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Decision making with Conditional Value-at-Risk and spectral risk measures: The problem of comparative risk aversion

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We analyze spectral risk measures with respect to comparative risk aversion following ARROW (1965) and PRATT (1964) on the one hand, and ROSS (1981) on the other hand. The implications for two standard financial decision problems, namely the willingness to pay for insurance and portfolio selection, are studied. Within the framework of Arrow and Pratt, we show that the widely-applied spectral Arrow-Pratt-measure is not a consistent measure of Arrow-Pratt-risk aversion. A decision maker with a greater spectral Arrow-Pratt-measure may only be willing to pay less for insurance or to invest more in the risky asset than a decision maker with a smaller spectral Arrow-Pratt-measure. We further show how a proper measure of Arrow-Pratt-risk aversion should look like instead. Within the framework of Ross, we show that the popular subclasses of Conditional Value-at-Risk, and exponential and power spectral risk measures cannot be completely ordered with respect to Ross-risk aversion. As a consequence, these subclasses also exhibit counter-intuitive comparative static results. In the insurance problem, the willingness to pay for insurance may be decreasing with increasing risk parameter. In the portfolio selection problem, the investment in the risky asset may be increasing with increasing risk parameter. These shortcomings have to be considered before spectral risk measures can be applied for the purpose of optimal decision making and regulatory issues.

JEL-classification: C44, D81, G11, G21

Keywords: Spectral risk measures, Conditional Value-at-Risk, Exponential spectral risk measures, Power spectral risk measures, Arrow-Pratt-risk aversion, Ross-risk aversion

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1. Introduction

The theory of risk measurement has faced a paradigm shift in recent times. Following an axiomatic – rather than an ad hoc – approach, ARTZNER ET AL. (1999) have proposed a set of properties that any so-called coherent risk measure in the field of bank regulation and solvency capital requirements should satisfy. Among them, the property of subadditivity has gained particular relevance, as it ensures that issues of diversification are adequately taken into account.

In the course, Conditional Value-at-Risk has been introduced as the most important representative of coherent risk measures (e.g., ACERBI/TASCHE (2002), ROCKAFELLAR/URYASEV (2002)), and has been canonically extended towards spectral risk measures (ACERBI (2002)). Since then, spectral risk measures have emerged as the most important subclass of coherent risk measures. Spectral risk measures are characterized by a so-called risk spectrum that assigns subjective weights to the quantiles of a profit and loss (P&L) distribution in order to represent subjective risk aversion. To this end, DOWD ET AL. (2008) have derived the specific subclasses of exponential and power spectral risk measures.

In the recent literature, spectral risk measures and their specific subclasses have also been applied to numerous fields that go beyond the determination of solvency capital requirements, such as portfolio selection (e.g., ADAM ET AL. (2008), BRANDTNER (2013)), the analysis of optimal (re-)insurance contracts (e.g., CAI ET AL. (2008)), and problems in operations management (e.g., JAMMERNEGG/KISCHKA (2007)), to name but a few. These recent applications extend the scope of spectral risk measures from pure regulation-based tools towards a “new” class of models of decision making. Hence, the analysis of spectral risk measures from the perspective of decision theory as, for example, addressing standard questions such as the measurement and the comparative statics of risk aversion, ought to become an increasingly relevant issue.

Unfortunately, that kind of analysis of “modern” spectral risk measures received only little attention so far. The lack of attention is at least unsatisfactory, if not dangerous, as we do not know under what conditions these “new” decision models provide reasonable predictions. It is well-known, for example, that even classical expected utility (EU-)theory may fail in providing consistent comparative static results in standard decision problems such as the willingness to pay for insurance or portfolio selection (e.g., KIHLMSTROM ET AL. (1981), ROSS (1981), and more recently, WILHELM (2008) and LIU/MEYER (2013)).

In this paper, we aim to fill this research gap by analyzing spectral risk measures, including their popular subclasses of Conditional Value-at-Risk, and exponential and power spectral risk measures, with respect to comparative risk aversion. Our contribution is twofold:

First, we argue within the classical ARROW (1965) and PRATT (1964) (AP) framework of (comparative) risk aversion, which has dominated the discussion of risk aversion in hundreds

of published papers of economic decision making in the EU-framework. Inspired by this classical (“old”) setting and the classical AP-measure of risk aversion, the so-called “spectral” AP-measure is regularly employed as a measure of AP-risk aversion in the literature on the “new” framework of spectral risk measures. In this paper, we show that the spectral AP-measure, contrary to intuition, is not a consistent measure of AP-risk aversion and how a proper definition should look like.

In terms of application, we demonstrate that the concept of AP-risk aversion is consistent with the two standard financial decision problems of the willingness to pay for insurance and the portfolio selection between a risk free and a risky asset. By contrast, applying the spectral AP-measure to these problems as is done in the literature fails: A decision maker with a greater spectral AP-measure may only be willing to pay less for insurance or to invest more in the risky asset than a decision maker with a smaller spectral AP-measure.

Second, we argue within the extended framework of ROSS (1981) (R), who has offered another concept of (comparative) risk aversion for a more general situation where the initial wealth is random and the risk of the final wealth can only be eliminated partly. Accordingly, this framework addresses the shift from “more” to “less” risk, which appears to be the more realistic setting, in particular from the point of view of the recent financial crises. We show that neither Conditional Value-at-Risk, nor exponential and power spectral risk measures can be completely ordered with respect to R-risk aversion. We further provide a general “destructive” result of non-consistency between spectral risk measures and R-risk aversion.

Thus, these popular subclasses of spectral risk measures exhibit counter-intuitive comparative static results with respect to risk aversion in the two standard financial decision problems: In the insurance problem, the willingness to pay for insurance may be decreasing with increasing risk parameter. Likewise, in the portfolio selection problem, the investment in the risky asset may be increasing with increasing risk parameter. Decision makers and regulators should be aware of these shortcomings before applying spectral risk measures.

The paper proceeds as follows. Section 2 introduces spectral risk measures and the specific subclasses under consideration. Section 3 addresses the concept of risk aversion following ARROW (1965) and PRATT (1964), while Section 4 addresses the concept of ROSS (1981). Section 5 concludes.

2. Setting: Spectral risk measures

Let \mathcal{X} denote the set of all real valued (P&L) random variables X on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $F_X(x) = F(x) = P(X \leq x)$ be the cumulative distribution function of X with corresponding quantile function $F_X^{-1}(p) = F^{-1}(p) = \sup\{x \in \mathbb{R} | F(x) < p\}$, $p \in (0, 1]$ and $F^{-1}(0) = \lim_{t \rightarrow 0^+} F^{-1}(t)$.

2.1. Properties and representation

Spectral risk measures originally have been introduced for determining solvency capital requirements in bank regulation. In order to satisfy this purpose adequately, they have to fulfill a set of properties (or axioms) (ACERBI (2002), ACERBI (2004), Proposition 3.26).

Definition 2.1. *A mapping $\rho_\phi : \mathcal{X} \rightarrow \mathbb{R}$ is called spectral risk measure if it satisfies the following properties for all $X, Y, Z \in \mathcal{X}$:*

- *Monotonicity:* $X \leq Y \Rightarrow \rho_\phi(X) \geq \rho_\phi(Y)$.
- *Translation invariance:* $\rho_\phi(X + c) = \rho_\phi(X) - c, c \in \mathbb{R}$.
- *Subadditivity:* $\rho_\phi(X + Y) \leq \rho_\phi(X) + \rho_\phi(Y)$.
- *Positive homogeneity:* $\rho_\phi(\lambda \cdot X) = \lambda \cdot \rho_\phi(X), \lambda \geq 0$.
- *Comonotonic additivity:* X, Y comonotonic (i.e., $\exists Z$ and g, h non-decreasing such that $X = g(Z)$ and $Y = h(Z)$) $\Rightarrow \rho_\phi(X + Y) = \rho_\phi(X) + \rho_\phi(Y)$.
- *Law invariance:* $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R} \Rightarrow \rho_\phi(X) = \rho_\phi(Y)$.

For a thorough discussion of these properties see ACERBI (2004). In the literature, the property of subadditivity has gained particular attention, as it serves as a preliminary that spectral risk measures may, at least principally, reflect desirable effects of diversification. Nevertheless, a counter-intuitive tendency towards corner solutions is inherent to spectral risk measures, as will be shown below. Accordingly, besides subadditivity, the linearity of spectral risk measures induced by the properties of translation invariance and positive homogeneity

$$\rho_\phi(\lambda \cdot X + c) = \lambda \cdot \rho_\phi(X) - c, c \in \mathbb{R}, \lambda \geq 0 \quad (1)$$

will become relevant for our analyses below.

Spectral risk measures have the following representation.

Theorem 2.2. *Any spectral risk measure $\rho_\phi : \mathcal{X} \rightarrow \mathbb{R}$ is of the form*

$$\rho_\phi(X) = - \int_0^1 F^{-1}(p) \cdot \phi(p) dp, \quad (2)$$

where the so-called risk spectrum $\phi : [0, 1] \rightarrow \mathbb{R}_+$ is a non-increasing density function.

For the proof, see ACERBI (2004), Proposition 3.4. Spectral risk measures through the risk spectrum ϕ assign subjective weights to the p -quantiles of a random variable X with

smaller quantiles receiving relatively greater weights and vice versa. The antiderivative of the risk spectrum $\phi(p)$ given by

$$\Phi(p) = \int_0^p \phi(t) dt \quad (3)$$

is a concave cumulative distribution function on $[0, 1]$ with $\Phi(0) = 0$, $\Phi(1) = 1$.

In the recent literature, spectral risk measures have also been applied to numerous fields beyond the determination of solvency capital requirements such as portfolio selection (e.g., ADAM ET AL. (2008), BRANDTNER (2013)), optimal (re-)insurance contracts (e.g., CAI ET AL. (2008)), and problems in operations management (e.g., JAMMERNEGG/KISCHKA (2007)), to name but a few.

These recent applications extend the scope of spectral risk measures from pure regulation-based tools of risk towards models of economic decision making that are used to find optimal solutions over a set of alternatives. In order to capture this new scope, we make use of the following notion.

Definition 2.3. *If a decision maker decides according to*

$$X \text{ is preferred to } Y \Leftrightarrow \rho_\phi(X) \leq \rho_\phi(Y), \quad (4)$$

he is called a spectral risk measure (SRM)-decision maker.

