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# MONITORING STATIONARITY AND COINTEGRATION

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## Abstract

We propose a monitoring procedure to detect a structural change from stationary to integrated behavior. When the procedure is applied to the errors of a relationship between integrated series it thus monitors a structural change from a cointegrating relationship to a spurious regression. The cointegration monitoring procedure is based on residuals from modified least squares estimation, using either Fully Modified, Dynamic or Integrated Modified OLS. The procedure is inspired by Chu et al. (1996) in that it is based on parameter estimation only on a pre-break “calibration” period rather than being based on sequential estimation over the full sample. We investigate the asymptotic behavior of the procedures under the null, for (fixed and local) alternatives and in case of parameter changes. We also study the finite sample performance via simulations. An application to credit default swap spreads illustrates the potential usefulness of the procedure.

**Keywords:** Stationarity, Cointegration, Unit Roots, Structural Change, Monitoring

**JEL Code:** C22, C32, C52

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# 1. INTRODUCTION

It is common practice in time series econometrics to investigate the stationarity, unit root and cointegration properties of time series and a plethora of tests for stationarity, unit roots and cointegration is available. In relation to this practice, however, it may well be reasonable to investigate the question whether the stationarity or cointegration behavior of time series changes over time. In particular, a time series may change its behavior from stationarity, or being  $I(0)$ , to being integrated and a cointegrating relationship between several time series may break down and turn into a spurious relationship.<sup>1</sup> Examples where one may be concerned about this type of structural change include deviations from purchasing power parity after a period of international economic stability or nonstationarity of credit default swap (CDS) spreads after the onset of a financial crisis.<sup>2</sup>

Our monitoring procedure is inspired by the monitoring procedure for linear regression models of Chu et al. (1996) in that parameter estimation, for estimating trend and – when monitoring cointegration – slope parameters, is based solely on a “calibration” period at the beginning of the sample that is known or assumed to be free of structural change.<sup>3</sup> Based on the parameter estimates, computed using only calibration period data, the detrended observations (or the residuals of a cointegrating relationship) are the key ingredient for the monitoring procedure. The monitoring procedure is based on sequentially computing the differences of scaled partial sums of squared residuals over the growing monitoring period and the calibration period.<sup>4</sup> The detection time, defined in the following section, serves as an immediately available estimate of the break-point.

In order to obtain nuisance parameter free limiting distributions of the test statistics

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<sup>1</sup>As discussed below, the approach also allows to monitor whether the coefficients of a cointegrating relationship change over time.

<sup>2</sup>Related issues are analyzed in tests for so-called asset price bubbles, see e.g. Phillips et al. (2011), where a bubble is associated with a structural change towards explosive behavior. Our approach can be applied in this context as well, as discussed later.

<sup>3</sup>This approach to monitoring, based on estimation in a pre-break sample period only, has been extended to the multivariate linear regression case by Groen et al. (2013) and has been applied to monitor changes in the correlation structure by Wied and Galeano (2013).

<sup>4</sup>This, of course, immediately implies consistency of the procedure against any “more explosive” alternative, like higher order integration or explosive behavior.

when applying the principle to monitor cointegrating relationships, parameter estimation on the calibration sample is performed using any of the available modified least squares estimators that lead to nuisance parameter free limiting distributions of the parameters of the cointegrating relationship. In particular we consider here Fully Modified OLS (FM-OLS) of Phillips and Hansen (1990), Dynamic OLS (D-OLS) of Saikkonen (1991), Phillips and Loretan (1993) and Stock and Watson (1993) or Integrated Modified OLS (IM-OLS) of Vogelsang and Wagner (2014).

The asymptotic properties of the monitoring procedures are derived under both the null as well as under (fixed and local) alternatives and for the case of breaks in trend parameters. In case of cointegration monitoring we additionally consider the asymptotic behavior of the monitoring procedure in case of breaks in the slope parameters. Based on the asymptotic results, the performance of the proposed methods is investigated by means of local asymptotic power analysis. Furthermore, finite sample simulations are performed to consider empirical size and power for a variety of scenarios. Again, in addition to studying the power against the alternative of integrated behavior we also assess the performance in case of parameter changes in the trend and/or slope coefficients.

We briefly illustrate our monitoring procedure using daily CDS spreads series for Austria, Cyprus, France and Germany over the period April 3, 2009 to August 1, 2012. The null hypothesis of no structural change from stationary to integrated behavior is rejected for all countries and all five considered maturities. For most of the series the estimated break date is found to be in summer or fall 2010, i.e. already almost a year before the CDS spreads series started to be dramatically high starting in summer 2011.

The paper is organized as follows: Section 2 develops the stationarity monitoring procedure and studies its asymptotic properties under the null, for fixed and local alternatives and in case of trend breaks. In Section 3 the approach is extended to monitor cointegration and the asymptotic properties of the monitoring procedure are discussed in detail for the above mentioned three estimation procedures for cointegrating relationships. In addition we discuss here also the properties in case of structural change in the slope

parameters. Section 4 provides some simulation results investigating the finite sample properties of the proposed monitoring procedures and Section 5 contains a brief illustration of the monitoring procedure using CDS spreads data for four European countries. Finally, Section 6 briefly summarizes and concludes. All proofs are deferred to Appendix A and Appendix B provides some tables with critical values.

## 2. MODEL AND ASSUMPTIONS FOR MONITORING STATIONARITY

The starting point of our considerations is to monitor (trend-)stationarity of

$$y_t = D_t' \theta_D + u_t, \quad (1)$$

for which a sample of observations for  $t = 1, \dots, T$  is available and where  $D_t \in \mathbb{R}^p$  is a deterministic trend function for which the following assumption is made.

**Assumption 1.** *There exists a  $p \times p$  matrix  $G_D$  and a vector of functions,  $D(z)$ , such that*

$$T^{-1} G_D^{-1} \sum_{t=1}^{[sT]} D_t \rightarrow \int_0^s D(z) dz \text{ with } \int_0^s D(z) D(z)' dz > 0. \quad (2)$$

If e.g.  $D_t = (1, t, t^2, \dots, t^{p-1})'$ , then  $G_D = \text{diag}(1, T, T^2, \dots, T^{p-1})$  and  $D(z) = (1, z, z^2, \dots, z^{p-1})'$ .

Under the null hypothesis of trend stationarity we posit the assumption that the process  $u_t$  fulfills a functional central limit theorem (FCLT):

**Assumption 2.** *The stationary process  $u_t$  fulfills*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t \Rightarrow \omega^{1/2} W(s), \quad (3)$$

where  $[sT]$ , with  $0 \leq s \leq 1$ , denotes the integer part of  $sT$ ,  $W(s)$  denotes standard Brownian motion and where  $0 < \omega < \infty$  is the long-run variance of  $u_t$ ,

$$\omega := \sum_{j=-\infty}^{\infty} \mathbb{E}(u_t u_{t-j}). \quad (4)$$

In this paper we refer to a stochastic process fulfilling Assumption 2 as an I(0) process. Requiring  $\omega > 0$  excludes over-differenced process, e.g.  $u_t = \varepsilon_t - \varepsilon_{t-1}$  for some white noise process  $\varepsilon_t$ , has long-run variance equal to 0.

**Remark 1.** Note that we do not need stationarity in our I(0) definition, but only the mentioned functional central limit result (3), with  $\omega = \lim_{T \rightarrow \infty} \mathbb{E} \left( \frac{1}{T} \sum_{t=1}^T u_t \right)^2$ . All results hold without the extra requirement of stationarity in Assumption 2. We include stationarity in the assumption, since we consider later monitoring of cointegration, i.e. linear combinations of series that are stationary (in the usual definition of cointegration).  $\square$

Under the alternative we consider the situation that there exists some time point  $[rT]$ , with  $m \leq r < 1$  such that the process behaves like an I(1) process from  $[rT] + 1$  onwards, i.e. that Assumption 2 is violated from  $[rT] + 1$  onwards in a specific way. For our paper we define a process  $x_t$  to be an I(1) process, in accordance with our I(0) definition, if

$$\frac{1}{\sqrt{T}} x_{[sT]} \Rightarrow \omega^{1/2} W(s), \quad (5)$$

for some  $0 < \omega < \infty$  and  $W(s)$  again standard Brownian motion. It is clear that the (partial) sum process of an I(0) process is an I(1) process.

Thus, our null and alternative hypotheses are in formal terms given by

$$H_0 : \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t \Rightarrow \omega^{1/2} W(s), \quad \text{for all } 0 < s \leq 1 \quad (6)$$

$$H_1 : \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} u_t = O_p(T^{1/2}) \text{ and } \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} u_t \neq o_p(T^{1/2}) \quad (7)$$

for some  $1 > r \geq m > 0$  and for all  $s > r$

The above formulation is to be understood in the sense that also under the alternative the process  $u_t$  fulfills Assumption 2 up to  $[rT] \geq [mT]$ . Thus, we want to detect a change from I(0) to I(1) behavior under the alternative that occurs at time point  $[rT]$

with  $m \leq r < 1$ , i.e. a change that occurs only after some pre-break fraction of length  $[mT]$  with  $0 < m < 1$ .

As we shall see below, we need such a pre-break sample fraction  $m$  in particular in order to consistently estimate several quantities required to obtain a null limiting distribution of our detector that is a function only of the included deterministic components and standard Brownian motions and for which thus critical values can be simulated. These quantities include, depending upon situation considered, the long-run variance  $\omega$ , the trend parameters  $\theta_D$  and in the following section, dealing with cointegration, also slope parameters  $\theta_X$  corresponding to the I(1) regressors in the monitored cointegrating relationship.

**Remark 2.** As mentioned in the introduction, our approach is inspired by Chu et al. (1996), albeit we frame the problem slightly differently. As is standard in the unit root and cointegration literature we map the (full set of) observations  $1, \dots, T$  in the interval  $[0, 1]$ , as  $T \rightarrow \infty$ , and thus in the limit our pre-break estimation period corresponds to the interval  $[0, m]$ , as we use observations  $1, \dots, [mT]$  for parameter estimation. Chu et al. (1996) consider as their “historical” period observations  $1, \dots, m$ . In their asymptotic analysis  $m$ , with  $m \rightarrow \infty$ , is considered a fraction of the overall sample size  $T = \lambda m$ . Thus, the observations  $1, \dots, m$  are mapped into the interval  $[0, 1]$  and the whole set of observations into  $[0, \lambda]$ .  $\square$

Considering for the moment  $u_t$  to be observable and  $\omega$  known suffices to discuss the approach of the paper. In this idealized case the detector for  $u_t$  is given by

$$H^m(s) := \frac{1}{\omega} \left( \frac{1}{T} \sum_{i=[mT]+1}^{[sT]} \left( \frac{1}{\sqrt{T}} S_i \right)^2 - \frac{1}{T} \sum_{i=1}^{[mT]} \left( \frac{1}{\sqrt{T}} S_i \right)^2 \right), \quad (8)$$

for  $m \leq s \leq 1$  and with  $S_i = \sum_{t=1}^i u_t$  denoting the partial sums of  $u_t$ .

Under Assumption 2 it holds under the null hypothesis of no structural change that

$$H^m(s) \Rightarrow \mathcal{H}^m(s) := \left( \int_m^s W(z)^2 dz - \int_0^m W(z)^2 dz \right), \quad (9)$$

whereas the detector will diverge under the alternative (see the discussion below for details). The detector is inspired by the KPSS stationarity test of Kwiatkowski et al. (1992), with the (idealized) test statistic given by  $\frac{1}{\omega} \left( \frac{1}{T} \sum_{i=1}^T \left( \frac{1}{\sqrt{T}} S_i \right)^2 \right)$ , which converges to  $\int_0^1 W(z)^2 dz$  under the null of stationarity. In case of I(1) behavior the scaled sum diverges and our detector exploits these differences by comparing the convergent pre-break quantity with the potentially diverging post-break quantity.<sup>5</sup>

A related procedure is provided by Steland (2007), who bases his monitoring procedure on sequential kernel-weighted variance ratios, i.e. his detector to monitor a change from I(0) to I(1) behavior is, using our notation and for known  $\omega$ , given by  $\tilde{U}_T(s) = \frac{T^{-1} \sum_{i=1}^{\lfloor sT \rfloor} S_i^2 K_h(i - \lfloor sT \rfloor)}{\omega}$  for some kernel function  $K_h(\cdot)$ . Kernel weighting is important in his approach, as he also considers monitoring a change from I(1) to I(0) behavior (with a differently scaled detector than the one given above). In addition to the different construction of the detector, another difference is that Steland does not consider parameter estimation on a calibration period and in particular neither he nor any of the other available procedures consider monitoring of cointegrating relationships.<sup>6</sup>

We define the detection time  $\tau_m(H^m(s), w(s), c)$ , often only written as  $\tau_m$  if the context is clear, as

$$\tau_m := \min \left\{ \lfloor mT \rfloor + 1 \leq \lfloor sT \rfloor \leq T : \left| \frac{H^m(s)}{w(s)} \right| > c \right\}, \quad (10)$$

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<sup>5</sup>Let us note already here that in the cointegration monitoring situation our detector is similarly based on the cointegration test statistics of Shin (1994) that extends the KPSS test from a stationarity to a cointegration test.

<sup>6</sup>Another related procedure is discussed in Kim (2000), who considers the properly scaled ratio sums of squared partial sums of residuals before and after the hypothesized break. Note that Steland (2007) does not consider deterministic components. This restriction is relaxed to a certain extent in Steland (2008), where he considers polynomial trends. This situation is also considered in Qi et al. (2013). Chen et al. (2010) modify the approach of Steland and also use a ‘‘calibration’’ period at the beginning of the sample for which it is known whether the series is I(0) or I(1). Chen et al. (2012) use the bootstrap to detect multiple changes of persistence.



i.e. the null hypothesis is declared rejected when the standardized detector,  $\frac{H^m(s)}{w(s)}$ , exceeds a critical value  $c$  in absolute value for the first time. In case that  $\left| \frac{H^m(s)}{w(s)} \right| \leq c$  for all  $m \leq s \leq 1$  we write  $\tau_m = \infty$ . Consequently, the value  $\tau_m$  can be interpreted in two different ways. On the one hand, it tells us if the null hypothesis is rejected or not. On the other hand, it gives information about the position of the potential break point.

The properties of such a monitoring procedure hinge, by construction, upon the threshold function  $0 < w(s) < \infty$  and the constant  $c$ , which itself depends upon  $w(s)$ . These two have to be chosen in order to ensure that under the null hypothesis

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathbb{P}(\tau_m < \infty) &= \lim_{T \rightarrow \infty} \mathbb{P} \left( \sup_{[mT]+1 \leq [sT] \leq T} \left| \frac{H^m(s)}{w(s)} \right| > c \right) \\ &= \mathbb{P} \left( \sup_{m \leq s \leq 1} \left| \frac{\mathcal{H}^m(s)}{w(s)} \right| > c \right) = \alpha, \end{aligned} \quad (11)$$

with  $\alpha$  denoting the chosen significance level. The choice of  $w(s)$ , and  $c$ , is discussed in more detail below.

With the starting point in most applications being the observed time series  $y_t$  rather than  $u_t$ , partial sums of residuals are the input in the monitoring procedure. The residuals  $\hat{u}_{t,m}$  are given by detrended  $y_t$ , with the trend parameters  $\theta_D$  estimated from the pre-break sample  $t = 1, \dots, [mT]$ . Thus,

$$\begin{aligned} \hat{u}_{t,m} &:= y_t - D_t' \hat{\theta}_{D,m} \\ &= u_t - D_t' \left( \hat{\theta}_{D,m} - \theta_D \right) \\ &= u_t - D_t' \left( \sum_{i=1}^{[mT]} D_i D_i' \right)^{-1} \sum_{i=1}^{[mT]} D_i u_i. \end{aligned} \quad (12)$$

Under Assumptions 1 and 2 it follows immediately that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m} = \frac{1}{\sqrt{T}} \hat{S}_{[sT]} \Rightarrow \omega^{1/2} \widehat{W}_m(s), \quad (13)$$

with  $\widehat{S}_i = \sum_{t=1}^i \hat{u}_{t,m}$  and  $\widehat{W}_m(s) := W(s) - \int_0^s D(z)' dz \left( \int_0^m D(z)D(z)' dz \right)^{-1} \int_0^m D(z) dW(z)$ . Given the FCLT (13) for the partial sum of the detrended observations and the continuous mapping theorem, the asymptotic behavior of the detector based on  $\hat{u}_{t,m}$ ,  $\widehat{H}^m(s)$  say, under the null hypothesis can be established.

**Lemma 1.** *Let  $y_t$  be generated by (1) with Assumptions 1 and 2 in place and let  $\hat{\omega}_m$  denote a consistent long-run variance estimator based on  $\hat{u}_{t,m}$ , for  $t = 1, \dots, [mT]$ . Then it holds under the null hypothesis for  $T \rightarrow \infty$  and  $m \leq s \leq 1$  that*

$$\widehat{H}^m(s) := \frac{1}{\hat{\omega}_m} \left( \frac{1}{T} \sum_{i=[mT]+1}^{[sT]} \left( \frac{1}{\sqrt{T}} \widehat{S}_i \right)^2 - \frac{1}{T} \sum_{i=1}^{[mT]} \left( \frac{1}{\sqrt{T}} \widehat{S}_i \right)^2 \right) \quad (14)$$

$$\Rightarrow \int_m^s \widehat{W}_m^2(z) dz - \int_0^m \widehat{W}_m^2(z) dz =: \widehat{\mathcal{H}}^m(s). \quad (15)$$

In order to show consistency of the detector later, it is of key importance here that all parameters, i.e. the trend slopes  $\theta_D$  and the long-run variance  $\omega$ , are estimated only from the pre-break sample up to  $[mT]$ . Consistent long-run variance estimation is a well studied problem in the econometrics literature and has been established for a variety of primitive or high level assumptions. For simplicity in this paper we merely assume that the sufficient assumptions on  $u_t$ , the kernel function and bandwidth given in Jansson (2002) are fulfilled.<sup>7</sup>

Under the stated assumptions it can be shown that under the null hypothesis, for given weighting function  $w(s)$ , there exist critical values  $c = c(\alpha, w(s))$ , such that the detection time is finite with probability equal to the pre-specified level  $\alpha$ .

**Proposition 1.** *Let the data be generated by (1) with Assumptions 1 and 2 in place and let  $\hat{\omega}_m$  denote a consistent long-run variance estimator and consider  $0 < w(s) < \infty$ . Then, under the null hypothesis it holds that for any given  $0 < \alpha < 1$  and  $0 < w(s) < \infty$*

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<sup>7</sup>To be precise this means to specify more detailed assumptions on  $u_t$  than just the FCLT formulated in Assumption 2, compare Assumptions A1 and A2 in Jansson (2002). Let us note again, that any other set of assumptions that allows for consistent long-run variance estimation also would serve our purposes. In relation to Remark 1 it has to be noted that without the stationarity assumption one has to resort to other conditions, since Jansson (2002) considers stationary processes.

there exists a  $c = c(\alpha, w(s))$ , such that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{H}^m(s), w(s), c(\alpha, w(s))) < \infty) = \alpha. \quad (16)$$

Given the behavior under the null hypothesis the next result shows that the monitoring procedure is consistent against both fixed and local alternatives, defined precisely below. As fixed alternative we consider the case that  $u_t$  changes behavior from I(0) to I(1) at some point after  $[mT]$ , i.e. that  $H_1$  as given above holds. To understand the properties of our procedure in more detail we also consider local alternatives of the following form (inspired by Cappuccio and Lubian, 2005). There exists an  $r$ , with  $m \leq r < 1$  such that for all  $t \leq [rT]$  we have  $u_t = u_t^0$ , while for all  $t > [rT]$  it holds that

$$u_t = u_t^0 + \frac{\delta}{T} \sum_{i=[rT]+1}^t \xi_i, \quad (17)$$

with  $u_t^0$  and  $\xi_t$  independent processes both fulfilling Assumption 2, with long-run variances  $\omega$  and  $\omega_\xi$ , and  $\delta > 0$ . I.e. under the considered local alternatives the process  $u_t$  is, from time point  $[rT] + 1$  onwards, the sum of an I(0) process and an independent I(1) process divided by the sample size. The local alternatives imply that the partial sum process of  $\hat{u}_{t,m}$ , i.e.  $\widehat{S}_t$  converges to the following expression (for details see the proofs in Appendix A):

$$\frac{1}{\sqrt{T}} \widehat{S}_{[sT]} \Rightarrow \omega^{1/2} \widehat{W}_m(s) + \delta \omega_\xi^{1/2} \int_r^s (W_\xi(z) - W_\xi(r)) dz, \quad (18)$$

where integrals (and sums) with the lower boundary larger than the upper are defined to be equal to 0. Here  $W_\xi(s)$  is a standard Brownian motion independent of  $W(s)$ .

