

Kaivanto, Kim

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ALTERNATION BIAS AND THE PARAMETERIZATION OF
CUMULATIVE PROSPECT THEORY*

Kim Kaivanto^{†‡}

Department of Economics
Lancaster University Management School
Lancaster LA1 4YX, UK

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Abstract

Two recently published studies argue that conventional parameterizations of cumulative prospect theory (CPT) fail to resolve the St. Petersburg Paradox. Yet as a descriptive theory CPT is not intended to account for the local representativeness effect, which is known to induce ‘alternation bias’ on binary iid sequences such as those generated by coin tossing in St. Petersburg gambles. Once alternation bias is controlled for, conventional parameterizations of CPT yield finite certainty equivalents for the St. Petersburg gamble, negating the suggested need for reparameterization. Moreover, the associated willingness to pay estimates fall within the generally accepted empirical range.

Keywords: cumulative prospect theory, St. Petersburg Paradox, local representativeness effect, alternation bias, law of small numbers

JEL classification: D81, D84

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[†]tel +44(0)1524594030; fax +44(0)1524594244; e-mail k.kaivanto@lancaster.ac.uk

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1 Introduction

Several authors have recently addressed the question of whether cumulative prospect theory (CPT) resolves the St. Petersburg Paradox (Blavatsky, 2005; Rieger and Wang, 2006). These authors show that *direct application* of CPT to the St. Petersburg gamble fails to resolve the paradox under most conventional CPT parameterizations. They also propose a number of remedial fixes to CPT, central among which is a constraint on the value function exponent to be smaller than the probability weighting function exponent ($\alpha < \gamma$). As this constraint is violated by most experimentally determined CPT parameterizations,¹ the remedy amounts to a fundamental reparameterization of CPT.

Tversky and Kahneman's (1992) CPT is a descriptive theory. It is consistent with stochastic dominance and accounts for framing effects, nonlinear probability preferences,² source dependence, risk seeking behavior,³ loss aversion,⁴ and uncertainty aversion. Nowhere has it been suggested that CPT's descriptive power extends to local representativeness effects. As Tversky and Kahneman (1992) stress, "Theories of choice are at best approximate and incomplete." Like numerous other heuristics, the operation of the representativeness heuristic (Tversky and Kahneman, 1974) depends in a complex fashion not only on the structural formulation of the decision problem, but also on its context and manner of presentation. Hence, explicit incorporation of the representativeness heuristic into the formal structure of CPT would limit its applicability to a narrow range of problems. Incorporation of the binary sequence variant of the representativeness heuristic, the local representativeness heuristic, into the formal structure of CPT would limit its applicability still further. Thus, in order for CPT to function as a straightforwardly implementable general purpose descriptive theory, the local representativeness effect must remain outside its formal structure and specification.

Nevertheless, it is evident that the coin-tossing sequence found in the St. Petersburg gamble is precisely the sort of context where the local representativeness effect is likely to be operative. Indeed many psychological studies of randomness perception and the local representativeness

¹e.g. Abdellaoui (2000), Abdellaoui, Vossman and Weber (2005), Bleichrodt and Pinto (2000), Gonzalez and Wu (1999), Tversky and Fox (1995), and Tversky and Kahneman (1992)

²the certainty effect; overweighting small probabilities, underweighting large probabilities

³i.e. the reflection effect; risk seeking in losses and risk aversion in gains; diminishing sensitivity, whereby individuals are more sensitive to changes near their status quo than to changes that are more remote from their status quo

⁴losses are weighed more heavily than gains

effect in particular have utilized coin-tossing experiments (e.g. Rapoport and Budescu, 1997; Kareev, 1995). Moreover, these experimental studies of local representativeness have been designed in such a fashion⁵ so as to allow ‘clean’ estimates of the alternation bias—i.e. estimates that are free from the confounding of conditional probability distortion with outcome value weighting.

Under the alternation bias, subjects perceive negatively autocorrelated sequences as maximally random, while the runs that are characteristic of unbiased memoryless Bernoulli processes are perceived as being excessively regular to be random. Therefore alternation bias leads to the subjective association of a *negative autocorrelation* with known *memoryless* and unbiased Bernoulli processes. This may be viewed as a subjective distortion of conditional probability. As phenomena ranging from the Gambler’s Fallacy⁶ to behavior in the Monty Hall problem⁷ attest, people without specialist training in probability theory generally process conditional probability information differently than probability calculus intimates.

This note contends that once alternation bias is controlled for, conventional parameterizations of CPT do indeed succeed in resolving the St. Petersburg Paradox. The suggestion, made by Blavatsky (2005) and Rieger and Wang (2006), to constrain the value function exponent to be smaller than the probability weighting function exponent ($\alpha < \gamma$), confounds the subjective distortion of conditional probability with (i) the subjective distortion of unconditional probability and (ii) the subjective valuation of outcomes. Reparameterization of CPT on the basis of the St. Petersburg Paradox is not only unnecessary, but would also disturb the theory’s internal consistency and narrow its scope of applicability.

This note builds upon insights derived from Rabin (2002) on local representativeness. The following section briefly summarizes the local representativeness effect and presents empirical estimates of first-order and higher-order alternation bias. Section 3 uses these estimates to show: (subsection 3.1) that alternation bias on its own is sufficient for the subjective (mathematical) expectation of the St. Petersburg gamble to be rendered finite and within conventionally accepted empirical bounds; (subsection 3.2) that alternation bias relaxes the Blavatsky-Rieger-Wang

⁵by not invoking or bundling monetary payoffs with ‘Heads’ or ‘Tails’ realizations

⁶See e.g. Clotfelter and Cook (1993), Terrel (1994) and Croson and Sundali (2005).

