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# Estimation of NAIRU with Inflation Expectation Data

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# Estimation of NAIRU with Inflation Expectation Data\*

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## Abstract

Estimating natural rate of unemployment (NAIRU) is important for understanding the joint dynamics of unemployment, inflation, and inflation expectation. However, existing literature falls short in endogenizing inflation expectation together with NAIRU in a model consistent way. We develop and estimate a structural model with forward and backward looking Phillips curve. Inflation expectation is treated as a function of state variables and we use survey data as its observations. We find out that the estimated NAIRU using our methodology tracks the unemployment process closely except for the high inflation period around 1970. Moreover, the estimated Bayesian credible sets are narrower and our model leads to better inflation and unemployment forecasts. These results suggest that monetary policy was very effective during the sample periods and there was not much room for policy improvement.

*Keywords:* NAIRU; Inflation Expectation;

*JEL Classification:* C32; E23; E24

## 1 Introduction

There is one long-lasting idea in macroeconomics since [Friedman \(1968\)](#): inflation will increase if unemployment is below the natural rate, or known as NAIRU (non-accelerating inflation rate of unemployment). Numerous work, such as [Staiger, Stock, and Watson \(1997a,b\)](#),

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estimate NAIRU by using the so called Phillips curve with inflation, unemployment rate, and the NAIRU. Though the curve might take different specification, it targets the short-run trade-offs between inflation and unemployment due to nominal price and/or wage rigidities, i.e., nominal frictions.

Historically, the Phillips curve was one of the most controversial topic of the post-war period. Important contribution at least includes [Blanchard and Katz \(1997\)](#), [Gordon \(1997\)](#), [Staiger, Stock, and Watson \(1997a,b\)](#). As summarized by [King and Watson \(1994\)](#), there was a large division on whether there are such inflation-unemployment trade-offs. However, a general consensus seems to be that the trade-offs are significant and stable in business cycle frequencies. [Figure 1](#) made this point evident by decomposing unemployment rate and inflation into three frequencies. The plot with business cycle frequency (18 to 60 months) in the middle panel shows a significant negative correlation ( $-0.46$ ) between the two time series. In the long run, unemployment rate is not negatively correlated with inflation.

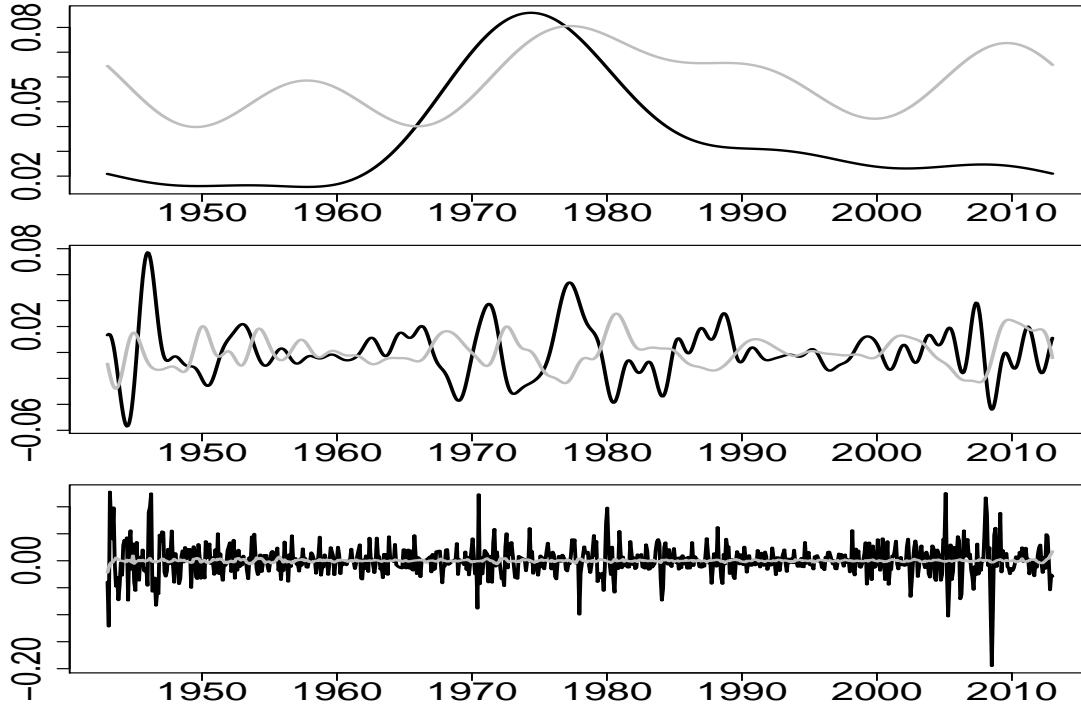
Given the different long-run trends of inflation and unemployment, one should account for the possibility of a time-varying NAIRU in estimation, either with a deterministic time trend or a stochastic trend. Second, as inflation has its own dynamics, inflation expectation should contain valuable information. Note that a surprised inflation will tend to reduce unemployment in the short run. However, the consideration of time-varying NAIRU and inflation expectation gives rise at least two key challenges: (1) What are the state variable(s)? (2) How do NAIRU, inflation, and inflation expectation depend on the state variable(s)?

