger would result. Kirman explains this phenomenon by a combination of pair-wise interaction (exchange of information by pheromons) and an autonomous switching probability due to stochastic search. With an increase of the strength of the herding component, the resulting stationary probability distribution of the model bifurcates from unimodality with an equal exploitation of both resources to a bi-modal one supporting the experimental findings.

In a long series of subsequent papers, this and similar mechanisms have been applied as formalizations of contagion effects in financial markets. Kirman (1991, 1996) replaces the binary choice of food sources by the ants by agents' choice of a particular rule for the formation of their expectations. In his adaption to foreign exchange market dynamics, agents can follow chartist and fundamentalist forecasts of future exchange rate movements. With the herding mechanism governing agents' distribution on these two behavioral types, the market switches between a dominance of fundamentalist and chartist behavior with periods in which the later prevails giving rise to speculative bubbles. Market equilibria are determined from a standard monetary exchange rate model in which the assumption of rational expectations of the original framework has been replaced by 'non-rational' expectations computed as the average of the expectations of the two groups of chartist and fundamentalist both weighted by a measure of their past success. Most relevant to our purpose, Kirman and Teissière (2002) have analyzed the temporal dependence of returns and volatility from simulated data using a variety of pertinent econometric tools and have found similar indication for long-term dependence in second moments as with real-life data.

A closely related variant of a herding model has been proposed by Lux (1995) who replaced the pair-interaction by an overall (mean-field) effect of the average attitude of the whole ensemble of traders on each individual's choice of strategy. This setting allowed to derive a number of analytical results via application of the

Master equation approach adopted from statistical physics (Lux, 1997, 1998). An extended version of this model (Lux and Marchesi, 1999) with switching between three different groups could be shown to give rise to realistic time series properties. Egenter et al. (1999) showed, however, that these results depended on the number of agents and that rising the number of market participants beyond several thousand individuals leads to a gradual fading away of all its 'interesting' dynamics. We will point out later what mechanism are responsible for this loss of stylized facts and under which assumptions 'interesting' solutions can be preserved even in large markets.

To set the stage, let us first restate the mechanics of the particular herding model proposed by Kirman (1993). The market (or ant colony) is populated by a fixed number of agents N, each of them being either in state 1 or 2. The number of agents in the first state shall be denoted by n so that $n \in \{0, 1, \ldots, N\}$ defines the state of system. The stochastic evolution of the system is governed by random meetings of two agents, after which one of them is converted to the second's 'view' with a probability b, and autonomous switches which occur with probability a. The stochastic population dynamics, then, evolves according to the probabilities of changing from state n at time t to some state n' at time $t + \Delta t$. Let these conditional probabilities be denoted by $\rho(n', t + \Delta t|n, t)$ which are related to the transition rates per unit time, $\pi(n \to n')$ by $\rho(n', t + \Delta t \mid n, t) = \pi(n \to n')\Delta t$ for small time increments Δt. Since in the limit of continuous time $\Delta t \to 0$, multiple switches during one incremental time unit become increasingly unlikely, one can confine the analysis to $n' = n \pm 1$ with transition rates

$$\begin{aligned} \pi(n \to n+1) &= (N-n)(a+bn), \\ \pi(n \to n-1) &= n[a+b(N-n)] \end{aligned} \tag{1}$$

Note that, in this setting, transition rates and transition probabilities are governed by the raw number of individuals in groups 1 or 2, and are, therefore, also affected by the overall number of agents N. In pertinent literature one, in fact, often finds a formulation in which transition rates depend on intensities rather than on raw group occupation numbers, i.e. on $\frac{n}{N}$ and $\frac{N-n}{N}$ instead of n and $N-n$ (Lux, 1995). Apparently, system size appears to become suppressed in such a formalisation and one might argue that this would make the model outcome insensitive with respect to the market size (as it turns out, the opposite, in fact, happens).

Would it make intuitive sense to replace raw occupation numbers by intensities in eq.(1)? First, as concerns the autonomous switching component, increasing the number of agents and keeping the Poisson probability a of an autonomous switch by any particular agent per time unit constant, larger n and $N-n$ would also lead to a higher probability of observing one such move within the time interval Δt. Since agents are independent in this autonomous component, the linearity in n and $N-n$, in fact, appears intuitively plausible. Replacing, for example, na by $\frac{n}{N}a$ in the second equation would amount to assuming that probability a does not apply to each individual agent herself but rather to a fraction of the population. One can argue in the same way with respect to the herding component: increasing the overall number of ants or agents (and assuming that they are distributed equally in the space or have the same probability of meeting other agents), the likelihood for agents to randomly bump into each other should also increase, presumably linearly, in the number of agents (if we were not to incorporate particular topological considerations into the model). Vice versa, a formulation in intensities, i.e. expressing the conversion probabilities by $b\frac{n}{N}$ and $b\frac{N-n}{N}$ in eq. (1) would amount to a constant probability of meeting agents from the other group independent of the overall number of agents which - at least in the literal interpretation of the model - would be implausible.

In order to investigate the effects of intensive vs. non-intensive probabilities later on, we can integrate this alternative approach into the framework of eqs. (1) by

replacing the constants a and b by flexible forms, $a = \frac{a_0 N_0}{N}$ and $b = \frac{b_0 N_0}{N}$ in which a_0, b_0 and N_0 are the benchmark values for some initial scenario with constant probabilities a_0 and b_0 and benchmark population N_0. The changes brought about through an increasing population in the intensive formulation would, then, simply amount to multiplying the original behavioral constants by $\frac{N_0}{N} < 1$ which obviously leads to an effective decrease of both the property of autonomous switches and the succeptability of agents to contagion effects with increasing population size.

In order to close the model, we embed the herding dynamics into a simple noise trader framework. In particular, we interpret group 1 and 2 as optimistic and pessimistic chartists, respectively, who expect the prices to increase or decrease in the near future. Optimists are assumed to buy a certain number T_c of additional units of the asset while pessimistis sell T_c units (T_c, thus, is a measure of average transaction volume of chartists). Besides these chartists or noise traders, whose belief dynamics are decribed by the above herding model, the model also hosts a second group of fundamentalist traders who buy/sell if the current price p is below their perceived fundamental value p_f. Assuming that this group has N_f members with average trading volume T_f and that they react on relative deviations between p and p_f, excess demand by this group amounts to $ED_f = N_f T_f ln \frac{p_f}{p}$.

Since Noise trader's excess demand mainly derives from a dominance of optimism and pessimism in this population, we can conveniently, trace it back to a quantity describing the population configuration: $x = \frac{2n}{N} - 1$ which if zero signals a balanced disposition (zero excess demand) and if positive or negative amounts to optimistic or pessimistic majorities. Noise traders' excess demand, therefore, becomes $ED_C = NT_c x$. Invoking a standard Walrasian price adjustment mechanism, relative price changes would depend on overall excess demand, i.e.

$$\frac{dp}{pdt} = \beta(N_f T_f ln \frac{p_f}{p} + NT_c x) \tag{2}$$

with β the assumed price adjustment speed. Except for the incorporation of the herding mechanism through x, this is a fairly standard asset price equation similar to what one finds in the legacy of models with chartist-fundamentalist interaction (e.g., Beja and Goldman, 1980; Day and Huang, 1990).

Setting (without loss of generality) $N_f T_f / NT_c = 1$ and assuming instantaneous market clearing ($\beta \to \infty$), we arrive at the equilibrium price driven both by fundamental information and the average mood of noise traders:

$$p(t) = p_f exp(x(t)). \tag{3}$$

Hence, derivation of the limiting distribution of the configuration x together with a specification of the fundamentals process would allow us to characterize the distribution of equilibrium prices. However, we are interested in returns rather than prices themselves as the later are known to be non-stationary in almost all real-life markets. From (3), returns over arbitrary time horizons are computed as:

$$\begin{aligned} r(t, \Delta t) &= \ln p(t + \Delta t) - \ln p(t) \\ &= \ln p_f(t + \Delta t) - \ln p_f(t) + x(t + \Delta t) - x(t) \end{aligned} \tag{4}$$

and volatility can be obtained as $V(t, \Delta t) = r^2(t, \Delta t)$.

Eqs. (1) to (3) define a simple noise trader/infection framework with heterogenous interacting agents which is quite similar in spirit but simpler than the formalisation of Kirman (1991) and Kirman and Teyssière (2002). Since the population configuration from the infection process enters directly on the demand side of the model it is, in fact, possible to infer the statistical properties of the returns and volatility processes under the influence of this ant-like contagion dynamics.