**Proposition 2. (Consistency and Local Asymptotic Power)**

*Let the data be generated by (1) with Assumption 1 in place and with  $u_t$  fulfilling Assumption 2 until  $[rT]$ , with  $m \leq r < 1$ . Furthermore, let  $\hat{\omega}_m$  again be a consistent long-run variance estimator and let  $0 < w(s) < \infty$ .*

(a) Let  $u_t$  be an  $I(1)$  process (as specified in  $H_1$ ) from  $[rT] + 1$  onwards. Then the monitoring procedure is consistent, i.e. for any  $0 < c < \infty$  it holds that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{H}^m(s), w(s), c) < \infty) = 1. \quad (19)$$

(b) Let  $u_t$  be as specified in (17) from  $[rT] + 1$  onwards. Then the monitoring procedure has non-trivial local power. That means that for any  $1 - \alpha \geq \epsilon > 0$  and the  $0 < c = c(\alpha, w(s)) < \infty$  from Proposition 1 there exists a  $0 < \delta = \delta(c, w(s)) < \infty$  such that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{H}^m(s), w(s), c(\alpha, w(s))) < \infty) \geq 1 - \epsilon. \quad (20)$$

The result underlying part (b) stems from the convergence result for  $\widehat{H}_m(s)$  under the considered local alternatives. For  $s > r \geq m$  it holds that:

$$\begin{aligned} \widehat{H}_m(s) \Rightarrow & \widehat{\mathcal{H}}^m(s) + 2\delta \left(\frac{\omega_\xi}{\omega}\right)^{1/2} \int_r^s \widehat{W}_m(z) \left( \int_r^z (W_\xi(g) - W_\xi(r)) dg \right) dz + \\ & + \delta^2 \frac{\omega_\xi}{\omega} \int_r^s \left( \int_r^z (W_\xi(g) - W_\xi(r)) dg \right)^2 dz. \end{aligned} \quad (21)$$

This result shows that the magnitude of the additional terms depends, in addition to  $\delta$ , upon the “signal-to-noise” ratio of  $\omega_\xi$  and  $\omega$ . As expected  $\omega$  enters with negative powers, i.e. a larger error variance decreases local asymptotic power and similarly a larger variance of the additional  $I(1)$  component increases local asymptotic power.<sup>8</sup>

**Remark 3.** It is clear and immediate from an inspection of the proof that the procedure is consistent not only against the  $I(1)$  alternative but also against the alternative of near-integrated processes, compare Phillips (1987). A near-integrated process

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<sup>8</sup>In the local asymptotic power simulations below we set the signal-to-noise ratio without loss of generality equal to one, since local asymptotic power depends only upon the product of  $\delta$  and the signal-to-noise ratio. Also note that it is sufficient to consider  $u_t^0$  and  $\xi_t$  independent, as asymptotic independence between the two components can always be achieved by redefining the two quantities correspondingly after “orthogonalization”.

is given by  $u_t = (1 - \frac{c}{T})u_{t-1} + \nu_t$ , with  $c > 0$  and  $\nu_t$  a stationary process with finite and positive long-run variance  $\omega$ . Clearly, in case  $c = 0$  we are back to the standard I(1) alternative. Consistency against near-integrated alternatives follows from the functional central limit theorem for near-integrated processes, i.e.  $\frac{1}{\sqrt{T}}u_{[rT]} \Rightarrow \omega V_c(r)$ , with  $V_c(r) = \int_0^r e^{-c(r-s)}dW(s)$ , which implies that near-integrated alternatives fulfill  $H_1$ .  $\square$

The above results do not pin down the threshold function  $w(s)$ , which could be specified in many ways. A natural candidate is to choose  $w(s)$  in relation to  $\mathbb{E}(\widehat{\mathcal{H}}^m(s))$ . In the special case of only an intercept ( $D_t = 1$ ), we obtain

$$\widehat{W}_m(s) = W(s) - \frac{s}{m} \int_0^m dW(z) = W(s) - \frac{s}{m}W(m). \quad (22)$$

Then, with the well-known covariance structure of a standard Brownian motion one obtains

$$\mathbb{E}(\widehat{W}_m^2(s)) = s - \frac{2s}{m} \min(s, m) + \frac{s^2}{m}, \quad (23)$$

from which, by changing the sequence of integration (Fubini), it follows that

$$\mathbb{E}(\widehat{\mathcal{H}}^m(s)) = \int_m^s \mathbb{E}(\widehat{W}_m^2(z))dz - \int_0^m \mathbb{E}(\widehat{W}_m^2(z))dz = -\frac{1}{2}s^2 + \frac{1}{3}\frac{s^3}{m}. \quad (24)$$

Thus, the order of the expected value is  $s^3$ , which motivates our choice  $w(s) = s^3$ . In case that the deterministic component consists of intercept and linear trend ( $D_t = (1, t)$ ), similar calculations lead to the order being  $s^5$ . Thus, in the linear trend case we consider  $w(s) = s^5$  in the simulations and application below.

For given or chosen weighting function critical values for the test procedure can be simulated by approximating the functionals of Brownian motions by the corresponding functions of random walks of length 1,000 generated from i.i.d. standard normal random variables. The available critical values are based on 1,000,000 replications. The critical values depend upon  $0 < m < 1$  and the specification of the deterministic component.

Detailed critical values for a grid of  $m$ -values ranging from 0.1 to 0.9 with mesh 0.01 are contained in Appendix B, in Table 2, for  $D_t = 1$  with  $w(s) = s^3$ , and in Table 3, for  $D_t = (1, t)'$  with  $w(s) = s^5$ .

The monitoring procedure can also be used to instead detect breaks in the trend parameters  $\theta_D$  that occur after  $[mT]$ , given that the trend parameters are also estimated only using the observations up to this time point. Again we consider fixed and local alternatives.<sup>9</sup> In the present context local alternatives are not described by a scalar parameter, but by  $\Delta_\theta \in \mathbb{R}^p$  and the appropriate local alternatives are given by  $\frac{1}{\sqrt{T}}G_D^{-1'}\Delta_\theta$ , reflecting the different rates of convergence of the trend parameters. E.g. for the constant term, the rate is the usual  $\frac{1}{\sqrt{T}}$ , whereas for the linear trend coefficient the rate is  $T^{-3/2}$ . It will also be seen that non-trivial local power is related to a certain constraint on the parameter change, see (26) below.

**Proposition 3. (Behavior in Case of Trend Breaks)**

Let the data be generated by (1) with Assumptions 1 and 2 in place and let  $\hat{\omega}_m$  denote again a consistent long-run variance estimator. Assume furthermore again that  $0 < w(s) < \infty$  and  $0 < c < \infty$ .

(a) (Fixed Alternative) Let  $\theta_D = \theta_{D,1}$  for  $t = 1, \dots, [rT]$  and  $\theta_D = \theta_{D,2}$ , with  $\theta_{D,1} \neq \theta_{D,2}$ , from  $t = [rT] + 1$  onwards, then

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\hat{H}^m(s), w(s), c) < \infty) = 1. \tag{25}$$

(b) (Local Alternative) Let  $\theta_D = \theta_{D,1}$  for  $t = 1, \dots, [rT]$  and  $\theta_D = \theta_{D,1} + \frac{1}{\sqrt{T}}G_D^{-1'}\Delta_\theta$  from  $t = [rT] + 1$  onwards with

$$\int_r^1 D(z)' dz \Delta_\theta \neq 0, \tag{26}$$

then the monitoring procedure has non-trivial local power. That means that for any

---

<sup>9</sup>The usage of the word alternative is sloppy here, since it is now not the stochastic component that leads to divergence of the detector. We are confident that this will not lead to any confusion.

$1 - \alpha \geq \epsilon > 0$  and the  $0 < c = c(\alpha, w(s)) < \infty$  from Proposition 1 there exists a  $\Delta_\theta = \Delta_\theta(c, w(s))$  fulfilling (26) such that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{H}^m(s), w(s), c(\alpha, w(s))) < \infty) \geq 1 - \epsilon. \quad (27)$$

The above condition (26) is *not* fulfilled only on an algebraic set. E.g. in case of  $D_t = (1, t)'$  it holds that  $\int_r^1 D(z)' dz = (1 - r, \frac{1}{2} - \frac{1}{2}r^2)$  and thus the set of values for which  $\Delta_\theta \in \mathbb{R}^2$  is orthogonal to this vector corresponds to a straight line in  $\mathbb{R}^2$ .

An analogous result as given above in (21) for the case of local trend breaks is, for the special case of  $D_t = 1$  and for  $s > r \geq m$ , given by:

$$\begin{aligned} \widehat{H}^m(s) \Rightarrow \widehat{\mathcal{H}}^m(s) + 2 \frac{\Delta_\theta}{\omega^{1/2}} \int_r^s \widehat{W}_m(z)(z - r) dz + \\ + \frac{\Delta_\theta^2}{\omega} \left( \frac{s^3}{3} - s^2 r + s r^2 - \frac{r^3}{3} \right) \end{aligned} \quad (28)$$

**Remark 4.** In case that breaks occur in both the deterministic and stochastic component of  $y_t$ , the behavior of our detector is a corresponding combination of the behavior discussed in Propositions 2 and 3. This implies that a rejection of the null hypothesis does not allow one to identify the source(s) of the break.  $\square$

**Remark 5.** Our approach can also be employed for detecting bubbles. In the recent literature, a bubble is often characterized as a period where the behavior of a time series has switched from integrated to explosive behavior, compare Phillips et al. (2011). Thus, our procedure allows to detect (the beginning of) a bubble by considering the first difference of the series, since in case of no bubble the first differences are stationary, whereas in case of explosive behavior also the first differences exhibit explosive behavior.  $\square$

**Remark 6.** In relation to the previous remark, with bubbles typically considered to be temporary rather than permanent phenomena, it has to be noted that our procedure will be consistent in detecting episodes of I(1) or explosive behavior, as long as these

episodes have asymptotically positive length. E.g. in the case of only one period under the alternative it has to hold that this period occurs over a sub-sample of the form  $[r_1T], \dots, [r_2T]$  with  $r_1 < r_2$ . It is immediate that consistency generalizes to multiple periods of this form.  $\square$

We close this section by looking at local asymptotic power (LAP) of the monitoring procedure. We consider LAP as discussed in Proposition 2(b) and against local-to-zero trend breaks as discussed in Proposition 3(b). The power curves are simulated similarly to the critical values. Discretized versions of the corresponding limiting distributions under the local alternatives are simulated. The limiting distribution for LAP against a unit root process is given in (21) and the limiting distribution in case of local trend breaks is given in (28). Again time series of length 1,000 are generated, with the number of replications given by 10,000. For each replication the errors  $u_t$ , and when considering Proposition 2(b) the  $\xi_t$ , are drawn as i.i.d. standard normal random variables independent of each other. The resulting values are then compared with the critical values and all test decisions are performed at the 5% level. All LAP curves are drawn for a grid of 21 equidistant values of  $\delta$ , starting at zero. Throughout we consider different combinations of the calibration period  $m \in \{0.25, 0.5, 0.75\}$  and break point  $r \geq m$  from the same set of values.

In Figure 1 we display local asymptotic power based on Proposition 2(b). The upper two plots consider the intercept only case and the lower two plots display the linear trend case. LAP is lower in the linear trend case compared to the intercept case. In each of the two sub-blocks of the figure the upper plot shows the effect of increasing  $r$  relative to fixed  $m = 0.25$ . As expected, LAP is decreasing with increasing  $r$ , since the period over which the integrated behavior can be detected is getting shorter and starts later. The lower two figures each display the effect of increasing the estimation sub-sample with increasing values of  $m = r$ . Here the effects are as expected for the intercept case, where LAP is decreasing with increasing values of  $m = r$ . The results are different in the linear trend case, where LAP is highest for  $m = r = 0.5$  and lowest for  $m = r = 0.25$ .

In Figure 2 we display, based on the result in Proposition 3(b), LAP against breaks in the



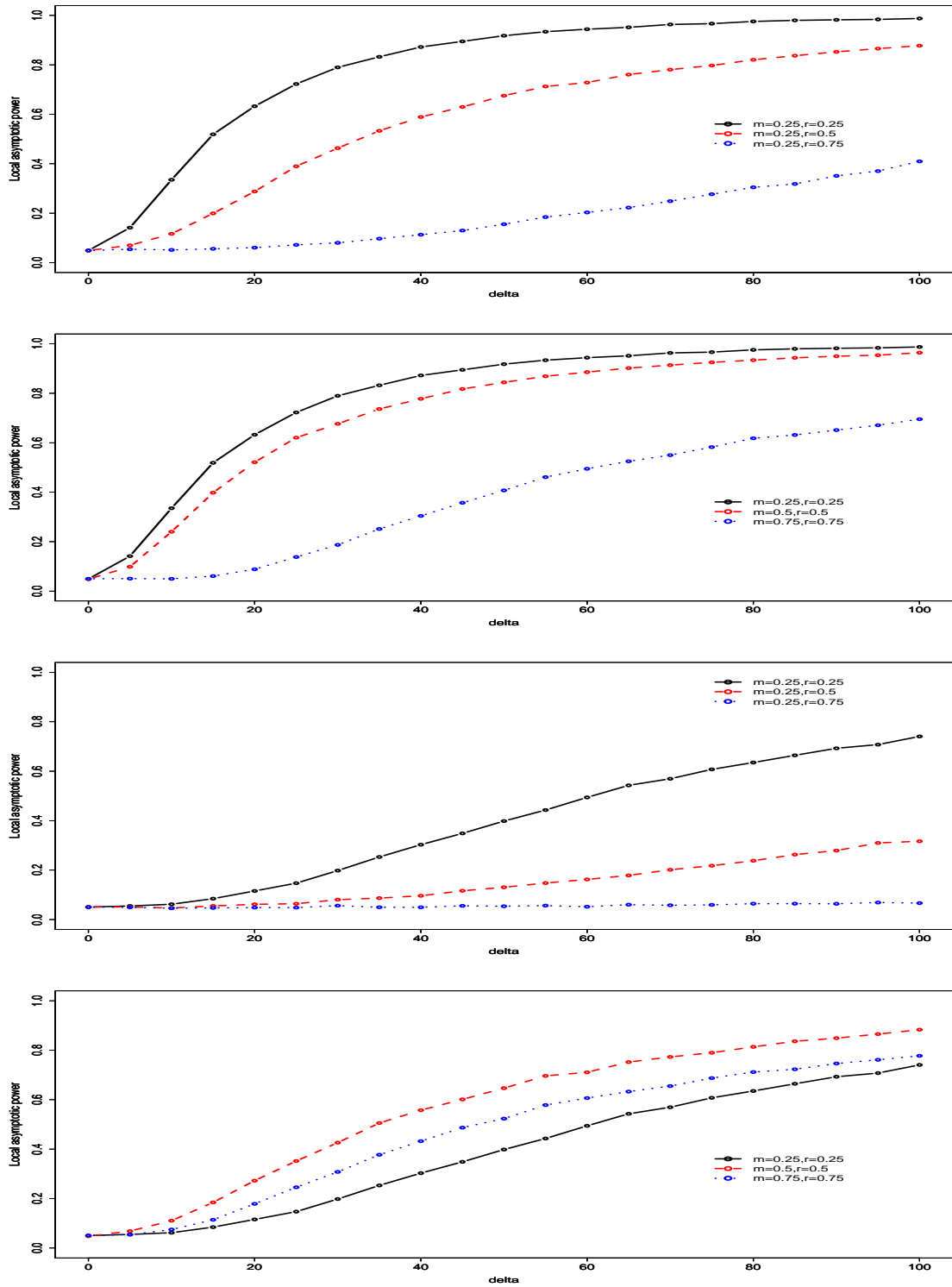


Figure 1: LAP for monitoring stationarity. The upper two plots display the intercept case and the lower two plots the linear trend case. The plots show results for different combinations of  $m$  and  $r$ .

intercept (upper two plots) and breaks in the linear trend parameter (lower two plots).<sup>10</sup> The structure of the plots is the same as in Figure 1 and also the findings are very similar. The only non-expected result again occurs in the lowest plot, where LAP for different values of  $m = r$  in case of a break in the linear trend is displayed. In this case LAP increases with the pre-break sample fraction, which is exactly opposite to the case of a break in the intercept.

### 3. MONITORING COINTEGRATION

In this section we use the same ideas as discussed before to monitor cointegrating relationships using the following setup:

$$y_t = D_t' \theta_D + X_t' \theta_X + u_t \quad (29)$$

$$X_t = X_{t-1} + v_t, \quad (30)$$

with  $y_t \in \mathbb{R}$  and  $D_t \in \mathbb{R}^p$  as before and  $X_t \in \mathbb{R}^k$ . The joint error vector  $\eta_t = (u_t, v_t)'$  fulfills a similar assumption as  $u_t$  in the previous section:

**Assumption 3.** *The stationary process  $\eta_t$  fulfills*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \eta_t = \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \begin{bmatrix} u_t \\ v_t \end{bmatrix} \Rightarrow \Omega^{1/2} W(s), \quad (31)$$

with  $W(s) = [W_{u,v}(s), W_v(s)]' \in \mathbb{R}^{1+k}$  a vector of standard Brownian motions and

$$\Omega = \begin{bmatrix} \Omega_{uu} & \Omega_{uv} \\ \Omega_{vu} & \Omega_{vv} \end{bmatrix} := \sum_{j=-\infty}^{\infty} \mathbb{E}(\eta_t \eta_{t-j}') > 0. \quad (32)$$

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<sup>10</sup>Note the different ranges of the horizontal axis for the two cases, with  $\Delta_\theta \in [0, 10]$  in case of an intercept break and  $\Delta_\theta \in [0, 100]$  in case of a break of the slope of the linear trend.

