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Hillebrand, Marten; Kikuchi, Tomoo

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A Mechanism for Booms and Busts in Housing Prices*

Marten Hillebrand† Tomoo Kikuchi‡

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Abstract

We study an exchange economy with overlapping generations of consumers who derive utility from consuming a non-durable commodity and housing. A banking sector offers loans to finance housing. We provide a complete characterization of the equilibrium dynamics which alternates between an expansive regime where housing prices increase and banks expand loans and a contractive regime associated with decreasing housing values and shrinking credit volume. Regime switches occur even under small but persistent income changes giving rise to large booms and busts in housing prices not reflecting changes in fundamentals.

JEL classification: C62, E32, G21

Keywords: OLG, Housing prices, Credit volume, Boom-bust scenarios, Regime switching

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†Karlsruhe Institute of Technology, Department of Economics, Germany, Email: marten.hillebrand@kit.edu, Tel.: +49 721 608 45667

‡National University of Singapore, Department of Economics, Singapore, Email: ecstk@nus.edu.sg

1 Introduction

For more than a decade from 1997 to 2007 the world had witnessed a long-lasting increase in U.S. residential values with housing prices in July 2007 being almost three times as high as in mid 1996.[1] In mid 2007, this increase came to an abrupt halt and housing prices dropped by almost 17% until June 2008 and by more than 30% between June 2007 and June 2009. In retrospect, most researchers today seem to agree that houses were overvalued in 2006/2007 (cf. Kocherlakota (2011b)), i.e., the increases in housing values were not backed by fundamentals. Hence, it seems justified to speak of a *housing bubble* which burst in mid 2007.

A closer inspection of the potential reasons also reveals that the build-up of the housing bubble was accompanied by a continual increase in mortgage loans and an ever increasing mortgage debt which dropped sharply after the burst in 2007. This observation points to a potential linkage between the lending activities of banks and the evolution of housing prices. As most residential investments are financed by mortgage loans, this seems fairly intuitive. As the U.S. housing market constituted the triggering event for the 2008 world financial crises, many recent papers study the role of housing and, more generally, of durable consumption goods for the macroeconomy (e.g.Bajari, Chan, Krueger & Miller (2010),Chen & Winter (2010), and Lustig & Nieuwerburgh (2010)).

The purpose of the present paper is to contribute to the previous body of research by developing a mechanism, which explains large movements in housing prices accompanied by corresponding changes in credit volume. Conceptually, we adopt an overlapping generations framework in order to assign a natural role to banks, which mediate intergenerational contracts that otherwise consumers themselves could not commit to. Unlike many studies in the literature, however, we completely abstract from the role of housing as collateral.

We demonstrate that the presence of a banking sector is capable of generating large and and recurrent booms and busts in housing prices. Our analysis uncovers a simple mechanism through which small but persistent income changes generate large movements in housing values and the aggregate credit volume. The mechanism generating these boom-bust cycles is due to a switch between two regimes each associated with a certain income level. In the first regime, consumers are eager to borrow and willing to pay a high return such that banks expand their credit volume over time and housing prices increase without bound. In the second regime, consumers are only willing to borrow at a lower return for which the credit volume shrinks over time and housing prices decrease to a lower bound. A switch between these

[1] These figures are taken from then Casey-Shiller housing index composed over 10 areas.

two regimes occurs under quite moderate income changes. If the change is persistent, the system tends to stay in one regime for a number of periods generating large movements in housing prices and the credit volume.

During boom phases, consumers buy houses at prices they could never have afforded and take loans they would be unable to repay from their income. The financial side (credit volume, housing prices) completely decouples from the real side represented by consumer incomes.

The paper is organized as follows. Section 2 introduces the model. Section 3 derives the forward-recursive structure of equilibria and studies the equilibrium dynamics under constant incomes. Section 4 generalizes the deterministic structure to the case with random incomes and analyzes the scope for bubbles to emerge due to persistent income changes. The theoretical findings are illustrated and quantified in Section 5 with the help of numerical simulations. Section 6 concludes, proofs for all results are relegated to the mathematical appendix.

2 The Model

We consider an exchange economy with discrete time periods $t \geq 0$ and a durable and a non-durable commodity. We refer to the durable commodity as 'housing' and the non-durable good as 'the consumption good', which is chosen as the numeraire.

Consumption sector

The consumption sector consists of overlapping generations of homogeneous, two-period lived consumers. Each member of the generation born in $t \geq 0$ is endowed with $e_t^y > 0$ units of the consumption commodity when young and $e_{t+1}^o > 0$ units when old. The following assumption specifies the probabilistic nature of their incomes.[2]

Assumption 1

The process $\{e_t\}_{t\geq 0}$, where $e_t := (e_t^y, e_{t+1}^o)$, $t \geq 0$, consists of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathcal{E} := [e_{\min}^y, e_{\max}^y] \times [e_{\min}^o, e_{\max}^o] \subset \mathbb{R}_{++}^2$. The process is adapted to some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $e_t : \Omega \to \mathcal{E}$ is $\mathcal{F}_t$–measurable.

A consequence of Assumption 1 is that young consumers observe their first *and* second period income when they make decisions. As the focus of our analysis is on how predictable income shifts affect housing prices and loans and not on the role of income uncertainty, this assumption seems innocuous.

[2]The notion of an adapted stochastic process $\{\xi_t\}_{t\geq 0}$ implies that each random variable ξ_t is $\mathcal{F}_t$-measurable and, therefore, can only depend on incomes e_τ observed during periods $\tau \leq t$. As a notational convention, inequalities involving random variables are understood to hold in the $\mathbb{P}$-almost sure sense without explicit notice.

Housing

Houses are retradable and in constant supply normalized to unity. The young purchase houses at the end of period t at the price $p_t > 0$, for which they incur a fixed cost $\kappa > 0$ per unit in the following period $t+1$. Housing investment provides a possibility to transfer wealth intertemporally and yields utility in the following period.

Banking Sector

The banking sector consists of a large number of banks which take deposits and offer loans at riskless rates. Assuming that there is no default risk, perfect competition implies that deposits and loans pay the same return $R_t > 0$. Let b_t denote the aggregate credit volume at time t. If $b_t > 0$, the young hold deposits while they take a loan if $b_t < 0$. Banks do not accumulate any resources and have a balanced budget such that

$$b_{t+1} = R_t b_t, \qquad t \geq 0. \tag{1}$$

Consumer Demand

The young choose (c^y, c^o, h) to maximize their expected lifetime utility based on some (von-Neumann Morgenstern) utility function U. Given $p_t > 0$, $R_t > 0$, and $p_{t+1} > 0$, the budget constraints are

$$c^y = e_t^y - b - p_t\, h \quad \text{and} \quad c^o = e_{t+1}^o + R_t\, b + (p_{t+1} - \kappa)\, h. \tag{2}$$

where b and h are the loan demand and housing investment respectively. Let $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_t]$ denote the expectations operator conditional on the information available at time t. Using (2), the objective function at time t is

$$V_t(b,h) := \mathbb{E}_t\Big[U(e_t^y - b - p_t\, h, e_{t+1}^o + R_t\, b + (p_{t+1} - \kappa)\, h, h)\Big]. \tag{3}$$

Note that the housing price p_{t+1} is the only potential source of uncertainty. The consumer's decision problem is

$$\max_{b,h}\Big\{V_t(b,h) \mid b + p_t\, h \leq e_t^y,\ e_{t+1}^o + b\, R_t + h(p_{t+1} - \kappa) \geq 0,\ h \geq 0\Big\} \tag{4}$$

Equilibrium.

The following definition of equilibrium reconciles market clearing and individual optimality under rational expectations.

Definition 2.1

Let incomes satisfy Assumption 1. Given an initial credit volume b_0, an equilibrium is an adapted stochastic process $\{b_t, h_t, R_t, p_t\}_{t\geq 0}$, which satisfies $p_t > 0$, $R_t > 0$, and the following conditions for each $t \geq 0$:

(i) The portfolio (b_t, h_t) maximizes utility (4) given prices and incomes.

(ii) Markets clear, i.e., $h_t = 1$ and b_t evolves according to (1).

Note that Walras' law implies consumption good market clearing at time $t \geq 0$, i.e, $c_t^1 + c_t^2 = e_t^y + e_t^o - \kappa$. From (1) and $R_t > 0$, it is clear that the credit volume b_t inherits the sign of b_{t-1} and is therefore uniquely determined by the sign of b_0. In the sequel, we restrict attention to the case where $b_0 \leq 0$, i.e., the banking sector offers loans to the young to finance their housing investments. Hence, the credit volume b_t at time t represents the outstanding payments on previous loans which are claims against old consumers.