We propose a simple framework that links inflation, inflation expectation, unemployment, and GDP together. The key of this framework is to solve inflation expectation endogenously from the model and then use a survey data of inflation expectation as the noisy observation of the state variables. Some recent work use inflation expectation similar to our approach (e.g. [Del-Negro and Eusepi \(2011\)](#) and [Del-Negro, Giannoni, and Schorfheide \(2014\)](#)), but information on unemployment is omitted. We find that unemployment with survey inflation expectation data suggest very small output gaps and unemployment gaps.

Recent studies use filtering to estimate hidden state variables such as [Apel and Jansson \(1999\)](#), [Laubach \(2001\)](#), and [Ferri, Greenberg, and Day \(2003\)](#). But inflation expectation is simply set to inflation realized in last period, as U.S. inflation is relatively stable after 1990. A notable exception is [Basistha and Nelson \(2007\)](#), where survey data of inflation expectation is treated as a control variable in the filtering problem. Nevertheless, this approach is subject to endogeneity issues. Expectation itself is endogenous and might have its own internal dynamics.

Specifically we extend the basic forward looking New Keynesian Phillips curve (see e.g.,

Figure 1: **Filtered Unemployment Rate and Inflation**



Note: filtered time series for both unemployment rate (grey) and inflation rate (black) 1948:01-2013:03 in zero frequency, business cycle frequency (18 months to 60 months), and the rest. The correlation is -0.46 in business cycle frequency. Inflation is annualized CPI growth rate.

Goodfriend and King (1997), Gali (1999) and Woodford (2003)) to allow for a subset of firms that set prices according to a backward looking rule (similar to Gali and Gertler (1999)). In addition, we incorporate unemployment by linking output gaps and unemployment gaps implied through the Okun’s Law. This setup facilitates us to have a rich model for NAIRU estimation, while keeping the common tractability of a standard new Keynesian model.

The model is cast into a state-space form after which Kalman filter and Bayesian estimation are used. We further apply Markov Chain Monte Carlo (MCMC) to approximate posteriors of the parameters. A Bayesian approach allows us to incorporate reasonable priors on the parameters and easily attain numerically stable estimates. Moreover, the induced Bayesian credible sets facilitate us to carry out inference for NAIRU.