Within the extended scope of spectral risk measures as a means of modeling optimal decisions of SRM-decision makers, the tradeoff between risk and reward becomes an important issue. Spectral risk measures regularly represent an implicit risk-reward tradeoff (e.g., ACERBI/SIMONETTI (2002), p. 10). In order to make this tradeoff explicit, assume that the risk spectrum satisfies $\phi(1) > 0$, which is common in the literature (e.g., DOWD ET AL. (2008)), and is also satisfied for the specific subclasses of spectral risk measures that will be introduced below. Then the risk spectrum can be decomposed by

$$\phi(p) = \phi(1) \cdot 1 + (1 - \phi(1)) \cdot \hat{\phi}(p), \text{ where } \hat{\phi}(p) = \frac{\phi(p) - \phi(1)}{1 - \phi(1)}, \quad (5)$$

such that the corresponding spectral risk measure can be rewritten as

$$\rho_\phi(X) = -(\phi(1) \cdot E(X) - (1 - \phi(1)) \cdot \rho_{\hat{\phi}}(X)). \quad (6)$$

Thus, spectral risk measures can be denoted as a (negative) linear combination of the expected value, $E(\cdot)$, as a reward measure and another spectral measure, $\rho_{\hat{\phi}}$, that captures “pure” risk.

Remark 2.4. Note that a close relationship prevails between spectral risk measures and YAARI (1987)'s dual theory of choice (see also ROELL (1987) and DENNEBERG (1988)). The dual theory of choice requires the existence of some dual utility function $v : [0, 1] \rightarrow [0, 1]$, $v(0) = 0, v(1) = 1$ such that the decision maker's preference over a set of risky positions is measured by

$$D_v(X) = \int_0^1 F^{-1}(p)dv(p) = \int_{-\infty}^{\infty} xdv(F(x)). \quad (7)$$

If a decision maker decides according to “ X is preferred to $Y \Leftrightarrow D_v(X) \geq D_v(Y)$ ”, he is called a dual theory of choice (DT)-decision maker. The representations of spectral risk measures (2) and the dual theory of choice (7) coincide (up to the sign) if one identifies the antiderivative of the risk spectrum Φ with the dual utility function v , with the exception that the dual theory of choice is more general in that v not necessarily needs to be concave. This close relationship allows us to rely on previous results on comparative risk aversion derived for the dual theory of choice below.

2.2. Examples

The most popular spectral risk measure is Conditional Value-at-Risk (e.g., ACERBI/TASCHE (2002), ROCKAFELLAR/URYASEV (2002)). Here, at the confidence level $\alpha \in [0, 1]$, the risk spectrum and its antiderivative are given by

$$\phi_\alpha(p) = \begin{cases} \frac{1}{\alpha} & \text{for } p \in [0, \alpha) \\ 0 & \text{for } p \in [\alpha, 1] \end{cases} \quad \text{and} \quad \Phi_\alpha(p) = \begin{cases} \frac{1}{\alpha} \cdot p & \text{for } p \in [0, \alpha) \\ 1 & \text{for } p \in [\alpha, 1] \end{cases} \quad (8)$$

(see Figure 1). Conditional Value-at-Risk assigns a constant weight of $\frac{1}{\alpha}$ to the $\alpha \cdot 100\%$ smallest outcomes, while the greater outcomes are not taken into account. As Conditional Value-at-Risk does not assign a positive weight of $\phi_\alpha(1) > 0$ to the $p = 1$ -quantile, it initially does not represent a risk-reward tradeoff along the decomposition (6). However, Conditional Value-at-Risk can be extended towards a (spectral) risk-reward tradeoff by forming a negative convex combination of the random variable's mean and Conditional Value-at-Risk itself, viz

$$\rho_\phi(X) = -(\lambda \cdot E(X) - (1 - \lambda) \cdot CVaR_\alpha(X)); \quad (9)$$

see ACERBI/SIMONETTI (2002) or JAMMERNEGG/KISCHKA (2007) in more detail.

As an alternative to popular Conditional Value-at-Risk, DOWD ET AL. (2008) have proposed exponential spectral risk measures, which in the course have become a relevant subclass of its own (e.g., BARBI/ROMAGNOLI (2013), COTTER/DOWD (2010), DOWD/BLAKE

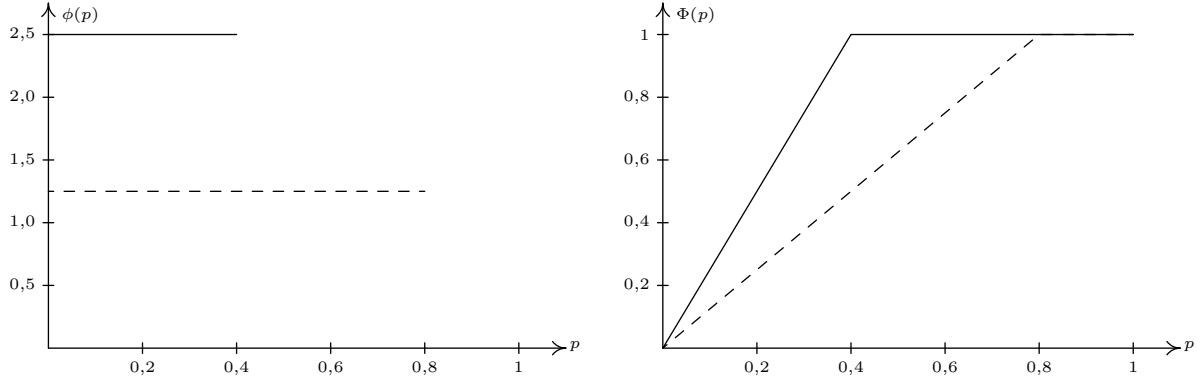


Figure 1: Conditional Value-at-Risk for $\alpha = 0,4$ (solid) and $\alpha = 0,8$ (dashed)

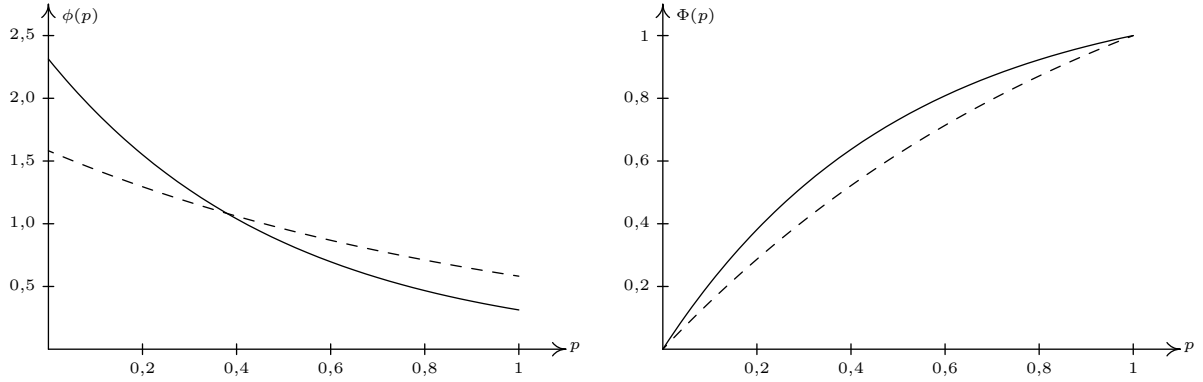


Figure 2: Exponential spectral risk measures for $a = 1$ (dashed) and $a = 2$ (solid)

(2006)). For exponential spectral risk measures, the risk spectrum and its antiderivative for $a > 0$ are given by

$$\phi_a(p) = \frac{a \cdot e^{-a \cdot p}}{1 - e^{-a}}, p \in [0, 1] \text{ and } \Phi_a(p) = \frac{1 - e^{-a \cdot p}}{1 - e^{-a}}, p \in [0, 1] \quad (10)$$

(see Figure 2). As $\phi_a(1) > 0$, exponential spectral risk measures represent an implicit risk-reward tradeoff as given by (6).

Moreover, DOWD ET AL. (2008) have introduced power spectral risk measures. Here, the risk spectrum and its antiderivative for $0 < b \leq 1$ are given by

$$\phi_b(p) = b \cdot p^{b-1}, p \in [0, 1] \text{ and } \Phi_b(p) = p^b, p \in [0, 1] \quad (11)$$

(see Figure 3). Again it holds that $\phi_b(1) = b > 0$, so for power spectral risk measures the risk-reward tradeoff (6) can be easily made explicit by

$$\rho_{\phi_b}(X) = -(b \cdot E(X) - (1 - b) \cdot \rho_{\hat{\phi}_b}(X)) \text{ with } \hat{\phi}_b(X) = \frac{b \cdot (p^{b-1} - 1)}{1 - b}. \quad (12)$$

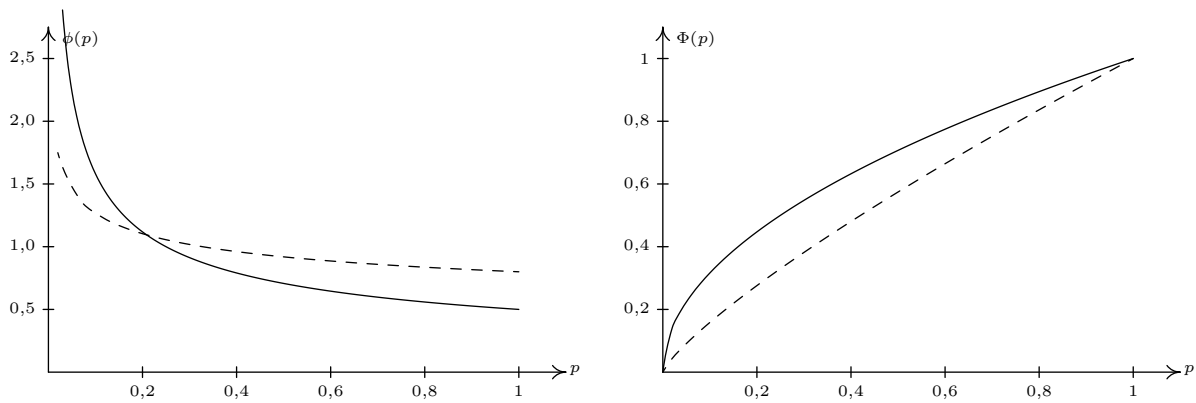


Figure 3: Power spectral risk measures for $b = 0,5$ (solid) and $b = 0,8$ (dashed)

3. Comparative AP-risk aversion: The case of deterministic initial wealth

3.1. Definitions

We rely on the general concept of (comparative) risk aversion following ARROW (1965) and PRATT (1964), which has been developed for the classical EU-theory first. Their framework is based on the idea of eliminating risk completely, i.e., it is assumed that a decision maker can switch from a risky position to a risk free position. In order to model this shift, the notion of the certainty equivalent is employed, and is related to the concepts of risk aversion and comparative risk aversion as follows:

Definition 3.1.

- a) The certainty equivalent of a position X , $c(X)$, indicates the certain position for which the decision maker is indifferent to the position X .
- b) A decision maker is said to be AP-risk averse if $c(X) \leq E(X)$ for all $X \in \mathcal{X}$.
- c) A decision maker 1 is said to be more AP-risk averse than a decision maker 2 if $c_1(X) \leq c_2(X) (\leq E(X))$ for all $X \in \mathcal{X}$.

For any risky position X and in order to avoid the risk induced thereby, an AP-risk averse decision maker is willing to accept a certain position that is less than the risky position's expectation $E(X)$. More AP-risk averse decision makers accept a smaller certainty equivalent than decision makers that are less AP-risk averse.