⁷Also known as the ‘three door problem’; it is mathematically equivalent to the ‘three prisoner problem’. Although bias is pervasive in these problems (Granberg and Brown, 1995; Granberg, 1999), nevertheless it is possible to devise schemes that allow subjects to learn how to overcome their anomalous initial biases (Friedman, 1998; Krauss and Wang, 2003). In a market setting, the presence of a small proportion of bias-free agents suffices to eliminate bias in prices (Kluger and Wyatt, 2004).

CPT finiteness constraint; and (subsection 3.3) that once alternation bias is controlled for, CPT yields a finite willingness to pay for the St. Petersburg gamble, which moreover falls within conventionally accepted empirical grounds. Section 4 concludes.

2 Local representativeness effect

That people display alternation bias in sequential randomization tasks was first hypothesized by Reichenbach (1934).⁸ Experimental and observational evidence consistent with this hypothesis amassed from the time that the hypothesis was first put to test. The effect is known by different labels in different contexts—such as the Gambler’s Fallacy in gambling, and the alternation bias in coin tossing—but these are specific examples of the local representativeness effect.⁹

People under the influence of the local representativeness effect attribute the salient properties of the population or generating process to short sequences. That is, such individuals do not adequately distinguish between local features and the properties of the whole, and they apply the latter to the former. For Bernoulli sequences this translates into local matching of outcome proportions with those of the long-run process, i.e. (.5, .5) for unbiased coins, and excessive local irregularity, i.e. a propensity to anticipate too many ‘reversals’ in short series.

This subjective predisposition to anticipate reversals is called the alternation bias, which equates to a negative subjective autocorrelation whereby people expect too few streaks in random sequences. Alternation bias effects have been documented up to sixth order (Budescu, 1987). Most empirical studies place the first-order effect at $P(H|T) = .6$ (see Budescu, 1987; Bar-Hillel and Wagenaar, 1991; Kareev, 1995). Still, some studies have found an even stronger first-order effect of $P(H|T) \in [.7, .8]$ (Gilovich, Vallone and Tversky, 1985). Taking higher-order effects into consideration, Rabin (2002) derives the following conditional probabilities from data presented in Rapoport and Budescu (1997): $P(H|T) = .585$, $P(H|HT) = .46$, $P(H|HHT) = .38$ and $P(H|HHH) = .298$, where this last expression refers to the conditional probability of a toss turning up ‘Heads’ given that the three immediately preceding tosses turned up ‘Heads’. After rounding we obtain the following higher-order transition probabilities (see Table 1) with an alternation bias that bridges, between first and third orders, the weaker and the stronger alternation bias magnitudes reported in the literature.

⁸In Reichenbach’s terminology, a ‘negative recency’ effect.

⁹or the ‘law of small numbers’

3 Application to the St. Petersburg gamble

In the modern variant of the St. Petersburg Paradox, a subject is offered a stochastic payout of $2^{\tilde{n}}$ dollars, where \tilde{n} is the index of the first toss on which a fair, memoryless coin turns up ‘Heads’.¹⁰ The paradox arises because although the mathematical expectation of the gross St. Petersburg gamble payout $G_{SP} = 2^{\tilde{n}}$ is infinite, people are typically willing to pay only a small, finite amount to obtain this gamble. The theoretical literature favors Willingness To Pay (WTP) estimates between \$2 and \$4. This accords with Bernoulli’s (1738) ‘expected moral worth’ solution which he formalized using the logarithmic function: abstracting from prior wealth, the St. Petersburg gamble is evaluated as $E[u(G_{SP})] = \sum_{n=1}^{\infty} 2^{-n} \log(2^n) = \sum_{n=1}^{\infty} \frac{n}{2^n} \log(2) = 2 \log(2) = \log(4)$, giving a certainty equivalent of \$4 (Schmeidler and Wakker, 1998).¹¹ With the exception of Bottom, Bontempo and Holtgrave (1989), formal experimental studies of the St. Petersburg gamble are thin on the ground, if for no other reason than the difficulty of bankrolling potentially very large (infinite in expectation) payouts. Some sources report that the typical WTP is no more than \$10 (Chernoff and Moses, 1959), while others, such as Camerer (2005), report that people typically disclose a WTP of approximately \$20.

The St. Petersburg Paradox was the earliest example of an ‘anomaly’ in choice behavior that led to a change in theory, insofar as it caused Bernoulli to supplant the Pascal-Fermat theory of Expected Monetary Value (EMV) with what has become known as Expected Utility (EU). Numerous alternative solutions to the paradox have subsequently been proposed.¹² Moreover, Yaari’s (1987) dual theory of choice under risk has shown that the concave utility function (distortion of outcomes) solution is observationally indistinguishable from a distortion of probabilities solution, and that as such, concave utility is therefore not a necessary precondition for solving the St. Petersburg Paradox. Yet ultimately it was the mounting evidence of experimentally demonstrated EU-violating ‘anomalies’—heuristics and biases of choice under risk and uncertainty—that allowed CPT to emerge as an alternative to EU. Although CPT serves as a

¹⁰In Daniel Bernoulli’s (1738) variant of the St. Petersburg gamble, the subject’s (Paul’s) payout is $2^{\tilde{n}-1}$ ducats.

¹¹If the payout is specified as $2^{\tilde{n}-1}$ dollars, then the certainty equivalent associated with the logarithmic utility function is 2 currency units: $E[u(2^{\tilde{n}-1})] = \sum_{n=1}^{\infty} 2^{-n} \log(2^{n-1}) = \sum_{n=1}^{\infty} \frac{n-1}{2^n} \log(2) = \log(2)$.