We compare two estimation exercises: one using the survey inflation expectation data (obtained from Michigan Consumer Survey data set), and the other without using expectation data. Once the expectation data is used, we show that (surprisingly different from the existing literature) unemployment gaps and output gaps shrink to 0.5% magnitude from 1% magnitude when no survey data is used. NAIRU thus shifts from relatively stable around 4.5% - 5% over time to moving closely to observed unemployment rate if we use the inflation

expectation data. More importantly, the 5%-95% intervals of NAIRU shrink to almost zero when we use the survey data. Nevertheless, when inflation expectation data is removed, we obtained large output and unemployment gaps (i.e., NAIRU is a smooth version of the unemployment rate), and the 5%-95% intervals amount to 1%.

Our findings suggest that expectation data indeed contains valuable information of the underlying economy. Given the existence of nominal rigidities in the model and in practice, monetary policy is very effective to dampen shocks such that the observed U.S. economy is very close to an economy without nominal rigidities. Therefore, NAIRU is not very different from observed unemployment, except for the accelerating inflation period in 1970s which reflects [Friedman \(1968\)](#)'s original idea.

Intuitively, private agents know more information than us who are performing econometric exercises. While the information is generally hidden, we still can extract some information by looking at their forecast of future variables (which in our case is inflation expectation). Importantly, we do not model monetary policy explicitly, but the inflation expectation can indicate private agents' belief on the direction of monetary policy as well as economic fundamental shocks.

The remaining sections of the paper are organized as follows. Section 2 describes the basic model and express inflation expectation as a function of the state variables. Section 3 transform the model into a state space form. In addition, we describe the data set and link that to the state space form. The basic results and discussions are in Section 4, where we compare estimation with and without using survey data. Section 5 concludes.

## 2 The Model

We start with model that incorporates inflation, unemployment, output, inflation expectation, together with unemployment and output gaps. The basic structure is in-line with previous studies such as [Basistha and Nelson \(2007\)](#).

### 2.1 Output and Unemployment

Denote  $Y_t$  as the real GDP and  $y_t = \log Y_t$  as the natural log of the real GDP. We label  $y_t$  as the realized output at time  $t$ . Denote  $y_t^n$  as the potential output, i.e., the natural log of GDP in absence of nominal price/wage rigidities. Then, the output gap  $y_t^g$  at time  $t$  is

the difference between the realized output and the potential output, which satisfies

$$y_t = y_t^n + y_t^g. \quad (1)$$

Note that  $y_t^g$  is approximately the percentage deviation of the real GDP to its potential level. To see this, denote the potential GDP as  $Y_t^n$ . Note that  $y_t^g = \log Y_t - \log Y_t^n = \log(1 + \frac{Y_t - Y_t^n}{Y_t^n}) \approx \frac{Y_t - Y_t^n}{Y_t^n}$ .

Following the tradition at least since [Watson \(1986\)](#), the potential output is assumed to follow a random walk with drift  $\mu$ , as real GDP exhibits a growth trend

$$y_t^n = \mu + y_{t-1}^n + \varepsilon_t^n. \quad (2)$$

where  $\varepsilon_t^g \sim N(0, \sigma_g^2)$ . By definition, output gaps can only be transitory. In the absence of any inflation surprise, potential output should incorporate all trend movement. To allow for sluggishness in the output dynamics, such as in [Watson \(1986\)](#), we assume that output gap follows an AR(2) process

$$y_t^g = \rho_1 y_{t-1}^g + \rho_2 y_{t-2}^g + \varepsilon_t^g. \quad (3)$$

where  $\varepsilon_t^g \sim N(0, \sigma_g^2)$ .