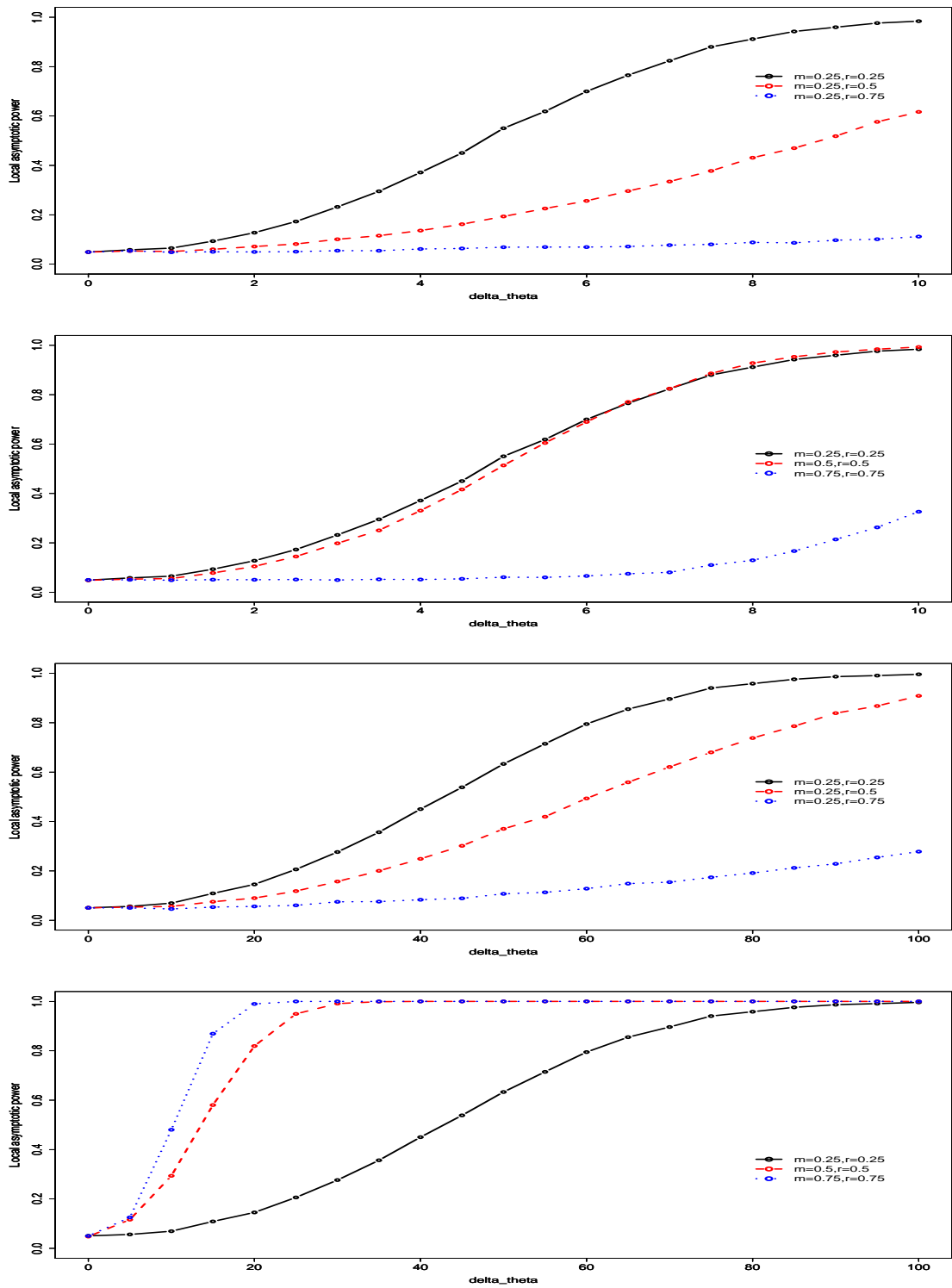


Figure 2: LAP against breaks in deterministic components. The upper two plots display the case of a break in the intercept and the lower two plots display the case of a break in the linear trend. The plots show results for different combinations of  $m$  and  $r$ .

For our purposes it is convenient to use

$$\Omega^{1/2} = \begin{bmatrix} \omega_{u \cdot v}^{1/2} & \lambda_{uv} \\ \mathbf{0} & \Omega_{vv}^{1/2} \end{bmatrix}, \quad (33)$$

where  $\omega_{u \cdot v} := \Omega_{uu} - \Omega_{uv}\Omega_{vv}^{-1}\Omega_{vu}$  and  $\lambda_{uv} := \Omega_{uv}(\Omega_{vv}^{1/2})^{-1}$ .

The assumption  $\Omega_{vv} > 0$  excludes cointegration amongst the regressors and is typically required for the modified OLS estimation techniques available, including Fully Modified OLS (FM-OLS) of Phillips and Hansen (1990), Dynamic OLS (D-OLS) of Saikkonen (1991) or Integrated Modified OLS (IM-OLS) of Vogelsang and Wagner (2014). It is well known that OLS estimation of  $\theta = [\theta'_D, \theta'_X]'$  in (29) is consistent, but that in general the limiting distribution of the OLS estimator depends on second order bias terms, which render asymptotic standard inference based on the OLS estimates infeasible. This problem occurs in particular when the regressors are not strictly exogenous, i.e. when the matrix  $\Omega$  is not block-diagonal.<sup>11</sup> The mentioned modified OLS estimators lead to limiting distributions of the parameters that are proportional to functionals of standard Brownian motions (which depend upon  $D_t$  and the number of integrated regressors  $k$ ) also in case of endogeneity. For brevity we abstain from explaining these well-known procedures here and just consider the residuals obtained from these estimation procedures as input in our monitoring procedure.<sup>12</sup>

We consider for illustration specifically the residuals of FM-OLS estimation

$$\begin{aligned} \hat{u}_{t,m}^+ &:= y_t - \Delta X'_t \hat{\Omega}_{vv}^{-1} \hat{\Omega}_{vu} - D'_t \hat{\theta}_{D,m} - X'_t \hat{\theta}_{X,m} \\ &= u_t - v'_t \hat{\Omega}_{vv}^{-1} \hat{\Omega}_{vu} - D'_t (\hat{\theta}_{D,m} - \theta_D) - X'_t (\hat{\theta}_{X,m} - \theta_X), \end{aligned} \quad (34)$$

where  $\hat{\theta}_{D,m}$  and  $\hat{\theta}_{X,m}$  denote the FM-OLS coefficient estimates and  $\hat{\Omega} (= \hat{\Omega}_m)$  denotes the

<sup>11</sup>In case of strict exogeneity, asymptotically valid inference can be based on the OLS estimates if serial correlation in  $u_t$  is handled appropriately using long-run variance estimates.

<sup>12</sup>Only the less well-known IM-OLS estimator is briefly discussed below.

long-run variance estimate, all computed from the pre-break sample  $1, \dots, [mT]$ .<sup>13</sup>

Using consistent estimates of the long-run variances, ensured again by assuming to be in the framework covered by Jansson (2002), leads to a FCLT for the modified OLS residuals  $\hat{u}_{t,m}^+$ .

**Lemma 2.** *Let the data be generated by (29) and (30) with Assumptions 1 and 3 in place and let  $\hat{\Omega}$  be a consistent long-run variance estimator (required only for FM-OLS). Then it holds under the null hypothesis and for  $m \leq s \leq 1$  for  $T \rightarrow \infty$  for FM-OLS and D-OLS that*

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m}^+ &\Rightarrow \omega_{u.v}^{1/2} \left( W_{u.v}(s) - \int_0^s J(z)' dz \left( \int_0^m J(z) J(z)' dz \right)^{-1} \int_0^m J(z) dW_{u.v}(z) \right) \\ &=: \omega_{u.v}^{1/2} \widehat{W}_{u.v}(s) \end{aligned} \quad (35)$$

with  $J(s) := [D(s)', W_v(s)']'$ .

Clearly, the process  $\widehat{W}_{u.v}(s)$  depends upon  $D_t$ , the number of integrated regressors  $k$  and the pre-break fraction  $m$ , with these dependencies neglected for notational brevity henceforth.

Given the FCLT (35) for the partial sum process of the modified residuals, the detector for cointegration, using either the FM-OLS or the D-OLS estimator, is defined by

$$\widehat{H}^{m,+}(s) := \frac{1}{\hat{\omega}_{u.v}} \left( \frac{1}{T} \sum_{i=[mT]+1}^{[sT]} \left( \frac{1}{\sqrt{T}} \widehat{S}_i^+ \right)^2 - \frac{1}{T} \sum_{i=1}^{[mT]} \left( \frac{1}{\sqrt{T}} \widehat{S}_i^+ \right)^2 \right), \quad (36)$$

where the scaling factor is now a consistent estimator  $\hat{\omega}_{u.v} = \hat{\Omega}_{uu} - \hat{\Omega}_{uv} \hat{\Omega}_{vv}^{-1} \hat{\Omega}_{vu}$  of the

<sup>13</sup>In case the procedure is implemented using the D-OLS estimator, the residuals are defined (using the same notation for the residuals and coefficient estimates) as  $\hat{u}_{t,m}^+ := y_t - D_t' \hat{\theta}_{D,m} - X_t' \hat{\theta}_{X,m} - \sum_{j=-k_1}^{k_2} \Delta X_{t-j}' \hat{\theta}_{j,m}$ , or equivalently  $\hat{u}_{t,m}^+ = u_t - D_t' (\hat{\theta}_{D,m} - \theta_D) - X_t' (\hat{\theta}_{X,m} - \theta_X) - \sum_{j=-k_1}^{k_2} \Delta X_{t-j}' \hat{\theta}_{j,m}$ , with the matrices  $\hat{\theta}_{D,m}$ ,  $\hat{\theta}_{X,m}$  and  $\hat{\theta}_{j,m}$  being the OLS estimates from the regression  $y_t = D_t' \theta_D + X_t' \theta_X + \sum_{j=-k_1}^{k_2} \Delta X_{t-j}' \theta_j + u_t$  estimated on the sample  $1, \dots, [mT]$ . Whereas in FM-OLS estimation bandwidth and kernel have to be chosen, D-OLS estimation requires choosing the number of lags  $k_1$  and leads  $k_2$ . Under appropriate assumptions concerning the asymptotic behavior of lag/lead choices the D-OLS residuals fulfill the same FCLT as the FM-OLS residuals. Asymptotically, therefore the usage of either estimator leads to the same monitoring procedure.

conditional long-run variance  $\omega_{u,v}$ .

All long-run variances and covariances required in the procedure (both for modified OLS parameter estimation as well as for scaling the detector) are based on the OLS residuals  $\hat{u}_{t,m}$  stacked on top of the first differences of the regressors, i.e. upon  $\hat{\eta}_t = [\hat{u}_{t,m}, v'_t]'$ . Again the OLS estimation from which the parameter estimates and long-run variance estimates are computed uses observations  $t = 1, \dots, [mT]$  only.

Given the definition of the detector for cointegration (36) the first result to be established is the asymptotic distribution of the detector under the null hypothesis.

**Lemma 3.** *Let the data be generated by (29) and (30) with Assumptions 1 and 3 in place and let  $\hat{\omega}_{u,v}$  denote a consistent long-run variance estimator. Then it holds under the null hypothesis and for  $m \leq s \leq 1$  for  $T \rightarrow \infty$  for FM-OLS and D-OLS that*

$$\hat{H}^{m,+}(s) \Rightarrow \int_m^s \widehat{W}_{u,v}^2(z) dz - \int_0^m \widehat{W}_{u,v}^2(z) dz =: \hat{\mathcal{H}}^{m,+}(s) \quad (37)$$

Alternatively, one can also base the cointegration monitoring procedure on the residuals of the recently proposed Integrated Modified OLS (IM-OLS) estimator of Vogelsang and Wagner (2014). A potential advantage of the IM-OLS estimator compared to FM-OLS and D-OLS is that for parameter estimation no kernel and bandwidth or lead and lag choices are required. The IM regression is given by

$$S_t^y = S_t^{D'} \theta_D + S_t^{X'} \theta_X + X_t' \gamma + S_t^u, \quad (38)$$

with  $S_t^y = \sum_{j=1}^t y_j$  denoting the partial sums, and similar definitions of  $S_t^D$  and  $S_t^X$ .

We denote the corresponding OLS residuals, with estimation based upon the pre-break sample  $1, \dots, [mT]$  by (using the same notation for the coefficient estimates as before)

$$\hat{S}_{t,m}^u := S_t^y - S_t^{D'} \hat{\theta}_{D,m} - S_t^{X'} \hat{\theta}_{X,m} - X_t' \hat{\gamma}_m \quad (39)$$

$$= S_t^u - X_t' \hat{\gamma}_m - S_t^{D'} (\hat{\theta}_{D,m} - \theta_D) - S_t^{X'} (\hat{\theta}_{X,m} - \theta_X) \quad (40)$$

Under the assumptions stated the following FCLT holds:

**Lemma 4.** *Let the data be generated by (29) and (30) with Assumptions 1 and 3 in place. Then it holds for  $T \rightarrow \infty$  that*

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=2}^{[sT]} \Delta \hat{S}_{t,m}^u &\Rightarrow \omega_{u,v}^{1/2} \left( W_{u,v}(s) - g(s)' \left( \int_0^m g(z)g(z)'dz \right)^{-1} \int_0^m [G(m) - G(z)] dW_{u,v}(z) \right) \\ &=: \omega_{u,v}^{1/2} \tilde{P}_m(s), \end{aligned} \quad (41)$$

where  $g(s) := [\int_0^s D(z)'dz, \int_0^s W_v(z)'ds, W_v(z)']'$  and  $G(s) := \int_0^s g(z)dz$ .

Based upon the above result, the IM-OLS based detector is defined analogously as:

$$\hat{I}^m(s) := \frac{1}{\hat{\omega}_{u,v}} \left( \frac{1}{T} \sum_{i=[mT]+1}^{[sT]} \left( \frac{1}{\sqrt{T}} \hat{S}_{i,m}^u \right)^2 - \frac{1}{T} \sum_{i=1}^{[mT]} \left( \frac{1}{\sqrt{T}} \hat{S}_{i,m}^u \right)^2 \right), \quad (42)$$

where the scaling is, as for the other detectors, based on a consistent estimator of  $\omega_{u,v}$ . Note that the same estimator as for FM-OLS or D-OLS, i.e. the estimator based on the OLS residuals  $\hat{u}_{t,m}$  stacked on top of the first differences of the regressors, is used. The asymptotic null behavior of the IM-OLS detector is given next.

**Lemma 5.** *Let the data be generated by (29) and (30) with Assumptions 1 and 3 in place and let  $\hat{\omega}_{u,v}$  denote a consistent long-run variance estimator. Then it holds under the null hypothesis and for  $m \leq s \leq 1$  for  $T \rightarrow \infty$  for IM-OLS that*

$$\hat{I}^m(s) \Rightarrow \int_m^s \tilde{P}_m(z)^2 dz - \int_0^m \tilde{P}_m(z)^2 dz =: \mathcal{I}^m(s) \quad (43)$$

As for the stationarity monitoring procedure, it can be shown that under the null hypothesis for given weighting function  $w(s)$ , there exist critical values  $c = c(\alpha, w(s))$ , such that the detection time is finite with probability equal to the pre-specified level  $\alpha$ .

**Proposition 4.** *Let the data be generated by (29) and (30) with Assumptions 1 and 3 in place, let long-run variance estimation be carried out consistently and consider  $0 <$*

$w(s) < \infty$ . Then, under the null hypothesis there exists for any given  $0 < \alpha < 1$  critical values  $c = c(\alpha, w(s))$ , depending upon estimation method, such that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{H}^{m,+}(s), w(s), c(\alpha, w(s))) < \infty) = \alpha, \quad (44)$$

in case that FM-OLS or D-OLS is used, respectively

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{I}^m(s), w(s), c(\alpha, w(s))) < \infty) = \alpha, \quad (45)$$

in case IM-OLS is used.

**Remark 7.** Note that for given weighting function  $w(s)$  the critical values are identical for the FM-OLS and D-OLS based detectors, but are different for the IM-OLS based detector.  $\square$

The critical values depend again also on the weighting function  $w(s)$ , which we again choose as  $w(s) = s^3$  in the intercept case  $D_t = 1$  and  $w(s) = s^5$  in the linear trend case  $D_t = (1, t)'$ . Critical values for these two cases are provided for both FM-OLS and D-OLS, in Tables 4 and 5, as well as for IM-OLS, in Tables 6 and 7, in Appendix B for one integrated regressor.<sup>14</sup> The simulations are performed analogously to the ones described above in Section 2 for the critical values for the stationarity monitoring procedure.

It remains to establish the behavior of the monitoring procedure under alternatives. In fact there are now three dimensions of structural change against which the procedures are shown to have power. First, changes in the behavior of  $u_t$ , where we consider exactly the same alternatives as above in Proposition 2. Second, again similar to before, we consider the behavior against breaks in the parameters corresponding to the deterministic component. Third, we now additionally consider the behavior against breaks in the slope coefficients corresponding to the integrated regressors. For all three cases we consider again fixed and local alternatives.

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<sup>14</sup>Critical values for up to four integrated regressors are available in supplementary material. Also computer code that implements the procedures is available upon request.



**Proposition 5. (Consistency and Local Asymptotic Power)**

Let the data be generated by (29) and (30) with Assumption 1 in place and  $\eta_t$  fulfilling Assumption 3 until  $[rT]$ , with  $m \leq r < 1$ . Furthermore, assume that long-run variance estimation is performed consistently using observations  $1, \dots, [mT]$  and let  $0 < w(s) < \infty$ . Denote with  $\widehat{F}^m(s)$  either  $\widehat{H}^{m,+}(s)$  or  $\widehat{I}^m(s)$ .

(a) Let  $u_t$  be an  $I(1)$  process (as specified in  $H_1$ ) from  $[rT] + 1$  onwards. Then the monitoring procedures are consistent, i.e. for any  $0 < c < \infty$  it holds that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{F}^m(s), w(s), c) < \infty) = 1. \quad (46)$$

(b) Let  $u_t$  be as specified in (17) from  $[rT] + 1$  onwards. Then the monitoring procedures have non-trivial local power. That means, for any  $1 - \alpha \geq \epsilon > 0$  and the  $0 < c = c(\alpha, w(s)) < \infty$  from Proposition 4 there exists a  $0 < \delta = \delta(c, w(s)) < \infty$  such that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{F}^m(s), w(s), c(\alpha, w(s))) < \infty) \geq 1 - \epsilon. \quad (47)$$

**Proposition 6. (Behavior in Case of Trend Breaks)**

Let the data be generated by (29) and (30) with Assumptions 1 and 3 in place. Furthermore, assume that long-run variance estimation is performed consistently using observations  $1, \dots, [mT]$  and let  $0 < w(s) < \infty$ . Denote with  $\widehat{F}^m(s)$  either  $\widehat{H}^{m,+}(s)$  or  $\widehat{I}^m(s)$ .