¹²Alternative solutions proposed for the St. Petersburg Paradox are too numerous to be discussed in detail here. For reviews see Samuelson (1977), Vlek and Wagenaar (1979) and Bottom, Bontempo and Holtgrave (1989).

descriptive model for a number of distinct behavioral biases and effects, alternation bias is not among them. Nevertheless, alternation bias is particularly relevant in the context of coin tossing sequences.

The next subsection shows that alternation bias is sufficient on its own to induce finite and moderate WTP for the St. Petersburg gamble. The following two subsections show in turn that by controlling for alternation bias, currently popular CPT parameterizations do in fact satisfy appropriately specified finiteness constraints for the St. Petersburg gamble, and moreover they yield Certainty Equivalents (CEs) and WTP within the accepted empirical range.

3.1 Mathematical expectation revisited

In the present context, alternation bias enters the formulation of subjective (mathematical) expectation by distorting the subject's perception of the probability distribution of \tilde{n} , the index of the first toss on which an unbiased memoryless coin turns up 'Heads'.

Objectively \tilde{n} follows a geometric distribution with parameter $p = \frac{1}{2}$, i.e. the objective probabilities are simply $p_n = \frac{1}{2}(1 - \frac{1}{2})^{n-1} = 2^{-n}$ for $n = 1, 2, \dots$.

Accounting for first-order alternation bias, under which $P(H|T) = .6$ and $P(T|T) = .4$, the subjectively perceived probability of the coin turning up 'Heads' for the first time on toss n then takes the form

$$p_n^{f.o} = \begin{cases} P(H) = \frac{1}{2} & \text{for } n = 1 \\ \frac{1}{2}P(H|T)P(T|T)^{n-2} = .3 \cdot .4^{n-2} & \text{for } n \geq 2 \end{cases}, \quad (3.1)$$

which gives a subjectively distorted mathematical expectation of

$$\begin{aligned} E^{f.o}(G_{SP}) &= \sum_{n=1}^{\infty} p_n^{f.o} 2^n = 1 + .3 \sum_{n=2}^{\infty} .4^{n-2} 2^n = 1 + \left(\frac{.3}{.4^2}\right) \sum_{n=2}^{\infty} .8^n \\ &= 1 + \left(\frac{.3}{.4^2}\right) \left[\frac{.8}{.2} - .8\right] = 7.0 \quad . \end{aligned} \quad (3.2)$$

So under first-order alternation bias alone, WTP is limited to \$7.0.

Accounting for higher-order alternation bias (see Table 1), the subjectively perceived prob-

ability of the coin turning up ‘Heads’ for the first time on toss n then takes the form

$$p_n^{h^o} = \begin{cases} P(H) = \frac{1}{2} & n = 1 \\ \frac{1}{2}P(H|T) = \frac{1}{2} \cdot .58 = .29 & n = 2 \\ \frac{1}{2}P(T|T)P(H|TT) = \frac{1}{2} \cdot .42 \cdot .62 = .1302 & n = 3 \\ \frac{1}{2}P(T|T)P(T|TT)P(H|TTT)P(T|TTT)^{n-4} = \frac{1}{2} \cdot .42 \cdot .38 \cdot .7 \cdot .3^{n-4} & n \geq 4 \\ = .05586 \cdot .3^{n-4} & \end{cases} \quad (3.3)$$

which gives a subjectively distorted mathematical expectation of

$$\begin{aligned} E^{h^o}(G_{SP}) &= \sum_{n=1}^3 p_n^{h^o} 2^n + \sum_{n=4}^{\infty} p_n^{h^o} 2^n = \sum_{n=1}^3 p_n^{h^o} 2^n + \left(\frac{.05586}{3^4} \right) \sum_{n=4}^{\infty} .6^n \\ &= \sum_{n=1}^3 p_n^{h^o} 2^n + \left(\frac{.05586}{3^4} \right) \left[\frac{.6^4}{.4} \right] = 5.436 . \end{aligned} \quad (3.4)$$

So under third-order alternation bias alone, WTP is limited to \$5.436.

Relative to the objective geometric distribution, first-order alternation bias induces a higher perceived probability of the coin-tossing sequence terminating on the second throw ($p_2 = .25 < .3 = p_2^{f^o}$) than on subsequent throws ($p_n > p_n^{f^o} \forall n \geq 3$), while third-order alternation bias induces a higher perceived probability of the coin-tossing sequence terminating on the second and third throws ($p_2 = .25 < .29 = p_2^{h^o}$, $p_3 = .125 < .1302 = p_3^{h^o}$) than on subsequent throws ($p_n > p_n^{h^o} \forall n \geq 4$).

Both the first-order (3.2) and higher-order (3.4) estimates of WTP induced by alternation bias alone fall within the generally accepted empirical range. Whereas Camerer (2005) shows that the ‘anomalies literature’—through loss aversion¹³ in particular—provides a solution to the St. Petersburg Paradox that requires neither a nonlinear value function nor a nonlinear (unconditional) probability weighting function, equations (3.2) and (3.4) show that the anomalies literature also gives rise to a second solution—based on alternation bias—which similarly makes no requirement for a nonlinear value function or a nonlinear (unconditional) probability weighting function.