It is interesting to compare the above framework to the way in which the ant model is incoporated into a monetary exchange rate model in Kirman (1991) and Kirman and Teyssière (2002). In their papers, the contagion dynamics occurs on a much faster time scale than price formation in the foreign exchange market. In particular, they extract the population configuration at integer times t as a snapshot of the distribution of agents after a large number of pairwise meetings (for example, 10,000 meetings in Kirman, 1991) at the"intra-daily level". The current distribution is then used to compute the weights of chartists and fundamentalists and defining the market expectation as the average of current chartist and fundamentalist forcasts, the equilibrium exchange rate is derived from a discrete-time monetary model.[2] The seperation of time scales allows to observe relatively large fluctuations of the average opinion at integer time steps, but it also impedes to take stock of the theoretical solution of the ant process for a formal analysis of the exchange rate dynamics. In contrast, in our approach, the time scales of the ant process and price formation are the same (although we will later dinstinguish between the continuous time dynamics of the process and discrete measurements thereof). With our Poisson probabilities, both occur in continuous time with changes of the configuration of noise traders coming along with changes in equlibrium prices.

3 Simulating the Herding Model

Following multi-particle simulations in statistical physics, different avenues exist for simulating the above model. The first, obvious choice would be a true microscopic simulation keeping track of the state of each individual agent and determining its switches over time by random number draws. Of course, the continuous-time framework would have to be simulated in discretised form. As an obvious restriction, these simulations would have to observe the condition that the sum of all transition rates has to remain smaller than unity, i.e. $\sum \pi(n \to n') \leq 1$ which translates into an upper bound for the usable micro time steps Δt_0 depending on the size of the population:

$$\triangle t_0 \leq \frac{2}{N(bN+2a)} \tag{5}$$

since the herding component in both transition rates assumes its maximum at $n = \frac{N}{2}$. The maximum admissible time increment obviously decreases hyperbolically with the population size, $\Delta t \propto N^{-2}$. Microscopic simulations, therefore, become increasingly more time consuming with increasing population size.

As an alternative, we could resort to simulating the stochastic dynamics of the population configuration which is summarised by the variable n. Rewriting (1) in terms of this state variable, one can see that transitions between states occur with probabilities:

$$\rho(n', t+\Delta t|n,t) = \begin{cases} 2\frac{(N-n)(a+bn)}{N^2b+2aN} & \text{if} \quad n' = n+1 \\ 2\frac{n(a+b(N-n))}{N^2b+2aN} & \text{if} \quad n' = n-1 \\ b\frac{N^2-4n(N-n)}{N^2b+2aN} & \text{if} \quad n' = n \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

In (6), Δt_0 had been chosen such that it allows the highest "efficiency" of the macroscopic simulations, i.e. such that it minimizes the probability to observe no change in n. This is equivelent to using (5) as an equality. Of course, only the

[2]There is, in fact, even a further intermediate step in which agents try to make an assessment of the prevailing majority opinion on the base of a noisy signal. The distribution of this observation rather than their own previous state is used in the computation of the weights of both groups.

smallest possible change in n can be observed during the micro-step Δt_0 of the simulation. Similarly as in Kirman and Teyssière (2002), we make a distinction between micro time steps Δt_0 and macro time increments Δt in which many increments of n may be observed. The difference is, however, that we also allow the price process to operate along the micro time scale and that together with the change of the population configuration we also aggregate over many small price changes at the macro time scale Δt. To illustrate the dynamics and to provide a justification of a "useful" macro time scale, consider the following scenario: for $n = 0$, the system can evolve like follows: it may remain unchanged with probability $\frac{N}{N+2\varepsilon}$, or it may change by one unit with the small probability $\frac{2\varepsilon}{N+2\varepsilon}, \varepsilon = \frac{a}{b}$. Therefore, the average number of iterations of (6) needed to observe a move of one agent is approximately equal to N. To observe larger increments of n (as changes by one unit are negligible, in particular for the case of large N), we need a multiple of N to define the number of micro time steps that make up one macro interval. From (5), a sensible choice appears to be:

$$\Delta t = \frac{b}{2} N^2 \Delta t_0 \tag{7}$$

In the simulations, one then iterates the process $\frac{b}{2}N^2$ times until one stores the current value of n as one realization at the macroscopic time scale Δt. The most interesting aspect of this approach is that it guarantees *invariance* of the dynamics of the macroscopic variable n with respect to the number of agents due to the flexibility of the chosen macro time scale. Fig. 1 provides an illustration of this feature in which we indeed observe no qualitative difference in the behavior of time series for different sizes of the population.

In terms of the intensive variable, (6) could be converted into

$$\pi(x \to x') = \begin{cases} \frac{1}{2}(1-x)(1+\gamma_N x) & \text{for} \quad x' = x + \frac{2}{N} \\ \frac{1}{2}(1+x)(1-\gamma_N x) & \text{for} \quad x' = x - \frac{2}{N} \\ \gamma_N x^2 & \text{for} \quad x' = x \\ 0 & \text{for} \quad otherwise \end{cases} \tag{8}$$

in which the maximum allowed micro time step Δt_0 has been factored into the parameter $\gamma_N = (1 + \frac{2\epsilon}{N})^{-1}$. Because of the dependency of Δt_0 and γ_N on system size, both (6) and (8) become very inefficient for large N. Note that if we introduce the intensity, $b = \frac{b_0 N_0}{N}$, the dependency on N in γ_N vanishes becoming $\gamma = (1 + 2\frac{a}{b_0 N_0})^{-1}$ and the macro time would become linear in N, $\Delta t = \frac{b_0 N_0}{2} N \Delta t_0$. Therefore, we would end up with "standard" linear time scaling, but the herding intensity would effectively decrease with population size N.

Keeping with the extensive formulation, one might, as a third alternative, simulate the model using a Langevin equation providing a Gaussian approximation to the stochastic dynamics over $\Delta t / \Delta t_0$ micro time steps per time unit Δt using drift and diffusion terms derived below (cf. eq 13):

$$x(t + \Delta t) = (1 - 2a\Delta t)x(t) + \sqrt{2b\Delta t(1 - x^2)}\eta(t + \Delta t) \tag{9}$$

In eq. (9), $\eta(t)$ is an iid noise variable following a Normal distribution with mean zero. In Appendix A1, we show how (9) can be derived from the Fokker-Planck equation if Δt is small enough to justify negligence of terms of order $(a\Delta t)^2$ and $(b\Delta t)^2$. Although for small step sizes Δt, the Langevin equation indeed provides a close approximation to the underlying agent-based model, it has the drawback that it might violate the built-in boundaries $x \in [-1, 1]$ of the population dynamics.

To keep x within meaningful bounds in simulations of (9), reflecting boundaries at $x = \pm 1$ need, therefore, to be imposed by hand.

Despite these difficulties, (9) has the important advantage that it not only facilitates numerical simulations, but also enables computation of conditional and unconditional moments of the price process.

4 Analytical Results: Unconditional Distributions and Moments

Sticking with our definition of transition rates in eq. (1), the group dynamics is characterized by the transition probabilities for agents' moves between the two states. These transition probabilities define a Markovian stochastic process which belongs to the class of so-called nonlinear "one-step processes" (cf. van Kampen, 1992). The nonlinearity of the process stems from the quadratic terms in eqs. (1) originating from the pair-wise infection pattern.

We can now analyse the dynamic evolution of the group configuration. The probabilities $\bar{\omega}(n,t)$ to have n agents in state one at time t, obey the so-called Master or backward Chapman-Kolmogorov equation, which gives the probability flux between states:

$$\bar{\omega}(n, t + \Delta t_0) = \sum_m (\rho_{n,m}\ \bar{\omega}(m,t) - \rho_{m,n}\ \bar{\omega}(n,t)) \tag{10}$$

with $\rho(n,m)$ the transition probabilities for a move of system from some state m to state n. Dividing both sides by Δt_0 and letting $\Delta t_0 \to 0$, we arrive at the so-called Master equation in continuous time:

$$\frac{\partial \bar{\omega}(n,t)}{\partial t} = \sum_m (\pi_{n,m}\ \bar{\omega}(m,t) - \pi_{m,n}\ \bar{\omega}(n,t)) \tag{11}$$

with $\pi_{n,m} = \frac{\rho_{n,m}}{\Delta t_0}$ the transition rates from states m to states n. In the limit $\Delta t_0 \to 0$, multiple jumps occur with probability zero, so that one only has to consider jumps to neighboring states, i.e. $m = n \pm 1$. In the following, we derive analytical approximations for various quantities of interest by means of a Fokker-Planck equation. Replacing n by the intensive variable x and treating it, for large N, as a continuous quantity, we can derive a Fokker-Planck-equation for the time change of the pertinent probability density (cf. Appendix A for details):

$$\frac{\partial \omega(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[2ax\ \omega(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\ \left[2b(1-x^2) + \frac{4a}{N}\right]\omega(x,t) \tag{12}$$