(a) (Fixed Alternative) Let  $\theta_D = \theta_{D,1}$  for  $t = 1, \dots, [rT]$  and  $\theta_D = \theta_{D,2}$ , with  $\theta_{D,1} \neq \theta_{D,2}$ , from  $t = [rT] + 1$  onwards, then

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{F}^m(s), w(s), c) < \infty) = 1. \quad (48)$$

(b) (Local Alternative) Let  $\theta_D = \theta_{D,1}$  for  $t = 1, \dots, [rT]$  and  $\theta_D = \theta_{D,1} + \frac{1}{\sqrt{T}} G_D^{-1'} \Delta_\theta$

from  $t = [rT] + 1$  onwards with

$$\int_r^1 D(z)' dz \Delta_\theta \neq 0, \quad (49)$$

then the monitoring procedure has non-trivial local power. That means, for any  $1 - \alpha \geq \epsilon > 0$  and the  $0 < c = c(\alpha, w(s)) < \infty$  from Proposition 4 there exists a  $\Delta_\theta = \Delta_\theta(c, w(s))$  fulfilling (49) such that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\tau_m(\widehat{F}^m(s), w(s), c(\alpha, w(s))) < \infty) \geq 1 - \epsilon. \quad (50)$$

Compared to the discussion in the previous section, there is now the additional possibility of breaks in the slope coefficients  $\theta_X$ , which are in a sense equivalent to changes in the behavior of the  $u_t$ . Consider for simplicity the case  $\theta_X = \theta_{X,1}$  for  $t = 1, \dots, [rT]$  and  $\theta_X = \theta_{X,2}$ , with  $\theta_{X,1} \neq \theta_{X,2}$ , for  $t = [rT] + 1, \dots, T$ . In this case we can write for  $t > [rT]$

$$y_t = D_t' \theta_D + X_t' \theta_{X,2} + u_t \quad (51)$$

$$= D_t' \theta_D + X_t' \theta_{X,1} + X_t' (\theta_{X,2} - \theta_{X,1}) + u_t \quad (52)$$

Clearly, this implies that in the residual process starting from  $[rT]$  onwards an integrated process given by  $X_t'(\theta_{X,2} - \hat{\theta}_{X,1,m})$  is present. This component remains present as an I(1) process also in the limit due to consistency of  $\hat{\theta}_{X,1,m} \rightarrow \theta_{X,1} \neq \theta_{X,2}$ . Consequently, in case of a break in the slope parameters, the residual process is an I(1) process. Therefore, the asymptotic behavior in case of slope breaks is similar to the case discussed in Proposition 5. We therefore have a very similar result, where local alternatives are now of the form  $\theta_{X,2} = \theta_{X,1} + \frac{1}{T} \Delta_\theta$ .

**Proposition 7. (Behavior in Case of Slope Breaks)**

Let the data be generated by (29) and (30) with Assumptions 1 and 3 in place. Furthermore, assume that long-run variance estimation is performed consistently using observations  $1, \dots, [mT]$  and let  $0 < w(s) < \infty$ . Denote with  $\widehat{F}^m(s)$  either  $\widehat{H}^{m,+}(s)$  or

$\widehat{I}^m(s)$ .

(a) *Considering fixed alternatives of the form  $\theta_X = \theta_{X,1}$  for  $t = 1, \dots, [rT]$  and  $\theta_X = \theta_{X,2}$  for  $t = [rT] + 1, \dots, T$ , with  $\theta_{X,1} \neq \theta_{X,2}$  leads to a similar result as in part (a) of Proposition 5.*

(b) *Considering local alternatives of the form  $\theta_X = \theta_{X,1}$  for  $t = 1, \dots, [rT]$  and  $\theta_X = \theta_{X,1} + \frac{1}{T}\Delta_\theta$ , with  $\Delta_\theta \neq 0$ , for  $t = [rT] + 1, \dots, T$  leads to a similar result as in part (b) of Proposition 5.*

**Remark 8.** If one considers the limiting distributions of the detectors under the various forms of alternatives in more detail one sees that for FM-OLS and D-OLS the additional terms related to the local alternatives involve integral terms ranging from the break fraction  $r$  to  $s$ . This is similar to the limit results that arise when considering local alternatives for stationarity monitoring, see again 21 or (28). In case of the IM-OLS based detector the corresponding terms involve integrals ranging from 0 to  $s$ . This is an effect of partial summing the observations before parameters estimation when using IM-OLS.

**Remark 9.** Remarks 3 to 6 apply analogously to the cointegration monitoring procedures as well. □

We again close this section by considering local asymptotic power which is simulated analogously to the LAP simulations in the previous section, i.e. the number of replications is 10,000 and the time series considered are of length 1,000. All random variables are i.i.d. standard normal. The limiting distribution for LAP as discussed in Proposition 5(b) is based on the FCLTs under local alternatives given in (59) for FM-OLS and D-OLS and (60) for IM-OLS in Appendix A. These are the input to obtain a limiting distribution similar to the limiting distribution given in (21) used above to study LAP when monitoring stationarity. Also local asymptotic power against trend breaks (Proposition 6) is simulated in the same way with the corresponding FCLTs given in (63) and (64) in

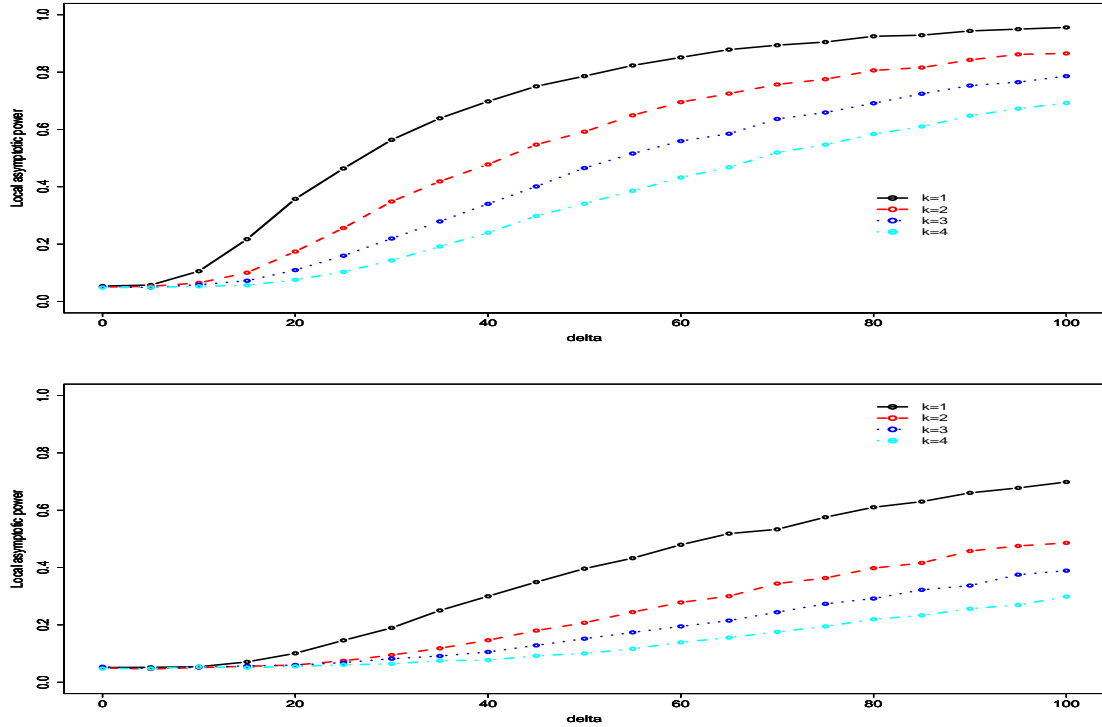


Figure 3: Local asymptotic power for  $k = 1, \dots, 4$  regressors for monitoring cointegration for the case with intercept. The upper plot corresponds to FM-OLS & D-OLS and the lower plot to IM-OLS. The plots show results for  $m = r = 0.25$ .

Appendix A that can be used to obtain limiting distributions similar to (28). Finally, local asymptotic power against slope breaks (Proposition 7) is based on the FCLTs given in (67) and (68) in Appendix A. Also for this case the FCLTs allow to establish the limiting distributions of the detectors under the local alternative. Considering these results in detail leads to exactly the same observations as in the previous section with respect to the dependence of LAP on the signal-to-noise ratio. As an extension of this fact it turns out that LAP against slope breaks increases with the variance of the regressors. Similarly to the simulations in the previous section we set all signal-to-noise ratios equal to one. It is also clear that in addition to the dependence upon the deterministic component, LAP now also depends upon the number of integrated regressors, as illustrated in Figure 3. As expected, LAP decreases with an increasing number of regressors. Consequently, all other results displayed are for the case of only one integrated regressor.

In Figure 4 we display LAP against local  $I(1)$  alternatives in case of intercept and linear trends included in the model, where we consider again the same combinations of  $m$  and  $r$  as in the previous section. The upper two plots correspond to FM-OLS and D-OLS and the lower two plots correspond to IM-OLS. The results show that LAP is lower for IM-OLS, which is as expected given the results of Vogelsang and Wagner (2014) concerning the relative conditional efficiency of FM-OLS over IM-OLS.<sup>15</sup> The practical usefulness of IM-OLS stems from the lower size distortions that it implies compared to FM-OLS, as illustrated in the following section where we consider finite sample simulations. With respect to changing values of  $m$  and  $r$  all methods have similar LAP rankings, which coincide with the rankings found in the previous section.

In Figure 5 we display local asymptotic power results against breaks in the intercept, with the same structure of the figure as in Figure 4. The ordering of LAP as a function of  $m = r$  (in the second and fourth plot) differs between FM-OLS/D-OLS and IM-OLS. For IM-OLS LAP increases with increasing  $m = r$ , whereas for the other two methods LAP is, as in the previous section, highest for  $m = r = 0.5$ . Also, as in the previous section the rankings differ between the intercept and the linear trend case.

#### 4. FINITE SAMPLE PERFORMANCE (OLD VERSION)

In this section we present some finite sample simulation results. For brevity we only consider cointegration monitoring. We consider a simulation design similar, with respect to endogeneity and serial correlation patterns, to Vogelsang and Wagner (2014), i.e. we

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<sup>15</sup>The results are similar, with the differences smaller, in case of the model with intercept only. Additional results, including also results for breaks in the slope parameter, are available in supplementary material. The findings are, as expected, very similar to the ones for local  $I(1)$  alternatives.

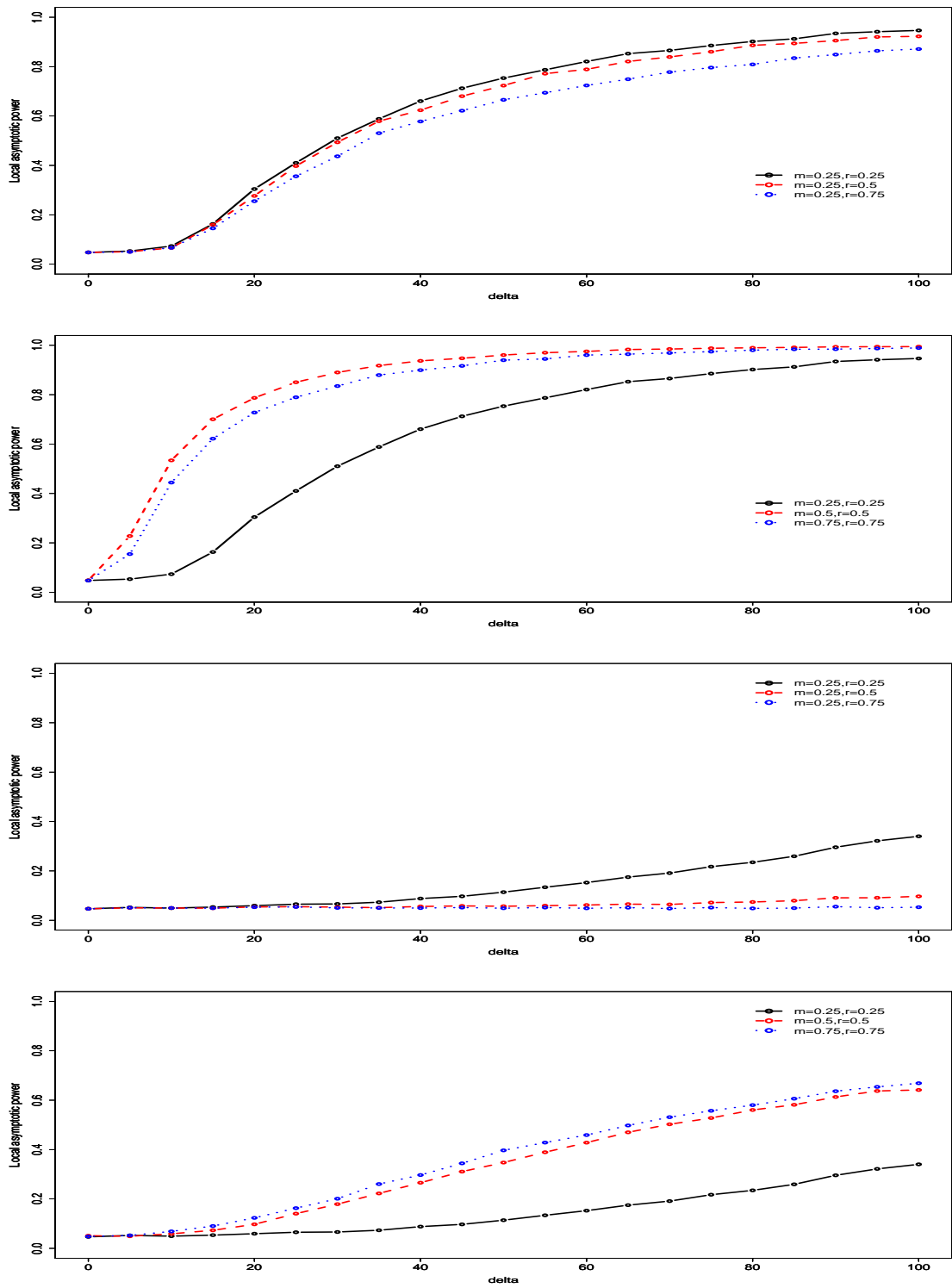


Figure 4: Local asymptotic power for monitoring cointegration for the case with intercept and linear trend. The upper two plots correspond to FM-OLS & D-OLS and the lower two plots to IM-OLS. The plots show results for different combinations of  $m$  and  $r$ .

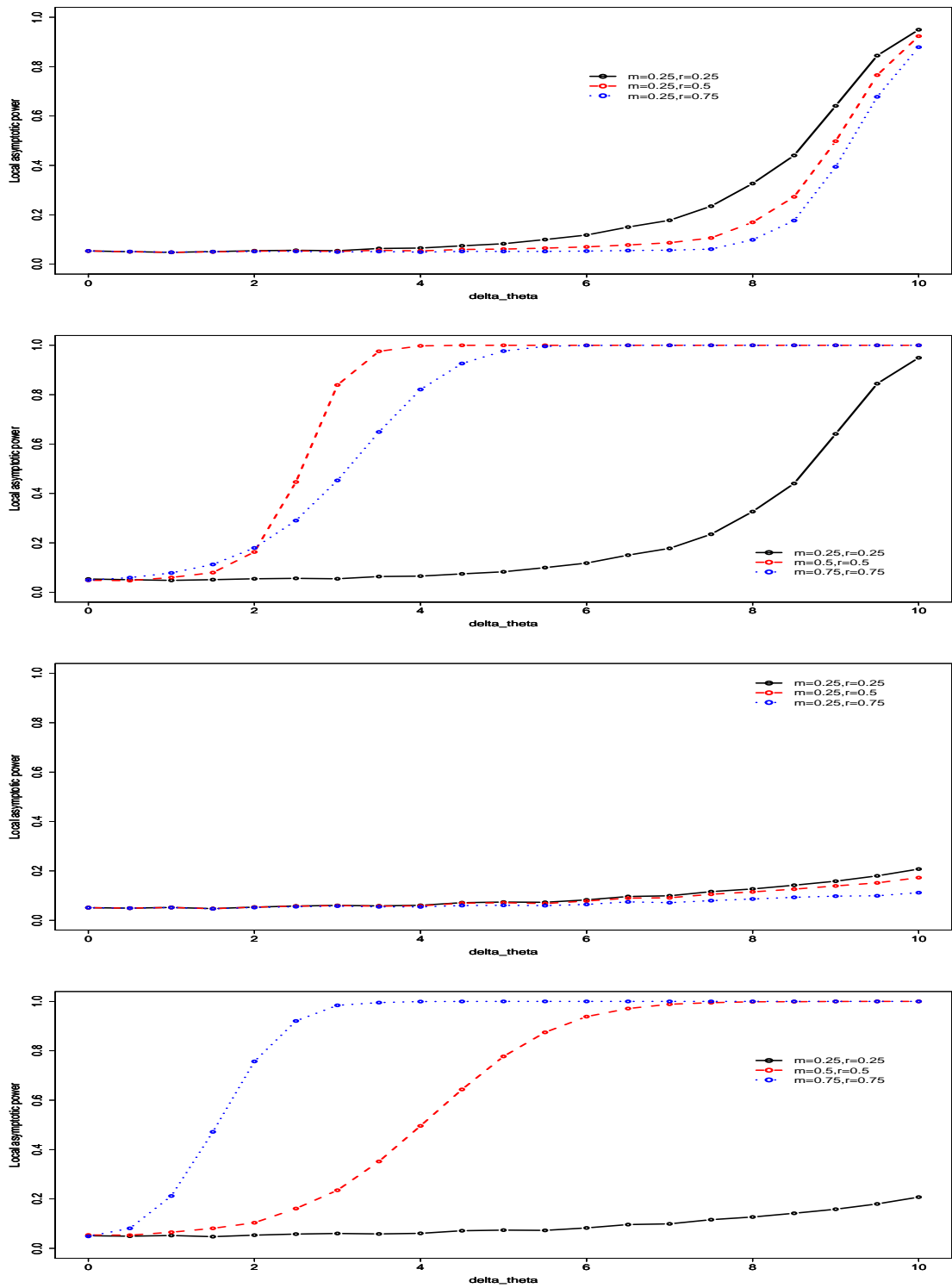


Figure 5: Local asymptotic power against break in intercept for monitoring cointegration for the case with intercept. The upper two plots correspond to FM-OLS & D-OLS and the lower two plots to IM-OLS. The plots show results for different combinations of  $m$  and  $r$ .

consider:

$$y_t = \mu + \theta t + x_t \beta + u_t$$

$$x_t = x_{t-1} + v_t$$

$$u_t = \rho_1 u_{t-1} + \epsilon_t + \rho_2 e_t$$

$$v_t = e_t + 0.5e_{t-1}$$

The errors  $\epsilon_t$  and  $e_t$  are i.i.d. standard normal random variables. The parameter  $\rho_1$  governs the extent of autocorrelation in the errors, and in case  $\rho_1 = 1$  we have a spurious relationship. The parameter  $\rho_2$  governs the extent of endogeneity. The parameters are set to  $\mu = 3$ ,  $\theta = 1$  and  $\beta = 1$ . We only display results here for  $T = 200$ , with 5,000 replications, and consider  $\rho_1, \rho_2 \in \{0, 0.3, 0.6, 0.9, 0.95\}$  and in addition  $\rho_1 = 1$  under the alternative. The values for  $m$  and  $r$  are again taken from the set  $\{0.25, 0.5, 0.75\}$ . All long-run variances are estimated using the Bartlett kernel with the bandwidth chosen according to Andrews (1991). The lead and lag length choices required for D-OLS estimation are performed following Kejriwal and Perron (2008). All tests are performed at a significance level of 5%.

We start in Figure 6 by displaying the empirical size of the monitoring procedures based on the three modified least squares estimators with  $m = 0.75$ . In this figure the points at the right end, where  $\rho_1 = 1$ , correspond to a spurious relationship, and thus do not display size but power against a spurious relationship prevalent over the whole sample period. Increasing the values of  $\rho_1, \rho_2$  leads to increasing size distortions. These size distortions are throughout most pronounced for FM-OLS and least pronounced for IM-OLS. D-OLS exhibits a performance in between the other two methods, but is in general closer to IM-OLS than to FM-OLS.