3.2. Comparative AP-risk aversion for SRM-decision makers

We now address the measurement of AP-risk aversion for SRM-decision makers. To this end, we first specify the certainty equivalent.

Theorem 3.2. *The certainty equivalent of a SRM-decision maker with risk spectrum ϕ , $c_\phi(X)$, is given by*

$$c_\phi(X) = -\rho_\phi(X). \quad (13)$$

The proof is straightforward: The certainty equivalent is defined by $\rho_\phi(c_\phi(X)) = \rho_\phi(X)$, and translation invariance yields $\rho_\phi(c_\phi(X)) = -c_\phi(X)$.

The next two Theorems 3.3 and 3.4 address the measurement of AP-risk aversion and comparative AP-risk aversion for SRM-decision makers. While the technical essentials by and large are known from the dual theory of choice, we for the first time apply these results to spectral risk measures in order to point out inconsistencies in the prevailing literature. There, and inspired by the classical AP-measure for EU-decision makers, the so-called spectral AP-measure defined by (14) is regularly used as measure of AP-risk aversion.

Theorem 3.3. *SRM-decision makers are AP-risk averse.*

For the proof, see ROELL (1987), Proposition II.2, who for the dual theory of choice proves that a DT-decision maker is AP-risk averse if and only if $v(p) \geq p$ for all $p \in [0, 1]$. In terms of spectral risk measures, this condition corresponds to $\Phi(p) \geq p$ for all $p \in [0, 1]$, which by definition is satisfied for all spectral risk measures, as Φ is a concave cumulative distribution function with $\Phi(0) = 0$ and $\Phi(1) = 1$.

Theorem 3.4. *A SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than a SRM-decision maker with risk spectrum ϕ_2 if and only if $\Phi_1(p) - \Phi_2(p) \geq 0$ for all $p \in [0, 1]$.*

For the proof, see again ROELL (1987), Proposition II.4. Theorem 3.4 tells that for a SRM-decision maker to be more AP-risk averse, the antiderivative of his risk spectrum has to lie above the one of the less AP-risk averse SRM-decision maker on the entire support.

For the subclasses of spectral risk measures introduced in Section 2.2, the respective parameters can be consistently interpreted as parameters of AP-risk aversion, as the following Theorem 3.5 shows.

Theorem 3.5.

1. *Let α_1 and α_2 be the confidence levels of two CVaR-decision makers with risk spectrum as given in (8). Then the CVaR-decision maker with confidence level α_1 is more AP-risk averse than the CVaR-decision maker with confidence level α_2 if and only if $\alpha_1 \leq \alpha_2$.*
2. *Let a_1 and a_2 be the parameters of two SRM-decision makers with exponential risk spectrum as given in (10). Then the SRM-decision maker with a_1 is more AP-risk averse than the SRM-decision maker with a_2 if and only if $a_1 \geq a_2$.*

3. Let b_1 and b_2 be the parameters of two SRM-decision makers with power risk spectrum as given in (11). Then the SRM-decision maker with b_1 is more AP-risk averse than the SRM-decision maker with b_2 if and only if $b_1 \leq b_2$.

The proof is given in the Appendix.

3.3. Comparative AP-risk aversion and the spectral AP-measure

Inspired by the well-known AP-measure from EU-theory,¹ the literature on AP-risk aversion in the context of spectral risk measures has proposed to measure AP-risk aversion by the so-called spectral AP-measure

$$R_\phi(p) = -\frac{\Phi''(p)}{\Phi'(p)} = -\frac{\phi'(p)}{\phi(p)}. \quad (14)$$

As examples for this literature approach, see, among many others, the statements by WÄCHTER/MAZZONI (2013), p. 490, “Thus, by defining (14) a local measure of risk aversion in terms of the risk spectrum $\phi(\dots)$ is defined.” (see also their Examples 1-3), or by BARBI/ROMAGNOLI (2013), p. 8, “(...) the exponential risk measure (ERM), whose weights are based on the exponential utility function, where $k > 0$ is the constant Arrow-Pratt coefficient of absolute risk aversion.”

As already shown in Theorem 3.4, AP-risk aversion for SRM-decision makers is measured by the difference of the antiderivatives of the risk spectra and not by the spectral AP-measure (14). More precisely, the following relationship holds:

Theorem 3.6. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers. If $R_{\phi_1}(p) \geq R_{\phi_2}(p)$ for all $p \in [0, 1]$, then the SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than the SRM-decision maker with risk spectrum ϕ_2 . The converse is not true.*

The proof is given in the Appendix. Note that the missing “Only if”-part in Theorem 3.6 constitutes a significant divergence of the spectral AP-measure, $R_\phi(p)$, in the context of spectral risk measures from the classical AP-measure, $R_u(x)$, in the context of EU-theory: While R_u is a consistent measure of AP-risk aversion in the EU-context, $R_\phi(p)$ is not a consistent measure of AP-risk aversion in the context of spectral risk measures. The following (counter-)example shows that the spectral AP-measure may incorrectly classify a

¹In the framework of EU-theory, the AP-risk aversion of EU-decision makers with utility function u is measured by the classical AP-measure

$$R_u(x) = -\frac{u''(x)}{u'(x)}.$$

An EU-decision maker is AP-risk averse if and only if $R_u(x) \geq 0$ for all $x \in \mathbb{R}$. Likewise, an EU-decision maker with utility function u_1 is more AP-risk averse than an EU-decision maker with utility function u_2 if and only if $R_{u_1}(x) \geq R_{u_2}(x) (\geq 0)$ for all $x \in \mathbb{R}$ (e.g., PRATT (1964), Theorem 1).

SRM-decision maker as more AP-risk averse although he is not. Note that the example is relevant also from a practical perspective, as it is based on exponential and power spectral risk measures, both of which are well-established in the literature on, e.g., quantile-based risk measures in insurance (e.g., DOWD/BLAKE (2006)).

Example 3.7. Consider two SRM-decision makers 1 and 2 with power risk spectrum Φ_1 and exponential risk spectrum Φ_2 , respectively,

$$\Phi_1(p) = p^b, \quad 0 < b \leq 1, p \in [0, 1] \quad (15)$$

$$\Phi_2(p) = \frac{1 - e^{-a \cdot p}}{1 - e^{-a}}, \quad a > 0, p \in [0, 1] \quad (16)$$

(see Figure 4). For $b = 0,5$ and $a = 1$, it holds that $\Phi_1(p) \geq \Phi_2(p)$ for all $p \in [0, 1]$, so SRM-decision maker 1 is uniformly more AP-risk averse due to Theorem 3.4.

By contrast, let us now apply the spectral AP-measure, as it is commonly proposed in the literature (e.g., WÄCHTER/MAZZONI (2013) or BARBI/ROMAGNOLI (2013)). We then observe

$$R_{\phi_1}(p) = \frac{1}{2 \cdot p} \begin{cases} > \\ = \\ < \end{cases} 1 = R_{\phi_2}(p) \text{ if } \begin{cases} 0 \leq p < 0,5 \\ p = 0,5 \\ 0,5 < p \leq 1 \end{cases}, \quad (17)$$

i.e., SRM-decision maker 1 is ranked (locally) more “spectral” AP-risk averse if $p < 0,5$, while he is ranked (locally) less “spectral” risk averse if $p > 0,5$. This split ranking, however, is misleading as can be seen from the position

$$X = \begin{cases} x_1 & p \\ x_2 & 1 - p \end{cases}, \quad x_1 < x_2, p \in [0, 1], \quad (18)$$

which covers the cases $p < 0,5$ as well as $p > 0,5$. In both cases the certainty equivalent of X is given by

$$c_\phi(X) = -\rho_\phi(X) = \Phi(p) \cdot x_1 + (1 - \Phi(p)) \cdot x_2 = \Phi(p) \cdot (x_1 - x_2) + x_2. \quad (19)$$

Hence, $\Phi_1(p) \geq \Phi_2(p)$ for all $p \in [0, 1]$ implies $c_{\phi_1}(X, p) \leq c_{\phi_2}(X, p)$ for all $p \in [0, 1]$ and confirms that the SRM-decision maker with risk spectrum ϕ_1 is uniformly more AP-risk averse. \square

3.4. Two standard financial decision problems

We now analyze two standard financial decision problems in order to show that the concept of AP-risk aversion is relevant from an economic point of view.

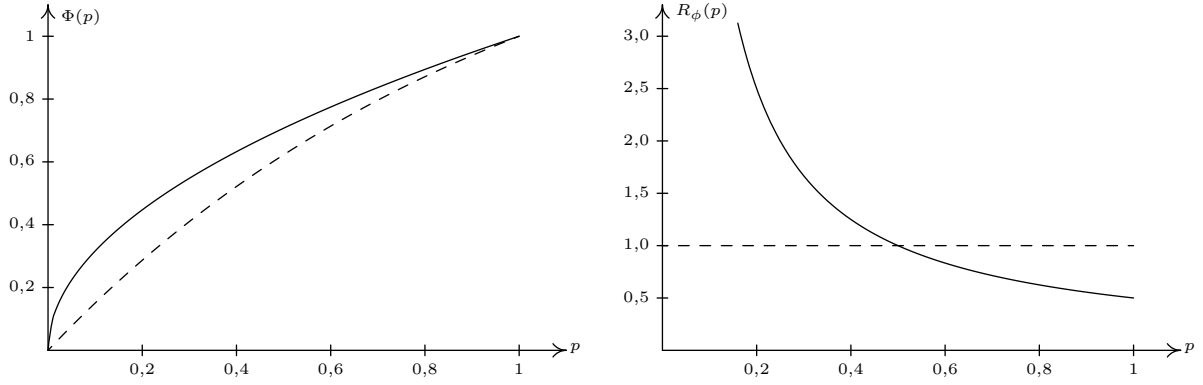


Figure 4: Power (solid) vs. exponential (dashed) risk spectrum and spectral AP-measure

The first problem addresses the willingness to pay for insurance. Here, a measure of risk aversion is consistent if it yields that the more risk averse SRM-decision maker, for any risk to insure, is willing to pay a greater premium. The second problem is the portfolio selection between a risk free and a risky asset. Here, a measure of risk aversion is consistent if it ensures that the more risk averse SRM-investor always invests less in the risky asset. It will turn out that the concept of AP-risk aversion is consistent with these two decision problems.

In the insurance problem, a SRM-decision maker with deterministic initial wealth w_0 is faced with an additional risk $X \in \mathcal{X}$, $X \neq E(X)$. By signing an insurance contract he can switch from the risky position $w_0 + X$ to the certain position $w_0 - \pi_\phi(w_0, X)$, where $\pi_\phi(w_0, X)$ denotes the insurance premium that the SRM-decision maker is willing to pay to cede X .