3.2 CPT finiteness constraint revisited

Blavatsky (2005) and Rieger and Wang (2006) contend that conventional parameterizations of CPT fail to yield finite valuations for the St. Petersburg gamble, and that in order to resolve

¹³in conjunction with piecewise linear utility and large but finite upper ceiling on the maximum payout

the St. Petersburg Paradox the parameterization of CPT must satisfy an additional constraint, namely $\alpha < \gamma$. Yet given what has been established above about the alternation bias—i.e. (i) that it is well-documented, (ii) that it affects sequence trials exemplified by coin-tossing sequences such as those found in St. Petersburg gambles, and (iii) that it has been established independently of unconditional probability distortion and non-linear outcome weighting—and given that CPT has been conceived as a descriptive theory to explain numerous heuristics and biases in choice under risk and uncertainty but *exclusive* of local representativeness effects, it is indeed no surprise at all that direct application of CPT to the St. Petersburg gamble proves problematic. For these very same reasons, however, it is neither necessary nor desirable to enforce the constraint $\alpha < \gamma$ even for the sole purpose of analyzing the St. Petersburg gamble. This holds with even more force for the parameterization of CPT for general use.

Incorporation of alternation bias into the analysis of the St. Petersburg gamble proceeds by way of distortion of conditional probabilities between coin tosses. Just as a casino player under the influence of the Gambler’s Fallacy believes that his probability of winning this hand is higher *because* of a long sequence of losing hands leading up to this hand, an individual contemplating the St. Petersburg gamble under the influence of alternation bias believes that the probability of a particular toss turning up ‘Heads’ is higher *because* of an unbroken string of preceding ‘Tails’.

As a consequence, the following two propositions may be proved using the estimates for first-order and higher-order alternation bias set out in Table 1. Proofs are collected in the appendix.

Proposition 3.1 (First-order constraint). *Once first-order alternation bias is controlled for, the finiteness constraint relaxes to*

$$\alpha < \frac{\log(5/2)}{\log(2)} \cdot \gamma \approx 1.32 \cdot \gamma . \tag{3.5}$$

Proposition 3.2 (Higher-order constraint). *Once alternation bias effects up to third order are controlled for, the finiteness constraint relaxes to*

$$\alpha < -\frac{\log(.3)}{\log(2)} \cdot \gamma \approx 1.737 \cdot \gamma . \tag{3.6}$$

As Figure 1 illustrates, popular conventional parameterizations of CPT comfortably satisfy the finiteness constraint once it is adjusted for alternation bias up to third order.

————— insert Figure 1 here —————

Mathematically, the Blavatsky-Rieger-Wang constraint without alternation bias, illustrated as (a) in Figure 1, is derived from the limit behavior of the gross payout from the St. Petersburg gamble (G_{SP}). Similarly, the constraints (3.5) and (3.6) above, illustrated as (b) and (c) in Figure 1, are also derived from the limit behavior of the gross payout from the St. Petersburg gamble (G_{SP}). Using numerical procedures it is possible to determine the Certainty Equivalent (CE) of this gross payout for each parameterization of CPT and for each of the three assumptions about alternation bias. The results of this numerical implementation are presented below in Table 2.

———— insert Table 2 here ————

For the single-parameter probability weighting function specification, higher-order alternation bias brings the CE of the gross payout down to within the range [5.64, 20.39]. The distance of the parameter pair (γ, α) from the finiteness constraint is one determinant of the magnitude of this CE, but so is its location along the length of the finiteness constraint. Figures 2 and 3 illustrate this by way of the CE=\$5, CE=\$10 and CE=\$20 contours for the gross payoff under first-order alternation bias and higher-order alternation bias respectively. The differences between these contour maps explain for instance why the Tversky and Fox (1995) parameterization yields a larger CE than the Gonzalez and Wu (1999) parameterization under first-order alternation bias ($74.81 > 56.53$) while the reverse is true under higher-order alternation bias ($10.15 < 13.05$).

———— insert Figure 2 here ————

———— insert Figure 3 here ————

3.3 WTP under CPT revisited

Nevertheless the above gross payout CE calculations should not be confused with WTP for the St. Petersburg gamble under CPT. Correct calculation of WTP under CPT must incorporate loss aversion over the shortfall between the gross payout G_{SP} and the up-front payment P exacted as the entry fee for participation in the St. Petersburg coin-tossing gamble. As Camerer (2005) points out,¹⁴ attention must be focused on the net gamble payout $G_{SP}-P$, which involves

¹⁴Camerer (2005) uses piecewise linear utility, loss aversion, and the realistic assumption of the existence of a finite maximum payout ceiling to show that risk aversion is not a necessary condition for resolution of the St. Petersburg Paradox.

an ex ante probable loss for $P > 2$. For each parameterization the maximum WTP will be less than the CE of the gross payout. Thus for any entry fee $P > 2$, the CPT evaluation occurs with respect to both gains and losses

$$V(G_{StP} - P) = V^+((G_{StP} - P)^+) + V^-((G_{StP} - P)^-) \quad (3.7)$$

where the ‘+’ superscript refers to gains and the ‘-’ superscript refers to losses. The maximum WTP is the entry fee P^* that solves

$$V^+((G_{StP} - P^*)^+) + V^-((G_{StP} - P^*)^-) = 0 \quad (3.8)$$

For the Tversky and Kahneman (1992) parameters $\gamma^{\text{gains}} = .61$, $\alpha^{\text{gains}} = .88$, $\lambda = 2.25$, $\gamma^{\text{loss}} = .69$, $\alpha^{\text{loss}} = .88$ and third-order alternation bias, equation (3.8) is solved by $P^* \approx \$9.95$, which is a finite maximum WTP that resolves the St. Petersburg Paradox.

4 Conclusion

If it could be shown that alternation bias is not operative in the St. Petersburg gamble setting, then present results would not diminish the case for restricting CPT parameterization in accordance with the Blavatsky-Rieger-Wang finiteness constraint. However, experimental studies suggest strongly that coin-flipping series are indeed an exemplar contexts where alternation bias is operative and for which reliable and replicated empirical estimates of alternation bias magnitude are available.