Neglecting the term of order N^{-1}, we, therefore, end up with a Fokker-Planck-equation with drift and diffusion terms $A(x)$ and $D(x)$ given by:

$$A(x) = -2ax, D(x) = 2b(1-x^2), \tag{13}$$

which are the mean and diffusion terms used in the Langevin eq (9). Since x is bounded between -1 and 1, we also have to add the 'natural' boundary conditions of the model, that the probability current:

$$j(x,t) = -2ax\omega(x,t) - \frac{\partial}{\partial x}(b(1-x^2)\omega(x,t)) \tag{14}$$

has to vanish at $|x| = 1$. It can be shown that (13) and (14) lead to an equilibrium distribution of x, $\omega_e(x)$, which only depends on the ratio $\varepsilon = \frac{a}{b}$: (cf. Appendix A):

$$\omega_e(x) = \frac{\Gamma(2\varepsilon)}{2^{2\varepsilon-1}\Gamma(\varepsilon)^2}(1-x^2)^{\varepsilon-1} \tag{15}$$

Inspection of (15) shows that it preserves the results of the ants model: $\varepsilon > 1$, i.e. dominance of autonomous switches, implies uni-modality with a peak at $x = 0$ and possibly small fluctuations of $ln(\frac{p_t}{p_f})$ about zero. Dominance of the herding component, $\varepsilon > 1$, however, leads to a bimodal distribution with probability mass reaching its maximum at ± 1 generating phases of overvaluation and undervaluation of the price compared to its fundamental value. In the knife-edge scenario $\varepsilon = 1$, the autonomous switching propensity is exactly compensated by the herding tendency, generating a uniform distribution of x.[3]

If one would like to investigate the case of pure herding, $\varepsilon = 0$, the boundary conditions would have to be modified in order to avoid absorbing states at $|x| = 1$. Following, for example, Lux and Marchesi (1999) reflecting boundary conditions can be imposed so that at least n_0 agents remain always optimistic or pessimistic. Let $\delta = n_0/N$, the boundary conditions then require the current to vanish at $|x| = 1-\delta$. The parameter δ can become arbitrary small for $N \to \infty$. As shown via numerical simulations in Alfarano and Lux (2003) the stochastic switches between the vicinity of $x = 1$ and $x = -1$, give rise to volatility clustering. In Appendix B we provide theoretical results for this particular variant of the herding model.

Using the equilibrium distribution (15) we can proceed by deriving a compact formula for the calculation of moments (cf. Appendix A):

$$E\left[\left(1-x^2\right)^k\right] = 2^{2k}\frac{\Gamma(2\varepsilon)}{\Gamma(2\varepsilon+2k)}\left(\frac{\Gamma(\varepsilon+k)}{\Gamma(\varepsilon)}\right)^2 \tag{16}$$

Equipped with this result, we can invoke the Langevin equation for the dynamics of x to characterize the ensuing dynamics of relative price changes. Neglecting changes of the fundamental value, we see that (4) simply leads to:

$$r(t,\Delta t) = x(t+\Delta t) - x(t) = -2ax(t)\Delta t + \sqrt{2b\left(1-x^2\right)\Delta t}\,\eta(t+\Delta t) \tag{17}$$

Eq. (17) allows to determine the expectation of moments r^{2k} from those of x^{2k}. In leading order Δt we get:

$$E\left[r(t,\Delta t)^{2k}\right] = (2b)^k E\left[\left(1-x^2\right)^k \eta^{2k}\right]\Delta t^k \tag{18}$$

The second moment and kurtosis, for example, are given by:

$$E\left[r(t,\Delta t)^2\right] = \frac{4a\Delta t}{2\varepsilon+1}, \quad \frac{E\left[r(t,\Delta t)^4\right]}{E\left[r(t,\Delta t)^2\right]^2} - 3 = \frac{3}{\varepsilon(2\varepsilon+3)} \tag{19}$$

Kurtosis is positive and vanishes only for $\varepsilon \to \infty$ so that even in the case of a uni-model distribution, the herding mechanism would lead to a leptokurtotic shape of returns which is in agreement with the ubiquitious empirical deviation of relative price changes from Gaussian behaviour. Since $\varepsilon \to \infty$ would either be due to

[3] Alfarano, Lux and Wagner (2004) derive closed-form solutions for an asymmetric herding model with different propensities for autonomous moves, a_1 and a_2, in one or the other direction.

abscence of herding or dominance of stochastic search due to very large a compared to b, we observe that it is the introduction of interpersonal communication which is crucial for emergence of leptokurtic returns. In the case $\varepsilon = 0$, we can show that, for small δ:

$$\begin{aligned} E\left[r\left(t,\Delta t\right)^2\right] &= 4b\Delta t\left(ln\left(\frac{2}{\delta}-1\right)\right)^{-1} \\ \frac{E\left[r\left(t,\Delta t\right)^4\right]}{E\left[r\left(t,\Delta t\right)^2\right]^2}-3 &= ln\left(\frac{2}{\delta}-1\right)-3 \end{aligned} \tag{20}$$

which again proves that also this particular variant of the model leads to a widening of the probability distribution of returns in its outer parts[4].

5 Time Scales and Correlations

Conditional properties of the population index x and returns can be obtained in various ways: first, using the general solution to the Fokker-Planck equation derived in Appendix B, we could provide a complete charaterization of conditional moments which is supplied in Appendix C. Alternatively, correlation functions could be determined recursively using the Langevin approximation of the Fokker-Planck equation. For discrete time steps $t = n\Delta t$, we consider:

$$F_n = E\left[x_t x_0\right], \tag{21}$$

for which the Langevian equation provides us with the recursion formula:

$$F_n = \left(1-2a\Delta t\right)F_{n-1} \tag{22}$$

Backward iterating leads to:

$$F_n = F_0\left(1-2a\Delta t\right)^n \tag{23}$$

with $F_0 = E[x_0^2]$, which can be used to determine the auto-correlation of $x(t), C_x(t)$:

$$C_x(t) = \left(1-2a\Delta t\right)^{\frac{t}{\Delta t}} \tag{24}$$

which for $\Delta t \to 0$ leads to $C_x(t) \sim exp(-2at)$.

The correlation of r_t can similarly be expressed in terms of F_n:

$$E\left[r_{t+1}r_0\right] = E\left[\left(x_{t+\Delta t}-x_t\right)\left(x_{\Delta t}-x_0\right)\right] = 2F_n - F_{n-1} - F_{n+1} \tag{25}$$

Therefore, for $n > 0$, we obtain:

$$C_r(n\Delta t) = \frac{2F_n - F_{n-1} - F_{n+1}}{E\left[\left(x_{\Delta t}-x_0\right)^2\right]} \tag{26}$$

Using $E\left[\left(x_{\Delta t}-x_0\right)^2\right] = 4a\Delta t F_0$ and inserting the solution for F_n, one ends up with the approximation[5]:

[4] A mathematically curious case is the one obtained for $\varepsilon = 0.5$, in which the time development of the population index x turns out to be the sine of a random walk. This can be seen by performing the transformation $x = sin(\theta)$ in which case the Langevin equation becomes: $\theta\left(t+\Delta t\right) = \theta\left(t\right) - \sqrt{2b\Delta t}\ \eta\left(t+\Delta t\right)$.

[5] Using the Taylor series expansion $e^x = 1 + x + \frac{x^2}{2} + o(x^3)$ for both F_{n+1} and F_{n-1}

$$C_r(n\Delta t) = -a\Delta t(1-2a\Delta t)^{n-1} \approx -a\Delta t\ exp(-2at) \tag{27}$$

In principle, the small negative correlation is at odds with the informational efficiency and random walk nature of financial markets. However, the small mean-reverting tendency of the population index and returns can easily be blurred by the noise level generated by the finiteness of the data sample and would not easily be detected if a were not too large.