By definition, the insurance premium is given by

$$\rho_\phi(w_0 + X) = \rho_\phi(w_0 - \pi_\phi(w_0, X)). \quad (20)$$

Due to the linearity property (1) of spectral risk measures, the premium does not depend on the initial wealth w_0 and shrinks to the negative certainty equivalent of X ,

$$\begin{aligned} \rho_\phi(X) - w_0 &= \pi_\phi(w_0, X) - w_0, \text{ i.e.,} \\ \pi_\phi(w_0, X) &= \rho_\phi(X) = -c_\phi(X). \end{aligned} \quad (21)$$

Hence, the insurance problem is consistent with the concept of AP-risk aversion. Following (21) and Definition 3.1, we immediately obtain the following comparative static result.

Corollary 3.8. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $\pi_\phi(w_0, X)$ the insurance premium as given in (21). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than SRM-decision maker with risk spectrum ϕ_2 .*

2. $\pi_{\phi_1}(w_0, X) \geq \pi_{\phi_2}(w_0, X)$ for all $X \in \mathcal{X}$, $X \neq E(X)$.

Due to Corollary 3.8 and Theorem 3.4, it is the non-negative difference of the antiderivatives of the risk spectra that is necessary and sufficient for consistent comparative static results in the insurance problem. Conversely, when making use of the spectral AP-measure instead, the more AP-risk averse SRM-decision maker may be ranked (locally) less risk averse, though he is willing to pay a greater insurance premium for at least some positions X , as has been shown by (counter-)example 3.7.

Let us proceed with the portfolio selection problem, where w_0 is again the deterministic initial wealth of a SRM-investor. Let further $z \in [0, w_0]$ and $w_0 - z$ be the amounts that are invested in the risk free and a risky asset, respectively. Finally, the returns of the risk free and the risky asset are given by $(r_f - 1)$ and $(r - 1)$, $r \in \mathcal{X}$, $r \neq E(r)$, $E(r) > r_f$, respectively. The SRM-investor's final wealth reads

$$(w_0 - z) \cdot r + z \cdot r_f. \quad (22)$$

The optimal amount that is invested in the risk free asset is given by

$$\begin{aligned} z_{\phi}^*(w_0, r) &= \arg \min_{z \in [0, w_0]} \rho_{\phi}((w_0 - z) \cdot r + z \cdot r_f) \\ &= \arg \min_{z \in [0, w_0]} (w_0 - z) \cdot \rho_{\phi}(r) - z \cdot r_f, \end{aligned} \quad (23)$$

where again the linearity property (1) has been used. The corresponding derivative

$$\frac{\partial \rho_{\phi}(\cdot)}{\partial z} = -\rho_{\phi}(r) - r_f \quad (24)$$

renders²

$$z_{\phi}^*(w_0, r) = \begin{cases} w_0 & \rho_{\phi}(r) > -r_f = \rho_{\phi}(r_f) \\ 0 & \rho_{\phi}(r) \leq -r_f = \rho_{\phi}(r_f) \end{cases}. \quad (25)$$

Spectral risk measures yield an all or nothing-decision: Instead of portfolio diversification, either the exclusive investment in the risk free asset, or the exclusive investment in the risky asset is optimal. This result fundamentally differs from classical results obtained for EU-theory or mean-variance-approaches, and is a first pitfall of spectral risk measures when they are used for portfolio selection.³ As has been recently shown by BRANDTNER

²Without loss of generality, we assume that the investor decides for a corner position if he is indifferent between the risk free and the risky asset.

³Note that due to the linearity property (1), spectral risk measures exhibit constant absolute and constant relative risk aversion at the same time. In classical EU-theory, the simultaneous presence of constant absolute and relative risk aversion only holds if and only if the utility function is linear, $u(x) = x$ (see BAMBERG/SPREMANN (1981), Theorem 5). In this case, EU-theory predicts plunging as well.

(2013), non-diversification also prevails in the extended setting of a risk free and more than one risky asset. On the other side, plunging will no longer occur when leaving out the risk free asset and considering risky assets only. As the recent financial crises question the existence of a risk free asset anyway, this setting comes without plunging and appears to be the more realistic scenario. We will address the case of “more” versus “less” risky assets in detail below.

Before, we come back to analyzing the relationship between AP-risk aversion and the above portfolio selection problem with one risk free and one risky asset. As the amount invested in the risk free asset solely depends on the (negative) certainty equivalent, $\rho_\phi(r) = -c_\phi(r)$, the portfolio selection problem is consistent with the concept of AP-risk aversion. Definition 3.1 together with (25) yields

Corollary 3.9. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $z_\phi^*(w_0, r)$ the optimal amount that is invested in the risk free asset as given in (25). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more AP-risk averse than SRM-decision maker with risk spectrum ϕ_2 .*
2. *$z_{\phi_1}^*(w_0, r) \geq z_{\phi_2}^*(w_0, r)$ for all $r \in \mathcal{X}, r \neq E(r), E(r) > r_f$.*

Due to Corollary 3.9 and Theorem 3.4 it is again the non-negative difference of the antiderivatives of the risk spectra that is necessary and sufficient for consistent comparative static results in the portfolio selection problem.

4. Comparative R-risk aversion: The case of random initial wealth

4.1. Definitions

The framework of ARROW (1965) and PRATT (1964) is restricted to decision situations with deterministic initial wealth and where the risk of the final wealth can be eliminated completely. ROSS (1981) has extended this framework towards a setting where the initial wealth is random and the risk of the final wealth can be reduced only by part. Accordingly, the framework of ROSS (1981) also addresses shifts from “more” to “less” risk, which appears to be the more realistic setting. In the recent financial crises, for example, even “no risk” senior tranches became risky after junior and mezzanine tranches defaulted.

In order to model increasing risk, ROSS (1981) has made use of the concept of “adding noise”.

Definition 4.1. *A random variable Z is called more risky than a random variable X if $Z = X + Y$ with $E(Y|X) = 0$.*

Definition 4.1 is closely related to ROTHSCILD/STIGLITZ (1970)'s concept of a mean preserving spread (see also WILHELM (2008), p. 488).

Theorem 4.2. *The following holds:*

1. $Z = X + Y$ with $E(Y|X) = 0 \Rightarrow Z$ is a mean preserving spread of X , i.e., $\int_{-\infty}^t (F_Z(x) - F_X(x))dx \geq 0$ for all $t \in \mathbb{R}$ and $\int_{-\infty}^{\infty} (F_Z(x) - F_X(x))dx = 0$.
2. Z is a mean preserving spread of $X \Rightarrow \exists Y$ such that $Z \stackrel{d}{=} X + Y$ and $E(Y|X) = 0$.

Theorem 4.2, Part 1, ensures that the form of increasing risk chosen in Definition 4.1 is consistent with SRM-decision makers, as all SRM-decision makers reject any mean preserving spread (for a proof, see ADAM ET AL. (2008), Appendix A or LEITNER (2005)). So “adding noise” is a suitable definition of increasing risk in the “new” context of SRM-decision makers as it has proven to be suitable in the “old” context of EU-decision makers before. The second part of Theorem 4.2 allows us to provide simple (counter-)examples that are based on a mean preserving spread between X and Z , as this guarantees the existence of some random variables X, Y and Z that satisfy the requirements of Definition 4.1.

In order to capture a reduction of risk by switching from Z to X , ROSS (1981) has introduced the incremental risk premium.

Definition 4.3. *For the positions $X, Y, Z = X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$, the incremental risk premium, $RP(X, Y)$, is defined by $RP(X, Y) = c(X) - c(X + Y)$.*

The incremental risk premium indicates the sure amount a decision maker is willing to pay if he is able to switch from Z to X and avoid Y . Based on the incremental risk premium, R-risk aversion and comparative R-risk aversion are defined as follows.

Definition 4.4.

- a) *A decision maker is said to be R-risk averse if $RP(X, Y) \geq 0$ for all $X, Y, Z = X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$.*
- b) *A decision maker 1 is said to be more R-risk averse than a decision maker 2 if $RP_1(X, Y) \geq RP_2(X, Y) (\geq 0)$ for all $X, Y, Z = X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$.*

An R-risk averse decision maker, starting from some random initial wealth X , is willing to make a non-negative payment to avoid the additional risk Y . For the more R-risk averse decision maker, this payment is greater than for a decision maker who is less R-risk averse.

4.2. Comparative R-risk aversion for SRM-decision makers

We now address the measurement of R-risk aversion for SRM-decision makers. To this end, we first specify the incremental risk premium.

Corollary 4.5. *For the positions $X, Y, Z = X + Y \in \mathcal{X}$, $X \neq E(X)$, $Y \neq E(Y)$, $E(Y|X) = 0$, the incremental risk premium of a SRM-decision maker with risk spectrum ϕ , $RP_\phi(X, Y)$, is given by*

$$RP_\phi(X, Y) = \rho_\phi(X + Y) - \rho_\phi(X). \quad (26)$$

The proof directly follows from the identity $c_\phi(X) = -\rho_\phi(X)$ (see Theorem 3.2).

The following Theorems 4.6 and 4.7 address the measurement of R-risk aversion and comparative R-risk aversion for SRM-decision makers. Note that the technical essentials again are based on previous results from the dual theory of choice; their implications for the issue of decision making under spectral risk measures, however, have not been made explicit as yet.

Theorem 4.6. *SRM-decision makers are R-risk averse.*

For the proof, see YAARI (1987), Theorem 2, who proves that a DT-decision maker is R-risk averse if and only if $v(p)$ is concave on the entire support. In terms of spectral risk measures, this condition corresponds to $\Phi(p)$ being concave on the entire support, which is satisfied by definition. Note that although the technical requirements for AP-risk aversion (i.e., $\Phi(p) \geq p$, see Theorem 3.4) and R-risk aversion (i.e., $\Phi(p)$ concave, see Theorem 3.9) differ, they are both satisfied for spectral risk measures by definition. Consequently, for SRM-decision makers, AP-risk aversion and R-risk aversion are equivalent concepts.

Theorem 4.7. *A SRM-decision maker with risk spectrum ϕ_1 is more R-risk averse than a SRM-decision maker with risk spectrum ϕ_2 if and only if $\Phi_1(p) - \Phi_2(p)$ is non-negative and concave on the entire support.*

For the proof, see ROELL (1987), Proposition II.5, again for the dual theory of choice. For a SRM-decision maker to be more R-risk averse, two conditions have to be satisfied. First, the antiderivative of the risk spectrum of the more R-risk averse SRM-decision maker has to lie above the one of the SRM-decision maker who is less R-risk averse, $\Phi_1(p) - \Phi_2(p) \geq 0$. Second, the difference of the antiderivatives of the risk spectra needs to be concave on the entire support. Hence, greater R-risk aversion implies greater AP-risk aversion, but the converse it not true.

4.3. Comparative AP-risk aversion vs. comparative R-risk aversion

We now specify the insights from Section 4.2 with respect to Conditional Value-at-Risk, and exponential and power spectral risk measures. While these popular subclasses could be completely ordered with respect to AP-risk aversion, we show by means of a simple counter-example that under R-risk aversion this useful property fails.