The Blavatsky (2005) and Rieger and Wang (2006) papers expose an important feature of conventional CPT parameterizations. Yet their elegantly straightforward remedy, namely the Blavatsky-Rieger-Wang finiteness constraint ($\alpha < \gamma$)—although mathematically unobjectionable and certainly a solution worthy of consideration per se—is not without logical and theoretical consequences of its own. It localizes the remedy to the popular and conventional parameterizations of CPT, and these parameterizations are singled out as the effective cause of the finiteness problem. Yet the substantial experimental literature on the alternation bias suggests strongly that the solution to the finiteness problem may in fact lie in the subjective distortion of conditional probabilities, rather than in the subjective distortion of unconditional probabilities. The former (distortion of conditional probabilities), is formally outside the scope of CPT, whereas the latter (distortion of unconditional probabilities), is a proper part and object of the analytical structure of CPT. To require CPT’s unconditional probability distortion

parameterization to reflect and incorporate the conditional probability distortion caused by alternation bias induced in the St. Petersburg gamble is to introduce a ‘foreign’ element into CPT (conditional probability distortion) and to do so in a way that confounds the magnitude of conditional probability distortion with the magnitude of unconditional probability distortion, as opposed to keeping the magnitudes of these two distinct effects separate and individually identifiable. Moreover, imposition of this constraint on parameterization limits the scope of applicability of CPT, insofar as the Blavatsky-Rieger-Wang constraint rules out most of the widely used conventional parameterizations, which are tuned to achieving descriptive accuracy in a variety of settings that do not share the St. Petersburg gamble’s sequential structure.

None of these concessions are necessary, though the cost of avoiding them is to bring more of the experimental and behavioral literature into the foreground. Recognizing the role of alternation bias in the St. Petersburg coin-tossing sequence allows the Paradox to be resolved, while preserving the distinction between conditional and unconditional probability distortion, and moreover preserving CPT’s scope of descriptive applicability that is embodied in its conventional parameterizations.

As CPT becomes increasingly popular and is adopted and applied ever more widely, the question that is at the root of the divergence between the approach of this paper and that of Blavatsky (2005) and Rieger and Wang (2006) will re-emerge with increasing frequency: How are we to apply, interpret and evaluate CPT? Is CPT a self-contained portable module that can be applied across the whole spectrum of problem settings without any need to anticipate complications, or is CPT essentially inseparable from the wider ‘heuristics and biases’ program? The special application studied in this paper lends weight to the latter. Although CPT has a concise, self-contained mathematical form, it should not be applied without giving due care and attention to the full range of behavioral effects that may arise. Some of these effects are captured by CPT, yet others require separate accommodation.

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Figure 1: Conventional parameterizations of cumulative prospect theory and the finiteness constraint computed (a) without alternation bias, (b) with first-order alternation bias, and (c) with higher-order alternation bias.

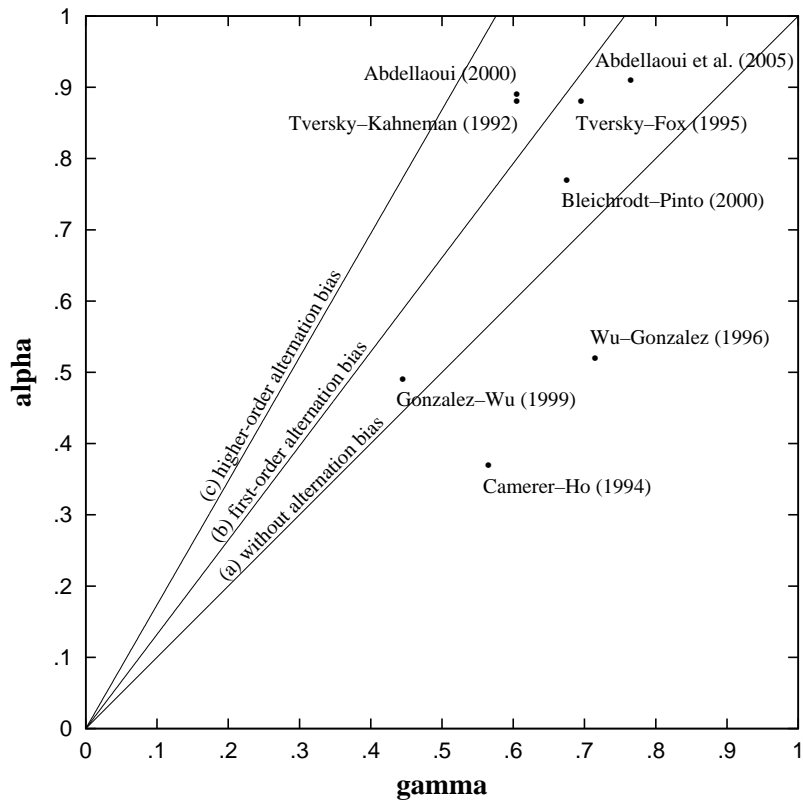


Figure 2: The \$5, \$10, and \$20 Certainty Equivalent contours of the gross St. Petersburg gamble payout under first-order alternation bias.