Now, we turn to unemployment. Denote the unemployment rate at time  $t$  as  $u_t$  and the NAIRU as  $u_t^n$ . Then, the unemployment gap  $u_t^g$  satisfies:

$$u_t = u_t^n + u_t^g. \quad (4)$$

A random walk stochastic process for the NAIRU process is reasonable, since the unemployment rate is very persistent, which follows in fact previous studies on the U.S. NAIRU (see e.g., [Laubach \(2001\)](#) and [Basistha and Nelson \(2007\)](#)),

$$u_t^n = u_{t-1}^n + \varepsilon_t^u. \quad (5)$$

where  $\varepsilon_t^u \sim N(0, \sigma_u^2)$ . Note that the natural rate reflects fundamental labor market outcome which might be involved with search and matching between firms and workers. We choose not to model a search and matching equilibrium as in [Shimer \(2005\)](#) to determine the transitory unemployment gaps, but simply link the unemployment gaps to output gaps through the *Okun's Law*, a statistical significant rule that links output and unemployment. Doing so simplifies the analysis, as we can focus on inflation and inflation expectation dynamics.

That is, unemployment gaps can be expressed as

$$u_t^g = \eta_0 y_t^g + \eta_1 y_{t-1}^g. \quad (6)$$

## 2.2 Inflation and The (New Keynesian) Phillips Curve

Our economy features a standard forward-looking New Keynesian Phillips Curve. In addition, we also incorporate backward-looking behavior to account for persistent inflation in the data. The following briefly describes the setting of our analysis with details in the Appendix.

There are intermediate goods firms and final goods firms. Assume that intermediate firms are identical at the beginning, but produce differentiated products for their pricing history. They discount future profits at a rate  $\beta \in (0, 1)$ . In addition, final goods firms are competitive and assemble intermediate goods to produce consumption goods.

Let  $P_t$  be the nominal price of final goods and  $P_{it}$  be the nominal price of intermediate good  $i$ . Naturally, the (gross) inflation rate at time  $t$  is:

$$\Pi_t = P_t/P_{t-1}.$$

Each intermediate firm can change price with a probability  $1 - \alpha$  where  $\alpha \in (0, 1)$ . For example, if  $\alpha = 0.75$  for a quarterly model, prices are fixed on average for  $1/(1 - \alpha) = 4$  quarters, or a year. With a probability  $\alpha$ , it must keep its price unchanged, except for adjustment indexed to past inflation and trend inflation.  $\alpha$  therefore represents the price rigidities. That is, if firm  $i$  cannot adjust its price, the price  $P_{it}$  will be

$$P_{it} = P_{it-1} \bar{\Pi}^{1-\zeta} \Pi_{t-1}^\zeta.$$

where  $\zeta$  measures the elasticity of the indexation to past inflation and  $\bar{\Pi}$  is the steady state (trend) inflation.

Firms' optimal price setting behaviors leads to a almost standard New Keynesian Phillips curve (NKPC) (details in the Appendix). Finally, we add exogenous push shocks  $\varepsilon_t^\pi \sim N(0, \sigma_\pi^2)$  to inflation such as monetary policy shocks or exogenous oil price movements that are outside of our model. Then, if we denote  $\pi_t$  as inflation's percentage deviation from its steady state  $\bar{\Pi}$ , the NKPC can be written as

$$\pi_t = \gamma_f \mathbf{E}_t [\pi_{t+1}] + \gamma_b \pi_{t-1} + \lambda \kappa y_t^g + \varepsilon_t^\pi, \quad (7)$$



where  $\kappa$  is some parameter related to household and  $\lambda$ ,  $\gamma_f$ , and  $\gamma_b$  are

$$\lambda = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+\zeta\beta)}, \quad \gamma_f = \frac{\beta}{1+\zeta\beta}, \quad \gamma_b = \frac{\zeta}{1+\zeta\beta}.$$

This NKPC curve is intuitive. Inflation depends on past inflation as some firms will adjust price relative to previous inflation. In addition, since some firms take into account of future inflation in adjusting prices, Inflation also depends on the expectation of future inflation. Besides past and future inflation, a demand driven rapid production today reflected by a higher output gap  $y_t^g$  will push up inflation.