4.3.1. An illustrative (counter-)example

Building on Theorem 4.7, we argue by contradiction and assume without loss of generality that there exists some $\bar{p} < 1$ such that $\Phi_1(p) - \Phi_2(p)$ is decreasing and convex on $p \in (\bar{p}, 1)$ (see Figure 5).⁴ Then a binary random variable can be constructed for which the more AP-risk averse SRM-decision maker is less R-risk averse.

Let X and $X + Y$ be two risky positions with

$$X = \begin{cases} 0 & q_1 \\ \frac{1-q_2}{1-q_1} & 1 - q_1 \end{cases}, \quad X + Y = \begin{cases} 0 & q_2 \\ 1 & 1 - q_2 \end{cases}, \quad 0 \leq q_1 < q_2 \leq 1; \quad (27)$$

obviously, $X + Y$ is constructed from X by a mean preserving spread. If SRM-decision maker 1 is more R-risk averse than SRM-decision maker 2, the incremental risk premiums (see Definition 4.4) fulfill

$$RP_{\phi_1}(X, Y) \geq RP_{\phi_2}(X, Y) \Leftrightarrow \frac{\Phi_1(q_2) - \Phi_2(q_2)}{1 - q_2} \geq \frac{\Phi_1(q_1) - \Phi_2(q_1)}{1 - q_1}. \quad (28)$$

However, as $\Phi_1(p) - \Phi_2(p) (\geq 0)$ has been assumed to be decreasing and convex on $(\bar{p}, 1)$, choosing $q_1, q_2 \in (\bar{p}, 1)$ yields

$$\frac{\Phi_1(q_2) - \Phi_2(q_2)}{1 - q_2} < \frac{\Phi_1(q_1) - \Phi_2(q_1)}{1 - q_1} \Leftrightarrow RP_{\phi_1}(X, Y) < RP_{\phi_2}(X, Y), \quad (29)$$

a contradiction (see Figure 5).

4.3.2. Conditional Value-at-Risk

We start by analyzing the relationship between comparative AP-risk aversion and comparative R-risk aversion for Conditional Value-at-Risk. Here, the difference of the antiderivatives

⁴A more general counter-example can be constructed by setting the less restrictive assumption that $\Phi_1(p) - \Phi_2(p)$ is convex on some interval $p \in (\underline{p}, \bar{p}) \subseteq [0, 1]$. For any of our subclasses of spectral risk measures, however, it will turn out that $\Phi_1(p) - \Phi_2(p)$ is decreasing and convex on some interval $(\bar{p}, 1)$, so for the sake of consistency we set up the corresponding assumptions in our counter-example.

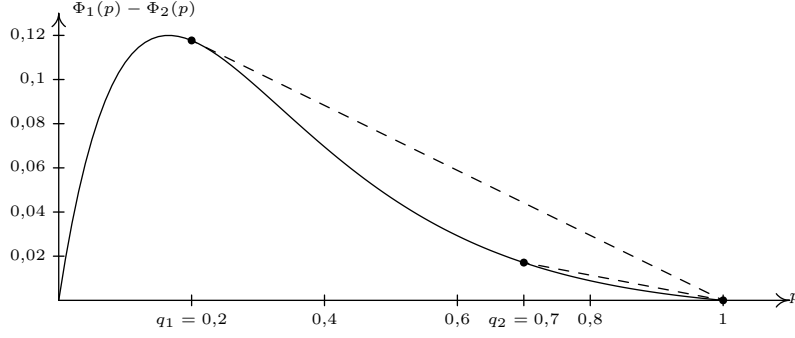


Figure 5: Comparative R-risk aversion and difference of the antiderivatives of the risk spectra

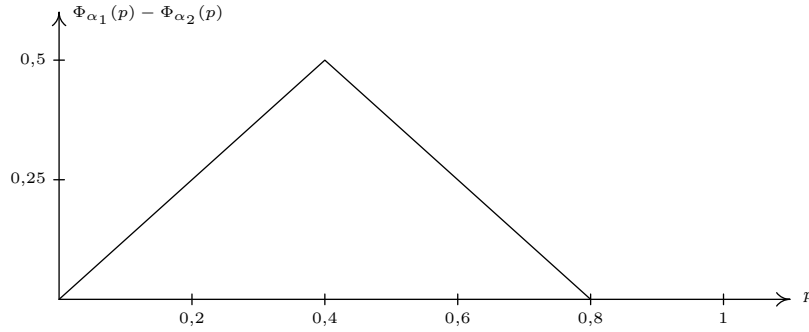


Figure 6: Difference of the antiderivatives of the risk spectra for Conditional Value-at-Risk and $\alpha_1 = 0,4$ and $\alpha_2 = 0,8$

of the risk spectra is given by

$$\Phi_{\alpha_1}(p) - \Phi_{\alpha_2}(p) = \begin{cases} \frac{p}{\alpha_1} - \frac{p}{\alpha_2} & 0 \leq p \leq \alpha_1 \\ 1 - \frac{p}{\alpha_2} & \alpha_1 < p \leq \alpha_2 \\ 0 & \alpha_2 < p \leq 1 \end{cases} \quad (30)$$

and is convex on $(\alpha_1, 1)$ for any two confidence levels $0 < \alpha_1 < \alpha_2 < 1$. Figure 6 illustrates the case for $\alpha_1 = 0,4$ and $\alpha_2 = 0,8$.

Accordingly, for any two confidence levels $\alpha_1 < \alpha_2$ one can construct the following (counter-)example where the less AP-risk averse CVaR-decision maker, at the same time, is more R-risk averse and exhibits a greater incremental risk premium.

Example 4.8. Let α_1 and α_2 , $\alpha_1 < \alpha_2$, be the confidence levels of two CVaR-decision makers. Let further the two risky positions from the general (counter-)example (27) with $q_1 = \alpha_1$ and $q_2 = \alpha_2$ be given by

$$X = \begin{cases} 0 & \alpha_1 \\ \frac{1-\alpha_2}{1-\alpha_1} & 1 - \alpha_1 \end{cases}, X + Y = \begin{cases} 0 & \alpha_2 \\ 1 & 1 - \alpha_2 \end{cases}. \quad (31)$$

We then have

$$RP_{\alpha_1}(X, Y) = CVaR_{\alpha_1}(X + Y) - CVaR_{\alpha_1}(X) = 0 - 0 = 0 \quad (32)$$

$$RP_{\alpha_2}(X, Y) = CVaR_{\alpha_2}(X + Y) - CVaR_{\alpha_2}(X) = 0 + \frac{1 - \alpha_2}{1 - \alpha_1} \cdot \frac{\alpha_2 - \alpha_1}{\alpha_2} > 0. \quad (33)$$

Contrary to intuition, the more AP-risk averse CVaR-decision maker 1 (with confidence level α_1) is not willing to make a positive payment to reduce risk by switching from $X + Y$ to X , as he only takes into account the $\alpha_1 \cdot 100\%$ worst outcomes, which for both positions correspond to zero. The less AP-risk averse CVaR-decision maker 2 (his confidence level is at $\alpha_2 > \alpha_1$) for the position X also (partly) takes into account the states of the world where the outcome is positive, so he is willing to make a positive payment to reduce his risk. \square

4.3.3. Exponential spectral risk measures

For the subclass of exponential spectral risk measures, Theorem 4.9 holds.

Theorem 4.9. *Let $\{\rho_{\phi_a}, a > 0\}$ be the subclass of exponential spectral risk measures as given in (10). Let further $a_1 > a_2$, i.e., SRM-decision maker 1 is more AP-risk averse than SRM-decision maker 2, and let $\bar{a} = 1,5937$.*

1. *Let $a_2 < a_1 < \bar{a}$. Then SRM-decision maker 1 is uniformly more R-risk averse than SRM-decision maker 2.*
2. *Let $a_1 > a_2 > \bar{a}$. Then SRM-decision maker 1 is not uniformly more R-risk averse than SRM-decision maker 2.*
3. *Let $a_2 < \bar{a} < a_1$. Then SRM-decision maker 1 is uniformly more R-risk averse than SRM-decision maker 2 if $\frac{e^{a_1} - 1}{a_1^2} \geq \frac{e^{a_2} - 1}{a_2^2}$, while otherwise he is not.*

The proof is given in the Appendix. In the first case, both SRM-decision makers are assumed to be “slightly” AP-risk averse, i.e., $a < 1,5937$. Then a greater a consistently represents greater AP- and R-risk aversion. In the second case, both SRM-decision makers are assumed to be “strongly” AP-risk averse, i.e., $a > 1,5937$. Then the decision maker with $a_1 > a_2$ is uniformly more risk averse in the sense of Arrow and Pratt, but not in the sense of Ross. That is, by making use of (27), for any two a_1 and a_2 one can construct a (counter-)example where the more AP-risk averse SRM-decision maker exhibits a smaller incremental risk premium than the less AP-risk averse SRM-decision maker. We illustrate this case in more detail in Example 4.10 below. Finally, when considering a “slightly” and a “strongly” AP-risk averse SRM-decision maker, the results are ambiguous. If the difference $a_1 - a_2 > 0$ is sufficiently large, the more AP-risk averse decision maker is not uniformly more R-risk averse, while the opposite is true if the difference is sufficiently small.

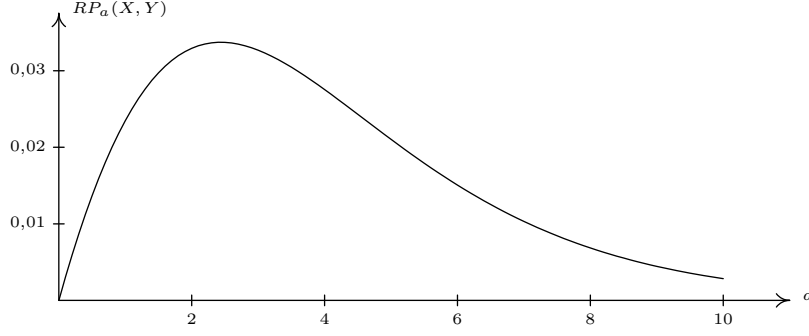


Figure 7: Incremental risk premium for exponential spectral risk measures

Example 4.10. Let ϕ_a be the exponential risk spectrum of a SRM-decision maker as given in (10). Let further be X and $X + Y$ two risky positions as in (27) with $q_1 = 0,5$ and $q_2 = 0,75$

$$X = \begin{Bmatrix} 0 & 0,5 \\ 0,5 & 0,5 \end{Bmatrix}, \quad X + Y = \begin{Bmatrix} 0 & 0,75 \\ 1 & 0,25 \end{Bmatrix}. \quad (34)$$

Now assume that the degree of AP-risk aversion is increased by increasing a . It holds that (see (29))

$$\frac{\partial \Phi_a(q_1)}{\partial a} \cdot \frac{1}{1 - q_1} \begin{cases} \leq \\ > \end{cases} \frac{\partial \Phi_a(q_2)}{\partial a} \cdot \frac{1}{1 - q_2} \text{ if } \begin{cases} a \leq 2,44 \\ a > 2,44 \end{cases}. \quad (35)$$

Accordingly, for $a \leq 2,44$, the incremental risk premium is increasing in a , while it is decreasing in a for $a > 2,44$: Thus, the SRM-decision maker becomes more AP-risk averse, while, at the same time, his incremental risk premium is decreasing. Figure 7 illustrates these results. \square

4.3.4. Power spectral risk measures

For the subclass of power spectral risk measures, the following holds.