3 Housing Price Dynamics

Predictable housing prices

To study the existence and dynamic properties of equilibria, we consider the case where housing prices are predictable, i.e., their realization can be predicted one period in advance. This assumption enables us to obtain an analytically tractable equilibrium. It imposes no restrictions in the case when incomes are deterministic. Formally, we assume that the housing price p_{t+1} is $\mathcal{F}_t$-measurable. Under this hypothesis, the young at time t solve a decision problem under certainty such that the expectations operator in (3) can be dropped.

Preferences

Determining the structure and properties of equilibrium necessitates additional restrictions on the consumers' preferences represented by the utility function U. Assume first that utility is additively separable over time, i.e.,

$$U(c^1, c^2, h) = u(c^1) + v(c^2, h). \tag{5}$$

The function u is taken to be of the isoelastic form

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}, \quad \alpha > 0. \tag{6}$$

Second period utility v is the composition of u and an aggregator function $g : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_+$ which aggregates durable and non-durable consumption to a composite commodity $g(c_t, h_t)$. Following Bajari, Chan, Krueger & Miller (2010) or Lustig & Nieuwerburgh (2010), we use a CES aggregator

$$g(c, h) = [\beta c^\rho + (1 - \beta)(\theta h)^\rho]^{\frac{1}{\rho}}, \ 0 < \beta < 1, \ \theta > 0, \ \rho < 1.$$

The young discount second-period utility by $\gamma > 0$ and thus v takes the form

$$v(c, h) = \gamma\, u(g(c, h)) = \gamma \frac{[\beta c^\rho + (1 - \beta)(\theta h)^\rho]^{\frac{1-\alpha}{\rho}} - 1}{1 - \alpha}. \tag{7}$$

If $\rho = 0$, v is Cobb-Douglas while it is additively separable in housing and consumption if $\rho = 1 - \alpha$.

Recursive Equilibrium

As a first step, we wish to unveil the forward-recursive structure of equilibrium and the state dynamics of the model. In this regard, we will show that the dynamics is essentially driven by the evolution of the variable

$$m_t := b_t + p_t, \quad t \geq 0. \tag{8}$$

Recalling that $h_t \equiv 1$, the quantity m_t represents the net investment of the young at time t consisting of housing investment and the loan taken. If $m_t > 0$, housing investment at time t is not fully financed by loans while these loans exceed housing investment if $m_t < 0$. Since housing prices are predictable and banks offer riskless returns , the process $\{m_t\}_{t\geq 0}$ is predictable as well due to (1).

To derive the recursive equilibrium structure, consider the solution to a young consumer's decision problem (4). Since no sign-restrictions are imposed on b at the individual level, it is clear that the first order conditions must be satisfied in equilibrium. Exploiting this and (5) and (8), the following Euler equations have to hold for each period $t \geq 0$:

$$u'(e_t^y - m_t) = R_t\, v_c(e_{t+1}^o - \kappa + m_{t+1}, 1) \tag{9a}$$

$$p_t\, u'(e_t^y - m_t) = (p_{t+1} - \kappa)\, v_c(e_{t+1}^o - \kappa + m_{t+1}, 1) + v_h(e_{t+1}^o - \kappa + m_{t+1}, 1). \tag{9b}$$

Given $e = (e^y, e^o) \in \mathcal{E}$, let $F(\cdot, \cdot; e) : (\kappa - e^0, \infty) \times (-\infty, e^y) \longrightarrow \mathbb{R}$ where

$$F(m_1, m; e) := m\, u'(e^y - m) - v_c(e^o - \kappa + m_1, 1)\,(m_1 - \kappa) - v_h(e^o - \kappa + m_1, 1). \tag{10}$$

Then, adding (9a) and (9b) using (1) and (8) gives the following equilibrium condition

$$F(m_{t+1}, m_t; e_t) = 0 \tag{11}$$

which has to hold at each time $t \geq 0$. Condition (11) determines the value m_{t+1} implicitly as a function of m_t and e_t which will be key to derive the model's state dynamics. The following result states necessary and sufficient conditions under which a unique solution to (11) can be determined.

Lemma 1

Suppose the parameters in (7) satisfy $\rho \geq 0$ and $\alpha < 1$. Then, for each $e = (e^y, e^o) \in \mathcal{E}$ and $m < e^y$ there exists a unique value $m_1 > \kappa - e^o$ to satisfy $F(m_1, m; e) = 0$.

The restrictions $\rho \geq 0$ and $\alpha < 1$ are necessary and sufficient for $\lim_{c\to\infty} v_c(c, 1)\, c = \infty$ which is crucial for existence of a solution to (11) for arbitrary m_t and e_t.[3] Under the restriction $\rho \geq 0$ and $\alpha < 1$, Lemma 1 permits to define a

[3] Although the restriction $\alpha < 1$ excludes a logarithmic function u used in Bajari, Chan, Krueger & Miller (2010), this case can be approximated as the limiting case $\alpha \to 1$ in our setup.

map $f(\cdot; e^y, e^o) : (-\infty, e^y) \longrightarrow (\kappa - e^o, \infty)$ which determines the unique zero of $F(\cdot, m; e) = 0$ for each $m < e^y$. Thus, whenever $m_t < e_t^y$, the solution to (11) can be written as

$$m_{t+1} = f(m_t; e_t). \tag{12}$$

The next result establishes properties of f.

Lemma 2
Let $e = (e^y, e^0) \in \mathcal{E}$ be arbitrary and the hypotheses of Lemma 1 be satisfied. Then the map $f = f(\cdot; e)$ is continuously differentiable with derivative $f'(m) > 0$ for all $m < e^y$.

Using the result from (12) in (9a) and (9b), the equilibrium loan returns and housing prices at time t are determined for each $e_t \in \mathcal{E}$ and $m_t < e_t^y$ as

$$R_t = \mathcal{R}(m_t; e_t) \quad := \quad \frac{u'(e_t^y - m_t)}{v_c(e_{t+1}^o - \kappa + f(m_t; e_t^y), 1)} \tag{13}$$

$$p_{t+1} = \mathcal{P}(p_t, m_t; e_t) \quad := \quad \mathcal{R}(m_t, e_t)p_t + \kappa - \frac{v_h(e_{t+1}^o - \kappa + f(m_t; e_t), 1)}{v_c(e^o - \kappa + f(m; e), 1)} \tag{14}$$

while loans b_{t+1} follow from (1). We focus on the case $m_t > 0$ and $b_t \leq 0$ for all t. The latter restriction corresponds to the case when banks provide mortgage loans while the former implies that the young need to make a downpayment when taking a loan to purchase housing. By (1), $b_t \leq 0$ holds automatically if $b_0 \leq 0$. The remainder will establish conditions under which $m_t > 0$ for all t. By (8), these restrictions also ensure that $p_t > 0$ for all times $t \geq 0$.

Dynamics under constant incomes

Based on the previous results, the sequel studies the dynamic evolution of the model and establishes conditions for the existence of equilibrium. To this end, note that (12) does not yet define a dynamical system because we have not determined a suitable set $\mathbb{M} \subset (-\infty, e^y)$ on which the dynamics can live. This will be our next goal. As a first step, the remainder of this section studies the equilibrium dynamics under constant first-period income. The next section will extend this to the case where first-period income is random and time-dependent. Thus, fix $e_t \equiv e = (e^y, e^o) \in \mathcal{E}$ for all $t \geq 0$. For notational convenience, the dependence of variables and functions on e will be suppressed. For the above derivations to be valid for any period, we need to make sure that $m_t < e^y$ in each period. In addition, we require that $m_t > 0$ for all $t \geq 0$. Formally, we seek to determine a suitable open interval $\mathbb{M} \subset (0, e^y)$ which is self-supporting for the map $f = f(\cdot; e)$, i.e., $f(m) \in \mathbb{M}$ for all $m \in \mathbb{M}$. In this regard, fixed points of f, i.e., values $\bar{m}$ which satisfy $\bar{m} = f(\bar{m})$ will play a crucial role. Since f maps $(-\infty, e^y)$ into $(\kappa - e^o, \infty)$, it is clear that any such fixed point must lie in the open interval $(\kappa - e^o, e^y)$. Therefore, a necessary precondition for fixed points to exist is

$e^y + e^o > \kappa$. This condition simply says that the resources available in each period are large enough to cover housing costs which we will assume in the sequel. Then, it follows from (10) that fixed-points of f obtain as zeros of the map $G : (\kappa - e^o, e^y) \longrightarrow \mathbb{R}$ where

$$\begin{aligned} G(m) &:= F(m, m; e) \\ &= m\, u'(e^y - m) - v_c(e^o - \kappa + m, 1)\,(m - \kappa) - v_h(e^o - \kappa + m, 1). \end{aligned} \tag{15}$$

The following result states properties of the map G.

Lemma 3
Suppose $\rho \geq 0$ and $\alpha < 1$. Then, for each $e = (e^y, e^o) \in \mathcal{E}$ satisfying $e^y + e^o > \kappa$ the map $G = G(\cdot; e)$ is a strictly convex function and the derivative satisfies the boundary behavior $\lim_{m \to e^y} G'(m) = -\lim_{m \to \kappa - e^o} G'(m) = \infty$.