## 2.3 Solving Forward

Now, we solve the NKPC forward to express inflation as a function of state variables. This is the key step before expressing inflation expectation as a function of state variables. First, express inflation as a function of state variables and rewrite (16) as

$$\gamma_0\pi_t - \gamma_b\pi_{t-1} = \gamma_1 \mathbf{E}_t [\gamma_0\pi_{t+1} - \gamma_b\pi_t] + \lambda\kappa y_t^g + \varepsilon_t^\pi,$$

where

$$\gamma_0\gamma_1 = \gamma_f, \quad \gamma_0 + \gamma_1\gamma_b = 1,$$

from which we solve forward (assuming  $\gamma_1 < 1$  which is verified in the appendix)

$$\gamma_0\pi_t - \gamma_b\pi_{t-1} = \lim_{k \rightarrow \infty} (\gamma_1)^k \mathbf{E}_t [\gamma_0\pi_{t+k} - \gamma_b\pi_{t+k-1}] + \lambda\kappa \sum_{s=0}^{\infty} \gamma_1^s \mathbf{E}_t [y_{t+s}^g] + \varepsilon_t^\pi.$$

Second, notice that  $\lim_{k \rightarrow \infty} (\gamma_1)^k \mathbf{E}_t [\gamma_0\pi_{t+k} - \gamma_b\pi_{t+k-1}] \rightarrow 0$  *a.s.*, as we log-linearize around the steady state inflation, then

$$\pi_t = \frac{\gamma_b}{\gamma_0}\pi_{t-1} + \frac{\lambda\kappa}{\gamma_0} \sum_{s=0}^{\infty} \gamma_1^s \mathbf{E}_t y_{t+s}^g + \varepsilon_t^\pi.$$

This expression is useful because one can see directly what inflation indexation is needed to avoid exploding equilibrium. We know that  $0 < \gamma_f < 1$ ,  $0 < \gamma_b < 1$ , and  $\gamma_f + \gamma_b \leq 1$ , and we further prove that  $\gamma_f + \gamma_b < 1$  in the following.

### Proposition 1:

*In this framework, the necessary condition for steady state  $\bar{\Pi}$  to exist is that  $\gamma_f + \gamma_b < 1$ .*

That is,  $\zeta < 1$  and there cannot be full indexation to past inflation.

*PROOF.* See the Appendix. □

That is, we do not allow the coefficients of forward looking and backward looking inflation sum up to 1. In the previous empirical literature, such as [Sahuc \(2006\)](#), they found very unlikely that the coefficients sum up to 1 as well. Nevertheless, the proposition is specific to this type of framework. In a more complex economy, other aspects such as monetary and fiscal policy can help stabilize the economy and have a unique stable equilibrium.

In addition, according to [Blanchard and Kahn \(1980\)](#), one needs as many eigenvalues that are larger than 1 as the number of forward looking variables in the system. Nevertheless, there is a degree of freedom to classify the unit root inflation to be in the group with eigenvalues larger than 1 or in the group with eigenvalues smaller than 1. In general, we want to avoid this scenario and that is why  $\zeta \neq 1$ .

Finally, we further simplify the expression  $\sum_{s=0}^{\infty} \gamma_1^s \mathbf{E}_t [y_{t+s}^g]$ . Using the specification for  $y_t^g$  in (3) and stacking  $y_t^g$  and  $y_{t-1}^g$  together to be  $Y_t^g = [y_t^g, y_{t-1}^g]^\top$ ,

$$Y_t^g = A_1 Y_{t-1}^g + \xi_t^g,$$

where

$$A_1 = \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix}, \quad \xi_t^g = \begin{bmatrix} \varepsilon_t^g \\ 0 \end{bmatrix}.$$