One can proceed similarly in deriving the auto-correlation functions of higher moments. In order to compute C_{r^2} one starts with the relation[6]:

$$E\left[r^2_{t+\Delta t} r^2_{\Delta t}\right] = 4b^2\Delta t^2\left[1 - 2E(x_t^2) + E(x_t^2 x_0^2)\right] \tag{28}$$

The last term can be determined using the above recursive method:

$$E\left(x_t^2 x_0^2\right) = E[x_0^2]^2 + \left[E\left(x_0^4\right) - E\left(x_0^2\right)^2\right]\ \ exp\left[-2\left(2\varepsilon+1\right)bt\right] \tag{29}$$

so that we finally arrive at:

$$C_{r^2}(t) = \frac{1}{4\varepsilon^2+6\varepsilon+3}\ \ exp\left[-2\left(1+2\varepsilon\right)bt\right] \tag{30}$$

Obviously, $C_{r^2}(t) > 0$ for all $t > 0$ so that we have shown that volatility persistence is a natural outcome of this model.

It is interesting to complement the analysis of conditional properties by closed-form solutions of so-called mean first passage times (MFPT), the time it takes the system on average to switch from one state to another. Of greatest interest for the calculation of MFPTs are the modes and anti-modes $x = 0, -1$ and 1.

As shown in Appendix B, we get for $0 < \varepsilon < 1$ an MFTP for the time it needs for a transition between the two modes $x = 1$ and $x = -1$:

$$T\left(-1 \to 1\right) = T\left(1 \to -1\right) = \frac{1}{b}\ \frac{\pi}{1-2\varepsilon}\ \frac{cos(\pi\varepsilon)}{sin(\pi\varepsilon)} \tag{31}$$

As can be seen, $T(\cdot)$ diverges for $\varepsilon \to 0$ which corresponds to the emergence of two absorbing states at $|x| = 1$ in the case of a vanishing autonomous switching probability. The lock-in at one of the extremal modes from which the dynamics would not return with $a = \varepsilon = 0$ is mirrored in a divergence to infinity of its MFPT. For $\varepsilon \to 1$, divergence is obtained because for the uni-model case established for $\varepsilon > 1, \pm 1$ are reached with a vanishing probability. Interestingly and maybe running against the first intuition, the MFPT is not monotonic in ε, but has a minimum at $\varepsilon = 0.5$, i.e. $a = 0.5b$, and at $\varepsilon = 0.5$ an increase of either the autonomous component or the herding propensity would lead to an increase of $T(\cdot)$.

If we move to the intensive formulation, eqs $(\cdot)$ to $(\cdot)$ still apply but would contain a dependency on the number of agents via $b = \frac{b_0 N_0}{N}$, i.e.

...

Obviously, volatility dependence gets lost with an increasing population size as illustrated in Fig. ?.

The loss of volatility dependence corresponds to the approach of the unconditional distribution towards the Gaussian pointed out in Sec. 4. It also shows up in a dependency of the MFPT on system size in the intensive setting. As can be seen from inserting the flexible b into eq $(\cdot)$, the MFPT can be written as:

[6] Additional details are to be found in Appendix A4

$$T\left(-1 \to 1\right) = \frac{4}{a}\left(1 - \frac{N}{N_c}\right)^{-1} \tag{32}$$

with a critical number of agents $N_c = \frac{b_0 N_0}{a}$ at which the MFPT diverges because of the transition to uni-modality of the unconditional distribution. Time series of x_t and r_t for various numbers of agents under the intensive formalization are exhibited in Fig. ?. The approach to the trivial Gaussian behaviour is clearly visible. Similar findings have been reported for the more involved model of Lux and Marchesi (1999) with three different groups by Egenter, Lux and Stauffer (1999). The triviality of results in the large N limit in this model as well as related ones, hence, is an immediate consequence of the at first view innocuous intensive framework used for modelling the herding mechanism. We have already argued in sec. 3 that the intensive alternative leads to an effective decrease of the herding propensity with larger N and might, therefore, not match the intuition of the underlying approach. If one accepts this view, the realistic time series properties of fat tails and volatility clustering would, in fact, be immune to the number of agents within the artificial market.

It is interesting to note that dependency on system size could also be brought about in the case of reflecting boundaries with $\varepsilon = 0$ mentioned in sec. 4. As shown in Appendix C, using fixed boundaries n_0 for the minimum number of agents in each group, the correlation of any odd function of x_t decays at large t exponentially with a decay constant

$$t_D = \frac{1}{2b}\ ln\left(\frac{2N}{n_0}\right) = \frac{1}{2b}\ ln\left(\frac{2}{\delta}\right) \tag{33}$$

which obviously diverges for $N \to \infty$. The same behaviour is reflected in the first passage time from the outer mode, $x = 1 - \delta$ in this case, to the anti-mode at $x = 0$:

$$T\left(1 - \delta \to 0\right) = \frac{1}{2b}\ ln\left(\frac{1}{2\delta}\right) + o(\delta) \tag{34}$$

The reversed transition from 0 to $1 - \delta$, however, occurs with a natural time scale $\frac{1}{b}$:

$$T\left(0 \to 1 - \delta\right) = \frac{ln2}{b} + o(\delta) \tag{35}$$

Adding up both MFPTs, one exactly recovers the decay constant t_0 which, hence, also charaterizes the average time needed for transitions from $x \sim 1$ to $x \sim -1$. Hence, for large N, even in the extensive formulation, the system would spend more and more time in the vicinity of the extremal regions $|x| \sim 1$ before eventually switching to the opposite state. Limited simulation runs would, then, not necessarily reveal the bi-modal nature of the equilibrium distribution, but would rather lock in at one particular region of high probability. Although we were not able to derive the autocorrelation of even moments in this case, one might argue that the less frequent switching at large N would also reduce volutility auto correlations. Simulations, in fact, confirm this conjecture. Of course, it is obvious from eqs $(\cdot)$ that a re-scaling of the boundaries n_0 as a constant fraction of N would restore independence of all unconditional properties from system size. This underlines the importance of the precise design of multi-agent models for the emergent time series properties of artificial markets.

A Appendix A: Solution of the Kirman Model

A.1 Derivation of the Fokker-Planck

In this appendix we derive the Fokker-Plank equation introduced in eq. (**??**) as a second-order Taylor approximation to the continuous limit of our population dynamics. The first step is the description of the transition probabilities, as in eq. (**??**), governing the agents' switching process between the two states[7]:

$$\rho_{n-1,n} = \Delta t_0 n[a + b(N-n)] \tag{36}$$
$$\rho_{n+1,n} = \Delta t_0 (N-n)(a+bn) \tag{37}$$
$$\rho_{n,n} = 1 - \rho_{n,n-1} - \rho_{n,n+1} \tag{38}$$

where a and b are constant; Δt_0 is fixed to satisfy eq. (**??**) with equality. The previous transition probabilities define a Markovian stochastic process, that belongs to the so-called non-linear "one-step processes" [**?**]. The quadratic term in the transitions gives rise to the non-linear nature of the process.

We can formulate the dynamic equation for $\bar{\omega}_n(t)$[8], i.e. the probability to have n agents in state one at the time $t + \Delta t_0$:

$$\bar{\omega}(n, t+\Delta t_0) = \bar{\omega}_{n+1}(t)\rho_{n,n+1} + \bar{\omega}_{n-1}(t)\rho_{n,n-1} + \bar{\omega}_n(t)\rho_{n,n} \tag{39}$$

The equation (39) simply states that the probability to find n agents in the state 1 at the time $t + \Delta t_0$ is given by three contributions:

- the switch of one agent from state 1 to state 2, and $n+1$ agents in the state 1 at time t;
- the switch of one agent from state 2 to state 1, and $n-1$ agents in the state 1 at time t;
- no switch, and n agents in the state 1 at time t.