A consequence of the lemma is the existence of a unique value $m_{\min} \in (\kappa - e^o, e^y)$ at which $G'(m_{\min}) = 0$ and G attains its global minimum. Based on this insight, the next result states conditions for fixed points to exist and characterizes their properties.

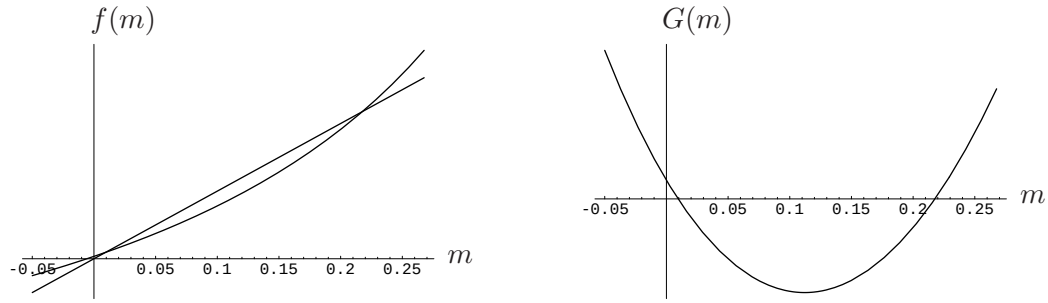
Lemma 4
In addition to the hypotheses of Lemma 3, let $G(m_{\min}) < 0$. Then, the following holds:

(i) The map f has precisely two fixed points $\bar{m} \in (\kappa - e^o, m_{\min})$ and $\bar{\bar{m}} \in (m_{\min}, e^y)$.

(ii) The fixed point $\bar{\bar{m}}$ is locally unstable while $\bar{m}$ is asymptotically stable. Moreover, $f(m) > m$ for all $m \in (-\infty, \bar{m}\,) \cup (\,\bar{\bar{m}}, e^y)$ and $f(m) < m$ for all $m \in (\bar{m}, \bar{\bar{m}})$.

Setting aside the non-generic case where $G(m_{\min}) = 0$, the boundary behavior of G implies that the condition $G(m_{\min}) < 0$ is not only sufficient but also necessary for fixed points to exist. Moreover, it can be shown that $G(m_{\min}) > 0$ would imply $f(m; e) > m$ for all m. In this case, for any $m_0 < e^y$ a repeated iteration of the forward-recursion (12) would produce a value $m_t > e^y$ after finitely many periods $t \geq 1$. Thus, $G(m_{\min}) < 0$ is also a necessary condition for the dynamics to be viable. The finding from Lemma 4 is illustrated in the following figures which depict the map f and the fixed point map G. Note that the zeros of G in Figure 1(b) correspond to intersections of (the graph of) f with the principal diagonal in Figure 1(a). Lemma 4(ii) reveals that the set $(-\infty, \bar{\bar{m}})$ is self-supporting under f. Thus, the map f restricted to this set becomes the time-one map of a one-dimensional dynamical system in discrete time.[4] Moreover, for any initial value $m_0 \in (-\infty, \bar{\bar{m}})$ the sequence $\{m_t\}_{t \geq 0}$ defined recursively as $m_{t+1} = f(m_t)$, $t \geq 0$ converges monotonically

[4]For convenience, we denote the restriction of f to a subset $\mathbb{M} \subset (-\infty, e^y)$ by f as well.



(a) The map f

(b) Fixed point map G

Figure 1: Time-one map of state dynamics

to $\bar{m}$. By contrast, for any initial value $m_0 \in (\bar{\bar{m}}, e^y)$ the forward-recursion $m_{t+1} = f(m_t)$ exceeds e^y after finitely many periods. Thus, the forward dynamics are well-defined if and only if $m_0 \in (-\infty, \bar{\bar{m}})$ and it would be possible to restrict the map f to the interval $(-\infty, \bar{\bar{m}})$ which is the largest subset on which the m-dynamics defined by (12) are viable.

To ensure $m_t > 0$, we seek to impose additional restrictions under which the dynamics can further be restricted to a subset of $\mathbb{R}_{++}$. For this purpose, assume that $e^o - \kappa > 0$ and suppose that the smaller fixed point satisfies $\bar{m} > 0$. Then, Lemma 4(ii) in conjunction with Lemma 2 imply that the interval $\mathbb{M} := (0, \bar{\bar{m}})$ is self-supporting under f as well. Thus, restricting f to $\mathbb{M}$ defines a discrete dynamical system which governs the evolution of the variable $m_t \in \mathbb{M}$ over time. The long-run behavior is now characterized in the following result.

Lemma 5
Suppose $\rho \geq 0$ and $\alpha < 1$ and let $e = (e^y, e^0) \in \mathcal{E}$ be given and satisfy $e^o > \kappa$. In addition, let $G(m_{\min}) < 0 < \bar{m}$ and define $\mathbb{M} := (0, \bar{\bar{m}})$. Then, the following holds:

(i) The restricted map $f : \mathbb{M} \longrightarrow \mathbb{M}$ has $\bar{m}$ as its unique fixed point.

(ii) This fixed point is globally stable and for each $m_0 \in \mathbb{M}$ the sequence $\{m_t\}_{t\geq 0}$ defined recursively as $m_{t+1} = f(m_t; e^y)$, $t \geq 0$ converges monotonically to $\bar{m}$.

In addition to the parameter restrictions $\rho \geq 0$ and $\alpha < 1$, the previous findings show that the main restrictions needed for the dynamics to be well-defined are that $G(m_{\min}) < 0$ and, in addition, $\bar{m} > 0$. The latter is necessary and sufficient for the dynamics to live in a subset of $\mathbb{R}_{++}$ and satisfied if and only if $G'(0) < 0 < G(0)$. Using (15) and (A.2), these two conditions hold if and only if

$$\kappa > \frac{v_h(e^o - \kappa, 1)}{v_c(e^o - \kappa, 1)} = \frac{1-\beta}{\beta}\theta^\rho [e^o - \kappa]^{1-\rho} \wedge u'(e^y) < (1-\alpha) v_c(e^o - \kappa, 1) - e^o v_{cc}(e^o - \kappa, 1).$$

The first condition is violated for $\kappa = 0$ and, therefore, requires housing costs to exceed a critical level which depends on e^o. As this condition is independent of first-period income, the value e^y can always be chosen sufficiently large to satisfy the second condition. Notice, however, that the condition $G(m_{\min}) < 0$ also depends on these choices as well. Given that $m_{\min}$ is only implicitly defined by $G'(m) = 0$, it is difficult to fully characterize the underlying parameter sets. The numerical simulation of the following section shows that all three conditions are satisfied for a broad range of economically reasonable parameterizations.

Equilibrium under constant incomes

Employing the previous findings, we are now in a position to characterize the complete equilibrium dynamics for a fixed income process $e_t \equiv e = (e^y, e^o) \in \mathcal{E}$ for which all hypotheses of Lemma 5 are satisfied. Fix an initial value (p_0, b_0) which satisfies $b_0 \leq 0$, $p_0 > 0$, and $m_0 := p_0 + b_0 \in \mathbb{M} = (0, \bar{\bar{m}})$. Then, combining our previous results with (1), (13), and (14) one obtains the following three-dimensional system which governs the evolution of all equilibrium variables:

$$m_{t+1} = f(m_t; e) \tag{16a}$$

$$b_{t+1} = \mathcal{B}(b_t, m_t, e) := \mathcal{R}(m_t; e) b_t \tag{16b}$$

$$p_{t+1} = \mathcal{P}(p_t, m_t, e) = \mathcal{R}(m_t, e) p_t + \kappa - \frac{v_h(e^o - \kappa + f(m_t; e), 1)}{v_c(e^o - \kappa + f(m_t; e), 1)}. \tag{16c}$$

The dynamics (16a) of m_t is decoupled from the other two variables and converge monotonically to a unique steady state $\bar{m}$ by Lemma 5. It is clear from (16b) and (16c) that the qualitative long-run dynamic behavior of housing prices p_t and the credit volume b_t depend on the steady state loan return $\mathcal{R}(\bar{m}; e)$. If $\mathcal{R}(\bar{m}; e) < 1$, the credit volume asymptotically converges to zero while, by (8) prices converge to $\bar{p} = \bar{m}$. Conversely, if $\mathcal{R}(\bar{m}; e) > 1$ and $b_0 < 0$, both the credit volume and housing prices grow without bound and converge to plus and minus infinity, respectively. Notice, however, that the equilibrium dynamics are well-defined in either case. The following final theorem of this section summarizes these insights and establishes the existence and properties of equilibrium.