Figure 7 displays size corrected power for the same set of  $\rho_1$  and  $\rho_2$  values as displayed in Figure 6. Size corrected power is slightly larger for FM-OLS than for D-OLS and



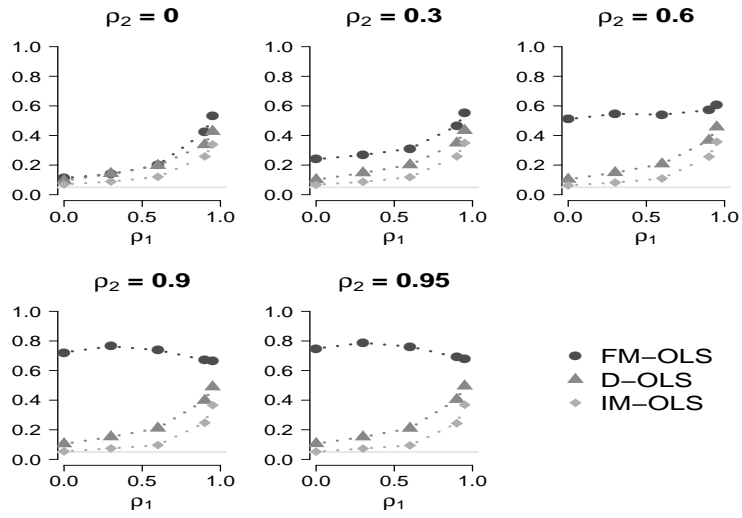


Figure 6: Empirical size for the case with intercept and linear trend with  $m = 0.75$ .

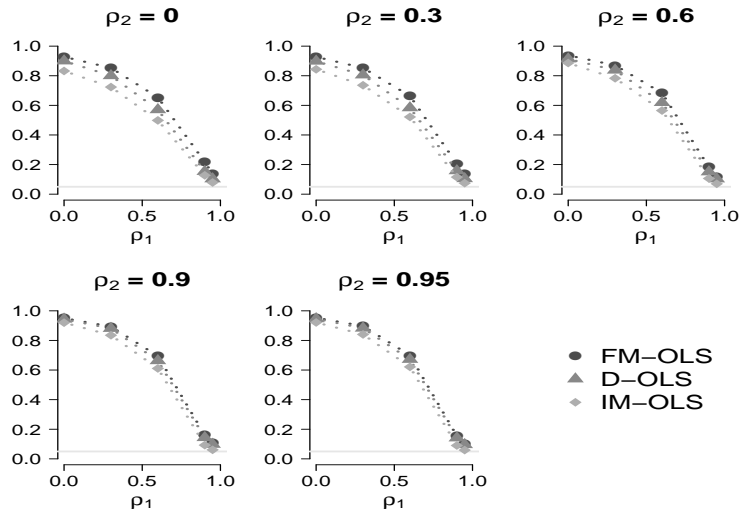


Figure 7: Size corrected power for the case with intercept and linear trend with  $m = r = 0.5$  and  $\rho_1, \rho_2 = 0.3$ .

IM-OLS.<sup>16</sup> This finding is well in line with the findings of Vogelsang and Wagner (2014), who also find that FM-OLS has highest power but also the largest size distortions. The figure also shows that power crucially depends upon the serial correlation in  $u_t$ , but is hardly affected by the parameter  $\rho_2$ , i.e. by regressor endogeneity. Of course, this finding has to be interpreted with caution as the figures displays size corrected power.

In Figure 8 we display histograms of the estimated break point for  $m = 0.25$  and  $r = 0.5$  (indicated by the vertical lines) in relation to the power results displayed in Figure 7. The left column displays the results for  $\rho_1, \rho_2 = 0.3$  and the right column for  $\rho_1, \rho_2 = 0.9$ . Two main observations can be made: First, the break point is estimated with delay. Second, this delay decreases for increasing  $\rho_1, \rho_2$ . The second observation is not in line with expectations, given that increasing values of  $\rho_1, \rho_2$  lead to deteriorating performance of the monitoring procedures.

Finally, Figure 9 displays size corrected power against breaks in the slope parameter  $\beta$  for different combinations of  $m$  and  $r$  with  $\rho_1, \rho_2 = 0.3$ . The parameter  $\beta$  is chosen from the interval  $[1, 2]$  using a grid of mesh 0.1, with the point  $\beta = 1$  corresponding to the null hypothesis. As expected, the results are – in terms of ranking – similar to the power results discussed before, with FM-OLS displaying the best results and D-OLS and IM-OLS showing quite similar performance.

## 5. EMPIRICAL ILLUSTRATION

We apply the monitoring procedure to investigate daily CDS spreads data for Austria, Cyprus, France and Germany for maturities one, three, five, seven and ten years, see Figure 10. Stationary or nonstationary behavior of CDS spreads during financial crisis has been investigated e.g. by Dieckmann and Plank (2011) who find mixed evidence in this respect. The mixed evidence may not least be driven by changing stationarity behavior over time such that an application of our monitoring procedure appears potentially useful. The time span chosen ranges from April 3, 2009 to August 1, 2012, giving a total of

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<sup>16</sup>The points at the right end of the plots are by construction at the chosen significance level of 5%.

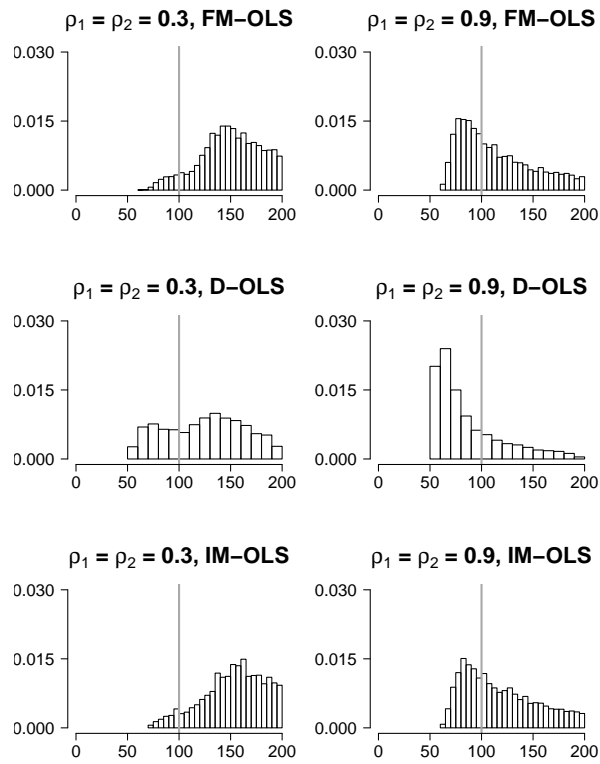


Figure 8: Estimated detection times for a shift from  $I(0)$  to  $I(1)$  behavior of  $u_t$  at  $T = 100$  for  $m = 0.25$ .

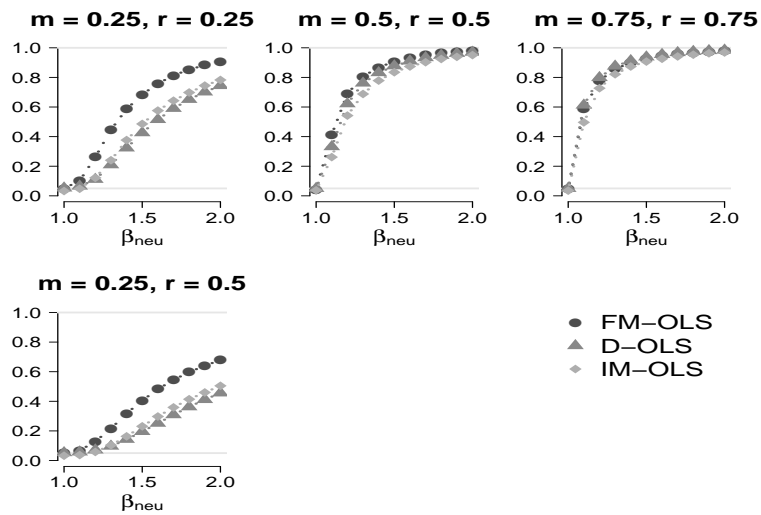


Figure 9: Size corrected power against a break in the trend parameter for  $\rho_1, \rho_2 = 0.3$

$T = 869$  observations. We try to exclude the immediate consequences of the Lehman brothers bankruptcy on September 15, 2008 and therefore take as starting date the day after a G20 summit held in London on April 1–2, 2009 that is considered to have had stabilizing influences on the financial markets. Our sample period ranges until August 1, 2012, which is the date when Cyprus was downgraded from investment grade BB+ to BB by Standard & Poor’s. The calibration period consists of the first 10% of the observations, i.e. ranges from April 3, 2009 to July 31, 2009 and is indicated by the vertical lines in Figure 10.<sup>17</sup>

Graphical inspection of the data already allows for a few important observations. First, within each country the series corresponding to the different maturities move together quite closely, displaying something like a term structure of CDS spreads. In this respect it is interesting to note that for Cyprus the “term structure” is inverse after the two big jumps occurring on July 27 and August 26, 2011.<sup>18</sup> After these jumps the level of the series is much larger for Cyprus than for the other three countries that also experience upward jumps around the same time. This, of course, illustrates the well-known fact that there are not only within-country dependencies across different maturities, but that there is also a large amount of cross-country co-movements of CDS spreads series, compare also Dieckmann and Plank (2011).<sup>19</sup>

The test results are displayed in Table 1 and indicate a rejection of the null hypothesis of stationarity at the 5% significance level for all series considered. By construction, the estimated break dates are later for lower values of  $\alpha$ . The break dates (at the 5% significance level) are between April 27, 2010 (France, ten years) and February 1, 2011 (Germany, one year). All these dates precede the discussed major level shifts in the CDS

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<sup>17</sup>The results are very robust and do not change at all when considering  $m = 0.2$  instead. Note that for values of  $m$  larger than 0.2 the KPSS stationarity test leads to rejection of the null of stationarity on the calibration sample for all countries and maturities at the 1% level. From this perspective therefore, the choice of the calibration period appears to be in line with data properties.

<sup>18</sup>On July 27, 2011 Moody’s downgraded Cyprus to Baa1 after an explosion at a marine base.

<sup>19</sup>The strong nature of co-movements of series indicates that an extension of our monitoring procedures to monitor also structural change in the cross-sectional co-movements of potentially (co-)integrated series, in addition to changing (idiosyncratic) time series properties, may be relevant. Changes in the cross-sectional dependence structure may be seen as an indicator of contagion phenomena.

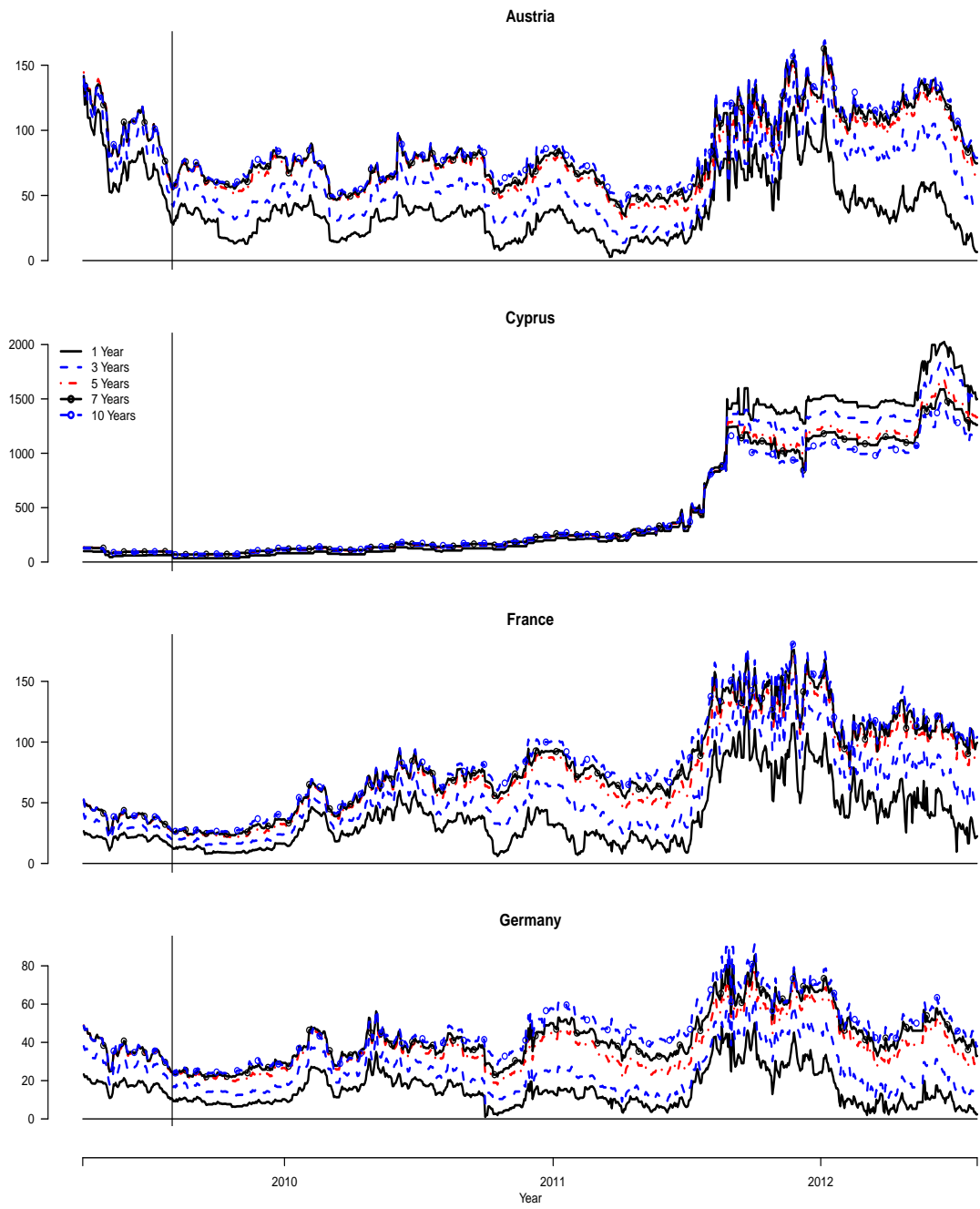


Figure 10: Daily CDS spreads series.

Country	Maturity	Test statistic	$\tau_m(\alpha = 0.1)$	$\tau_m(\alpha = 0.05)$
Austria	1	4317.35	09.08.2010	03.11.2010
	3	4647.26	06.08.2010	22.10.2010
	5	5582.71	02.07.2010	07.09.2010
	7	5492.46	05.07.2010	09.09.2010
	10	5643.04	28.06.2010	02.09.2010
Cyprus	1	24474.22	07.07.2010	10.09.2010
	3	20564.81	16.07.2010	24.09.2010
	5	18220.25	14.07.2010	21.09.2010
	7	17693.05	30.06.2010	02.09.2010
	10	16749.96	09.06.2010	06.08.2010
France	1	3527.84	10.09.2010	12.01.2011
	3	12371.54	11.05.2010	03.06.2010
	5	14299.06	21.04.2010	17.05.2010
	7	15142.24	13.04.2010	07.05.2010
	10	16221.89	31.03.2010	27.04.2010
Germany	1	3450.41	02.09.2010	01.02.2011
	3	3905.52	18.08.2010	18.11.2010
	5	7042.27	21.05.2010	29.06.2010
	7	8139.72	03.05.2010	04.06.2010
	10	9083.49	06.05.2010	10.06.2010

Table 1: Results of stationarity monitoring for the daily CDS spreads data for  $m = 0.1$ . The third column displays  $\sup_{m \leq s \leq 1} \left| \frac{\hat{H}^m(s)}{w(s)} \right|$  and the fourth and fifth columns the associated detection times  $\tau_m(\hat{H}^m(s), w(s), c(\alpha, w))$  for  $\alpha = \{0.1, 0.05\}$ . Intercept and linear trend are included, hence  $w(s) = s^5$ . The null hypothesis is rejected throughout. The 10% critical value is 1252.59 and the 5% critical value is 1777.80.

series that occurred in summer 2011. It is interesting to note that for the longer maturities the breaks are dated earlier for the big EU member states France and Germany, whereas it is the opposite for the short maturities. An in-depth exploration of the economic or political reasons underlying these findings and break dates is beyond the purpose of the present paper where we merely intend to illustrate our procedure.

**Remark 10.** For financial data collected at higher frequencies, e.g. the considered daily CDS spreads series, the assumption of time-constant variances under the null hypothesis may be too restrictive. Similarly to observations made in e.g. Cavaliere and Taylor (2008), non-constant second moments of a form that lead to functional central limit theorems invoking time deformed Brownian motions – in this literature typically referred to as nonstationary volatility – will lead to size distortion of our monitoring procedure, but

will not invalidate consistency against fixed alternatives.<sup>20</sup> To be precise, consistency of the detection procedure hinges on the fact that the long-run variance estimators computed over the pre-break period do not diverge in this more general setting.  $\square$

## 6. SUMMARY AND CONCLUSIONS (PRELIMINARY VERSION)

We have proposed monitoring procedures for stationarity and cointegration that are based on parameter estimation on a pre-break calibration period. The key ingredients of our detectors are properly scaled squared partial sums of residuals that are compared between the calibration and the successively increasing monitoring period. Thus, the detectors are based on the Kwiatkowski et al. (1992) respectively Shin (1994) tests. In case of cointegration monitoring we have investigated the properties of the monitoring procedure when using FM-OLS, D-OLS or IM-OLS residuals. Modified least squares estimation is necessary to correct the effects of serial correlation and endogeneity in order to obtain nuisance parameter free limiting distributions. The procedures are shown to be consistent against  $I(1)$  alternatives, breaks in the parameters corresponding to the deterministic components and, in case of cointegration monitoring, breaks in the slope parameters.

The performance of the procedures has been investigated both in terms of local asymptotic power as well as by means of finite sample simulations. LAP is lower for the IM-OLS based procedure compared to LAP when using either FM-OLS or D-OLS. The finite sample simulations, however, indicate that the price for the higher power of the latter two methods is their larger biases.

A brief empirical application to CDS spreads data of four European countries indicates the usefulness of the proposed method for the case of stationarity monitoring. The break dates all precede the turbulent period of summer 2011 by between a year and half a year. Several extensions of the approach are conceivable: First, it may be relevant to flip null and alternative hypothesis, i.e. to monitor changes from  $I(1)$  to  $I(0)$  behavior. Second,

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<sup>20</sup>Cavaliere and Taylor (2005) consider the asymptotic behavior of the KPSS statistic in this context and Cavaliere and Taylor (2006) consider cointegration testing with variance breaks.

the empirical application, with the clearly visible co-movements across maturities but also across countries, indicates that multivariate monitoring procedures may be important for applied research. Third, especially important for monitoring data collected at higher frequencies, the effects of non-constant variances need to be investigated in detail.