Theorem 4.11. Let $\{\rho_{\phi_b}, 0 < b \leq 1\}$ be the subclass of power spectral risk measures as given in (11). Let further $b_1 < b_2$, i.e., SRM-decision maker 1 is more AP-risk averse than SRM-decision maker 2, and let $\bar{b} = 0,5$.

1. Let $\bar{b} < b_1 < b_2$. Then SRM-decision maker 1 is uniformly more R-risk averse than SRM-decision maker 2.
2. Let $b_1 < b_2 < \bar{b}$. Then SRM-decision maker 1 is not uniformly more R-risk averse than SRM-decision maker 2.
3. Let $b_1 < \bar{b} < b_2$. Then SRM-decision maker 1 is uniformly more R-risk averse than SRM-decision maker 2 if $b_2 < 1 - b_1$, while otherwise he is not.

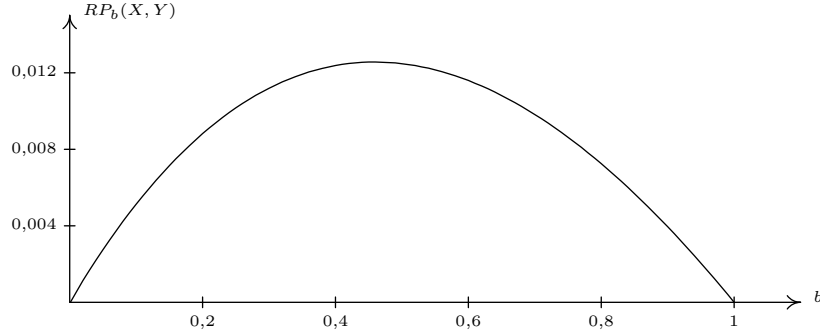


Figure 8: Incremental risk premium for power spectral risk measures

The proof is given in the Appendix. Again, if both SRM-decision makers are “slightly” AP-risk averse with $b > 0,5$, a smaller b consistently represents greater AP- and R-risk aversion. By contrast, if both SRM-decision makers are “strongly” AP-risk averse, with $b < 0,5$, the SRM-decision maker with $b_1 < b_2$ is uniformly more AP-risk averse, while he is not uniformly more R-risk averse. The counter-example (27) again allows to illustrate this general incompatibility between AP-risk aversion and R-risk aversion for SRM-decision makers. Finally, if SRM-decision maker 1 is “strongly” risk averse and SRM-decision maker 2 is “slightly” risk averse, the results are ambiguous. The following example is given for illustration.

Example 4.12. Let ϕ_b be the power risk spectrum of a SRM-decision maker as given in (11). As in the previous Example 4.10, let X and $X + Y$ be two risky positions as in (27) with $q_1 = 0,5$ and $q_2 = 0,75$

$$X = \begin{Bmatrix} 0 & 0,5 \\ 0,5 & 0,5 \end{Bmatrix}, \quad X + Y = \begin{Bmatrix} 0 & 0,75 \\ 1 & 0,25 \end{Bmatrix}. \quad (36)$$

Now assume that the degree of AP-risk aversion is increased by decreasing b . It holds that (see (29))

$$\frac{\partial \Phi_b(q_1)}{\partial b} \cdot \frac{1}{1 - q_1} \begin{cases} \leq \\ > \end{cases} \frac{\partial \Phi_b(q_2)}{\partial b} \cdot \frac{1}{1 - q_2} \text{ if } \begin{cases} b \geq 0,4593 \\ b < 0,4593 \end{cases}. \quad (37)$$

Accordingly, for $b \geq 0,4593$, the incremental risk premium is decreasing in b , i.e., is going to fall with decreasing AP-risk aversion, while for $b < 0,4593$ the incremental risk premium is increasing in b , i.e., is going to rise with decreasing AP-risk aversion: Thus, the SRM-decision maker may become less AP-risk averse and more R-risk averse at the same time. Figure 8 illustrates these results. \square

4.3.5. Completeness, or: A non-consistency result

In this section, we show that those subclasses of spectral risk measures that are able to cover any degree of AP-risk aversion between AP-risk neutrality and infinite AP-risk aversion cannot be completely ordered with respect to R-risk aversion.

The property of completeness is defined as follows.

Definition 4.13. *Let $\{\rho_{\phi_\theta}, \theta \in (\underline{\theta}, \bar{\theta})\}$, be a one-parameter family of spectral risk measures. Let further X be a uniformly distributed random variable on $[0, 1]$. $\{\rho_{\phi_\theta}\}$ is said to be complete if $\rho_{\phi_\theta}(X)$ can take any value in $[-0,5, 0]$.*

Note that, if $\rho_{\phi_\theta}(X) = -0,5 = -E(X)$, the SRM-decision maker is risk neutral. If $\rho_{\phi_\theta}(X) = 0$, the SRM-decision maker is infinitely risk averse, as in this case the spectral risk measure corresponds to the negative maximum loss, $\rho_{\phi_\theta}(X) = -\inf\{X(\omega) : \omega \in \Omega\}$. Moreover, whatsoever spectral risk $\bar{\rho} \in [-0,5, 0]$ the SRM-decision maker wants to assign to X , for complete subclasses there exists a bijective mapping $\theta = \theta(\bar{\rho})$ between θ and ρ that yields this risk. Note that the subclasses of Conditional Value-at-Risk, and exponential and power spectral risk measures are complete.

Although completeness is a desirable property of a family of spectral risk measures, it does not coincide with R-risk aversion:

Theorem 4.14. *Let $\{\rho_{\phi_\theta}, \theta \in (\underline{\theta}, \bar{\theta})\}$ be a one-parameter family of spectral risk measures that satisfies the completeness property as given in Definition 4.13. Then there exists at least one pair of risk spectra $\theta_1, \theta_2 \in (\underline{\theta}, \bar{\theta})$ and $\underline{p}, \bar{p} \in [0, 1]$ such that $\Phi_{\theta_1}(p) - \Phi_{\theta_2}(p)$ is convex on $(\underline{p}, \bar{p}) \subseteq [0, 1]$.*

For the proof it is sufficient to note that the antiderivative of the risk spectrum of the negative maximum loss is given by $\Phi_{ML}(p) = 1, p \in [0, 1]$, and that $\Phi_{ML}(p) - \Phi_{\theta_2}(p)$ is convex on $(0, 1)$ for any $\theta_2 \in (\underline{\theta}, \bar{\theta})$.

Theorem 4.14 further confirms our observations from Section 4.3.2 that “strongly” AP-risk averse SRM-decision makers face inconsistencies with R-risk aversion: In the extreme case of infinite AP-risk aversion, the (negative) Maximum Loss-decision maker has the risk spectrum $\Phi_{ML}(p) = 1, p \in [0, 1]$, and the difference $\Phi_{ML}(p) - \Phi_{\theta_2}(p)$ is convex on $(0, 1)$ for any $\theta_2 \in (\underline{\theta}, \bar{\theta})$. Conversely, in the extreme case of zero AP-risk aversion, the (negative) Mean-decision maker has the risk spectrum $\Phi_\mu(p) = p, p \in [0, 1]$, so the difference $\Phi_{\theta_2}(p) - \Phi_\mu(p)$ is concave on $(0, 1)$ for any $\theta_2 \in (\underline{\theta}, \bar{\theta})$.

4.4. Two standard financial decision problems

In order to demonstrate the economic relevance of the concept of R-risk aversion, we again analyze an insurance problem and a portfolio selection problem. Unlike our previous study

of these problems in Section 3.4, the decision maker's initial wealth now is assumed to be random, and the risk of the final wealth can only be eliminated partly.

In the insurance problem with some random initial wealth, a measure of risk aversion is consistent if it yields that the more risk averse SRM-decision maker is willing to pay a greater insurance premium for any reduction in risk in the sense of Definition 4.1. Likewise, in a portfolio selection problem with a risk free and a risky asset in the presence of some random initial wealth, a measure of risk aversion is consistent if it yields that the more risk averse SRM-investor always invests less in the risky asset. In both decision problems, the concept of R-risk aversion will prove to be consistent.

We start with the insurance problem. A SRM-decision maker with random initial wealth $w_0 + X \in \mathcal{X}, X \neq E(X)$, is faced with an additional risk $Y \in \mathcal{X}, Y \neq E(Y), E(Y|X) = 0$. By signing an insurance contract, he can switch from the risky position $w_0 + X + Y$ to the less risky position at $w_0 + X - \pi_\phi(X, Y)$, where $\pi_\phi(X, Y)$ denotes the insurance premium that the SRM-decision maker is willing to pay to cede Y .

By definition, the insurance premium satisfies

$$\rho_\phi(w_0 + X + Y) = \rho_\phi(w_0 + X - \pi_\phi(X, Y)). \quad (38)$$

Due to the linearity property of spectral risk measures (1), the premium does not depend on the initial wealth w_0 and coincides with the incremental risk premium

$$\begin{aligned} \rho_\phi(X + Y) - w_0 &= \rho_\phi(X) - w_0 + \pi_\phi(X, Y) \Leftrightarrow \\ \pi_\phi(X, Y) &= \rho_\phi(X + Y) - \rho_\phi(X) = RP_\phi(X, Y). \end{aligned} \quad (39)$$

Hence, the insurance problem is consistent with the concept of R-risk aversion. Based on (39) and Definition 4.4, we immediately obtain the following comparative static result.

Corollary 4.15. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $\pi_\phi(X, Y)$ the insurance premium as given in (39). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more R-risk averse than SRM-decision maker with risk spectrum ϕ_2 .*
2. *$\pi_{\phi_1}(X, Y) \geq \pi_{\phi_2}(X, Y)$ for all $w_0 + X, Y, X + Y \in \mathcal{X}, X \neq E(X), Y \neq E(Y), E(Y|X) = 0$.*

According to Corollary 4.15 and Theorem 4.7, a non-negative and concave difference of the antiderivatives of the risk spectra is necessary and sufficient for consistent comparative static results in the insurance problem.⁵ So, Conditional Value-at-Risk, and exponential

⁵Note that due to the property of translation invariance, Theorem 4.15 also holds for $E(Y) \neq 0$, when assuming $E(Y - E(Y)|X) = 0$.

and power spectral risk measures regularly yield counterintuitive results as soon as they are applied to questions of optimal insurance. In all three subclasses, the willingness to pay for insurance is non-monotonic with respect to the respective AP-risk aversion parameters (see again the (counter-)examples 4.8, 4.10, and 4.12). Especially, for “strongly” AP-risk averse SRM-decision makers one can always find (counter-)examples where the willingness to pay for insurance is decreasing though the SRM-decision maker’s AP-risk aversion is increasing at the same time.