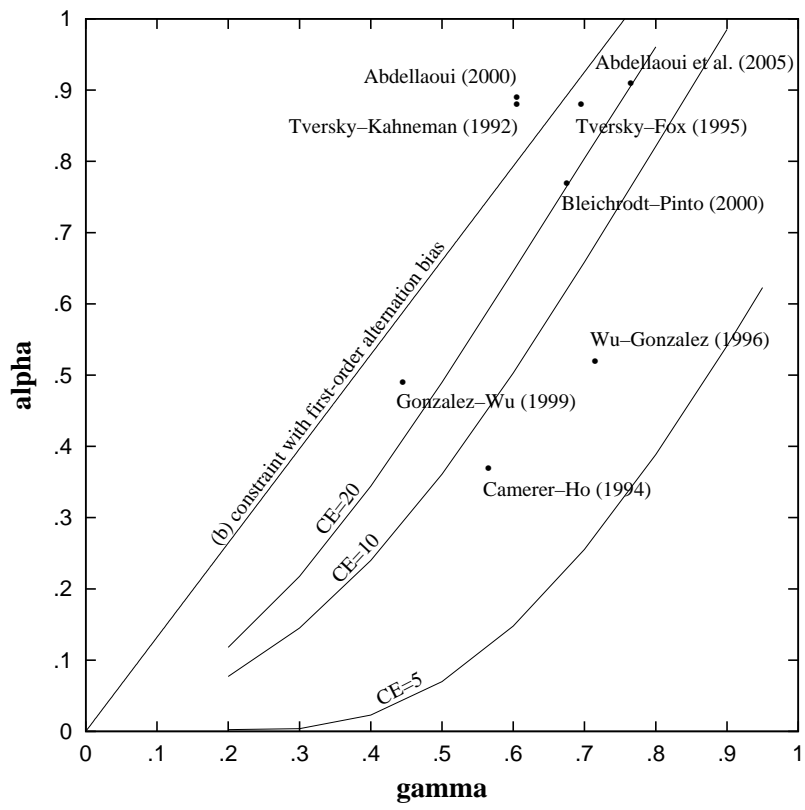


Figure 3: The \$5, \$10, and \$20 Certainty Equivalent contours of the gross St. Petersburg gamble payout under higher-order alternation bias.

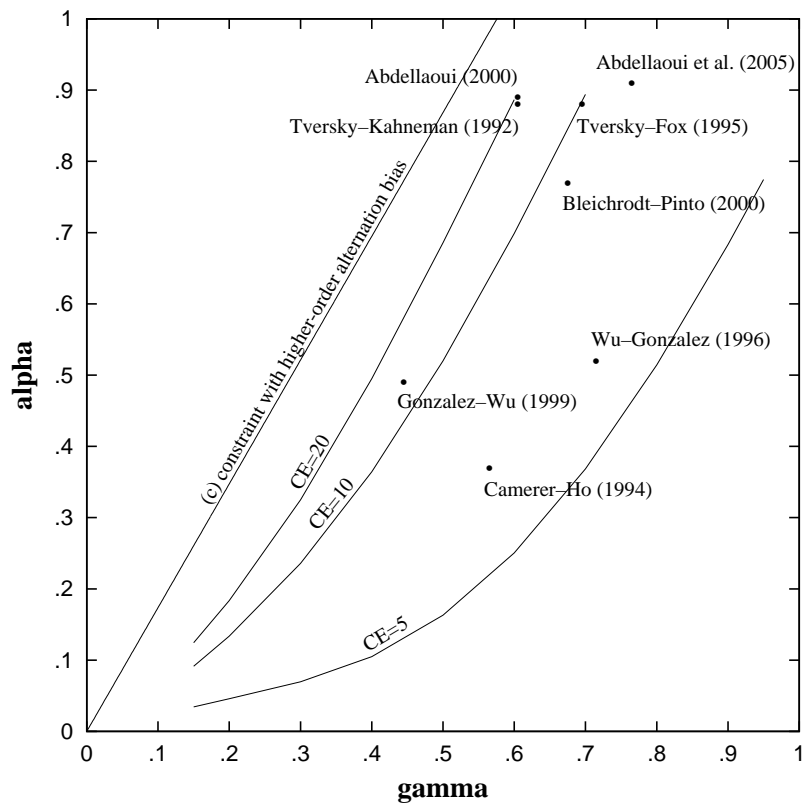


Table 1: Alternation bias estimates from the literature expressed as transition probabilities

First order	Higher order		
$P(H T)$	$P(H T)$	$P(H TT)$	$P(H TTT)$
.6	.58	.62	.70

Table 2: Certainty Equivalents of the gross payout from the St. Petersburg gamble under conventional parameterizations of CPT computed (a) without alternation bias, (b) with first-order alternation bias, and (c) with higher-order alternation bias.

	$(\gamma, \alpha)^a$ $(\delta, \gamma, \alpha)^b$	(a) without alternation bias	(b) first-order alternation bias	(c) higher-order alternation bias
Wu & Gonzalez (1996)				
1-param $w^+(p)$	(.71, .52)	16.00	6.95	5.64
2-param $w^+(p)$	(.84, .68, .52)	17.18	7.00	5.58
Camerer & Ho (1994)				
1-param $w^+(p)$	(.56, .37)	19.32	7.98	6.07
Abdellaoui et al (2005)				
1-param $w^+(p)$	(.76, .91)	∞	22.08	8.18
2-param $w^+(p)$	(.98, .83, .91)	∞	11.39	6.75
Bleichrodt & Pinto (2000)				
1-param $w^+(p)$	(.67, .77)	∞	22.21	8.72
2-param $w^+(p)$	(.82, .55, .77)	∞	∞	16.23
Gonzalez & Wu (1999)				
1-param $w^+(p)$	(.44, .49)	∞	56.53	13.05
2-param $w^+(p)$	(.77, .44, .49)	∞	46.00	12.30
Tversky & Fox (1995)				
1-param $w^+(p)$	(.69, .88)	∞	74.81	10.15
2-param $w^+(p)$	(.76, .69, .88)	∞	56.15	8.53
Tversky & Kahneman (1992)				
1-param $w^+(p)$	(.61, .88)	∞	∞	17.39
Abdellaoui (2000)				
1-param $w^+(p)$	(.60, .89)	∞	∞	20.39

$${}^a w^+(p) = p^\gamma / (p^\gamma + (1-p)^\gamma)^{1/\gamma}, u^+(x) = x^\alpha$$

$${}^b w^+(p) = \delta p^\gamma / (\delta p^\gamma + (1-p)^\gamma), u^+(x) = x^\alpha$$

A Mathematical appendix

The St. Petersburg gamble pays out $2^{\tilde{n}}$ where $\tilde{n} \in \mathbb{N}$ is the index of the first toss on which an unbiased memoryless coin turns up ‘Heads’. Alternation bias alters the subjective perception of the distribution of \tilde{n} .