Given eq. (38), we rewrite eq. (39):

$$\bar{\omega}_n(t+\Delta t_0) - \bar{\omega}_n(t) = \bar{\omega}_{n+1}(t)\rho_{n,n+1} + \bar{\omega}_{n-1}(t)\rho_{n,n-1} - \bar{\omega}_n(t)\rho_{n+1,n} - \bar{\omega}_n(t)\rho_{n-1,n} \tag{40}$$

and dividing both sides by Δt_0, we have:

$$\frac{\bar{\omega}_n(t+\Delta t_0) - \bar{\omega}_n(t)}{\Delta t_0} = \bar{\omega}_{n+1}(t)\pi_{n,n+1} + \bar{\omega}_{n-1}(t)\pi_{n,n-1} - \bar{\omega}_n(t)\pi_{n+1,n} - \bar{\omega}_n(t)\pi_{n-1,n} \tag{41}$$

where $\pi_{n,n+1} = \frac{\rho(n,n+1)}{\Delta t_0}$ are the transition rates associated to the above transition probabilities. We express the previous equation as a **Master equation**:

$$\frac{\partial \bar{\omega}_n(t)}{\partial t} = \sum_{m=n\pm 1} (\pi_{n,m}\bar{\omega}_m - \pi_{m,n}\bar{\omega}_n)$$

where the index m takes only two value $n+1$ and $n-1$. This equation states that the rate of change of the probability in time is given by a competition between two terms: the probability outflow from and the probability inflow in a particular state. The probability current is given by:

$$j_n(t) = \bar{\omega}_{n-1}(t)\pi_{n,n-1} - \bar{\omega}_n(t)\pi_{n-1,n} \tag{42}$$

[7] $\rho(n-1, n)$ reads as $\rho(n \to n-1)$

[8] $\bar{\omega}_n(t+\Delta t_0)$ represents the probability function of the *discrete* variable n.

that allows to rewrite the Master Equation as a continuity equation[9]:

$$\frac{\partial \bar{\omega}_n(t)}{\partial t} = -[j_{n+1}(t) - j_n(t)] \tag{43}$$

The configuration of the system can be conveniently described by an intensive variable:

$$x = \frac{2n - N}{N}, \qquad \Delta x = \frac{2}{N}$$

Let us now rewrite eq. (42) in terms of the new variable x:

$$\begin{aligned} j(x) &= a[1-(x-\Delta x)]\frac{N}{2}\bar{\omega}(x-\Delta x) - a(1+x)\frac{N}{2}\bar{\omega}(x) + \\ &\quad + b\frac{N}{2}\{[1+(x-\Delta x)][1-(x-\Delta x)]\frac{N}{2}\bar{\omega}(x-\Delta x) - (1-x^2)\frac{N}{2}\bar{\omega}(x)\} \end{aligned}$$

Eq. (43) can be reformulated as a function of x:

$$\begin{aligned} \frac{\partial \bar{\omega}(x,t)}{\partial t} = -[j(x+\Delta x) - j(x)] &= +a\frac{N}{2}\bar{\omega}(x+\Delta x) - 2a\frac{N}{2}\bar{\omega}(x) + a\frac{N}{2}\bar{\omega}(x-\Delta x) + \\ &\quad + a[(x+\Delta x)\frac{N}{2}\bar{\omega}(x+\Delta x) - x\frac{N}{2}\bar{\omega}(x)] + \\ &\quad + a[x\frac{N}{2}\bar{\omega}(x) - (x-\Delta x)\frac{N}{2}\bar{\omega}(x-\Delta x)] + \\ &\quad + b\frac{N}{2}\{[1-(x+\Delta x)^2]\frac{N}{2}\bar{\omega}(x+\Delta x) - 2(1-x^2)\frac{N}{2}\bar{\omega}(x) + \\ &\quad + [1-(x-\Delta x)^2]\frac{N}{2}\bar{\omega}(x-\Delta x)\} \end{aligned}$$

that can be transformed into a more opportune form:

$$-\Delta x\frac{j(x+\Delta x,t) - j(x,t)}{\Delta x} =$$

$$a\Delta x^2\frac{[\frac{N}{2}\bar{\omega}(x+\Delta x,t) - 2\frac{N}{2}\bar{\omega}(x,t) + \frac{N}{2}\bar{\omega}(x-\Delta x,t)]}{\Delta x^2} +$$

$$a\Delta x\frac{[(x+\Delta x)\frac{N}{2}\bar{\omega}(x+\Delta x,t) - x\frac{N}{2}\bar{\omega}(x,t)]}{\Delta x} + a\Delta x\frac{[x\frac{N}{2}\bar{\omega}(x,t) - (x-\Delta x)\frac{N}{2}\bar{\omega}(x-\Delta x,t)]}{\Delta x} +$$

$$b\Delta x^2\frac{N}{2}\{\frac{[1-(x+\Delta x)^2]\frac{N}{2}\bar{\omega}(x+\Delta x,t) - 2(1-x^2)\frac{N}{2}\bar{\omega}(x,t) + [1-(x-\Delta x)^2]\frac{N}{2}\bar{\omega}(x-\Delta x,t)}{\Delta x^2}\}$$

For $\Delta x \to 0$, the first derivative of a '*well behaved*' function can be approximated in the following way:

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x} \tag{44}$$

and its second derivative can be written as:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+\Delta x,y) - 2f(x,y) + f(x-\Delta x,y)}{\Delta x^2} \tag{45}$$

[9]The term continuity equation indicates that a *quantity*, characterized by a density function $\rho(x,t)$ depending on time and space, is conserved, i.e. if it holds that $\int \rho(x,t)dx = Q$, where Q is constant, then the density function satisfies the continuity equation $\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial j(x,t}{\partial x} = 0$.

Given the approximations (44) and (45), we can express the previous equation in a more compact way:

$$\frac{\partial\bar{\omega}(x,t)}{\Delta x\partial t} \simeq -\frac{\partial j(x,t)}{\partial x} = +\frac{2a}{N}\frac{\partial^2[\frac{N}{2}\bar{\omega}(x,t)]}{\partial x^2} + \\ +2a\frac{\partial[x\frac{N}{2}\bar{\omega}(x,t)]}{\partial x} + b\frac{\partial^2[(1-x^2)\frac{N}{2}\bar{\omega}(x,t)]}{\partial x^2}$$

and rearranging the terms, we end up with:

$$\frac{\partial\frac{N}{2}\bar{\omega}(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} = 2a\frac{\partial x\frac{N}{2}\bar{\omega}(x,t)}{\partial x} + b\frac{\partial^2[(1-x^2)\frac{N}{2}\bar{\omega}(x,t)]}{\partial x^2} + \frac{2a}{N}\frac{\partial\frac{N}{2}\bar{\omega}(x,t)]}{\partial x} \tag{46}$$

For large N, we can approximate the discrete variable x by a continuous variable; therefore instead of the probability function $\bar{\omega}(x,t)$, we describe the system in terms of:

$$\omega(x,t) = \lim_{\Delta x\to 0}\frac{\bar{\omega}(x,t)}{\Delta x} = \lim_{N\to+\infty}\frac{N}{2}\bar{\omega}(x,t)$$

Eq. (46) becomes:

$$\frac{\partial\omega(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} = \frac{\partial}{\partial x}\left[2ax\omega(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[2b(1-x^2) + \frac{4a}{N}\right]\omega(x,t) \tag{47}$$

Eq. (47), is a so-called Fokker-Plank equation, i.e. a linear partial differential equation of second order, with drift and diffusion functions given by:

$$A(x) = -2ax \qquad D(x) = 2b(1-x^2) + \frac{4a}{N} \tag{48}$$

For finite b the term proportional to a/N can be neglected, which proves eq. (**??**). However, it can not be ignored when we discuss the vanishing herding component in appendix B1.

A.2 Unconditional distribution of x and returns

The following textbook formula allows to compute the equilibrium distribution $\omega_e(x)$, given in eq. (**??**) in the main text, if we know the diffusion and drift functions:

$$\omega_e(x) = \frac{K}{D(x)}\exp\left(\int^x\frac{2A(y)}{D(y)}dy\right) \tag{49}$$

(see for instance Van Kampen [**?**]). The first step consists in solving the following integral:

$$\int\frac{2A(x)}{D(x)}dx = -\frac{2a}{b}\int\frac{x}{1-x^2}dx$$

where we take the drift and diffusion functions from eq. (48), neglecting the last term in $D(x)$. With the definition of ε from eq. (**??**) and a little effort, the solution is:

$$\int\frac{2A(x)}{D(x)}dx = \varepsilon\ln(1-x^2)$$

that plugged into (49), gives the equilibrium distribution $\omega_e(x)$:

$$\omega_e(x) = K(1-x^2)^{\varepsilon-1}$$

The constant K is computed via normalization condition:

$$K\int_{-1}^{+1}\omega_e(x)dx = 1 \tag{50}$$

Recalling the definition of beta function:

$$B(\varepsilon_1, \varepsilon_2) := \int_0^1 z^{\varepsilon_1 - 1}(1-z)^{\varepsilon_2 - 1} dz = \frac{\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)}{\Gamma(\varepsilon_1 + \varepsilon_2)} \tag{51}$$

that, for symmetric coefficients $\varepsilon_1 = \varepsilon_2$ and with the substitution $z = \frac{1}{2}(1+x)$, becomes:

$$\int_{-1}^{+1} (1-x^2)^{\varepsilon - 1} dx = 2^{2\varepsilon - 1} \frac{\Gamma(\varepsilon)^2}{\Gamma(2\varepsilon)} \tag{52}$$

help us to solve the integral (50), obtaining $K = \frac{\Gamma(2\varepsilon)}{2^{2\varepsilon-1}\Gamma(\varepsilon)^2}$ and eq.(**??**) in the main text.