We proceed with the portfolio selection problem. Let w_0 denote the SRM-decision maker’s deterministic initial wealth and $X \in \mathcal{X}, X \neq E(X)$ be his random initial wealth. The amounts $z \in [0, w_0]$ and $w_0 - z$, are invested in the risk free and a risky asset, respectively. Finally, the returns of the risk free and the risky asset are given by $(r_f - 1)$ and $(r - 1), r \in \mathcal{X}, r \neq E(r), E(r) > r_f, r_0 := r - E(r), E(r_0|X) = 0$, respectively. The SRM-investor’s final wealth comes to

$$X + (w_0 - z) \cdot r + z \cdot r_f, \quad (40)$$

and the optimal amount of risk free investment is given by

$$\begin{aligned} z_\phi^*(X, r) &= \arg \min_{z \in [0, w_0]} \rho_\phi(X + (w_0 - z) \cdot r + z \cdot r_f) \\ &= \arg \min_{z \in [0, w_0]} \rho_\phi(X + (w_0 - z) \cdot r_0) - (w_0 - z) \cdot E(r) - z \cdot r_f. \end{aligned} \quad (41)$$

The linearity in (41) again is induced by the linearity property (1) of spectral risk measures. The corresponding first order condition reads

$$\begin{aligned} \frac{\partial \rho_\phi(\cdot)}{\partial z} &= \frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) + (E(r) - r_f) \stackrel{!}{=} 0, \text{ or} \\ &\frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) \stackrel{!}{=} \rho_\phi(E(r) - r_f) \end{aligned} \quad (42)$$

The left-hand side of (42) is consistent with the concept of R-risk aversion and thus yields the following Theorem 4.16 (the proof is given in the Appendix).

Theorem 4.16. *Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers and $z_\phi^*(X, r)$ the optimal risk free investment as given by (42). The following statements are equivalent:*

1. *SRM-decision maker with risk spectrum ϕ_1 is more R-risk averse than SRM-decision maker with risk spectrum ϕ_2 .*
2. *$z_{\phi_1}^*(X, r) \geq z_{\phi_2}^*(X, r)$ for all $X, r \in \mathcal{X}, r \neq E(r), X \neq E(X), E(r) > r_f, r_0 = r - E(r), E(r_0|X) = 0$.*

According to Theorems 4.7 and 4.16, a non-negative and concave difference of the antiderivatives of the risk spectra is necessary and sufficient for consistent comparative static results in the portfolio selection problem. Unfortunately, for Conditional Value-at-Risk, and exponential and power spectral risk measures this difference is regularly convex. We thus obtain counterintuitive results in the portfolio selection problem, as the following (counter-)examples show.

Example 4.17. Let α be the confidence level of a CVaR-decision maker. Let further be $w_0, X, r_0, E(r)$ and r_f be given by

$$w_0 = 80, X = \begin{pmatrix} 8 & 0,1 \\ 8 & 0,1 \\ 10 & 0,2 \\ 10 & 0,6 \end{pmatrix}, r_0 = \begin{pmatrix} 0 & 0,1 \\ 0 & 0,1 \\ -0,05 & 0,2 \\ 0,0167 & 0,6 \end{pmatrix}, E(r) = 1,06, r_f = 1,05. \quad (43)$$

The optimal investment in the risk free asset, z_α^* , as a function of the confidence level α is given in Figure 9 and exhibits two counter-intuitive characteristics. First, we again observe a tendency towards corner solutions. While the entire investment opportunity set $z \in [0, 80]$ itself is efficient in that none of the investment opportunities is dominated in terms of a mean preserving spread, the set of optima shrinks to only three possible solutions $z_\alpha^* = \{0, 40, 80\}$. Once again, this is a consequence of the linearity property (1), which has already been responsible for the all-or-nothing decisions in the portfolio selection problem with deterministic initial wealth.

Second, the optimal investment in the risk free asset is initially increasing when decreasing the CVaR-investor's confidence level α : We observe $z_\alpha^* = 0$ for $\alpha \in (0,63, 1]$ and $z_\alpha^* = 80$ for $\alpha \in (0,25, 0,63)$, i.e., the optimum jumps from the exclusive investment in the risky asset to the exclusive investment in the risk free asset as soon as the confidence level falls below $\alpha = 0,63$. However, beyond a confidence level of $\alpha = 0,25$, the investment in the risk free asset starts to decrease again. Instead of investing in the risk free asset exclusively, the CVaR-investor for $\alpha < 0,25$ now invests $z^* = 40$ in the risk free asset and $w_0 - z^* = 80 - 40 = 40$ in the risky asset. Technically, this counter-intuitive result follows from the piecewise convex difference of the antiderivatives of the risk spectra of Conditional Value-at-Risk (see Figure 5). \square

Example 4.18. Let a be the parameter of a SRM-decision maker with exponential risk

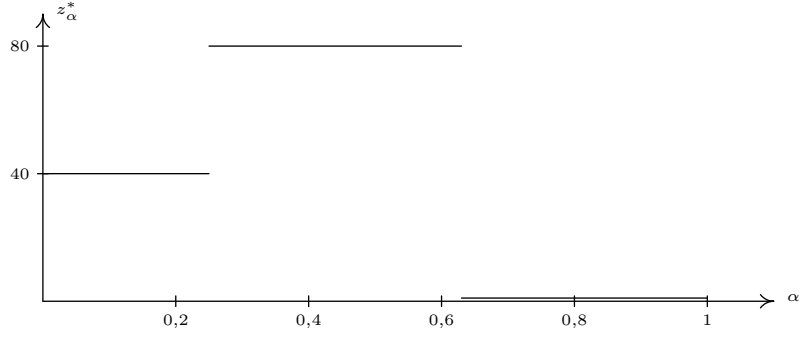


Figure 9: The optimal investment in the risk free asset as a function of the confidence level α

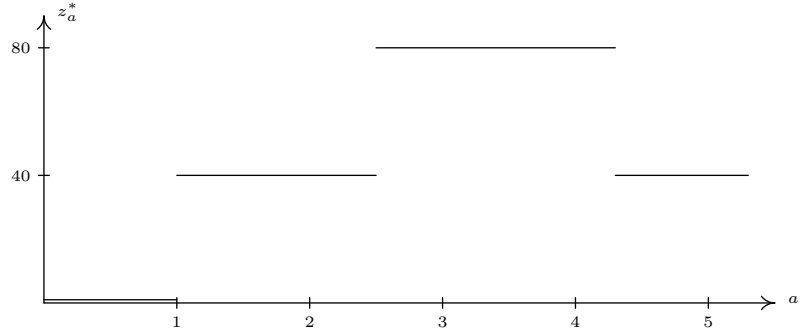


Figure 10: The optimal investment in the risk free asset as a function of the parameter a

spectrum. Let further be $w_0, X, r_0, E(r)$ and r_f be given by

$$w_0 = 80, X = \begin{cases} 8 & 0,1 \\ 8 & 0,1 \\ 10 & 0,6 \\ 10 & 0,2 \end{cases}, r_0 = \begin{cases} 0 & 0,1 \\ 0 & 0,1 \\ -0,05 & 0,6 \\ 0,15 & 0,2 \end{cases}, E(r) = 1,06, r_f = 1,043. \quad (44)$$

The optimal investment in the risk free asset, z_a^* , as a function of the parameter a is given in Figure 10. Again, we observe a tendency towards corner solutions, as the set of optimal solutions only consists of $z_a^* = \{0, 40, 80\}$. Moreover, the investment in the risk free asset is decreasing as the SRM-decision maker becomes “strongly” AP-risk averse: For $a \in (2,5, 4,3)$, the SRM-decision maker invests his entire wealth in the risk free asset, i.e. $z_a^* = 80$, while for $a \in [4,3, \infty)$ he switches back to a 50/50-investment, i.e. $z_a^* = w_0 - z_a^* = 40$. \square

Example 4.19. Let b be the parameter of a SRM-decision maker with power risk spectrum.

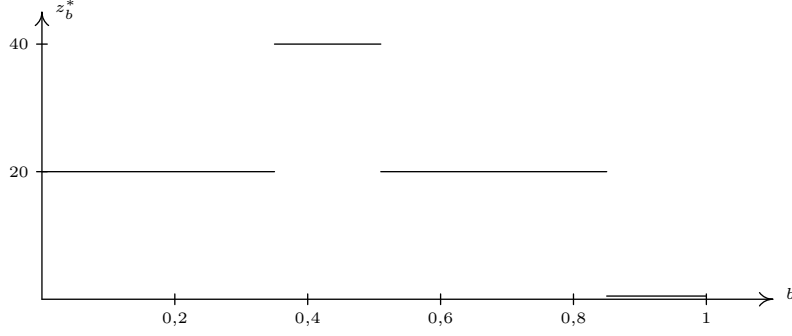


Figure 11: The optimal investment in the risk free asset as a function of the parameter b

Let further be $w_0, X, r_0, E(r)$ and r_f be given by

$$w_0 = 40, X = \begin{pmatrix} 8 & 0,1 \\ 8 & 0,1 \\ 10 & 0,6 \\ 10 & 0,2 \end{pmatrix}, r_0 = \begin{pmatrix} 0 & 0,1 \\ 0 & 0,1 \\ -0,10 & 0,6 \\ 0,30 & 0,2 \end{pmatrix}, E(r) = 1,06, r_f = 1,047. \quad (45)$$

The optimal investment in the risk free asset, z_b^* , as a function of the parameter b is given in Figure 11. Again, the set of optimal solutions only consists of three elements, namely $z_b^* = \{0, 20, 40\}$. Moreover, the investment in the risk free asset is decreasing as the SRM-decision maker becomes “strongly” AP-risk averse: For $b \in (0,35, 0,51)$, the SRM-decision maker invests his entire wealth in the risk free asset, i.e. $z_a^* = 40$, while for $b \in (0, 0,35]$ he switches back to a 50/50-investment, i.e. $z_a^* = w_0 - z_a^* = 20$. \square

5. Conclusions

In this paper, we have studied the concepts of comparative risk aversion following ARROW (1965) and PRATT (1964) on the one hand, and ROSS (1981) on the other hand, together with their implications for the willingness to pay for insurance and portfolio selection in the context of spectral risk measures.

In the framework following ARROW (1965) and PRATT (1964), we have shown that the difference of the antiderivatives of the risk spectra is the consistent measure of AP-risk aversion, and not the spectral AP-measure as commonly used in the literature. By analogy, as long as the (wrong) spectral AP-measure is applied, the less AP-risk averse SRM-decision maker may be ranked more AP-risk averse although he is not. We have further shown that the framework of ARROW (1965) and PRATT (1964) is consistent with the standard financial decision problems of the willingness to pay for insurance and portfolio selection, respectively. Consequently, the spectral AP-measure cannot be applied to these problems either, as a SRM-decision maker with a greater spectral AP-measure may only be willing

to pay less for insurance or to invest more in the risky asset than a SRM-decision maker with a smaller spectral AP-measure.