Theorem 1

Let incomes $e = (e^y, e^o) \in \mathcal{E}$ be constant and satisfy the hypotheses of Lemma 5. Then, the following holds:

(i) Each $p_0 > 0$ and $b_0 \leq 0$ for which $p_0 + b_0 \in \mathbb{M} =]0, \bar{\bar{m}}[$ defines an equilibrium where the evolution of the equilibrium variables follows (16a–c) and $\lim_{t\to\infty} m_t = \bar{m}$.

(ii) If $b_0 < 0$ and $\mathcal{R}(\bar{m}; e) > 1$, then $\lim_{t\to\infty} p_t = -\lim_{t\to\infty} b_t = \infty$.

(iii) If $b_0 = 0$ or $\mathcal{R}(\bar{m}; e) < 1$, then $\lim_{t\to\infty} p_t = \bar{m}$ while $\lim_{t\to\infty} b_t = 0$.

4 Housing Booms and Busts

Dynamics under random incomes

We now analyze the case where incomes fluctuate randomly over time. For ease of exposition, we will confine attention to the case where only first-period incomes change over time while second-period incomes are assumed to be constant. Thus, assume as in the previous section that $e_t^o \equiv e^o > \kappa$ while e_t^y fluctuates over time taking values in the set $\mathcal{E}^y := [e^y_{\min}, e^y_{\max}] \subset \mathbb{R}_{++}$. In the sequel, we will therefore drop the argument e^o writing e.g. $f(m; e^y)$ instead of $f(m; e^y, e^o)$. Consider first how the process $\{m_t\}_{t\geq 0}$ defined in (8) evolves over time. It is clear from (12) that this process is well-defined if and only if $m_t < e_t^y$ for all $t \geq 0$ $\mathbb{P}$–a.s., i.e., $m_t < e^y_{\min}$ for all $t \geq 0$ $\mathbb{P}$–a.s. Suppose this can be satisfied. Then, the forward-recursive structure of the model is well-defined and generated by randomly mixing the family of mappings $(f\cdot; e^y)_{e^y \in \mathcal{E}^y}$. That is, given m_t, the value $e_t^y \in \mathcal{E}$ realized at time t 'selects' a particular map $f(\cdot; e_t^y)$ which determines the next value $m_{t+1} = f(m_t; e_t^y)$. For this forward-recursion to be well-defined, we seek to determine a stable interval $\mathbb{M} \subset]0, e^y_{\min}[$ which is self supporting under the family $(f\cdot; e^y)_{e^y \in \mathcal{E}^y}$, i.e., $m \in \mathbb{M}$ implies $f(m; e^y) \in \mathbb{M}$ for all $e^y \in \mathcal{E}^y$. To accomplish this, assume that the hypotheses of Lemma 5 are satisfied for all $e^y \in \mathcal{E}^y$. Then, each map $f(\cdot; e^y)$ has precisely two fixed points in $(0, e^y)$ which we denote by $\bar{m}(e^y)$ and $\bar{\bar{m}}(e^y)$, respectively as a way of stressing their dependence on e^y. The next result describes how these fixed points vary with income.

Lemma 6

Let the hypotheses of Lemma 5 be true for each $e^y \in \mathcal{E}^y$. Then, the following holds:

(i) For each $m > 0$ the map $e^y \mapsto f(m; e^y)$ is continuously differentiable (on the interior of $\mathcal{E}^y$) and strictly decreasing.

(ii) The fixed point maps $e^y \mapsto \bar{m}(e^y)$ and $e^y \mapsto \bar{\bar{m}}(e^y)$ are both continuously differentiable. Moreover, $\bar{m}(\cdot)$ is strictly decreasing while $\bar{\bar{m}}(\cdot)$ is strictly increasing.

Using the previous result, define

$$\bar{m}_{\min} := \min_{e^y \in \mathcal{E}^y} \left\{ \bar{m}(e^y) \right\} = \bar{m}(e^y_{\max}) \tag{17a}$$

$$\bar{m}_{\max} := \max_{e^y \in \mathcal{E}^y} \left\{ \bar{m}(e^y) \right\} = \bar{m}(e^y_{\min}) \tag{17b}$$

$$\bar{\bar{m}}_{\min} := \min_{e^y \in \mathcal{E}^y} \left\{ \bar{\bar{m}}(e^y) \right\} = \bar{\bar{m}}(e^y_{\min}). \tag{17c}$$

Note that the values defined in (17a–c) satisfy $0 < \bar{m}_{\min} < \bar{m}_{\max} < \bar{\bar{m}}_{\min}$. Thus, defining $\bar{\mathbb{M}} := [\bar{m}_{\min}, \bar{m}_{\max}]$ and $\bar{\bar{\mathbb{M}}} := (0, \bar{\bar{m}}_{\min})$ we have the inclusions

$\emptyset \neq \bar{\mathbb{M}} \subsetneqq \bar{\bar{\mathbb{M}}} \subset \mathbb{R}_{++}$. The following result essentially extends Lemma 5 to the more general stochastic case.

Lemma 7
Let the hypotheses of Lemma 5 be satisfied for each $e^y \in \mathcal{E}^y$. Then, the following holds:

(i) Both intervals $\bar{\mathbb{M}}$ and $\bar{\bar{\mathbb{M}}}$ are self-supporting for the family $(f\cdot : e^y)_{e^y \in \mathcal{E}^y}$.

(ii) For each $m_0 \in \bar{\bar{\mathbb{M}}}$, the dynamics generated by randomly mixing $(f\cdot; e^y)_{e^y \in \mathcal{E}^y}$ converge to the set $\bar{\mathbb{M}}$ $\mathbb{P}$–a.s..

It follows from Lemma 7 that asymptotically, the process $\{m_t\}_{t\geq 0}$ will take values in the set $\bar{\mathbb{M}}$. Thus, if $\{e_t^y\}_{t\geq 0}$ is sufficiently regular, e.g., follows a Markov process, standard results from the literature (cf. Brock & Mirman (1972), Wang (1993)) imply the existence of a unique invariant distribution supported on $\bar{\mathbb{M}}$ which governs the long-run probabilistic behavior of the process $\{m_t\}_{t\geq 0}$. In particular, this process will be asymptotically stationary.[5] Figure 2 illustrates the finding from Lemma 7 for the case with two shocks where $e_t^y \in \{e_{\min}^y, e_{\max}^y\}$ for all t.

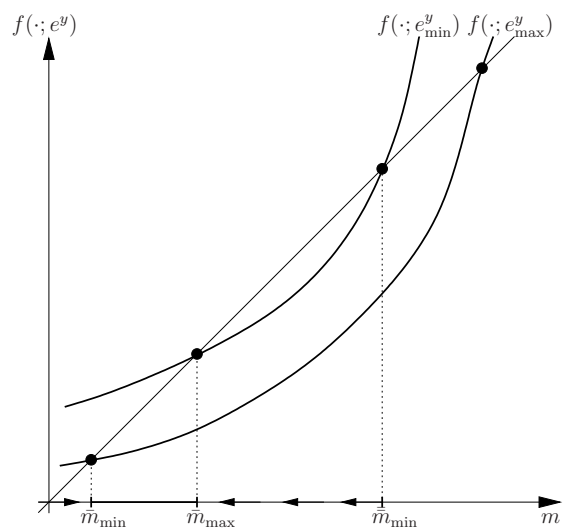


Figure 2: Time-one maps generating the dynamics under two shocks

Equilibrium under random incomes
Based on the previous result, the following theorem generalizes Theorem 1 to the case with stochastic first-period incomes. Note that Theorem 1 obtains as a special case where $e_{\min}^y = e_{\max}^y = e^y$.

[5]Lemma 4(ii) and the definitions (17a–c) imply that the family $f = (f\cdot : e^y)_{e^y \in \mathcal{E}}$ restricted to the interval $\bar{\bar{\mathbb{M}}}$ possesses a *stable fixed-point configuration* in the sense of Brock & Mirman (1972). Thus, the assertion follows from their results, see also Wang (1993).

Theorem 2
Let the hypotheses of Lemma 5 be satisfied for each $e^y \in \mathcal{E}^y$. Then, each $p_0 > 0$ and $b_0 \leq 0$ for which $m_0 := p_0 + b_0 \in \bar{\bar{\mathbb{M}}} = (0, \bar{\bar{m}}_{\min})$ defines an equilibrium.

Since we are interested in the long-run properties of equilibrium, we can confine attention to the set $\bar{\mathbb{M}} := [\bar{m}_{\min}, \bar{m}_{\max}]$ by virtue of Lemma 7(ii). Analogously to the previous section, the behavior of loan returns is crucial for the long-run behavior of equilibrium housing prices and the credit volume. The following result characterizes how the loan return changes along with the shocks and the value of m.