It is interesting to show that for $\varepsilon >> 1$, the equilibrium distribution is well approximated by a Gaussian with mean zero and variance $1/2\varepsilon$. Eq. (**??**) can be written as:

$$\omega_e(x) \propto \exp\left[(\varepsilon - 1)\ln(1-x^2)\right] \approx \exp\left[-\frac{1}{2} 2\varepsilon x^2\right] \tag{53}$$

where the last step follows the approximation:

$$\ln(1-x^2) \approx -x^2$$

or $|x| \ll 1$.

Using eq. (**??**) we can derive a compact formula for the calculation of moments:

$$E[(1-x^2)^k] = \frac{\Gamma(2\varepsilon)}{2^{2\varepsilon-1}\Gamma(\varepsilon)^2} \int_{-1}^{1} \left(1-x^2\right)^{\varepsilon - 1} (1-x^2)^k dx \tag{54}$$

Using (51) to evaluate the integral, eq. (54) becomes:

$$E[(1-x^2)^k] = 2^{2k} \frac{\Gamma(2\varepsilon)}{\Gamma(2\varepsilon + 2k)} \left[\frac{\Gamma(\varepsilon + k)}{\Gamma(\varepsilon)}\right]^2 \tag{55}$$

Using the Langevin approximation eq. (**??**), we can now easily compute the unconditional properties of returns, assuming that the fundermental value remains constant:

$$r(t, \Delta t) = x(t + \Delta t) - x(t) = -2ax(t)\Delta t + \sqrt{2b\Delta t(1-x^2)}\eta(t + \Delta t) \tag{56}$$

Eq. (56) relates the expectation of r^{2n} to that of x^{2n}. In leading order Δt, we get

$$E(r(t, \Delta t)^n) = (2b)^{2n} E[(1-x^2)^{2n}\eta^{2n}]\Delta t^n \tag{57}$$

Since $\eta(t)$ is independent of $x(t)$ we can simply insert the Gaussian moments for η^{2n}. Using eq. (55) for the moments of $1 - x^2$, we obtain (**??**).

A.3 Integration of the Fokker-Planck equation

To integrate the Fokker-Planck equation, we can use the separation variable technique, since the drift and diffusion terms are independent on t. We assume, then, a solution of the form:

$$\omega(x, t) = T(t)\omega_e(x)P(x) \tag{58}$$

The equation (**??**) separates into:

$$\frac{T'(\tau)}{T(\tau)} = \frac{\Theta[P''(x), P'(x), P(x)]}{P(x)} = -\lambda \tag{59}$$

where $\tau = bt$ is a re-scaled time variable, and Θ is a function of $P(x)$, its first and second derivative. Eq. (59) is satisfied for every τ and x only in the case that both, the left and right hand sides, are equal to an arbitrary constant $-\lambda$. Therefore, the time dependence is of the form:

$$T(\tau) = T(0)e^{-\lambda\tau} \tag{60}$$

and, making all the calculations, $P(x)$ turns out to satisfy the adjoint differential equation:

$$(1-x^2)P''(x,\lambda) - 2\epsilon x P'(x,\lambda) + \lambda P(x,\lambda) = 0 \tag{61}$$

The previous equation is a well-known ordinary differential equation of the second order [**?**], whose general solutions are *hypergeometric* functions $F(x;\varepsilon,\lambda)$. If we plug the series expansion of the solution $F = \sum_{k=0}^{\infty} a_k x^k$ in eq. (61), we obtain the recursion formula:

$$a_{k+2} = a_k \frac{k(k+2\varepsilon-1)-\lambda}{(k+2)(k+1)} \tag{62}$$

that provides the values of the coefficients a_k, given the two initial values a_0 and a_1. Then, given the linear structure of the F-P, its general solution will be a linear combination[10] of the type:

$$w(x,t) = w_e(x)\sum_n c_n F_n(x;\varepsilon,\lambda)\exp(-\lambda bt) \tag{63}$$

Using eq. (**??**), the current is given by

$$j = -b\sum_n c_n\ w_e(x)(1-x^2)F_n'(x;\varepsilon,\lambda)\exp(-\lambda bt) \tag{64}$$

To fully characterize the solution of (**??**), we have to set the boundary conditions. Since the conservation of probability, the current has to vanishes at the borders $x = 1$ and $x = -1$. However, for $\varepsilon > 0$, the hypergeometric functions $F_n(x;\varepsilon,\lambda)$ have a singular derivative at $|x| = 1$. To show this, we expand the functions in a power series around the two points $x = \pm 1$ is[11]:

$$F(x;\varepsilon,\lambda) = (x \mp 1)^{(1-\varepsilon)}\sum_k a_k (x \mp 1)^k \tag{65}$$

Using eq. (65), the first order in the expansion of $F_n'(x;\varepsilon,\lambda)$ around those two points is in fact:

$$F'(x;\varepsilon,\lambda) = a_0(1-\varepsilon)(x \pm 1)^{-\varepsilon} \tag{66}$$

Plugging eq. (66) and eq. (**??**) in eq. (64), the current is never zero, unless $F_n(x;\varepsilon,\lambda)$ are polynomials, which lead to vanishing coefficients up to index n in the recursion formula (62). This implies that λ is discrete and it satisfies the relation:

$$\lambda_n = n(n+2\varepsilon-1) \tag{67}$$

These polynomials are the **Gegenbauer polynomials**[12] $C_n(x,\varepsilon)$ with eigenvalues $n(n+2\varepsilon-1)$. These polynomials are orthogonal with respect to the metric $w_e(x)$, that means:

$$\int_{-1}^{1} C_n(x,\varepsilon)C_m(x,\varepsilon)w_e(x)dx = h_n(\varepsilon)\delta_{n,m} \tag{68}$$

[10] A more rigorous mathematical derivation would involve an integral rather than a summation over the integer index n; however we anticipate the discreetness of the solution, given by the boundary conditions, in order to leave out some unnecessary technicalities.

[11] The two points $x = \pm 1$, for which the coefficient function of the second order term in (61) vanishes, are called nonessential singularities. The series expansion around those points is easily calculated using the theorem of Fuchs [**?**].

[12] Plugging the relation (67) in (61), we obtain, in fact, the so-called Gegenbauer differential equation, whose solutions are the Gegenbauer polynomials [**?**].

where $h_n(\varepsilon)$ is a normalization factor[13]. They form a complete set of basis functions in $x\epsilon[-1,+1]$, which allows to express *every* other function in the same interval as a linear combination of them. The first three polynomials are:

$$C_0 = 1 \quad , \quad C_1 \propto x \quad , \quad C_2 \propto x^2 - \frac{1}{2\epsilon + 1} \tag{69}$$

In conclusion, the solution of the F-P (**??**), with the appropriate boundary conditions, assume the final form:

$$w(x,t) = w_e(x) \sum_n c_n C_n(x,\varepsilon) \exp\left[-n(n+2\varepsilon-1)bt\right] \tag{70}$$

where the coefficients c_n are fixed by the initial distribution $w(x_0,t_0)$. Let us derive the solution with a initial condition picked[14] at x_0:

$$\omega_0(x) = \delta(x - x_0) = w_e(x) \sum_n c_n C_n(x,\varepsilon) \tag{71}$$

Multiplying both sides by $J_m(x,\varepsilon)$ and integrating, we have:

$$\int_{-1}^{1} \delta(x-x_0) C_m(x,\varepsilon) dx = \sum_n \int_{-1}^{1} c_n C_n(x,\varepsilon) C_m(x,\varepsilon) w_e(x) dx \tag{72}$$

And using the orthogonality (68), we end up with:

$$c_n = \frac{C_n(x_0)}{h_n(\varepsilon)} \tag{73}$$

that fix the value of the coefficient for the chosen starting distribution.

Special values of ε lead to more simple functions. E.g. at $\varepsilon = 1/2$, expressing x in terms of an angle $x = \sin(\phi)$, we have

$$T_{2n} = \cos(2n\phi) \quad , \quad T_{2n+1} = \sin((2n+1)\phi) \tag{74}$$

and they are called **Chebyshev** polynomials.

A.4 Conditional properties of x and returns: autocorrelation functions

The general solution (70) of the F-P allows for a complete characterization of the conditional moments of x and returns, such as the autocorrelation functions.