In the framework following ROSS (1981), it can be shown that the difference of the antiderivatives of the risk spectra has to be non-negative and concave on the entire support in order to provide consistent comparative static results with respect to R-risk aversion. Neither Conditional Value-at-Risk, nor exponential and power spectral risk measures satisfy these requirements, so none of these three subclasses can be completely ordered with respect to R-risk aversion. In general, spectral risk measures and R-risk aversion are incompatible concepts. As a consequence, these subclasses of spectral risk measures also exhibit counter-intuitive comparative static results with respect to the respective AP-risk aversion parameters, both in the insurance problem and the portfolio selection problem: In the insurance problem, the willingness to pay for insurance may be decreasing with increasing AP-risk aversion. Likewise, in the portfolio selection problem, the investment in the risky asset may be increasing with increasing AP-risk aversion. This will regularly happen when SRM-decision makers are assumed to be “strongly” AP-risk averse.

The paper gives directions for future research. One open question is how a comparative static “increasing risk” of the underlying random variable affects the willingness to pay for insurance or the optimal risky investment. EECKHOUDT ET AL. (1991), in the framework of classical EU-theory, have shown that the demand for insurance may be decreasing when increasing the risk of the underlying random variable. It might be interesting to see whether similar results also prevail for decision makers under the “new” setting of spectral risk measures. Another open issue is the tendency towards corner solutions in portfolio selection problems, or plunging, which is induced by the properties of translation invariance and positive homogeneity of spectral risk measures. In the recent literature, the extended notion of convex risk measures has been introduced to tackle this problem (e.g., FOELLMER/SCHIED (2002)). Convex risk measures do not need to satisfy the property of positive homogeneity and thus regularly predict portfolio diversification. It is an interesting research question whether and under which conditions convex risk measures are able to provide consistent comparative static results with respect to risk aversion, both in the frameworks of ARROW (1965) and PRATT (1964), and ROSS (1981), respectively.

A. Proof of Theorem 3.5

For Conditional Value-at-Risk, the difference of the antiderivatives of the risk spectra

$$\Phi_{\alpha_1}(p) - \Phi_{\alpha_2}(p) = \begin{cases} \frac{p}{\alpha_1} - \frac{p}{\alpha_2} & 0 \leq p \leq \alpha_1 \\ 1 - \frac{p}{\alpha_2} & \alpha_1 < p \leq \alpha_2 \\ 0 & \alpha_2 < p \leq 1 \end{cases} \quad (46)$$

for any two confidence levels $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ is non-negative on the entire support.

We next come to exponential spectral risk measures. First note that applying the spectral AP-measure defined by (14) yields

$$R_{\phi_1}(p) = -\frac{\Phi_1''(p)}{\Phi_1'(p)} \geq R_{\phi_2}(p) = -\frac{\Phi_2''(p)}{\Phi_2'(p)} \text{ for all } p \in [0, 1] \Rightarrow \Phi_1(p) \geq \Phi_2(p) \text{ for all } p \in [0, 1] \quad (47)$$

(see the proof of Theorem 3.6 below). For exponential spectral risk measures,

$$R_{\phi_a}(p) = a, \quad (48)$$

which for $a_1 \geq a_2$ yields the assertion.

Likewise, for power spectral risk measures it holds that

$$R_{\phi_b}(p) = \frac{1-b}{p}, \quad (49)$$

which for $b_2 \geq b_1$ yields the assertion. \square

B. Proof of Theorem 3.6

If-part: We first show that $R_{\phi_1}(p) \geq R_{\phi_2}(p)$ for all $p \in [0, 1] \Leftrightarrow \Phi_1(p) = g(\Phi_2(p))$ with $g(0) = 0$, $g(1) = 1$ and $g' > 0$, $g'' \leq 0$: It holds that

$$R_{\phi_1}(p) = -\frac{\Phi_1''(p)}{\Phi_1'(p)} = -\frac{\Phi_2''(p)}{\Phi_2'(p)} - \frac{g''(\Phi_2(p))}{g'(\Phi_2(p))} \cdot \Phi_2'(p) = R_{\phi_2}(p) - \frac{g''(\Phi_2(p))}{g'(\Phi_2(p))} \cdot \Phi_2'(p). \quad (50)$$

Hence, $R_{\phi_1}(p)$ is uniformly greater than $R_{\phi_2}(p)$ if and only if g satisfies $g' > 0$, $g'' \leq 0$.

Finally, for spectral risk measures we have $\Phi_1(p) = g(\Phi_2(p))$ with $g(0) = 0$, $g(1) = 1$ and $g' > 0$, $g'' \leq 0 \Rightarrow \Phi_1(p) - \Phi_2(p) \geq 0$ for all $p \in [0, 1]$ due to the concavity of Φ . This proves the assertion.

Example 3.9 shows that the Only if-part does not hold. \square

C. Proof of Theorem 4.9

The difference $\Phi_{a_1}(p) - \Phi_{a_2}(p)$, $a_1 > a_2$ is convex on $(\bar{p}, 1)$ if

$$\Phi_{a_1}''(p) - \Phi_{a_2}''(p) > 0 \quad (51)$$

$$\Leftrightarrow \frac{-a_1^2 \cdot e^{-a_1 \cdot p}}{1 - e^{-a_1}} > \frac{-a_2^2 \cdot e^{-a_2 \cdot p}}{1 - e^{-a_2}} \quad (52)$$

$$\Leftrightarrow p > \bar{p} = \frac{1}{a_2 - a_1} \cdot \ln \left(\frac{a_2^2 \cdot (1 - e^{-a_1})}{a_1^2 \cdot (1 - e^{-a_2})} \right) \quad (53)$$

It holds that $\bar{p} < 1$ if and only if

$$\frac{e^{a_1} - 1}{a_1^2} > \frac{e^{a_2} - 1}{a_2^2}. \quad (54)$$

The function

$$f(a) = \frac{e^a - 1}{a^2} \quad (55)$$

is decreasing on $(0, 1,5937]$ and increasing on $(1,5937, \infty)$, which proves 1. and 2., while 3. follows from (53) and (54). \square

D. Proof of Theorem 4.11

The difference $\Phi_{b_1}(p) - \Phi_{b_2}(p)$, $0 < b_1 < b_2 \leq 1$ is convex on $(\bar{p}, 1)$ if

$$\Phi_{b_1}''(p) - \Phi_{b_2}''(p) > 0 \quad (56)$$

$$\Leftrightarrow (b_1 - 1) \cdot b_1 \cdot p^{b_1} > (b_2 - 1) \cdot b_2 \cdot p^{b_2} \quad (57)$$

$$\Leftrightarrow p > \bar{p} = \left(\frac{b_2 \cdot (1 - b_2)}{b_1 \cdot (1 - b_1)} \right)^{\frac{1}{b_1 - b_2}} \quad (58)$$

It holds that $\bar{p} < 1$ if and only if

$$b_2 \cdot (1 - b_2) > b_1 \cdot (1 - b_1). \quad (59)$$

The function

$$f(b) = b \cdot (1 - b) \quad (60)$$

is increasing on $[0, 0,5]$ and decreasing on $(0,5, 1)$, which proves 1. and 2., while 3. follows from (58) and (59). \square

E. Proof of Theorem 4.16

The first order condition for the optimal risk free investment is given by

$$\begin{aligned} \frac{\partial \rho_\phi(\cdot)}{\partial z} &= \frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) + (E(r) - r_f) \stackrel{!}{=} 0 \Leftrightarrow \\ &\frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0) \stackrel{!}{=} \rho_\phi(E(r) - r_f) \end{aligned} \quad (61)$$

First note that $\rho_\phi(X + (w_0 - z) \cdot r_0)$ is decreasing and convex in z for $z \in [0, w_0]$: $\rho_\phi(X + (w_0 - z) \cdot r_0)$ is decreasing in z , as decreasing z constitutes a mean preserving spread, which is rejected by any SRM-decision maker. $\rho_\phi(X + (w_0 - z) \cdot r_0)$ is convex in z , as the

convexity of spectral risk measures (implied by subadditivity and positive homogeneity) for $z \in [0, w_0]$ yields

$$\begin{aligned}\rho_\phi(X + (w_0 - z) \cdot r_0) &= \rho_\phi\left(\frac{z}{w_0} \cdot X + \frac{w_0 - z}{w_0} \cdot (X + w_0 \cdot r_0)\right) \\ &\leq \frac{z}{w_0} \cdot \rho_\phi(X) + \frac{w_0 - z}{w_0} \cdot \rho_\phi(X + w_0 \cdot r_0).\end{aligned}\quad (62)$$

As $\rho_\phi(X + (w_0 - z) \cdot r_0)$ itself is decreasing and convex in z for $z \in [0, w_0]$, the left-hand side of (61), $\frac{\partial}{\partial z} \rho_\phi(X + (w_0 - z) \cdot r_0)$, is negative and increasing in z for $z \in [0, w_0]$. The right-hand side of (61) is a negative constant. The optimal risk free investment is given at the intersection of the left-hand side and the right-hand side.

Let ϕ_1 and ϕ_2 be the risk spectra of two SRM-decision makers. Then $z_{\phi_1}^*(X, r) \geq z_{\phi_2}^*(X, r)$ for all $X, r \in \mathcal{X}, r \neq E(r), X \neq E(X), E(r) > r_f, r_0 = r - E(r), E(r_0|X) = 0$ if and only if

$$\frac{\partial}{\partial z} \rho_{\phi_1}(X + (w_0 - z) \cdot r_0) \leq \frac{\partial}{\partial z} \rho_{\phi_2}(X + (w_0 - z) \cdot r_0) \text{ for all } z \in [0, w_0], \quad (63)$$

i.e., if and only if the left-hand side of (61) for the SRM-decision maker with risk spectrum ϕ_1 is greater than (or equal to) the one of the SRM-decision maker with risk spectrum ϕ_2 for all $z \in [0, w_0]$. Condition (63) is satisfied if and only if

$$\begin{aligned}RP_{\phi_1}(X + (w_0 - z_2), (z_2 - z_1) \cdot r_0) \\ &= \rho_{\phi_1}(X + (w_0 - z_1) \cdot r_0) - \rho_{\phi_1}(X + (w_0 - z_2) \cdot r_0) \\ &\geq \rho_{\phi_2}(X + (w_0 - z_1) \cdot r_0) - \rho_{\phi_2}(X + (w_0 - z_2) \cdot r_0) \\ &= RP_{\phi_2}(X + (w_0 - z_2), (z_2 - z_1) \cdot r_0)\end{aligned}\quad (64)$$

for all $X, r \in \mathcal{X}, r \neq E(r), X \neq E(X), E(r) > r_f, r_0 = r - E(r), E(r_0|X) = 0$ and for all $0 \leq z_1 \leq z_2 \leq w_0$. This proves the assertion. \square

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