Lemma 8
Let the hypotheses of Lemma 5 be satisfied for each $e^y \in \mathcal{E}^y$. Then, the map $\mathcal{R}$ defined in (13) is continuously differentiable with partial derivatives $\mathcal{R}_{e^y}(m; e^y) < 0 < \mathcal{R}_m(m; e^y)$ for all $e^y > 0$ and $m < e^y$.

For $e^y \in \mathcal{E}^y$, let $\bar{\mathcal{R}}(e^y) := \mathcal{R}(\bar{m}(e^y); e^y)$ and set

$$R^{\min} := \min\{\mathcal{R}(m; e^y) \mid m \in \bar{\mathbb{M}}, e^y \in \mathcal{E}\} = \mathcal{R}(\bar{m}_{\min}, e^y_{\max}) = \bar{\mathcal{R}}(e^y_{\max}) \tag{18a}$$

$$R^{\max} := \max\{\mathcal{R}(m; e^y) \mid m \in \bar{\mathbb{M}}, e^y \in \mathcal{E}\} = \mathcal{R}(\bar{m}_{\max}, e^y_{\min}) = \bar{\mathcal{R}}(e^y_{\min}) \tag{18b}$$

Thus, the minimum (maximum) loan return observed in the long-run is attained at the maximum (minimum) fixed point of f. We now have the following result which provides a first characterization of equilibrium.

Theorem 3
Let the hypotheses of Lemma 5 be satisfied for each $e^y \in \mathcal{E}^y$. Then, the following holds:

(i) If $b_0 < 0$ and $R^{\min} > 1$, then $\lim_{t\to\infty} p_t = -\lim_{t\to\infty} b_t = \infty$, $\mathbb{P}$–a.s.

(ii) If $b_0 = 0$ or $R^{\max} < 1$, then $\lim_{t\to\infty} b_t = 0$ $\mathbb{P}$–a.s. while $\lim_{t\to\infty} |p_t - m_t| = 0$ $\mathbb{P}$–a.s.

Theorem 3 describes two cases where the long-run behavior of housing prices and the credit volume is either expansive (i) or stationary (ii). We also observe that if $b_0 = 0$, i.e., in the absence of a banking sector, the housing price coincides with the process $\{m_t\}_{t\geq 0}$ which is stationary and well-behaved. Thus, any possible long run expansion of the credit volume and the housing price is exclusively due to the banking sector. Excluding the non-generic cases of either $R^{\min} = 1$ or $R^{\max} = 1$, recurrent housing booms and busts emerge only if $R^{\min} < 1 < R^{\max}$.

The mechanism for booms and busts

To illustrate the mechanism that generates booms and busts of housing prices, consider the simplest case where e^y takes two values $e^y_{\min}$ and $e^y_{\max}$ with positive probability. Suppose that $b_0 < 0$ and $R^{\min} < 1 < R^{\max}$ and incomes

initially take the lower value $e_t^y = e_{\min}^y$. Then, the dynamics generated by the map $f(\cdot; e_{\min}^y)$ start converging to the associated steady state $\bar{m}(e_{\min}^y) = \bar{m}_{\max}$ and we have $R_t > 1$ for t sufficiently large as $\mathcal{R}(\bar{m}_{\max}, e_{\min}^y) = R^{\max} > 1$. By (1), the credit volume starts to expand (in absolute value) and so do housing prices as their sum m_t is stationary. Intuitively, the low first period income increases the need for consumption smoothing and the demand for credit, for which the young are willing to pay a high interest rate. Although the supply of credit expands over time as well, this is absorbed by a corresponding higher demand due to the increase in housing prices. Thus, we see that as long as the low income regime prevails, both housing prices and credit volume increase whereas their sum converges to $\bar{m}_{\max}$.

Now, suppose that at some time $\tilde{t} > 0$, income take the higher value $e_{\max}^y$. The corresponding dynamics is generated by the map $f(\cdot; e_{\max}^y)$ which has $\bar{m}_{\min}$ as its unique steady state to which the variable m_t start to converge. For sufficiently large $t > \tilde{t}$, we will have $R_t < 1$ implying that both the credit volume and housing prices will contract. Combining these observations, it is clear that the system will alternate between an expansionary regime and a contractive regime. These changes are most profound if $\mathcal{R}(m; e_{\min}^y) > 1$ and $\mathcal{R}(m; e_{\max}^y) < 1$ for all $m \in \bar{\mathbb{M}}$. The first requirement is equivalent to $\mathcal{R}(\bar{m}_{\min}, e_{\min}^y) > 1$ and implies that the credit volume starts to expand immediately when $e_t = e_{\min}^y$. The second condition is equivalent to $\mathcal{R}(\bar{m}_{\max}, e_{\max}^y) < 1$ and implies that the credit volume starts to contract immediately when $e_t = e_{\min}^y$. Now if the income process is persistent, then long periods of credit expansion will follow long periods of credit contraction. This mechanism offers a potential to generate large movements in housing prices simply due persistent income changes.

This mechanism straightforwardly generalizes to the case where incomes are continuously distributed on the interval $[e_{\min}^y, e_{\max}^y]$ as long as the dynamics alternate between the expansive regime $\mathbb{M}^x := \{(m, e) \in \bar{\mathbb{M}} \times \mathcal{E}^y \,|\, \mathcal{R}(m; e) > 1\}$ and the contractive regime $\mathbb{M}^c := \{(m, e) \in \bar{\mathbb{M}} \times \mathcal{E}^y \,|\, \mathcal{R}(m; e) < 1\}$. In the following section we will employ numerical simulations to show that the previous scenario occurs under realistic parameter choices and the switch between the two regimes is triggered by relatively small income changes.

5 Simulation Results

Parameters

We calibrate our model based on various sources in the literature. Bajari, Chan, Krueger & Miller (2010) use a logarithmic function u which we approximate in our setup by choosing α close to unity. They also devise a an elasticity of substitution between housing and second-period consumption slightly larger than unity (about 1.3) corresponding to $\rho = 0.24$. For sim-

plicity, we follow Li & Yao (2007) by confining ourselves to the case of unit elasticity setting $\rho = 0$ which yields a Cobb-Douglas function for second-period utility. For this choice, the parameter $1-\beta$ can be interpreted as the share of housing expenditure in consumer income and Bajari, Chan, Krueger & Miller (2010) choose a value of $\beta \approx .77$. Given that housing is confined to the second period of life in our setup, we choose a smaller value $\beta = \frac{2}{3}$. As in Lustig & Nieuwerburgh (2010), the scaling parameter θ is set to unity and we also assume, again for simplicity, that $\gamma = 1$.

As for the income processes, survey data on household incomes typically exhibit a hump-shaped distribution over the life cycle, cf. Bajari, Chan, Krueger & Miller (2010). We choose (expected) incomes slightly higher in the first than in the second period. This seems broadly consistent with cf. Bajari, Chan, Krueger & Miller (2010) and the values in their Table 3. Finally, the housing cost parameter κ was chosen equal to about $\frac{1}{3}$ of second period income e^o. This may be justified given that housing consumption is confined to the second period of life. Table 1 lists the set of parameter values employed.

Parameter	Value	Parameter	Value	Parameter	Value
α	.97	γ	1	β	.67
θ	1	ρ	0	κ	0.35
$e^y_{\min}$	1.06	$e^y_{\max}$	1.09	e^o	1.05

Table 1: Parameters

Under this parametrization, the hypotheses of Lemma 4 hold for all $e^y \in \mathcal{E}^y$. Thus, for each fixed income stream $e^y_t \equiv e^y \in \mathcal{E}^y$, the dynamics (16a) converges to a unique steady state $\bar{m}(e^y) > 0$.[6] In particular, the above parametrization implies that $R^{\min} = \mathcal{R}(\bar{m}(e^y_{\max}), e^y_{\max}) < 1 < \mathcal{R}(\bar{m}(e^y_{\min}), e^y_{\min}) = R^{\max}$ such that the necessary conditions for booms and busts of housing prices to occur are satisfied. For these to be sufficient, it is essentially required to specify the probabilistic nature of the income process such that the contractive and the expansionary regime both occur with a sufficient degree of persistence.