## REFERENCES

- ANDREWS, D. W. K. (1991): “Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation,” *Econometrica*, 59, 817–854.
- CAPPUCCIO, N. AND D. LUBIAN (2005): “Local Asymptotic Distributions of Stationarity Tests,” *Journal of Time Series Analysis*, 27, 323–345.
- CAVALIERE, G. AND A. M. R. TAYLOR (2005): “Stationarity Tests under Time Time-Varying Second Moments,” *Econometric Theory*, 21.
- (2006): “Testing the Null of Cointegration in the Presence of Variance Breaks,” *Journal of Time Series Analysis*, 27.
- (2008): “Testing for a Change in Persistence in the Presence of Non-Stationary Volatility,” *Journal of Econometrics*, 147, 84–98.
- CHEN, Z., Z. JIN, Z. TIAN, AND P. QI (2012): “Bootstrap Testing Multiple Changes in Persistence for a Heavy-Tailed Sequence,” *Computational Statistics and Data Analysis*, 56, 2303–2316.
- CHEN, Z., Z. TIAN, AND Y. WEI (2010): “Monitoring Change in Persistence in Linear Time Series,” *Statistics & Probability Letters*, 80, 1520–1527.
- CHU, C.-S. J., M. STINCHCOMBE, AND H. WHITE (1996): “Monitoring Structural Change,” *Econometrica*, 64, 1045–1065.
- DIECKMANN, S. AND T. PLANK (2011): “Default Risk of Advanced Economies: An



- Empirical Analysis of Credit Default Swaps during the Financial Crisis,” *Review of Finance*, forthcoming, doi: 10.1093/rof/rfr015.
- GROEN, J. J. J., G. KAPETANIOS, AND S. PRICE (2013): “Multivariate Methods for Monitoring Structural Change,” *Journal of Applied Econometrics*, 28, 250–274.
- JANSSON, M. (2002): “Consistent Covariance Matrix Estimation for Linear Processes,” *Econometric Theory*, 18, 1449–1459.
- KEJRIWAL, M. AND P. PERRON (2008): “Data Dependent Rules for Selection of the Number of Leads and Lags in the Dynamic OLS Cointegrating Regressions,” *Econometric Theory*, 24, 1425–1441.
- KIM, J.-Y. (2000): “Detection of Change in Persistence of a Linear Time Series,” *Journal of Econometrics*, 95, 97–116.
- KWIATKOWSKI, D., P. C. B. PHILLIPS, P. SCHMIDT, AND Y. SHIN (1992): “Testing the Null Hypothesis of Stationarity against the Alternative of a Unit Root,” *Journal of Econometrics*, 54, 159–178.
- PHILLIPS, P. C. B. (1987): “Towards a Unified Asymptotic Theory for Autoregression,” *Biometrika*, 74, 535–547.
- PHILLIPS, P. C. B. AND B. HANSEN (1990): “Statistical Inference in Instrumental Variables Regression with I(1) Processes,” *Review of Economic Studies*, 57, 99–125.
- PHILLIPS, P. C. B. AND M. LORETAN (1993): “Estimating Long Run Economic Equilibria,” *Review of Economic Studies*, 58, 407–436.
- PHILLIPS, P. C. B., Y. WU, AND J. YU (2011): “Explosive Behavior in the 1990s Nasdaq: When did Exuberance escalate Asset Values?” *International Economic Review*, 52, 201–226.

- QI, P., Z. JIN, Z. TIAN, AND Z. CHEN (2013): “Monitoring Persistent Change in a Heavy-Tailed Sequence with Polynomial Trends,” *Journal of the Korean Statistical Society*, 42, 497–506.
- SAIKKONEN, P. (1991): “Asymptotically Efficient Estimation of Cointegrating Regressions,” *Econometric Theory*, 7, 1–21.
- SHIN, Y. (1994): “A Residual-Based Test of the Null of Cointegration against the Alternative of No Cointegration,” *Econometric Theory*, 10, 91–115.
- STELAND, A. (2007): “Monitoring Procedures to Detect Unit Roots and Stationarity,” *Econometric Theory*, 23, 1108–1135.
- (2008): “Sequentially Updated Residuals and Detection of Stationary Errors in Polynomial Regression Models,” *Sequential Analysis: Design Methods and Applications*, 27, 304–329.
- STOCK, J. H. AND M. W. WATSON (1993): “A Simple Estimator of Cointegrating Vectors in Higher Order Integrated Systems,” *Econometrica*, 61, 783–820.
- VOGELSANG, T. J. AND M. WAGNER (2014): “Integrated Modified OLS Estimation and Fixed- $b$  Inference for Cointegrating Regressions,” *Journal of Econometrics*, 148, 741–760.
- WIED, D. AND P. GALEANO (2013): “Monitoring Correlation Change in a Sequence of Random Variables,” *Journal of Statistical Planning and Inference*, 143, 186–196.

## A. PROOFS

*Proof of Lemma 1:*

The result follows directly from the assumptions using consistency of OLS detrending (used already for the FCLT in (13)) and of long-run variance estimation. ■

*Proof of Proposition 1:*

The result follows from Lemma 1 since the limit of  $\widehat{H}^m(s)$ ,  $\widehat{\mathcal{H}}^m(s)$  is well defined and the continuous mapping theorem, compare (11) with  $\widehat{H}^m(s)$  instead of  $H^m(s)$ . ■

*Proof of Proposition 2:*

(a): Start by decomposing the partial sum process, for  $1 \geq s > r \geq m$  into

$$\frac{1}{\sqrt{T}} \widehat{S}_{[sT]} = \frac{1}{\sqrt{T}} \widehat{S}_{[rT]} + \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} \widehat{u}_{t,m}. \quad (53)$$

The first term above converges to  $\omega^{1/2} \widehat{W}_m(r)$  and the second term above is unbounded under the considered alternative, since

$$\frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} \widehat{u}_{t,m} = \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} u_t - \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} D'_t \left( \sum_{i=1}^{[mT]} D_i D'_i \right)^{-1} \sum_{i=1}^{[mT]} D_i u_i, \quad (54)$$

where the first term is  $O_p(T)$ , as it converges, when scaled by  $T^{-1}$ , towards the integral of a Brownian motion, and the second term converges to  $\int_r^s D(z)' \left( \int_0^m D(z) D(z)' dz \right)^{-1} \int_0^m D(z) dW(z)$ . Consequently, for  $s > r$ , the first term in  $\widehat{H}^m(s)$  and thus  $\widehat{H}^m(s)$  diverges, which establishes the result.

(b): Straightforward calculations similar to the one above establish for  $1 \geq s > r \geq m$  that

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m} &= \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} D'_t \left( \sum_{i=1}^{[mT]} D_i D'_i \right)^{-1} \sum_{i=1}^{[mT]} D_i u_i \\
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t^0 + \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \frac{\delta}{T} \sum_{i=[rT]+1}^{[sT]} \xi_i - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} D'_t \left( \sum_{i=1}^{[mT]} D_i D'_i \right)^{-1} \sum_{i=1}^{[mT]} D_i u_i^0 \\
&\Rightarrow \omega^{1/2} \widehat{W}_m(s) + \delta \omega_\xi^{1/2} \int_r^s (W_\xi(z) - W_\xi(r)) dz.
\end{aligned} \tag{55}$$

This shows that the partial sum process can be made arbitrarily large (with arbitrarily large probability) by choosing  $\delta$  large enough. This in turn makes the probability of a finite  $\tau_m$  arbitrarily large.  $\blacksquare$

*Proof of Proposition 3:*

(a): Again the starting point is the partial sum process of the OLS residuals. Consider again  $1 \geq s > r \geq m$ :

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m} = \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} D'_t \left( \sum_{i=1}^{[mT]} D_i D'_i \right)^{-1} \sum_{i=1}^{[mT]} D_i u_i - \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} D'_t (\theta_{D,1} - \theta_{D,2}).$$

The first two terms above converge to  $\omega^{1/2} \widehat{W}_m(s)$  and the third term diverges due to Assumption 1.<sup>21</sup>

(b): When considering local alternatives, the partial sum process is as follows (for  $1 \geq s > r \geq m$ ):

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<sup>21</sup>To be precise, divergence occurs for all trend functions such that  $\sqrt{T}G_D$  diverges.

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m} &= \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t \\
&\quad - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} D'_t \left( \sum_{i=1}^{[mT]} D_i D'_i \right)^{-1} \sum_{i=1}^{[mT]} D_i u_i \\
&\quad + \frac{1}{T} \sum_{t=[rT]+1}^{[sT]} D'_t G_D^{-1'} \Delta_\theta.
\end{aligned} \tag{56}$$

Under the stated assumptions the limit is given by

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m} \Rightarrow \omega^{1/2} \widehat{W}_m(s) + \int_r^s D(z)' dz \Delta_\theta, \tag{57}$$

from which the result follows. ■

*Proof of Lemmata 2, 3, 4 and 5:*

These lemmata all follow immediately from the convergence properties of the underlying estimation methods in conjunction with consistent long-run variance estimation and the continuous mapping theorem. ■

*Proof of Proposition 4:*

The argument is similar to the argument used in Proposition 1 and follows from the FCLT provided in Vogelsang and Wagner (2014, Lemma 2). ■

*Proof of Proposition 5:*

(a): Again, the limiting behavior of the partial sum process of the residuals is the key to the result, where we have to now distinguish between two cases, FM-OLS and D-OLS estimation on the one hand and IM-OLS estimation on the other. Again we outline the arguments here for the FM-OLS estimator, noting that the limit process is similar for D-OLS estimation, and the IM-OLS estimator.

For FM-OLS, the partial sum process of the residuals is given by (again for  $1 \geq s > r \geq m$ ):

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m}^+ &= \frac{1}{\sqrt{T}} \sum_{t=1}^{[rT]} \hat{u}_{t,m}^+ + \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} \hat{u}_{t,m}^+ \\
&= \frac{1}{\sqrt{T}} \sum_{t=1}^{[rT]} \hat{u}_{t,m}^+ + \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} u_t - \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} v_t' \hat{\Omega}_{vv}^{-1} \hat{\Omega}_{vu} - \\
&\quad - \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} D_t' (\hat{\theta}_{D,m} - \theta_D) - \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} X_t' (\hat{\theta}_{X,m} - \theta_X).
\end{aligned} \tag{58}$$

The first term above converges to  $\omega_{u.v}^{1/2} \widehat{W}_{u.v}(r)$ , according to Lemma 2 and the second term diverges since for the sample period considered  $u_t$  is an I(1) process. The remaining three terms converge in distribution. Thus, the partial sum process is in this case  $O_p(T)$ . The argument is analogous for the IM-OLS partial sum process, i.e. for  $\hat{S}_{t,m}^u$ , with Lemma 4 replacing Lemma 2.

(b): Also for this result similar arguments as in Proposition 2 apply. In particular it follows now for FM-OLS that (for  $1 \geq s > r \geq m$ ):

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m}^+ \Rightarrow \omega_{u.v} \widehat{W}_{u.v}(s) + \delta \omega_{\xi}^{1/2} \int_r^s (W_{\xi}(z) - W_{\xi}(r)) dr. \tag{59}$$

Thus, the same argument as in Proposition 2 applies. For IM-OLS the result is of the same type and given by

$$\frac{1}{\sqrt{T}} \hat{S}_{[sT],m}^u \Rightarrow \omega_{u.v}^{1/2} \tilde{P}_m(s) + \delta \omega_{\xi}^{1/2} \int_r^s (W_{\xi}(z) - W_{\xi}(r)) dr. \tag{60}$$

The results follow, since the identical second term can be again made arbitrarily large by choice of  $\delta$  for both cases. ■

*Proof of Proposition 6:*

The arguments are similar to the arguments in Proposition 3, with the only difference

being the exact form of the terms, due to the presence of integrated regressors  $X_t$ . For brevity we again just look at the FM-OLS expressions and note again that the result for D-OLS coincides asymptotically.

(a): Under the fixed alternative, the partial sum process of the FM-OLS residuals, again for  $1 \geq s > r \geq m$ , can be written as:

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m}^+ &= \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} v_t' \hat{\Omega}_{vv}^{-1} \hat{\Omega}_{vu} \\ &\quad - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} D_t' (\hat{\theta}_{D,m} - \theta_{D,1}) - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} X_t' (\hat{\theta}_{X,m} - \theta_X) \\ &\quad - \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} D_t' (\theta_{D,1} - \theta_{D,2}). \end{aligned} \quad (61)$$

From this expression it is seen that the result is similar to the result in Proposition 3, since the last term diverges and all other terms together converge to  $\omega_{u,v}^{1/2} \widehat{W}_{u,v}(s)$ .

(b): Also here the result is analogous, with the only difference being that under local alternatives the last term in (61) changes, as in part (b) of Proposition 3, and is instead given by

$$\frac{1}{T} \sum_{t=[rT]+1}^{[sT]} D_t' G_D^{-1'} \Delta_\theta \Rightarrow \int_r^s D(z)' dz \Delta_\theta \quad (62)$$

This shows the result as under the considered local alternatives it holds that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_{t,m}^+ \Rightarrow \omega_{u,v}^{1/2} W_{u,v}(s) + \int_r^s D(z)' dz \Delta_\theta. \quad (63)$$

The arguments are analogous for IM-OLS with the corresponding limit of the residual process given by

$$\frac{1}{\sqrt{T}} \hat{S}_{[sT],m}^u \Rightarrow \omega_{u,v}^{1/2} \tilde{P}_m(s) + \int_0^s D(z)' dz \Delta_\theta. \quad (64)$$

Note that for the IM-OLS result the integral in the second term is over the interval  $[0, s]$  rather than over  $[r, s]$ . This stems from the partial summing of the regressors in the IM regression, see also the proof of Proposition 7 where this effect is displayed in more detail for the case of slope breaks.  $\blacksquare$

*Proof of Proposition 7:*

The proof is analogous to the proof of Proposition 5 with the changes following from the discussion before Proposition 7.

(a): For FM-OLS we get for  $1 \geq s > r \geq m$ :

$$\begin{aligned} \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m}^+ &= \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} v_t' \hat{\Omega}_{vv}^{-1} \hat{\Omega}_{vu} \\ &\quad - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} D_t' (\hat{\theta}_{D,m} - \theta_{D,1}) - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} X_t' (\hat{\theta}_{X,m} - \theta_X) \\ &\quad - \frac{1}{\sqrt{T}} \sum_{t=[rT]+1}^{[sT]} X_t' (\theta_{D,1} - \theta_{D,2}). \end{aligned} \quad (65)$$

The first four terms together converge to  $\omega_{u,v}^{1/2} \widehat{W}_{u,v}(s)$  and the last term is  $O_p(T)$ .

For IM-OLS we get for  $1 \geq s > r \geq m$ :

$$\begin{aligned} \frac{1}{\sqrt{T}} \hat{S}_{[sT],m}^u &= \frac{1}{\sqrt{T}} S_{[sT]}^u - \frac{1}{\sqrt{T}} S_{[sT]}^{D'} (\hat{\theta}_{D,m} - \theta_D) - \frac{1}{\sqrt{T}} S_{[sT]}^{X'} (\hat{\theta}_{X,m} - \theta_{X,1}) \\ &\quad - \frac{1}{\sqrt{T}} X'_{[sT]} (\hat{\gamma}_m - \gamma) - \frac{1}{\sqrt{T}} S_{[sT]}^{X'} (\theta_{X,1} - \theta_{X,2}), \end{aligned} \quad (66)$$

from which it follows that the first four terms together converge to  $\omega_{u,v}^{1/2} \tilde{P}_m(s)$  and the last term is  $O_p(T)$ .

(b): The changes implied by the local alternatives considered are similar to the previous



propositions and we obtain for FM-OLS for  $1 \geq s > r \geq m$ :

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} \hat{u}_{t,m}^+ &= \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} u_t - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} v_t' \hat{\Omega}_{vv}^{-1} \hat{\Omega}_{vu} \\
&\quad - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} D_t' (\hat{\theta}_{D,m} - \theta_{D,1}) - \frac{1}{\sqrt{T}} \sum_{t=1}^{[sT]} X_t' (\hat{\theta}_{X,m} - \theta_X) \\
&\quad + \frac{1}{T^{3/2}} \sum_{t=[rT]+1}^{[sT]} X_t' \Delta_\theta \\
&\Rightarrow \omega_{u \cdot v}^{1/2} \widehat{W}_{u \cdot v}(s) + \int_r^s W_v(z)' dz \Omega_{vv}^{1/2'} \Delta_\theta.
\end{aligned} \tag{67}$$

The result for IM-OLS follows analogously to the result for FM-OLS from the result in part (a) and we obtain for  $1 \geq s > r \geq m$ :

$$\frac{1}{\sqrt{T}} \hat{S}_{[sT],m}^{u \cdot} \Rightarrow \omega_{u \cdot v}^{1/2} \tilde{P}_m(s) + \int_0^s W_v(z)' dz \Omega_{vv}^{1/2'} \Delta_\theta, \tag{68}$$

since under the considered local alternative the last term from (66) is given by  $T^{-3/2} S_{[sT]}^{X'} \Delta_\theta$ .

## B. CRITICAL VALUES

$m$	90%	95%	97.5%	99%	$m$	90%	95%	97.5%	99%
0.10	8.41	11.37	14.54	18.94	0.51	0.79	1.01	1.24	1.55
0.11	7.50	10.10	12.94	16.84	0.52	0.76	0.98	1.20	1.51
0.12	6.73	9.08	11.61	15.09	0.53	0.74	0.95	1.17	1.46
0.13	6.08	8.21	10.46	13.62	0.54	0.72	0.92	1.13	1.42
0.14	5.54	7.45	9.50	12.36	0.55	0.69	0.90	1.10	1.38
0.15	5.06	6.81	8.66	11.29	0.56	0.67	0.87	1.08	1.35
0.16	4.65	6.24	7.94	10.33	0.57	0.66	0.85	1.05	1.32
0.17	4.29	5.76	7.34	9.53	0.58	0.64	0.83	1.02	1.29
0.18	3.98	5.33	6.77	8.80	0.59	0.62	0.81	1.00	1.27
0.19	3.70	4.94	6.28	8.15	0.60	0.61	0.79	0.98	1.24
0.20	3.44	4.59	5.83	7.56	0.61	0.59	0.77	0.96	1.22
0.21	3.22	4.29	5.44	7.04	0.62	0.58	0.76	0.94	1.20
0.22	3.01	4.01	5.07	6.56	0.63	0.56	0.74	0.92	1.18
0.23	2.83	3.76	4.74	6.13	0.64	0.55	0.73	0.91	1.15
0.24	2.66	3.52	4.44	5.73	0.65	0.54	0.71	0.89	1.14
0.25	2.50	3.31	4.17	5.37	0.66	0.53	0.70	0.88	1.12
0.26	2.36	3.11	3.92	5.05	0.67	0.52	0.69	0.86	1.10
0.27	2.23	2.94	3.69	4.75	0.68	0.51	0.68	0.85	1.09
0.28	2.11	2.78	3.48	4.48	0.69	0.50	0.67	0.84	1.07
0.29	2.00	2.63	3.29	4.22	0.70	0.50	0.66	0.83	1.05
0.30	1.90	2.49	3.11	3.98	0.71	0.49	0.65	0.81	1.04
0.31	1.81	2.36	2.94	3.76	0.72	0.48	0.64	0.80	1.02
0.32	1.72	2.24	2.79	3.56	0.73	0.47	0.63	0.79	1.01
0.33	1.64	2.13	2.65	3.37	0.74	0.47	0.62	0.78	1.00
0.34	1.56	2.03	2.52	3.20	0.75	0.46	0.61	0.77	0.99
0.35	1.49	1.93	2.40	3.03	0.76	0.45	0.60	0.76	0.97
0.36	1.42	1.85	2.28	2.89	0.77	0.45	0.60	0.75	0.96
0.37	1.36	1.76	2.18	2.75	0.78	0.44	0.59	0.74	0.95
0.38	1.30	1.68	2.08	2.62	0.79	0.44	0.58	0.73	0.94
0.39	1.25	1.61	1.98	2.49	0.80	0.43	0.57	0.72	0.92
0.40	1.19	1.54	1.90	2.38	0.81	0.43	0.57	0.71	0.91
0.41	1.15	1.47	1.82	2.27	0.82	0.42	0.56	0.70	0.90
0.42	1.10	1.42	1.74	2.18	0.83	0.42	0.55	0.70	0.89
0.43	1.06	1.36	1.67	2.09	0.84	0.41	0.55	0.69	0.88
0.44	1.02	1.30	1.60	2.00	0.85	0.41	0.54	0.68	0.87
0.45	0.98	1.25	1.54	1.92	0.86	0.40	0.54	0.67	0.86
0.46	0.94	1.21	1.48	1.85	0.87	0.40	0.53	0.66	0.85
0.47	0.91	1.16	1.43	1.78	0.88	0.39	0.52	0.66	0.84
0.48	0.87	1.12	1.38	1.72	0.89	0.39	0.52	0.65	0.83
0.49	0.85	1.09	1.33	1.66	0.90	0.39	0.51	0.64	0.82
0.50	0.82	1.05	1.28	1.60					

Table 2: Critical values for stationarity monitoring for the intercept case.