By repeated iterations  $\mathbf{E}_t [Y_{t+1}^g] = A_1 Y_t^g$ ,  $\mathbf{E}_t [Y_{t+2}^g] = A_1^2 Y_t^g$ , ...,

$$\begin{aligned} \sum_{s=0}^{\infty} \gamma_1^s \mathbf{E}_t [Y_{t+s}^g] &= (I_{2 \times 2} - \gamma_1 A_1)^{-1} Y_t^g \\ &= \frac{1}{1 - \gamma_1 \rho_1 - \gamma_1^2 \rho_2} \begin{bmatrix} 1 & \gamma_1 \rho_2 \\ \gamma_1 & 1 - \gamma_1 \rho_1 \end{bmatrix} \begin{bmatrix} y_t^g \\ y_{t-1}^g \end{bmatrix}, \end{aligned}$$

from which we take the first component and write

$$\sum_{s=0}^{\infty} \gamma_1^s \mathbf{E}_t [y_{t+s}^g] = [1, 0] \sum_{s=0}^{\infty} \gamma_1^s \mathbf{E}_t [Y_{t+s}^g] = \frac{y_t^g + (1 - \gamma_1 \rho_1) y_{t-1}^g}{1 - \gamma_1 \rho_1 - \gamma_1^2 \rho_2}.$$

Following this result, we simplify the NKPC (16) to

$$\pi_t = \theta_0 y_t^g + \theta_1 y_{t-1}^g + \theta_2 \pi_{t-1} + \varepsilon_t^\pi, \quad (8)$$

where

$$\theta_0 = \frac{\lambda\kappa}{\gamma_0(1 - \gamma_1\rho_1 - \gamma_1^2\rho_2)}, \quad \theta_1 = \frac{\lambda\kappa(1 - \gamma_1\rho_1)}{\gamma_0(1 - \gamma_1\rho_1 - \gamma_1^2\rho_2)}, \quad \theta_2 = \frac{\gamma_b}{\gamma_0}.$$

Note that all the above derivation requires  $\gamma_1 < 1$ . In the Appendix, we show that the only solution with  $\gamma_1 < 1$  is

$$\gamma_1 = \frac{1 - \sqrt{1 - 4\gamma_f\gamma_b}}{2\gamma_b} < 1, \quad \gamma_0 = \frac{1 + \sqrt{1 - 4\gamma_f\gamma_b}}{2}.$$

## 2.4 Inflation Expectation

When inflation is expressed as a function of state variable, so can be inflation expectation. To see this, we begin with

$$\begin{aligned} \mathbb{E}_t[\pi_{t+1}] &= \mathbb{E}_t[[\theta_0, \theta_1](A_1 Y_t^g + \xi_{t+1}^\pi) + \theta_2 \pi_t + \varepsilon_{t+1}^\pi] \\ &= [\theta_0, \theta_1] A_1 Y_t^g + \theta_2 \pi_t \\ &= (\theta_1 + \rho_1 \theta_0) y_t^g + \rho_2 \theta_0 y_{t-1}^g + \theta_2 \pi_t. \end{aligned} \quad (9)$$

In our numerical analysis, the Michigan consumer survey data approximates expected inflation. The survey data asks  $k$ -step ahead inflation forecasts. For example, the Michigan consumer survey asks what is 1 year ahead and 5 year ahead inflation forecast. This fact implies that we need to derive a  $k$ -step ahead inflation expectation:

$$\mathbb{E}_t \left[ \frac{P_{t+k}}{P_t} \right] = \mathbb{E}_t \left[ \frac{P_{t+1}}{P_t} \frac{P_{t+2}}{P_{t+1}} \dots \frac{P_{t+k-1}}{P_{t+k-2}} \frac{P_{t+k}}{P_{t+k-1}} \right] = \mathbb{E}_t [\Pi_{t+1} \Pi_{t+2} \dots \Pi_{t+k}].$$

whose log-linearized version is  $\mathbb{E}_t[\pi_{t+1} + \pi_{t+2} + \dots + \pi_{t+k}]$ . Then, following (9), we have