Income processes

Assume that the income process $\{e^y_t\}_{t\geq 0}$ is governed by a two-state Markov process $\{s_t\}_{t\geq 0}$ where the random variable s_t takes values in $\{H, L\}$. In the state $s_t = H$ income is high, while it is low in the state $s_t = L$. Transitions between the two states occur with time-invariant probabilities given by the stochastic matrix

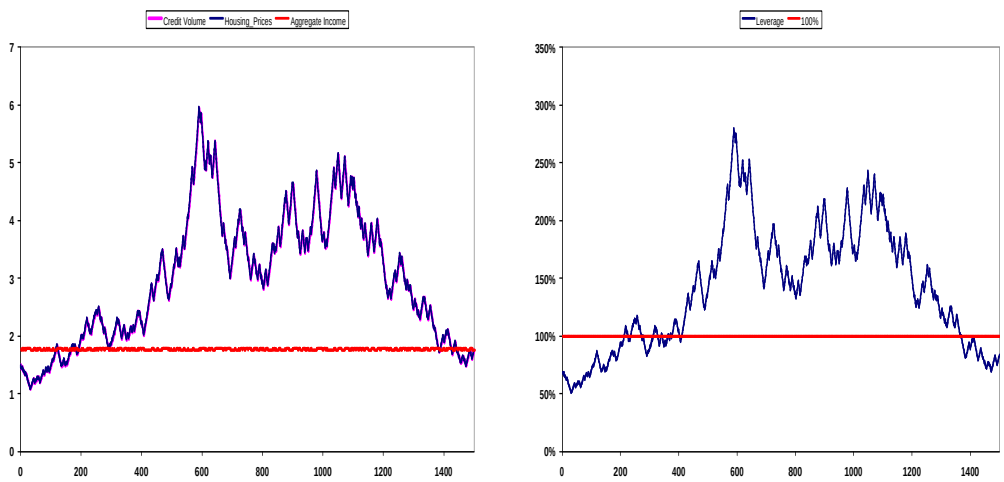
$$\Pi = \begin{bmatrix} \pi_{HH} & \pi_{HL} \\ \pi_{LH} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix}. \tag{19}$$

[6] We remark that the chosen parametrization guarantees positivity of steady states, which may fail to exist at all or $\bar{m}(e^y) < 0$ for some $e^y \in \mathcal{E}^y$ under other parametrizations.

Here $\pi_{ss'}$ is the probability of moving from state $s \in \{H, L\}$ to state $s' \in \{H, L\}$ and the initial distribution of s_0 is π_0. Thus, the process $\{s_t\}_{t\geq 0}$ is highly persistent with an 80% chance of retaining its current state and only a 20% chance of switching to the opposite state.

Two-state incomes

Suppose that the income process takes values in $\mathcal{E} = \{e^y_{\min}, e^y_{\max}\}$ such that $e^y_t = e^y_{\max}$ if $s_t = H$ and $e^y_t = e^y_{\min}$ if $s_t = L$. Thus, consumer income directly mirrors the two-state Markov process defined by (19). Figure 3 portrays time series' of the model's variables for $T = 1500$ periods. The left panel in Figure 3 shows a time window of housing prices p_t and the credit volume $|b_t|$. To relate movements in these variables to the fundamentals of the economy we also depict the aggregate net income $e^y_t + e^o - \kappa$ which represents the total resources available in period t net of housing costs. The right panel depicts the leverage ratio $\frac{|b_t|}{e^y_t + e^o R_t^{-1}}$ which measures the percentage share of loans backed by consumers' discounted lifetime incomes.



(a) p_t, $|b_t|$, and $e^y_t + e^o - \kappa$ (b) $\frac{|b_t|}{e^y_t + e^o R_t^{-1}}$

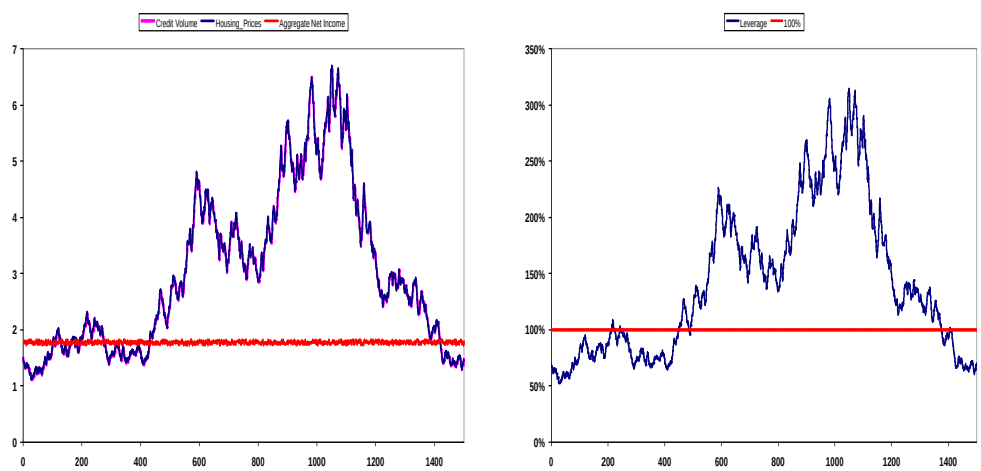
Figure 3: Two-state incomes

The figures confirm that the mechanism identified in the previous section generates large and persistent movements in housing prices and the credit volume. These two variables are intimately interconnected and are almost impossible to distinguish in Figure 3(a). By our previous results, we know that $p_t - |b_t|$ follows a stationary stochastic process. We also see that for most time periods both housing investments and the credit volume exceed the aggregate income by an order of magnitude. To provide a quantitative illustration of these phenomena, consider the situation in period $t_0 = 596$ where the bubble reaches its peak. In this period, the young are in the low-income state receiving $e^y_{t_0} = 1.06$ such that aggregate net income is $e^y_t + e^o - \kappa = 1.76$. The current credit volume supplied by banks is $|b_{t_0}| = 5.77$

and the interest on loans is 3%, i.e., $R_{t_0} = 1.03$. The young buy houses at the current price $p_{t_0} = 5.78$ which is financed by taking a loan equal to $|b_{t_0}|$. This corresponds to a leverage ratio of 275%, i.e., the loan taken exceeds young consumer's lifetime income by a factor in the ballpark of three. Moreover, the loan repayment $R_{t_0} b_{t_0} = -5.85$ is more than five time as large as second-period non-housing income e^o. However, the next period's housing price $p_{t_0+1} = 5.86$ allows consumers to repay their loan from the revenues of selling their houses at the end of period $t_0+1 = 597$. This confirms our earlier insight that a credit volume exceeding real incomes by an order of magnitude can still be sustained by corresponding increase in housing values. In fact, the net flow from young to old consumers is only equal to $p_{t_0} + b_{t_0} = m_{t_0} = 0.011$.

Continuously distributed income

We now consider the case where first period income has an absolutely-continuous and state dependent probability distribution. Thus, employing the previous model for the state variable s_t, we assume that the random variable e_t^y has probability distribution ν^H if $s_t = H$ and ν^L if $s_t = L$. Both distributions are uniform on $[e^L_{\min}, e^L_{\max}] = [1.03, 1.09]$ and $[e^H_{\min}, e^H_{\max}] = [1.06, 1.12]$, respectively. Observe that their supports overlap and that their mean values correspond to the choices for e^H and e^L in the previous case. Despite the fact that the income regimes overlap and , we see from Figure



(a) p_t, $|b_t|$ and $e_t^y + e^o - \kappa$ (b) Leverage $\frac{|b_t|}{e_t^y + e^o R_t^{-1}}$

Figure 4: Continuously distributed income

4 that all qualitative findings from the previous experiment remain intact. Thus, the previous phenomena are not due to the extreme specification of income following a two-state process and continue to hold under a more general and less restrictive specification.

6 Conclusions

In the absence of a banking sector the only intergenerational trade taking place is through the housing market. Consequently, housing values are bounded by young consumers' income. Introducing a banking sector adds a channel of intergenerational trade in the form of a credit market, which mediates an exchange of commodities from the old to young. The combination of these channels permits each flow of intergenerational trade to become arbitrarily large as long as the net flow remains bounded by consumer's income. The transfer of resources between generations amplifies small but persistent income changes into large movements of housing prices.

Housing prices and the credit volume are non-stationary while a linear combination of them follows a stationary stochastic process. The presence of such cointegration relationship between these variables is therefore a testable implication of the model that might be interesting to explore.