$m$	90%	95%	97.5%	99%	$m$	90%	95%	97.5%	99%
0.10	1252.59	1777.80	2327.20	3066.19	0.51	2.36	3.32	4.33	5.73
0.11	916.45	1300.66	1702.68	2242.99	0.52	2.13	2.99	3.90	5.15
0.12	685.24	977.06	1277.92	1684.26	0.53	1.92	2.70	3.52	4.64
0.13	526.01	748.01	978.33	1292.92	0.54	1.74	2.43	3.18	4.19
0.14	409.41	580.95	760.56	1003.24	0.55	1.58	2.20	2.86	3.78
0.15	323.20	459.38	599.14	792.11	0.56	1.43	1.99	2.59	3.42
0.16	259.06	366.74	480.88	634.05	0.57	1.30	1.79	2.33	3.08
0.17	209.66	297.55	388.27	512.91	0.58	1.18	1.62	2.11	2.77
0.18	171.47	243.49	318.84	421.52	0.59	1.07	1.47	1.90	2.50
0.19	141.61	200.86	263.26	347.20	0.60	0.98	1.32	1.71	2.25
0.20	117.97	167.16	218.50	287.23	0.61	0.89	1.20	1.55	2.04
0.21	99.05	140.26	183.14	241.95	0.62	0.81	1.09	1.40	1.83
0.22	83.56	118.52	155.38	204.50	0.63	0.74	0.99	1.26	1.66
0.23	71.00	100.76	131.56	173.22	0.64	0.68	0.90	1.14	1.49
0.24	60.64	85.92	112.21	148.82	0.65	0.63	0.82	1.03	1.34
0.25	51.96	73.73	96.37	126.62	0.66	0.58	0.75	0.94	1.21
0.26	44.71	63.46	82.84	109.29	0.67	0.53	0.68	0.85	1.09
0.27	38.77	55.02	71.90	94.71	0.68	0.49	0.63	0.77	0.98
0.28	33.62	47.80	62.36	82.22	0.69	0.46	0.58	0.71	0.89
0.29	29.32	41.64	54.36	71.81	0.70	0.42	0.53	0.65	0.81
0.30	25.61	36.39	47.56	62.62	0.71	0.40	0.49	0.60	0.74
0.31	22.45	31.85	41.67	54.92	0.72	0.37	0.46	0.55	0.68
0.32	19.74	27.97	36.70	48.32	0.73	0.35	0.43	0.51	0.63
0.33	17.42	24.64	32.24	42.61	0.74	0.32	0.40	0.48	0.58
0.34	15.36	21.83	28.51	37.57	0.75	0.30	0.38	0.45	0.55
0.35	13.60	19.30	25.22	33.24	0.76	0.29	0.35	0.42	0.51
0.36	12.09	17.13	22.40	29.55	0.77	0.27	0.34	0.40	0.49
0.37	10.73	15.21	19.88	26.23	0.78	0.26	0.32	0.38	0.46
0.38	9.57	13.54	17.68	23.33	0.79	0.25	0.30	0.36	0.44
0.39	8.53	12.07	15.79	20.76	0.80	0.23	0.29	0.35	0.42
0.40	7.61	10.77	14.08	18.56	0.81	0.22	0.28	0.33	0.41
0.41	6.80	9.61	12.57	16.63	0.82	0.22	0.27	0.32	0.39
0.42	6.09	8.61	11.25	14.92	0.83	0.21	0.26	0.31	0.38
0.43	5.46	7.75	10.09	13.32	0.84	0.20	0.25	0.30	0.36
0.44	4.91	6.96	9.06	11.96	0.85	0.19	0.24	0.29	0.35
0.45	4.41	6.24	8.13	10.71	0.86	0.19	0.23	0.28	0.34
0.46	3.97	5.61	7.33	9.67	0.87	0.18	0.22	0.27	0.33
0.47	3.57	5.05	6.61	8.73	0.88	0.17	0.22	0.26	0.32
0.48	3.21	4.55	5.96	7.87	0.89	0.17	0.21	0.25	0.31
0.49	2.90	4.10	5.36	7.04	0.90	0.16	0.20	0.24	0.30
0.50	2.61	3.70	4.81	6.34					

Table 3: Critical values for stationarity monitoring for the intercept and linear trend case.

$m$	90%	95%	97.5%	99%	$m$	90%	95%	97.5%	99%
0.10	155.37	296.84	502.54	902.43	0.51	1.33	2.14	3.25	5.25
0.11	124.53	238.91	403.91	723.99	0.52	1.22	1.96	2.97	4.76
0.12	101.35	193.34	330.54	589.85	0.53	1.13	1.79	2.70	4.31
0.13	83.57	159.61	269.82	483.63	0.54	1.05	1.63	2.45	3.90
0.14	69.99	133.19	224.75	401.60	0.55	0.97	1.50	2.23	3.53
0.15	58.93	111.90	188.55	333.77	0.56	0.91	1.37	2.03	3.19
0.16	50.22	94.76	160.22	283.59	0.57	0.84	1.26	1.84	2.87
0.17	42.90	81.12	136.42	240.58	0.58	0.79	1.16	1.67	2.62
0.18	36.94	69.60	117.13	207.64	0.59	0.73	1.07	1.53	2.35
0.19	32.13	60.17	101.28	178.28	0.60	0.69	0.99	1.40	2.14
0.20	28.06	52.42	87.90	154.94	0.61	0.64	0.92	1.28	1.93
0.21	24.62	45.98	76.70	134.76	0.62	0.61	0.86	1.18	1.74
0.22	21.67	40.30	67.25	118.09	0.63	0.57	0.80	1.08	1.59
0.23	19.16	35.53	59.29	103.42	0.64	0.54	0.75	1.00	1.45
0.24	17.00	31.43	51.89	90.86	0.65	0.51	0.70	0.93	1.33
0.25	15.13	27.87	46.01	79.96	0.66	0.48	0.66	0.87	1.21
0.26	13.49	24.75	40.78	70.51	0.67	0.46	0.62	0.81	1.12
0.27	12.10	22.17	36.50	63.01	0.68	0.43	0.59	0.77	1.05
0.28	10.87	19.82	32.65	56.42	0.69	0.41	0.56	0.72	0.97
0.29	9.78	17.75	29.23	50.12	0.70	0.40	0.53	0.69	0.92
0.30	8.81	15.99	26.18	44.75	0.71	0.38	0.51	0.66	0.87
0.31	7.98	14.39	23.45	40.20	0.72	0.36	0.49	0.63	0.83
0.32	7.21	12.98	21.05	36.15	0.73	0.35	0.47	0.61	0.80
0.33	6.51	11.68	18.94	32.28	0.74	0.34	0.46	0.58	0.77
0.34	5.91	10.55	17.09	29.00	0.75	0.33	0.44	0.57	0.75
0.35	5.37	9.54	15.39	26.04	0.76	0.32	0.43	0.55	0.73
0.36	4.88	8.65	13.87	23.28	0.77	0.31	0.42	0.54	0.71
0.37	4.45	7.83	12.57	21.05	0.78	0.31	0.41	0.53	0.70
0.38	4.06	7.08	11.37	19.04	0.79	0.30	0.40	0.52	0.69
0.39	3.70	6.45	10.30	17.28	0.80	0.29	0.40	0.51	0.68
0.40	3.38	5.85	9.31	15.62	0.81	0.29	0.39	0.51	0.67
0.41	3.10	5.33	8.44	14.15	0.82	0.29	0.39	0.50	0.66
0.42	2.82	4.85	7.69	12.79	0.83	0.28	0.38	0.49	0.65
0.43	2.59	4.42	6.96	11.56	0.84	0.28	0.38	0.49	0.64
0.44	2.38	4.03	6.33	10.47	0.85	0.27	0.37	0.48	0.64
0.45	2.18	3.67	5.74	9.50	0.86	0.27	0.37	0.48	0.63
0.46	2.00	3.36	5.22	8.63	0.87	0.27	0.36	0.47	0.62
0.47	1.84	3.06	4.75	7.78	0.88	0.27	0.36	0.46	0.62
0.48	1.70	2.80	4.33	7.06	0.89	0.26	0.36	0.46	0.61
0.49	1.56	2.55	3.93	6.38	0.90	0.26	0.35	0.45	0.60
0.50	1.44	2.34	3.57	5.80					

Table 4: Critical values for FM-OLS & D-OLS with one regressor for the intercept case.

$m$	90%	95%	97.5%	99%	$m$	90%	95%	97.5%	99%
0.10	3205.88	5325.61	8212.84	13425.47	0.51	4.42	6.85	9.95	15.33
0.11	2338.95	3866.29	5982.86	9753.77	0.52	3.93	6.10	8.85	13.59
0.12	1741.12	2875.63	4417.46	7231.43	0.53	3.51	5.44	7.88	12.07
0.13	1320.81	2181.13	3363.93	5486.46	0.54	3.14	4.85	6.99	10.67
0.14	1023.54	1683.19	2581.31	4222.53	0.55	2.81	4.32	6.21	9.50
0.15	803.93	1323.84	2030.97	3293.20	0.56	2.50	3.83	5.50	8.43
0.16	638.34	1048.89	1610.98	2613.38	0.57	2.23	3.42	4.90	7.44
0.17	514.48	845.20	1296.56	2104.27	0.58	1.99	3.05	4.36	6.58
0.18	417.80	687.25	1054.12	1711.22	0.59	1.78	2.72	3.88	5.83
0.19	343.32	562.83	860.87	1399.28	0.60	1.59	2.43	3.44	5.15
0.20	283.73	464.96	709.89	1149.95	0.61	1.42	2.15	3.05	4.57
0.21	236.56	386.21	591.37	954.52	0.62	1.26	1.91	2.71	4.06
0.22	198.19	323.69	494.83	800.37	0.63	1.12	1.69	2.39	3.56
0.23	167.04	272.48	416.71	676.20	0.64	1.00	1.50	2.11	3.15
0.24	141.49	230.35	351.74	568.17	0.65	0.89	1.33	1.87	2.78
0.25	120.65	195.58	297.08	481.41	0.66	0.79	1.18	1.66	2.45
0.26	102.92	166.91	253.84	409.60	0.67	0.71	1.04	1.46	2.15
0.27	88.33	142.87	217.42	349.75	0.68	0.63	0.92	1.29	1.89
0.28	76.20	123.56	186.78	300.96	0.69	0.56	0.81	1.13	1.66
0.29	65.95	106.60	161.16	259.16	0.70	0.51	0.72	0.99	1.45
0.30	57.22	92.54	139.93	223.76	0.71	0.46	0.64	0.87	1.26
0.31	49.82	80.61	121.45	194.05	0.72	0.41	0.57	0.76	1.10
0.32	43.65	70.07	105.78	168.11	0.73	0.37	0.51	0.67	0.95
0.33	38.07	61.16	91.95	146.09	0.74	0.34	0.45	0.59	0.82
0.34	33.36	53.51	80.53	128.04	0.75	0.31	0.41	0.52	0.72
0.35	29.27	46.81	70.38	111.98	0.76	0.28	0.37	0.47	0.63
0.36	25.70	41.15	61.75	98.46	0.77	0.26	0.33	0.42	0.55
0.37	22.72	36.37	54.33	86.55	0.78	0.24	0.31	0.38	0.49
0.38	20.04	32.00	47.60	75.75	0.79	0.23	0.28	0.35	0.44
0.39	17.75	28.28	42.28	66.64	0.80	0.21	0.26	0.32	0.40
0.40	15.70	25.03	37.47	58.94	0.81	0.20	0.25	0.30	0.37
0.41	13.92	22.13	32.91	51.92	0.82	0.19	0.23	0.28	0.35
0.42	12.37	19.60	29.07	45.88	0.83	0.18	0.22	0.26	0.33
0.43	11.01	17.40	25.75	40.46	0.84	0.17	0.21	0.25	0.31
0.44	9.82	15.47	22.79	35.75	0.85	0.16	0.20	0.24	0.30
0.45	8.74	13.72	20.29	31.72	0.86	0.16	0.19	0.23	0.29
0.46	7.77	12.19	17.92	28.27	0.87	0.15	0.19	0.22	0.28
0.47	6.92	10.86	15.93	24.84	0.88	0.14	0.18	0.22	0.27
0.48	6.20	9.68	14.21	22.00	0.89	0.14	0.17	0.21	0.26
0.49	5.54	8.63	12.61	19.58	0.90	0.14	0.17	0.20	0.25
0.50	4.95	7.67	11.22	17.35					

Table 5: Critical values for FM-OLS & D-OLS with one regressor for the intercept and linear trend case.

$m$	90%	95%	97.5%	99%	$m$	90%	95%	97.5%	99%
0.10	302.01	616.19	1123.15	2164.76	0.51	2.51	4.47	7.37	13.14
0.11	241.96	493.68	892.29	1746.33	0.52	2.31	4.09	6.72	11.88
0.12	196.96	401.15	727.88	1416.39	0.53	2.12	3.73	6.11	10.80
0.13	163.30	331.04	598.38	1168.42	0.54	1.95	3.42	5.57	9.79
0.14	136.16	275.69	499.38	964.93	0.55	1.79	3.13	5.07	8.91
0.15	114.69	232.45	417.34	807.86	0.56	1.65	2.86	4.62	8.08
0.16	97.31	196.84	354.39	684.46	0.57	1.51	2.61	4.20	7.29
0.17	83.27	167.74	301.22	577.64	0.58	1.39	2.38	3.82	6.59
0.18	71.82	144.46	258.47	495.75	0.59	1.27	2.18	3.47	5.96
0.19	62.20	124.50	222.98	430.48	0.60	1.17	1.99	3.15	5.39
0.20	54.42	108.62	193.99	371.84	0.61	1.07	1.81	2.85	4.90
0.21	47.71	94.86	169.98	324.57	0.62	0.98	1.65	2.59	4.44
0.22	41.90	83.56	149.20	284.74	0.63	0.90	1.50	2.36	4.01
0.23	37.10	73.76	131.23	248.97	0.64	0.82	1.37	2.14	3.61
0.24	32.97	65.32	115.76	220.59	0.65	0.75	1.25	1.94	3.25
0.25	29.34	57.81	102.68	195.35	0.66	0.69	1.13	1.76	2.93
0.26	26.15	51.51	91.22	173.74	0.67	0.63	1.03	1.59	2.63
0.27	23.37	45.91	81.20	154.67	0.68	0.57	0.93	1.43	2.37
0.28	20.96	41.03	72.56	137.34	0.69	0.52	0.84	1.29	2.11
0.29	18.88	36.74	64.76	122.15	0.70	0.47	0.76	1.16	1.90
0.30	17.01	33.02	57.93	108.59	0.71	0.43	0.69	1.04	1.70
0.31	15.35	29.73	52.00	97.75	0.72	0.39	0.62	0.93	1.51
0.32	13.90	26.90	46.73	87.99	0.73	0.35	0.56	0.84	1.35
0.33	12.60	24.30	42.13	78.89	0.74	0.32	0.50	0.75	1.20
0.34	11.41	21.97	38.08	70.78	0.75	0.29	0.45	0.67	1.06
0.35	10.38	19.85	34.37	63.50	0.76	0.26	0.40	0.59	0.94
0.36	9.44	18.01	31.16	57.60	0.77	0.23	0.36	0.53	0.83
0.37	8.59	16.31	28.20	51.74	0.78	0.21	0.32	0.47	0.73
0.38	7.82	14.79	25.47	46.81	0.79	0.19	0.29	0.41	0.64
0.39	7.13	13.43	23.03	42.45	0.80	0.18	0.26	0.37	0.56
0.40	6.51	12.21	20.90	38.47	0.81	0.16	0.23	0.32	0.49
0.41	5.96	11.12	18.96	34.86	0.82	0.15	0.21	0.29	0.43
0.42	5.46	10.14	17.22	31.48	0.83	0.14	0.19	0.25	0.37
0.43	4.99	9.23	15.63	28.61	0.84	0.13	0.17	0.23	0.33
0.44	4.57	8.42	14.21	25.91	0.85	0.12	0.16	0.21	0.29
0.45	4.19	7.68	12.96	23.53	0.86	0.11	0.15	0.19	0.26
0.46	3.85	7.00	11.78	21.32	0.87	0.11	0.14	0.18	0.24
0.47	3.53	6.41	10.75	19.38	0.88	0.10	0.13	0.17	0.22
0.48	3.24	5.86	9.79	17.55	0.89	0.10	0.13	0.16	0.21
0.49	2.97	5.35	8.91	15.96	0.90	0.10	0.12	0.16	0.20
0.50	2.73	4.89	8.11	14.45					

Table 6: Critical values for IM-OLS with one regressor for the intercept case.

$m$	90%	95%	97.5%	99%	$m$	90%	95%	97.5%	99%
0.10	5714.91	10161.34	16644.37	29142.29	0.51	7.86	12.94	20.08	33.62
0.11	4152.40	7378.48	12117.97	21252.75	0.52	7.03	11.55	17.83	29.76
0.12	3089.93	5464.02	8971.08	15675.90	0.53	6.29	10.31	15.89	26.53
0.13	2342.37	4146.13	6756.60	11835.05	0.54	5.62	9.21	14.14	23.58
0.14	1806.26	3197.50	5215.71	9125.17	0.55	5.03	8.21	12.61	20.91
0.15	1412.96	2501.02	4084.95	7165.94	0.56	4.50	7.33	11.27	18.47
0.16	1120.95	1980.70	3244.39	5689.81	0.57	4.03	6.54	10.03	16.43
0.17	902.07	1587.83	2604.16	4539.43	0.58	3.60	5.83	8.91	14.62
0.18	734.42	1290.14	2106.33	3677.08	0.59	3.21	5.21	7.94	12.98
0.19	602.14	1056.42	1729.21	2995.80	0.60	2.87	4.65	7.07	11.53
0.20	497.42	873.46	1420.37	2473.81	0.61	2.57	4.15	6.28	10.22
0.21	413.63	724.41	1179.08	2052.30	0.62	2.30	3.69	5.58	9.02
0.22	346.15	604.26	988.04	1708.32	0.63	2.06	3.29	4.94	7.94
0.23	291.81	511.02	832.81	1445.81	0.64	1.84	2.93	4.38	7.03
0.24	247.93	432.50	702.19	1222.98	0.65	1.64	2.60	3.88	6.19
0.25	210.91	367.02	596.79	1036.96	0.66	1.46	2.31	3.44	5.45
0.26	180.55	314.53	509.13	880.85	0.67	1.30	2.05	3.04	4.80
0.27	154.91	269.85	435.13	749.23	0.68	1.15	1.81	2.68	4.22
0.28	133.47	231.82	373.85	642.20	0.69	1.02	1.61	2.36	3.71
0.29	115.45	200.84	322.62	557.63	0.70	0.91	1.42	2.08	3.26
0.30	100.24	173.63	279.36	481.82	0.71	0.81	1.25	1.83	2.85
0.31	87.31	150.80	243.53	417.67	0.72	0.71	1.10	1.61	2.49
0.32	76.18	131.32	211.84	362.84	0.73	0.63	0.97	1.41	2.17
0.33	66.58	114.85	184.51	316.22	0.74	0.55	0.85	1.23	1.89
0.34	58.44	100.30	161.13	276.70	0.75	0.49	0.75	1.08	1.64
0.35	51.38	87.90	140.91	242.11	0.76	0.42	0.65	0.94	1.42
0.36	45.19	77.32	123.37	211.84	0.77	0.37	0.57	0.81	1.23
0.37	39.92	68.12	108.66	186.27	0.78	0.32	0.49	0.70	1.06
0.38	35.23	60.01	95.72	164.54	0.79	0.28	0.43	0.61	0.91
0.39	31.21	52.92	84.43	144.50	0.80	0.24	0.37	0.52	0.78
0.40	27.67	46.93	74.67	127.68	0.81	0.21	0.32	0.45	0.66
0.41	24.54	41.62	65.99	112.02	0.82	0.18	0.27	0.38	0.56
0.42	21.81	36.86	58.36	99.02	0.83	0.16	0.23	0.32	0.47
0.43	19.37	32.76	51.63	87.88	0.84	0.14	0.20	0.27	0.39
0.44	17.26	29.06	45.94	78.07	0.85	0.12	0.17	0.23	0.33
0.45	15.39	25.83	40.86	69.45	0.86	0.11	0.15	0.19	0.27
0.46	13.71	23.00	36.30	61.27	0.87	0.10	0.13	0.17	0.23
0.47	12.26	20.53	32.13	54.29	0.88	0.09	0.12	0.15	0.19
0.48	10.97	18.24	28.52	47.92	0.89	0.09	0.11	0.13	0.17
0.49	9.82	16.25	25.36	42.53	0.90	0.08	0.10	0.12	0.15
0.50	8.77	14.50	22.56	37.91					

Table 7: Critical values for IM-OLS with one regressor for the intercept and linear case.