An individual with CPT preferences evaluates the gross St. Petersburg gamble payout G_{SP} as

$$V^+(G_{SP}) = \sum_{n=1}^{\infty} u^+(2^n) \cdot \left[w^+\left(\sum_{i=n}^{\infty} p_i\right) - w^+\left(\sum_{i=n+1}^{\infty} p_i\right) \right] \quad (\text{A.1})$$

where

$$u^+(x) \equiv x^\alpha \quad x \geq 0, \alpha \in (0, 1) \quad (\text{A.2})$$

is the value function for gains ($x \geq 0$) and

$$w^+(p) \equiv \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \quad \gamma \in (0, 1), p \in [0, 1] \quad (\text{A.3})$$

is the Tversky and Kahneman (1992) inverse S-shaped probability weighting function for gains. Blavatsky (2005) shows that as $n \rightarrow \infty$ the denominator of the probability weighting function $w^+(p)$ converges to 1, and as attention may be restricted to the limit tail behavior, the approximation $w^+(p) \approx p^\gamma$ is valid, and thus in the case computed *without* alternation bias (A.1) simplifies to

$$V_w^+(G_{SP}) = (2^\gamma - 1) \sum_{n=1}^{\infty} 2^{(\alpha-\gamma)n} . \quad (\text{A.4})$$

In order to ensure that the geometric series in (A.4) is convergent so that $V_w^+(G_{SP})$ remains finite, the constraint $\alpha < \gamma$ must be imposed (see finiteness constraint (a) in Figure 1). However as Figure 1 illustrates, this constraint is violated by most conventional parameterizations of CPT.

A.1 Finiteness constraint with first-order alternation bias

Consider the form of (A.1) with the first-order alternation bias that Kareev (1995) reports as being a standard finding in the literature: $P(H|T) = .6$ and $P(T|T) = .4$. The probability of the coin turning up ‘Heads’ for the first time on toss n then takes the form

$$p_n^{f_o} = \begin{cases} P(H) = \frac{1}{2} & \text{for } n = 1 \\ \frac{1}{2}P(H|T)P(T|T)^{n-2} = .3 \cdot .4^{n-2} & \text{for } n \geq 2 \end{cases} \quad (\text{A.5})$$

and $\sum_{n=1}^{\infty} p_n^{f_o} = .5 + .3 \sum_{j=0}^{\infty} .4^j = .5 + (.3/.6) = 1$. Therefore the first term in the outside sum of (A.1) is

$$a_1^{f_o} = u^+(2^1) \left[w^+ \left(\sum_{i=1}^{\infty} p_i^{f_o} \right) - w^+ \left(\sum_{i=2}^{\infty} p_i^{f_o} \right) \right] = u^+(2^1) [1 - w^+(2^{-1})] = 2^\alpha [1 - 2^{-\gamma}] \quad (\text{A.6})$$

and subsequent terms are of the form

$$a_n^{f_o} = u^+(2^n) \left[w^+ \left(\left(\frac{.3}{.4^{2.6}} \right) .4^n \right) - w^+ \left(\left(\frac{.3}{.4^{2.6}} \right) .4^{n+1} \right) \right] \quad \forall n \geq 2 \quad (\text{A.7})$$

$$= \left(\frac{.3}{.4^{2.6}} \right)^\gamma 2^{\alpha n} [.4^{\gamma n} - .4^{\gamma(n+1)}] \quad \forall n \geq 2 \quad (\text{A.8})$$

giving a CPT evaluation of the gross St. Petersburg gamble payout G_{StP} under first-order alternation bias of

$$V_{f_o}^+(G_{StP}) = a_1^{f_o} + \sum_{n=2}^{\infty} \left(\frac{.3}{.4^{2.6}} \right)^\gamma 2^{\alpha n} [.4^{\gamma n} - .4^{\gamma(n+1)}] \quad (\text{A.9})$$

$$= a_1^{f_o} + \left(\frac{.3}{.4^{2.6}} \right)^\gamma \left[\sum_{n=2}^{\infty} \frac{2^{(\alpha+\gamma)n}}{5^{\gamma n}} - \left(\frac{2}{5} \right)^\gamma \sum_{n=2}^{\infty} \frac{2^{(\alpha+\gamma)n}}{5^{\gamma n}} \right] \quad (\text{A.10})$$

$$= a_1^{f_o} + \left(\frac{.3}{.4^{2.6}} \right)^\gamma \left(1 - \frac{2^\gamma}{5^\gamma} \right) \sum_{n=2}^{\infty} \left(\frac{2^{(\alpha+\gamma)}}{5^\gamma} \right)^n \quad (\text{A.11})$$

which is finite if the parameterization satisfies the constraint

$$\frac{2^{\alpha+\gamma}}{5^\gamma} < 1 \quad (\text{A.12})$$

$$\alpha < \frac{\log(5/2)}{\log(2)} \cdot \gamma \approx 1.32 \cdot \gamma . \quad (\text{A.13})$$

This result is formalized as Proposition 3.1 and illustrated as finiteness constraint (b) in Figure 1.