$$\begin{aligned} \mathbb{E}_t[\pi_{t+2}] &= \mathbb{E}_t[[\theta_0, \theta_1](A_1^2 Y_t^g + A_1 \xi_{t+1}^\pi + \xi_{t+2}^\pi) + \theta_2 \pi_{t+1} + \varepsilon_{t+2}^\pi] \\ &= [\theta_0, \theta_1](A_1^2 + \theta_2 A_1) Y_t^g + \theta_2^2 \pi_t, \\ &\quad \dots \\ \mathbb{E}_t[\pi_{t+k}] &= [\theta_0, \theta_1](A_1^k + \theta_2 A_1^{k-1} + \theta_2^2 A_1^{k-2} + \dots + \theta_2^{k-1} A_1) Y_t^g + \theta_2^k \pi_t. \end{aligned}$$

Now, the  $k$ -step ahead inflation forecast is

$$\begin{aligned}
\mathbf{E}_t [\pi_{t+1} + \dots + \pi_{t+k}] &= [\theta_0, \theta_1] A_1 [(I + A_1 + A_1^2 + \dots + A_1^{k-1}) + \theta_2(I + A_1 + \dots + A_1^{k-2}) \\
&\quad + \theta_2^2(I + A_1 + \dots + A_1^{k-3}) + \dots + \theta_2^{k-2}(I + A_1) + \theta_2^{k-1}I] Y_t^g + \frac{1 - \theta_2^k}{1 - \theta_2} \theta_2 \pi_t \\
&= A_1 (I - A_1)^{-1} [(I - A_1^k) + \theta_2(I - A_1^{k-1}) + \dots + \theta_2^{k-1}(I - A_1)] Y_t^g \\
&\quad + \frac{1 - \theta_2^{k+1}}{1 - \theta_2} \theta_2 \pi_t,
\end{aligned}$$

where we use matrix geometric summation to simplify the expression. By using

$$\begin{aligned}
(I - A_1^k) + \theta_2(I - A_1^{k-1}) + \dots + \theta_2^{k-1}(I - A_1) &= \frac{1 - \theta_2^k}{1 - \theta_2} I - \theta_2^{k-1} A_1 \left\{ I + \theta_2^{-1} A_1 + \dots + \theta_2^{-(k-1)} A_1^{k-1} \right\} \\
&= \frac{1 - \theta_2^k}{1 - \theta_2} I - \theta_2^{k-1} A_1 (I - \theta_2^{-1} A_1)^{-1} (I - \theta_2^{-k} A_1^k),
\end{aligned}$$

we obtain the final expression for the  $k$ -step ahead inflation forecasts

$$\begin{aligned}
\mathbf{E}_t [\pi_{t+1} + \dots + \pi_{t+k}] &= [\theta_0, \theta_1] A_1 (I - A_1)^{-1} \left\{ \frac{1 - \theta_2^k}{1 - \theta_2} I - \theta_2^{k-1} A_1 (I - \theta_2^{-1} A_1)^{-1} (I - \theta_2^{-k} A_1^k) \right\} Y_t^g \\
&\quad + \frac{1 - \theta_2^k}{1 - \theta_2} \theta_2 \pi_t. \tag{10}
\end{aligned}$$

As a comparison, the simplest scenario is when  $k = 1$ . Then, (10) becomes (9). For 1 year ahead inflation forecasts, the model's 4 quarters ahead inflation expectation is

$$\mathbf{E}_t [\pi_{t+1} + \dots + \pi_{t+k}] = \xi_0^e y_t^g + \xi_1^e y_{t-1}^g + \frac{(1 - \theta_2^5) \theta_2}{1 - \theta_2} \pi_t,$$