Let us start with a function $f(x)$. The auto-covariance is:

$$< f(t)f(0) >_t = E[f(x)\omega(x,t|x_0,0)f(x_0)\omega_e(x_0)] \tag{75}$$

$$= \int_{-1}^{1}\int_{-1}^{1} f(x)\omega(x,t|x_0,0)f(x_0)\omega_e(x_0)dxdx_0 \tag{76}$$

$$= \sum_n \frac{\exp(-b\lambda_n t)}{h_n(\varepsilon)} \int_{-1}^{1} f(x)C_n(x)\omega_e(x)dx \int_{-1}^{1} f(x_0)C_n(x_0)\omega_e(x)dx_0 \tag{77}$$

where we replace the average over time, labelled with the subscript t, with the average over the distribution $\omega(x,t|x_0,0)$, which represents the distribution of x

[13] Since the arbitrariness in the normalization choice, we keep $h_n(\varepsilon)$ as a general function.

[14] The defining property of the Dirac's delta is $\int_a^b f(x)\delta(x-x_0)dx = f(x_0)$ if $x_0 \in [a,b]$.

at the time t, given the starting distribution (71). The starting value x_0 is, then, averaged over the equilibrium distribution $\omega_e(x_0)$. We arrive to:

$$\sum_n E[f(x)C_n]^2 \frac{\exp(-\lambda_n bt)}{h_n(\varepsilon)} \tag{78}$$

Then:

$$< f(t) >_t < f(0) >_t = < f(0) >_t^2 = E[f(x)\omega_0(x)]^2 = \left[\int_{-1}^{1} f(x_0)\omega_e(x_0)dx_0\right]^2 \tag{79}$$

And the variance:

$$\{Var[f(t)]Var[f(0)]\}^{\frac{1}{2}} = Var[f(0)] = \int_{-1}^{1} f^2(x_0)\omega_e(x_0)dx_0 - \left\{\int_{-1}^{1} f(x_0)\omega_e(x_0)dx_0\right\}^2 \tag{80}$$

Finally, auto-correlation of a function $f(x)$ can be obtained from the general formula[15]

$$C_f(t) = \sum_{n>0} \frac{E[C_n f]^2}{Var(C_n)Var(f)} \cdot \exp(-\lambda_n bt) \tag{81}$$

For $f = x$, due to the orthogonality of the Jacobi polynomials[16], only the term $n = 1$ contributes to the sum in (81). With the moments (55), formula (**??**) is reproduced.

Alternatively, correlation functions can be determined recursively using the Langevin approximation of the Fokker-Plank. For the discrete time $t = n\Delta t$, we consider

$$F_n = E[x_t x_0] \qquad F_0 = E(x^2) \tag{82}$$

From the Langevin equation (56), we obtain a recursion for F_n:

$$F_n = (1 - 2a\Delta t)F_{n-1} \tag{83}$$

Inserting the solution

$$F_n = F_0(1 - 2a\Delta t)^n \tag{84}$$

into the auto-correlation formula of x, we get

$$C_x(t) \;=\; (1 - 2a\Delta t)^{t/\Delta t} \sim \exp(-2at) \tag{85}$$

where the exponential form is exact in the limit $\Delta t \to 0$. Eq. (85) agrees with eq. (**??**) obtained previously from the Fokker-Planck equation.

The correlation of r can also be expressed in terms of F_n:

$$E(r_{t+\Delta t} r_t) = E[(x_{t+\Delta t} - x_t)(x_1 - x_0)] = 2F_n - F_{n-1} - F_{n+1} \tag{86}$$

Therefore for $n > 0$:

$$C_r(n\Delta t) = \frac{2F_n - F_{n-1} - F_{n+1}}{E[(x_{\Delta t} - x_0)^2]} \tag{87}$$

Using $E[(x_{\Delta t} - x_0)^2] = 4a\Delta t F_0$ and inserting F_n into (87), one proves (**??**) given in the main text[17].

[15] We introduce the term $Var(C_n)$ to avoid the dependence on $h_n(\varepsilon)$.

[16] Note the the first polynomial is x, see (69).

[17] For F_{n+1} and F_{n-1} use the Taylor expansion $e^x = 1 + x + \frac{x^2}{2} + o(x^3)$.

Computation of C_{r^2} is slightly more complicated. One begins with the relation:

$$E(r^2_{t+\Delta t} r^2_1) = 4b^2 \Delta t^2 [1 - 2E(x^2) + E(x^2_t x^2_0)] \tag{88}$$

where the last term can be easily derived using the same recursive method. Then we have:

$$E(x^2_t x^2_0) = [E(x^4_0) - E(x^2_0)^2] \exp -2(2\varepsilon + 1)bt \tag{89}$$

Then, the auto-correlation of squared returns is given by:

$$C_{r^2}(t) = \frac{1}{4\epsilon^2 + 6\epsilon + 3} \exp\left[-2(1 + 2\varepsilon)bt\right] \tag{90}$$

In order to compute the auto-correlation function of the second moment, we use eq. (9) to substitute for the autocovariances of r^2_t:

$$\begin{aligned} E[r^2_{t+\Delta t} r^2_{\Delta t}] &= E[(x_{t+\Delta t} - x_t)^2 (x_{\Delta t} - x_0)^2] \\ &= E[(-2a\Delta t x_t + \sqrt{2b(1 - x^2_t)\Delta t}\eta_t)^2 (-2a\Delta t x_0 + \sqrt{2b(1 - x^2_0)\Delta t}\eta_0)^2] \\ &= E[2^2 b^2 \Delta t^2 (1 - x^2_t)(1 - x^2_0) + \theta(\Delta t^2)] \\ &= E[4b^2 \Delta t^2 (1 - x^2_t - x^2_0 - x^2_t x^2_0) + \theta(\Delta t^2)] \end{aligned}$$

It is obvious, that this expression is identical in leading order Δt^2 to eq. (28) in the main text. It can be easily derived from (16) that

$$E[x^2_t] = \frac{1}{2\epsilon + 1}.$$

In order to obtain a closed-form solution for $E[x^2_t x^2_0]$, we again make use of a recursive determination of the auto-covariances. Starting with a unit micro step Δt we have

$$\begin{aligned} E[x^2_{\Delta t} x^2_0] &= E[((1 - 2a\Delta t)x_0 + \sqrt{2b(1 - x^2_0)\Delta t}\eta_0)^2) x^2_0] \\ &= E[(1 - 4a\Delta t)x^4_0 - 2b\Delta t \eta^2_0 x^4_0 + 2b\Delta t \eta^2_0 x^2_0 + \theta(\Delta t^2)] \end{aligned}$$

Neglecting high-order terms in Δt, we can write the recursive law for autocovariances of x^2_t:

$$G_{n+1} = [1 - 2b(2\epsilon + 1)]G_n + E[2b\Delta t x^2_0]$$

for $G_n = E[x^2_{n\Delta t} x^2_0]$.
We arrive at the following closed-form solution:

$$G_n = (1 - 2b(2\epsilon + 1)\Delta t)^n (G_0 - E[X^2_0]^2) + E[x^2_0]^2$$

with $G_0 = E[x^4_0]$ which via the Taylor series expansion of the time - dependent prefactor can be translated into eq. (29).

From (16) we obtain the fourth moment of x_0: $G_0 = E[x_0^4] = \frac{3}{(2\epsilon+3)(2\epsilon+1)}$.
In order to solve for $C_{r^2}(t)$, we need some further ingredients:

$$\begin{aligned} E[r_{\Delta t}^2] = E[(x_{\Delta t} - x_0)^2)] &= E[(-2a\Delta t x_0 + \sqrt{2b(1-x_0^2)\Delta t}\eta_0)^2] \\ &= E[2b\Delta t(1-x_0^2)\eta_0^2 + \theta(\Delta t^2)] \simeq 2b\Delta t \frac{2\epsilon}{2\epsilon+1} \end{aligned}$$

We, therefore, get for the moments of $C_{r^2}(t)$:

$$\begin{aligned} & E[r_{t+\Delta t}^2 r_{\Delta t}^2] - E[r_{\Delta t}^2] \\ = \; & 4b^2\Delta t^2 \left(1 - 2E[x_0^2] + (1-2b(2\epsilon+1)\Delta t)^n (E[x_0^4] - E[x_0^2]^2) + E[x_0^2]^2 - \frac{(2\epsilon)^2}{(2\epsilon+1)^2}\right) \\ = \; & 4b^2\Delta t^2 (1-2b(2\epsilon+1)\Delta t)^n (E[x_0^4] - E[x_0^2]^2) \end{aligned}$$

Finally, in order to pin down the denominator, consider

$$\begin{aligned} E[r_{\Delta t}^4] &= E[(-2a\Delta t x_0 + \sqrt{2b(1-x_0^2)\Delta t}\eta_0)^4] \\ &= E[(2b\Delta t)^2 (1-x_0^2)^2 \eta^4 + \theta(\Delta t^{2.5})] \\ &\simeq E[(2b\Delta t)^2 (1 - 2x_0 + x_0^4)\eta_0^4] \\ &= 3(2b\Delta t)^2 \frac{4\epsilon(\epsilon+1)}{(2\epsilon+1)(2\epsilon+3)} \end{aligned}$$

It, then, turns out that:

$$E[r_{\Delta t}^4] - E[r_{\Delta t}^2]^2 = (2b\Delta t)^2 \frac{(4\epsilon^2+6\epsilon+3)4\epsilon}{(2\epsilon+1)^2(2\epsilon+3)}$$

One solves for

$$C_{r^2}(t) = \frac{E[r_{t+\Delta t}^2 r_{\Delta t}^2] - E[r_t^2]^2}{E[r_t^4] - E[r_t^2]^2}$$

and obtains eq. (30).