A.2 Finiteness constraint with higher-order alternation bias

Using the transition probabilities up to third order presented in Table 1, the probability of the coin turning up ‘Heads’ for the first time on toss n then takes the form

$$p_n^{ho} = \begin{cases} P(H) = \frac{1}{2} & n = 1 \\ \frac{1}{2} P(H|T) = \frac{1}{2} \cdot .58 = .29 & n = 2 \\ \frac{1}{2} P(T|T) P(H|TT) = \frac{1}{2} \cdot .42 \cdot .62 = .1302 & n = 3 \\ \frac{1}{2} P(T|T) P(T|TT) P(H|TTT) P(T|TTT)^{n-4} = \frac{1}{2} \cdot .42 \cdot .38 \cdot .7 \cdot .3^{n-4} & n \geq 4 \\ & = .05586 \cdot .3^{n-4} \end{cases} \quad (\text{A.14})$$

and $\sum_{n=1}^{\infty} p_n^{ho} = .5 + .29 + .1302 + .05586 \sum_{j=0}^{\infty} .3^j = .9202 + \frac{.05586}{.7} = 1$. The first, second and third terms in the outside sum of (A.1) are

$$a_1^{ho} = u^+(2^1) \left[w^+ \left(\sum_{i=1}^{\infty} p_i^{ho} \right) - w^+ \left(\sum_{i=2}^{\infty} p_i^{ho} \right) \right] = u^+(2^1) [1 - w^+(2^{-1})] = 2^\alpha [1 - 2^{-\gamma}] , \quad (\text{A.15})$$

$$\begin{aligned} a_2^{ho} &= u^+(2^2) \left[w^+ \left(\sum_{i=2}^{\infty} p_i^{ho} \right) - w^+ \left(\sum_{i=3}^{\infty} p_i^{ho} \right) \right] = u^+(2^2) [w^+(2^{-1}) - w^+(1 - .5 - .29)] \\ &= 2^{2\alpha} [2^{-\gamma} - .21^\gamma] , \end{aligned} \quad (\text{A.16})$$

and

$$\begin{aligned} a_3^{ho} &= u^+(2^3) \left[w^+ \left(\sum_{i=3}^{\infty} p_i^{ho} \right) - w^+ \left(\sum_{i=4}^{\infty} p_i^{ho} \right) \right] = u^+(2^3) [w^+ (.21) - w^+(1 - .5 - .29 - .1302)] \\ &= 2^{3\alpha} [.21^\gamma - .0798^\gamma] . \end{aligned} \quad (\text{A.17})$$

Subsequent terms ($\forall n \geq 4$) are of the form

$$a_n^{ho} = u^+(2^n) \left[w^+ \left(\sum_{i=n}^{\infty} p_i^{ho} \right) - w^+ \left(\sum_{i=n+1}^{\infty} p_i^{ho} \right) \right] \quad (\text{A.18})$$

$$= u^+(2^n) \left[w^+ \left(.05586 \sum_{i=n}^{\infty} .3^{i-4} \right) - w^+ \left(.05586 \sum_{i=n+1}^{\infty} .3^{i-4} \right) \right] \quad (\text{A.19})$$

$$= u^+(2^n) \left[w^+ \left(\frac{.05586}{.3^4} \sum_{i=n}^{\infty} .3^i \right) - w^+ \left(\frac{.05586}{.3^4} \sum_{i=n+1}^{\infty} .3^i \right) \right] \quad (\text{A.20})$$

$$= u^+(2^n) \left[w^+ \left(\frac{.05586}{.3^4 \cdot .7} .3^n \right) - w^+ \left(\frac{.05586}{.3^4 \cdot .7} .3^{n+1} \right) \right] \quad (\text{A.21})$$

$$= \left(\frac{.05586}{.3^4 \cdot .7} \right)^\gamma 2^{\alpha n} [.3^{\gamma n} - .3^{\gamma(n+1)}] . \quad (\text{A.22})$$

Thus the CPT evaluation of the gross St. Petersburg gamble payout G_{StP} under higher-order alternation bias may be written as

$$V_{ho}^+(G_{StP}) = \sum_{n=1}^3 a_n^{ho} + \sum_{n=4}^{\infty} \left(\frac{.05586}{.3^4 \cdot .7} \right)^\gamma 2^{\alpha n} [.3^{\gamma n} - .3^{\gamma(n+1)}] \quad (\text{A.23})$$

$$= \sum_{n=1}^3 a_n^{ho} + \left(\frac{.05586}{.3^4 \cdot .7} \right)^\gamma \left[\sum_{n=4}^{\infty} (2^\alpha .3^\gamma)^n - .3^\gamma \sum_{n=4}^{\infty} (2^\alpha .3^\gamma)^n \right] \quad (\text{A.24})$$

$$= \sum_{n=1}^3 a_n^{ho} + \left(\frac{.05586}{.3^4 \cdot .7} \right)^\gamma (1 - .3^\gamma) \sum_{n=4}^{\infty} (2^\alpha .3^\gamma)^n \quad (\text{A.25})$$

which converges to a finite value if the parameterization satisfies the constraint

$$2^\alpha .3^\gamma < 1 \quad (\text{A.26})$$

$$\alpha < -\frac{\log(.3)}{\log(2)} \cdot \gamma \approx 1.737 \cdot \gamma . \quad (\text{A.27})$$

This result is formalized as Proposition 3.2 and illustrated as finiteness constraint (c) in Figure 1.