A Mathematical Proofs

A.1 Proof of Lemma 1

Let $e = (e^y, e^o) \in \mathcal{E}$ and $m < e^y$ be arbitrary but fixed. For brevity, set $\underline{m} := \kappa - e^o$ and

$$H(m_1) := v_c(m_1 - \underline{m}, 1)\,(m_1 - \kappa) + v_h(m_1 - \underline{m}, 1), \quad m_1 > \underline{m}. \tag{A.1}$$

Since v in (7) is homogeneous of degree $1-\alpha$, Euler's theorem for homogeneous functions implies $v_c(c,1)\,c + v_h(c,1) = (1-\alpha)v(c,1)$ for all $c > 0$ permitting us to write

$$H(m_1) = (1-\alpha)\,v(m_1 - \underline{m}, 1) - v_c(m_1 - \underline{m}, 1)\,e^o, \quad m_1 > \underline{m}. \tag{A.2}$$

Since $\rho \geq 0$, the function v satisfies the Inada condition $\lim_{c\to 0} v_c(c,1) = \infty$. Thus,

$$\lim_{m_1 \to \underline{m}} H(m_1) = (1-\alpha)\,v(0,1) - e^o \lim_{m_1 \to \underline{m}} v_c(m_1 - \underline{m}, 1) = -\infty. \tag{A.3}$$

Furthermore, the restrictions $\rho \geq 0$ and $\alpha < 1$ together imply $\lim_{c\to\infty} c\,v_c(c,1) = \infty$. Using this in (A.1) yields the right limit as

$$\lim_{m_1 \to \infty} H(m_1) \geq \lim_{m_1 \to \infty} v_c(m_1 - \underline{m}, 1)\,(m_1 - \kappa) = \infty. \tag{A.4}$$

Existence of the desired solution thus follows from (A.3), (A.4), and continuity of H. Uniqueness is a consequence of (A.2) and the concavity of v which give the derivative

$$H'(m_1) = (1-\alpha)v_c(m_1 - \underline{m}, 1) - v_{cc}(m_1 - \underline{m}, 1)\,e^o > 0. \tag{A.5}$$

■

A.2 Proof of Lemma 2

Since $F_{m_1}(m_1, m; e) = -H'(m_1) < 0$ by (10) and (A.5) and F is continuously differentiable, so is the implicit function f by the Implicit Function Theorem. The partial derivative of (10) with respect to m computes

$$F_m(m_1, m; e) = u'(e^y - m) - mu''(e^y - m) = (e^y - m)^{-\alpha}\frac{e^y - (1-\alpha)m}{e^y - m} > 0. \tag{A.6}$$

By the implicit function theorem $f'(m) = -\frac{\partial_m F(m_1,m;e)}{\partial_{m_1} F(m_1,m;e)} > 0$ where $m_1 = f(m; e)$. ■

A.3 Proof of Lemma 3

By (15), the function G can be written as $G(m) = D(m) - H(m)$ with H being defined as in (A.2) and $D(m) := mu'(e^y - m) = m(e^y - m)^{-\alpha}, \quad m < e^y$. Consider first the behavior of the function D whose derivatives satisfy

$$D'(m) = \frac{e^y - (1-\alpha)m}{(e^y - m)^{1+\alpha}} > (1-\alpha)\frac{e^y - m}{(e^y - m)^{1+\alpha}} = \frac{1-\alpha}{(e^y - m)^{\alpha}} > 0 \tag{A.7}$$

$$D''(m) = \frac{\alpha}{(e^y - m)^{2+\alpha}}\Big(2e^y - (1-\alpha)m\Big) > \frac{\alpha(1-\alpha)}{(e^y - m)^{2+\alpha}}\Big(e^y - m\Big) > 0 \tag{A.8}$$

The second inequality shows that D is a strictly convex function while the first one implies that D is strictly increasing with boundary behavior $\lim_{m\to e^y} D'(m) = \infty$.

As shown in the proof of Lemma 1, the derivative of H is given by (A.5) and, therefore, satisfies $H'(m) > 0$ and $\lim_{m\to\kappa-e^o} H'(m) \geq (1-\alpha)\lim_{m\to\kappa-e^o} v_c(e^o - \kappa + m, 1) = \infty$. We claim that H' is a strictly decreasing function implying that $-H$ is strictly convex. The first term in (A.5) is strictly decreasing by strict concavity of v. It therefore suffices to show that $c \mapsto -v_{cc}(c, 1)$ is decreasing as well. Defining g as in (3), direct calculations reveal that the second derivative of v can be written as

$$\begin{aligned} -v_{cc}(c,1) &= \frac{\partial_c v(c,1)}{c}\left[1 - \rho - (1-\rho-\alpha)\frac{\beta c^\rho}{g(c,1)^\rho}\right] \\ &= \partial_c v(c,1)\left[\frac{\alpha}{c^{1-\rho}}\frac{\beta}{g(c,1)^\rho} + \frac{1-\rho}{c}\cdot\frac{(1-\beta)\theta^\rho}{g(c,1)^\rho}\right]. \end{aligned} \tag{A.9}$$

Recalling that $1 - \rho \geq 0$, all three terms in (A.9) are positive and strictly decreasing functions of c which implies that $c \mapsto -v_{cc}(c, 1)$ is decreasing as claimed.

Thus, $-H$ is a strictly convex function as claimed and G being the sum of two (strictly) convex functions is also strictly convex. The boundary behavior of G' follows directly from the limits computed above and the monotonicity properties of D and $-H$. ■

A.4 Proof of Lemma 4

(i) Using (15) in conjunction with (A.2), a routine calculation shows that $\lim_{m\to e^y} G(m) = \lim_{m\to\kappa-e^o} G(m) = \infty$. Thus, $G(m_{\min}) < 0$ implies that G has a fixed point in each of the intervals $(\kappa - e^o, m_{\min})$ and $(m_{\min}, e^y)$. By strict convexity and the boundary behavior of the first derivative (cf. Lemma 3), the map G is strictly decreasing on $(\kappa - e^o, m_{\min})$ and strictly increasing on $]m_{\min}, e^y[$. Thus, there can be at most one fixed point in each of the two intervals.

(ii) It is obvious from (i) that $G'(\bar{m}) < 0 < G'(\bar{\bar{m}})$. Utilizing the result from Lemma 2 and the definitions of D and H given in the proof of Lemma 4, this implies that $G'(\bar{m}) = D'(\bar{m}) - H'(\bar{m}) < 0$ and $G'(\bar{\bar{m}}) = D'(\bar{\bar{m}}) - H'(\bar{\bar{m}}) > 0$. Therefore,

$$0 < f'(\bar{m}) = \frac{D'(\bar{m})}{H'(\bar{m})} < 1 < \frac{D'(\bar{\bar{m}})}{H'(\bar{\bar{m}})} = f'(\bar{\bar{m}}) \tag{A.10}$$

which implies the local stability properties asserted. The remaining inequalities follow from this and the uniqueness of the fixed points on the respective intervals. ■

A.5 Proof of Lemma 5

Assertion (i) follows immediately from Lemma 4(i). The result in (ii) is a consequence of local stability of $\bar{m}$ and Lemma 4(ii). Monotonicity of the sequence $\{m_t\}_{t\geq 0}$ follows from this and Lemma 2. ■

A.6 Proof of Theorem 1

(i) Lemma 5 and $m_0 \in \mathbb{M}$ imply that $m_t \in \mathbb{M}$ for all t and $\lim_{t\to\infty} m_t = \bar{m}$. By (1) and (13), $b_t < 0$ for all t which implies $p_t > 0$ by (8) proving (i).
(ii) If $\mathcal{R}(\bar{m}; e) > 1$, then there exists $t_0 \geq 0$ such that $\mathcal{R}(m_t; e) > 1$ for all $t \geq t_0$. In fact, since $m \mapsto \mathcal{R}(m; e)$ is strictly increasing (cf. Lemma 8) and $\{m_t\}_{t\geq 0}$ converges monotonically, we have $\mathcal{R}(m_t; e) \geq R_{t_0} := \mathcal{R}(m_{t_0}; e) > 1$ for all $t \geq t_0$. Thus, $\lim_{t\to\infty} b_t \leq b_{t_0} \lim_{t\to\infty} R_{t_0}^{t-t_0} = -\infty$ and $p_t = m_t - b_t > -b_t$ for all t gives $\lim_{t\to\infty} p_t = \infty$.
(iii) If $b_0 = 0$, then $b_t = 0$ and $m_t = p_t$ for all t and the claim follows from (i). If $b_0 < 0$ and $\mathcal{R}(\bar{m}; e) < 1$, the same arguments as in the previous step show that $\mathcal{R}(m_t; e) \leq R_{t_0} := \mathcal{R}(m_{t_0}; e) < 1$ for all $t \geq t_0$. Thus, $0 \geq \lim_{t\to\infty} b_t \geq b_{t_0} \lim_{t\to\infty} R_{t_0}^{t-t_0} = 0$ and $\lim_{t\to\infty} p_t = \lim_{t\to\infty} m_t = \bar{m}$. ■

A.7 Proof of Lemma 6

(i) The proof of Lemma 2 revealed that $F_{m_1}(m_1, m; e) < 0$ with F defined in (10). Invoking the Implicit Function Theorem, the claim follows follows

from $\partial_{e^y} F(m_1, m; e) = m\, u''(e^y - m) < 0$ for all $m > 0$.
(ii) Recall that fixed points are solutions to $G(m; e) = F(m, m; e) = 0$. By (i), $G_{e^y}(m; e) = \partial_{e^y} F(m, m; e) < 0$. Also, the boundary behavior of G ($\lim_{m \to \kappa - e^o} G(m; e) = \lim_{m \to e^y} G(m; e) = \infty$) implies that $\partial_m G(\bar{m}; e) < 0 < \partial_m G(\bar{\bar{m}}; e)$. Thus, the claim follows again from the Implicit Function Theorem. ■