A.5 Mean First Passage Time

To compute the MFPT $T(x_0 \to x_2)$ one integrates the Fokker Planck equation with a reflecting boundary condition at $x = -1$ and an absorbing boundary condition at $x = +1$. Evoking the following general formula for MPFTs, we can try to find a closed-form solution for this quantity as well:

C.W.gardener Handbook of Stochastic Methods, Springerverlag, Berlin 1985

$$T(x_0 \to x_2) = \int_{x_0}^{x_2} dx (\psi(x))^{-1} \int_x^{x_1} \frac{2\psi(y)}{D(y)} dy \tag{91}$$

with the auxiliary $\Psi(x)$ function being defined as

$$\ln \psi(x) = \int_0^x dy \frac{2A(y)}{D(y)} \tag{92}$$

Note first that with drift and diffusion term $A(x) = -2ax$ and $D(x) = 2b(1-x^2)$, according to our previous computations $\varphi(x) = (1-x^2)^{\varepsilon}$.
Taking stock of this result we obtain:

$$T(-1 \to 1) = \int_{-1}^{1} (1-x^2)^{-\varepsilon} \left(\int_{-1}^{x} \frac{2(1-y^2)^{\varepsilon}}{2b(1-y^2)} dy \right) dx$$

Because of the symmetry of $(1-y^2)^{\varepsilon-1}$ we can solve the double integral in a few steps:

$$\begin{aligned} bT(-1 \to 1) &= \int_{-1}^{1} (1-x^2)^{-\varepsilon} \left(\int_{-1}^{1} (1-y)^{\xi-1} - \int_{-1}^{x} (1-y^2)^{\xi-1} dy \right) dx \\ &= \frac{1}{2} \int_{-1}^{1} (1-x^2)^{-\varepsilon} \left(\int_{-1}^{1} (1-y^2)^{\xi-1} dy \right) dx \\ &= \frac{1}{2} \int_{-1}^{1} 2^{2\varepsilon-1} \frac{\Gamma(\varepsilon)^2}{\Gamma(2\varepsilon)} (1-x^2)^{-\varepsilon} dx \\ &= \frac{1}{2} \cdot \frac{\Gamma(\varepsilon)^2 \cdot \Gamma(1-\varepsilon)^2}{\Gamma(2\varepsilon) \cdot \Gamma(2-2\varepsilon)} \end{aligned}$$

The identity $\Gamma(1-x) \cdot \Gamma(x) = \frac{\pi}{sin(\pi x)}$ finally leads to

$$T(-1 \to 1) = \frac{\pi}{b(1-2\varepsilon)} \cdot \frac{cos(\pi\varepsilon)}{sin(\pi\varepsilon)}$$

B Appendix B: Limiting Cases of the Kirman Model

B.1 Conventional scaling model

In many papers dealing with agent-based models of financial markets, the transition probabilities are express in terms of concentrations. If the number of agents is changed without contemporaneously modifying the time scale, this amounts to artificially creating a dependence in the herding component b, because the coefficient B is kept constant. Therefore:

$$b = b_0 N_0 / N \tag{93}$$

Since b decreases with N, the $1/N$ correction in the diffusion term in eq. (48) can be no longer neglected. The current density (47) can be rewritten as:

$$j(x,t) = -2axw(x,t) - \frac{1}{2} \frac{2aN_c}{N} \frac{\partial}{\partial x} \left(1 - x^2 + \frac{2}{N_c} \right) w(x,t) \tag{94}$$

where we have introduced a critical agent number:

$$N_c = N_0 \frac{b_0}{a} \tag{95}$$

Recalling the role of ε in eq. (**??**), we define an equivalent ε_N-parameter, where the subscript N indicates the N-dependence:

$$\varepsilon_N = \frac{a}{b} = \frac{aN}{bN_0} = \frac{N}{N_c} \tag{96}$$

Its magnitude determines the qualitative behavior of the model.

For small herding, that means $\varepsilon_N \geq 1$ or equivalently $N_c/N \leq 1$, the size of x^2 will be of order $1/N$[18] [**?**, **?**], we arrive to:

$$j(x,t) = -2axw(x,t) - \frac{1}{2}\frac{2a(N_c+2)}{N}\frac{\partial w(x,t)}{\partial x} \tag{97}$$

The equilibrium distribution, derived with formula (49), is Gaussian with variance:

$$\sigma^2 = \frac{(N_c+2)}{2N} = \frac{(N_0 b_0 + 2a)}{2aN} \tag{98}$$

that proves eq. (**??**).

For dominating herding, $\varepsilon_N \leq 1$ or equivalently $N_c/N \geq 1$, the $1/N$ corrections in eq. (48) can be neglected, and all the formulas hold by replacing ε with N/N_c. With this replacement, the equilibrium distribution reads as

$$w_0(x) \propto (1-x^2)^{N/N_c - 1} \tag{99}$$

At the critical agent number this distribution changes from a bimodal shape at $N < N_C$ to a uni-modal shape at $N > N_c$. For $N >> N_c$ we recover the Gaussian in eq. (**??**) or (98), recalling also the gaussian approximation illustrated at the end of appendix A2.

B.2 Alfarano-Lux model

I am still studying the paragraph.

If the autonomous parameter a vanishes we apply reflecting boundary conditions at $|x| = 1-\delta$. The equilibrium distribution is given by

$$w_0(x) = \left(\ln(\frac{2}{\delta} - 1)\right)^{-1} \frac{1}{1-x^2} \quad \text{for} \quad |x| \leq 1-\delta \tag{100}$$

The solutions of the adjoint equation (61) are either even (P_+) or odd (P_-) functions of x. They can be expressed in hypergeometric functions of x^2. A power expansion reads as:

$$P_\pm = \rho_\pm \sum_{n=0}^{\infty} \left\{ \begin{array}{c} 1 \\ x(n+1/2)^{-1} \end{array} \right\} \frac{x^{2n}}{n!} \frac{\Gamma(n+\gamma \mp 1/4)\Gamma(n-\gamma \mp 1/4)}{\Gamma(n+1/2)} \tag{101}$$

with $\gamma = \sqrt{4\lambda+1}\,/4$. The boundary conditions at $|x| = 1-\delta$ require $P'_\pm(1-\delta,\lambda) = 0$. For P'_+ its zeros are different from zero and close to the values $2n(2n-1)$ obtained from (67). In contrast the odd function exhibits a small eigenvalue. To find its analytical form in the limit of small δ we expand $P'_-(x,\lambda)$ up to terms vanishing with $\lambda \to 0$

$$P'_-(x,\lambda) = 2\rho_- \left(-\frac{2}{\lambda} - \ln\frac{1-x^2}{4}\right) \tag{102}$$

Its zero at $|x| = 1-\delta$ leads to a small eigenvalue for $\delta \to 0$ given by

$$\lambda_- = 2\left(\ln\frac{2}{\delta}\right)^{-1} \tag{103}$$

All other eigenvalues are of the form (67). From formula (81) we see, that even functions $f(x)$ exhibit no slow decay of correlations, whereas odd functions have a

[18] We obtain, in fact, an Ehrenfest-like urn model. The exact equilibrium distribution for this process is a binomial distribution $p(n,N) = \frac{1}{2}^N \binom{N}{n}$, that, in the approximation of large N and around the mean, reduces to a Gaussian with variance equal to $\frac{1}{N}$ [**?**].

component decaying with the long time constant $t_D = 1/(b\lambda_-)$ given by (103). MFPT's are obtained by the general formula (91) with $\psi = 1$ and $D = b(1 - x^2)$. The elementary integrals are expanded up vanishing terms in δ to obtain the results quoted in (**??**) and (**??**).