C. SUPPLEMENTARY MATERIAL: ADDITIONAL RESULTS (NOT FOR  
PUBLICATION)

Country	Maturity	Test statistic	$\tau_m(\alpha = 0.1)$	$\tau_m(\alpha = 0.05)$
Austria	1	4852.94	15.02.2010	25.02.2010
	3	4409.97	19.02.2010	04.03.2010
	5	3099.18	01.03.2010	17.03.2010
	7	2191.71	18.03.2010	20.04.2010
	10	1723.25	06.04.2010	18.05.2010
Cyprus	1	14751.56	11.03.2010	01.04.2010
	3	13686.61	04.03.2010	22.03.2010
	5	11018.63	10.03.2010	30.03.2010
	7	8978.85	16.03.2010	07.04.2010
	10	6431.65	25.03.2010	23.04.2010
France	1	4792.63	18.02.2010	25.02.2010
	3	7851.47	15.02.2010	19.02.2010
	5	7643.47	15.02.2010	22.02.2010
	7	5491.09	19.02.2010	01.03.2010
	10	3960.79	26.02.2010	10.03.2010
Germany	1	3994.32	23.02.2010	02.03.2010
	3	5026.07	17.02.2010	24.02.2010
	5	3462.37	25.02.2010	05.03.2010
	7	2345.01	08.03.2010	22.03.2010
	10	1962.23	01.04.2010	26.04.2010

Table 8: Results of stationarity monitoring for the daily CDS spreads data for  $m = 0.2$ . The third column displays  $\sup_{m \leq s \leq 1} \left| \frac{\hat{H}^m(s)}{w(s)} \right|$  and the fourth and fifth columns the associated detection times  $\tau_m(\hat{H}^m(s), w(s), c(\alpha, w))$  for  $\alpha = \{0.1, 0.05\}$ . Intercept and linear trend are included, hence  $w(s) = s^5$ . The null hypothesis is rejected throughout. The 10% critical value is 916.45 and the 5% critical value is 1300.66.

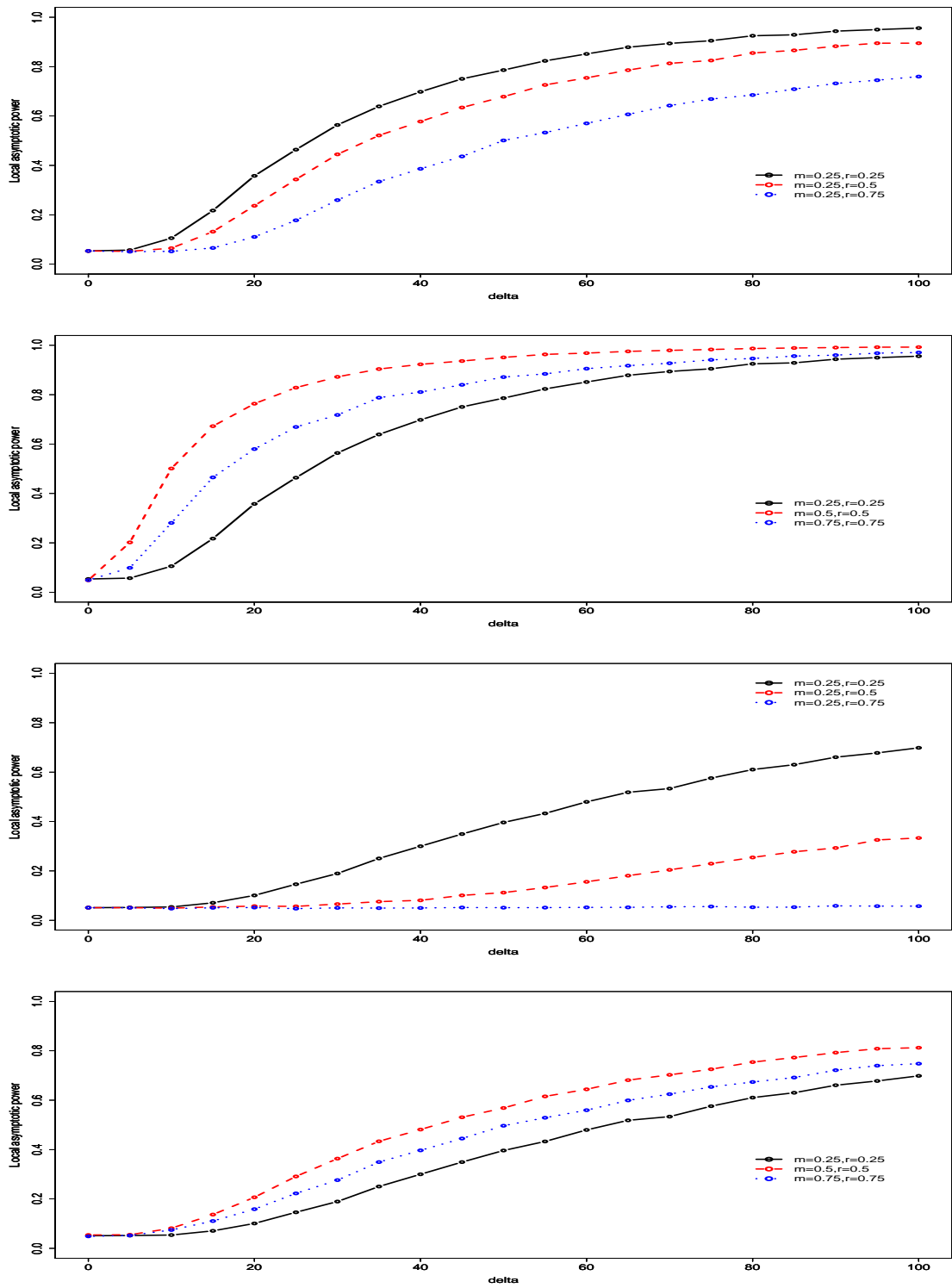


Figure 11: Local asymptotic power for monitoring cointegration for the case with intercept. The upper two plots correspond to FM-OLS & D-OLS and the lower two plots to IM-OLS. The plots show results for different combinations of  $m$  and  $r$ .

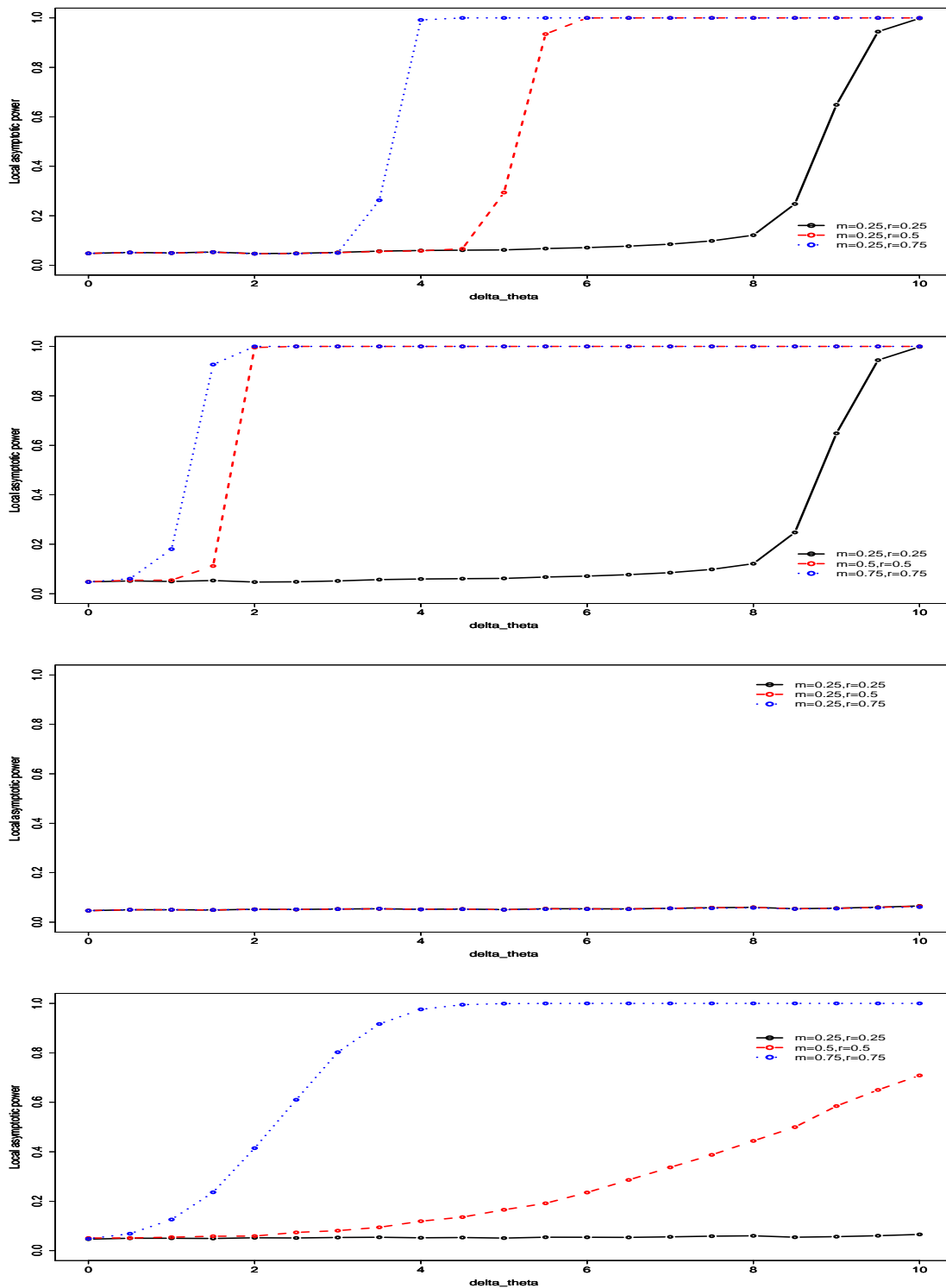


Figure 12: Local asymptotic power against break in trend parameter for monitoring cointegration for the case with intercept and linear trend. The upper two plots correspond to FM-OLS & D-OLS and the lower two plots to IM-OLS. The plots show results for different combinations of  $m$  and  $r$ .

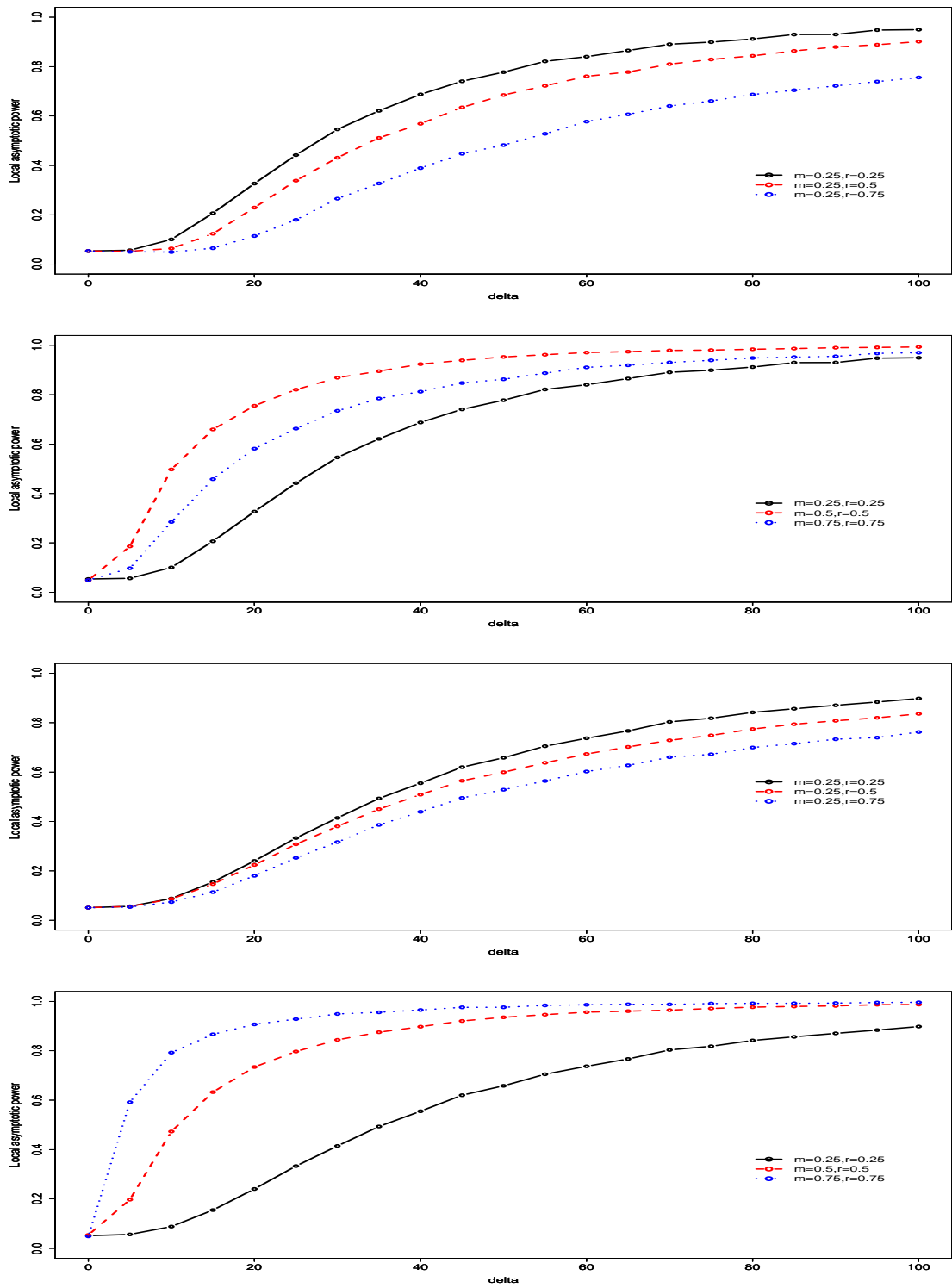


Figure 13: Local asymptotic power against break in slope parameter for monitoring cointegration for the case with intercept. The upper two plots correspond to FM-OLS & D-OLS and the lower two plots to IM-OLS. The plots show results for different combinations of  $m$  and  $r$ .

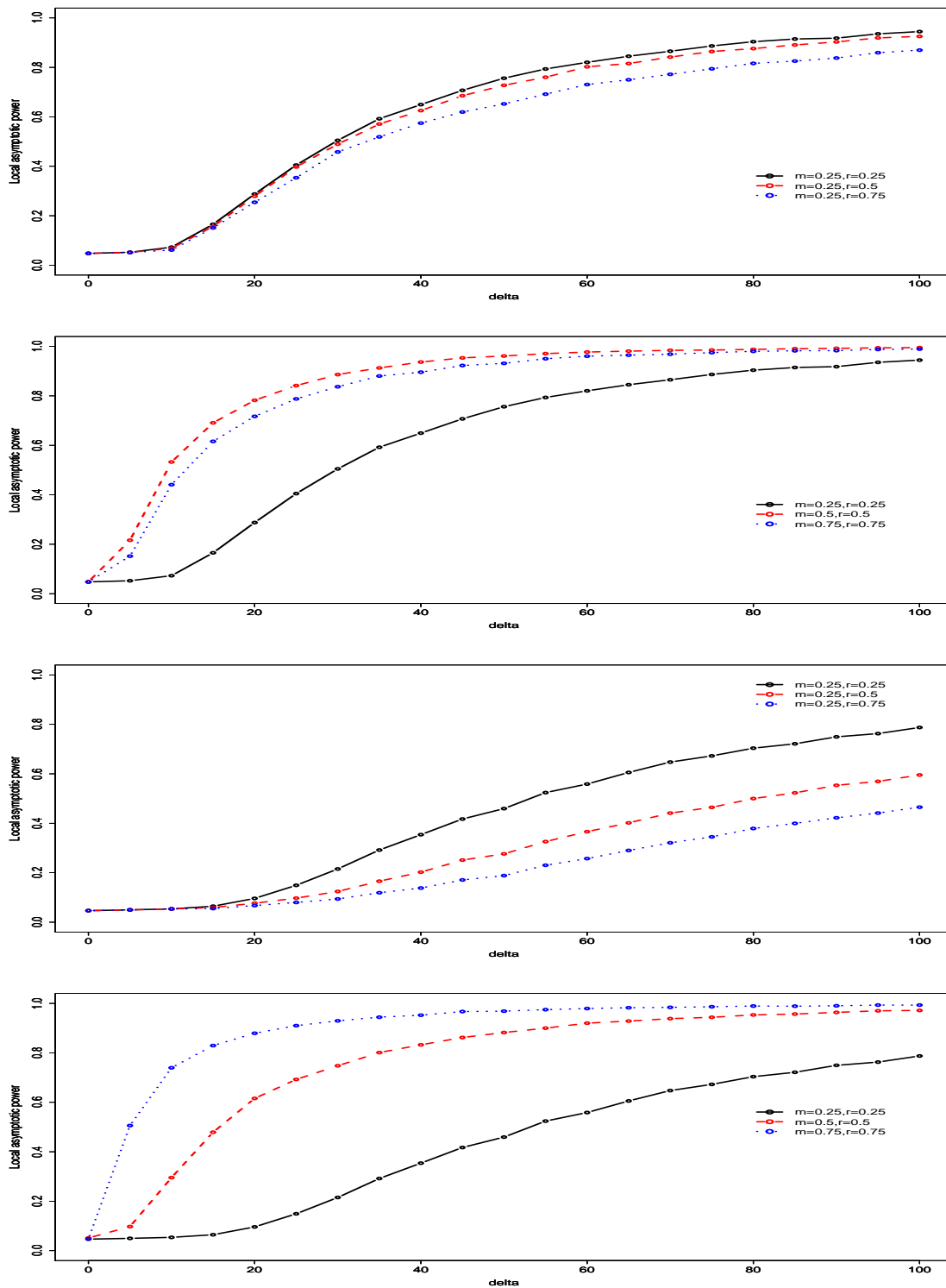


Figure 14: Local asymptotic power against break in slope parameter for monitoring cointegration for the case with intercept and linear trend. The upper two plots correspond to FM-OLS & D-OLS and the lower two plots to IM-OLS. The plots show results for different combinations of  $m$  and  $r$ .