A.8 Proof of Lemma 7

(i) We first show that $\bar{\mathbb{M}}$ is self-supporting. Let $m \in \bar{\mathbb{M}}$ be arbitrary. Then, using Lemma 4(ii) and the monotonicity properties of f together with the definitions (17a–c) we have for each $e^y \in \mathcal{E}$:

$$\bar{m}_{\min} = f(\bar{m}_{\min}; e^y_{\max}) \leq f(\bar{m}_{\min}; e^y) \leq f(m; e^y) \leq f(\bar{m}_{\max}; e^y) \leq f(\bar{m}_{\max}; e^y_{\min})) = \bar{m}_{\max}. \tag{A.11}$$

Thus, $f(m; e^y) \in \bar{\mathbb{M}}$. To prove that $\bar{\bar{\mathbb{M}}}$ is self-supporting, let $m \in \bar{\bar{\mathbb{M}}}$ and $e^y \in \mathcal{E}^y$ be arbitrary. The case $m \in \bar{\mathbb{M}}$ is evident, so suppose first that $m \in (\bar{m}_{\max}, \bar{\bar{m}}_{\min})$. Then, by (17a–c), $\bar{m}(e^y) \leq \bar{m}_{\max} < m < \bar{\bar{m}}_{\min} \leq \bar{\bar{m}}(e^y)$ which implies, by Lemma 4(ii) and monotonicity of f that $\bar{m}(e^y) < f(m; e^y) < m$. Thus, $f(m; e^y) \in \bar{\bar{\mathbb{M}}}$. Conversely, suppose $m \in (0, \bar{m}_{\min})$. Then, by (17a–c) $0 < m < \bar{m}_{\min} \leq \bar{m}(e^y)$ which implies $m < f(m; e^y) < \bar{m}(e^y)$ by Lemma 4(ii) and monotonicity of f. Thus, $f(m; e^y) \in \bar{\bar{\mathbb{M}}}$ again.
(ii) Let $m_0 \in \bar{\bar{\mathbb{M}}}$ be arbitrary. Define the sequences $\{\overline{m}_t\}_{t \geq 0}$ and $\{\underline{m}_t\}_{t \geq 0}$ by setting $\overline{m}_0 = \underline{m}_0 = m_0$ and $\overline{m}_{t+1} := f(\overline{m}_t; e^y_{\min})$ and $\underline{m}_{t+1} := f(\underline{m}_t; e^y_{\max})$ for each $t \geq 0$. Then, by the monotonicity properties of f, $\underline{m}_t \leq m_t \leq \overline{m}_t$ $\mathbb{P}$–a.s. for all $t \geq 0$ and the claim follows from $\lim_{t \to \infty} \underline{m}_t = \bar{m}(e^y_{\max}) = \bar{m}_{\min}$ and $\lim_{t \to \infty} \overline{m}_t = \bar{m}(e^y_{\min}) = \bar{m}_{\max}$. ■

A.9 Proof of Theorem 2

Lemma 7 ensures that $m_t \in \bar{\bar{M}} \subset (0, e^y_{\min})$ $\mathbb{P}$–a.s. for all $t \geq 0$. Thus, $b_0 \leq 0$ implies $b_t \leq 0$ by (1) and $p_t \geq m_t > 0$ by (8) $\mathbb{P}$–a.s. for all $t \geq 0$. ■

A.10 Proof of Lemma 8

The claim follows directly by taking the partial derivatives of (13) and using Lemmata 2 and 6(i). ■

A.11 Proof of Theorem 3

(i) Suppose $R^{\min} > 1$. Then, $\mathcal{R}(m; e) \geq R^{\min} > 1$ for all $m \in \bar{\mathbb{M}}$ and $e \in \mathcal{E}^y$. Let $\hat{R}^{\min}$ be a number between 1 and $R^{\min}$. By continuity of $\mathcal{R}$, we can choose an open neighborhood $\hat{\mathbb{M}}$ of $\bar{\mathbb{M}}$ such that $\mathcal{R}(m; e) > \hat{R}^{\min}$ for all $m \in \hat{\mathbb{M}}$ and $e \in \mathcal{E}^y$. Let $m_0 \in \mathbb{M}$ be arbitrary. By Lemma 7(ii), there exists $t_0 > 0$ such

that $m_t \in \hat{\mathbb{M}}$ for all $t > t_0$ $\mathbb{P}$–a.s. Hence, $R_t > \hat{R}^{\min} > 1$ for all $t > t_0$ $\mathbb{P}$–a.s. and it follows from (1) that $\lim_{t\to\infty} b_t \geq \lim_{t\to\infty} b_{t_0} \left(\hat{R}^{\min}\right)^{t-t_0} = \infty$. Since m_t remains uniformly bounded, the limit of the process $\{p_t\}_{t\geq 0}$ follows from (8).
(ii) Similar to the previous part, choose a number $\hat{R}^{\max}$ between $R^{\max}$ and 1 and an open neighborhood $\hat{\mathbb{M}}$ of $\bar{\mathbb{M}}$ such that $\mathcal{R}(m; e) < \hat{R}^{\max} < 1$ for all $m \in \hat{\mathbb{M}}$ and $e \in \mathcal{E}^y$. Let $m_0 \in \mathbb{M}$ be arbitrary. By Lemma 7(ii), there exists $t_0 > 0$ such that $m_t \in \hat{\mathbb{M}}$ for all $t > t_0$ $\mathbb{P}$–a.s. Hence, $R_t < \hat{R}^{\max} < 1$ for all $t > t_0$ $\mathbb{P}$–a.s. and it follows that $0 \leq \lim_{t\to\infty} b_t \leq \lim_{t\to\infty} b_{t_0} \left(\hat{R}^{\max}\right)^{t-t_0} = 0$. Finally, the previous result and (8) imply directly that $\lim_{t\to\infty} |p_t - m_t| = \lim_{t\to\infty} |b_t| = 0$. ■

References

Bajari, P., P. Chan, D. Krueger & D. Miller (2010): "A Dynamic Model of Housing Demand: Estimation and Policy Implications", Working paper.

Brock, W. A. & L. J. Mirman (1972): "Optimal Growth and Uncertainty: The Discounted Case", *Journal of Economic Theory*, 4, 479–513.

Caballero, R. & A. Krishnamurthy (2006): "Bubbles and Capital Flow Volatility: Causes and Risk Management", *Journal of Monetary Economics*, 53, 35–53.

Chen, H. & C. Winter (2010): "Mortgage Loans, the Decline of Household Saving, and the Increase in Risk-Sharing", Discussion paper, University of Zurich.

Duffie, D., J. Geanakoplos, A. Mas-Colell & A. McLennan (1994): "Stationary Markov Equilibria", *Econometrica*, 62(4), 745–781.

Farhi, E. & J. Tirole (2011): "Bubbly Liquidity", *Review of Economic Studies*, forthcoming.

Kiyotaki, N. & J. Moore (1997): "Credit Cycles", *Journal of Political Economy*, 105(2), 211–248.

Kocherlakota, N. R. (1992): "Bubbles and Constraints on Debt Accumulation", *Journal of Economic Theory*, 57, 245–256.

——— (2008): "Injecting Rational Bubbles", *Journal of Economic Theory*, 142, 218–232.

——— (2011a): "Bubbles and Unemployment", Working paper, Federal Reserve Bank of Minneapolis.

——— (2011b): "Two Models of Land Overvaluation and Their Implications", Working paper, Federal Reserve Bank of Minneapolis.

Li, W. & R. Yao (2007): "The Life-Cycle Effects of House Price Changes", *Journal of Money, Credit, and Banking*, 39, 1375–1409.

Lustig, H. & S. V. Nieuwerburgh (2010): "Housing Collateral, Consumption Insurance, and Risk Premia: An Empirical Perspective", *Journal of Finance*, 60(3), 1167–1219.

Michel, P. & B. Wigniolle (2003): "Temporary Bubbles", *Journal of Economic Theory*, 112, 173–183.

Rochon, C. & H. Polemarchakis (2006): "Debt, liquidity and dynamics", *Economic Theory*, 27, 179–211.

Santos, M. & M. Woodford (1997): "Rational Asset Pricing Bubbles", *Econometrica*, 65(1), 19–57.

Tirole, J. (1982): "On the Possibility of Speculation under Rational Expectations", *Econometrica*, 50, 1163–1182.

——— (1985): "Asset Bubbles and Overlapping Generations", *Econometrica*, 53(6), 1499–1528.

Ventura, J. (2011): "Bubbles and Capital Flows", *Journal of Economic Theory*, in press.

Wang, Y. (1993): "Stationary Equilibria in an Overlapping Generations Economy with Stochastic Production", *Journal of Economic Theory*, 61(2), 423–435.

Weil, P. (1987): "Confidence and the real value of money in an overlapping generations economy", *Quarterly Journal of Economics*, 102, 1–22.