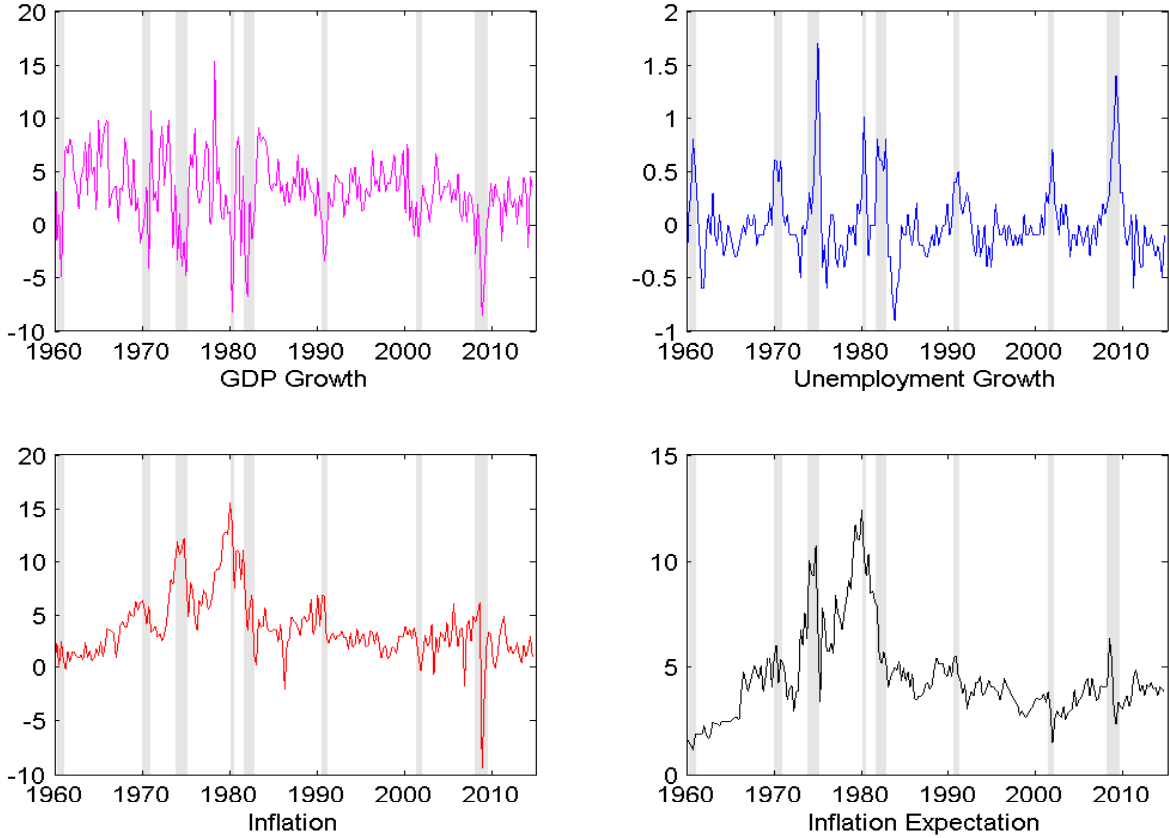
where  $\zeta_0$  and  $\zeta_1$  are the first and second elements of

$$[\theta_0, \theta_1] A_1 (I - A_1)^{-1} \left\{ \frac{1 - \theta_2^k}{1 - \theta_2} I - \theta_2^{k-1} A_1 (I - \theta_2^{-1} A_1)^{-1} (I - \theta_2^{-k} A_1^k) \right\}.$$

### 3 The State Space Form and the Data

We use quarterly (annualized) real GDP for  $Y_t$ , unemployment rate in percentage for  $u_t$ , consumer price level CPI for  $P_t$ , and one-year ahead inflation expectation for  $\Pi_t^{e4}$  which is the mean value taken from Michigan Consumer survey. All time series are from 1960:Q1 to 2014:Q3 to accommodate the survey data range. The raw data are presented in Figure 2.

Figure 2: **Raw Data**



Note: All time series are in percentage terms. Real GDP, unemployment, and CPI are from FRED data set maintained by Federal Reserve at St. Louis. One year ahead inflation expectation is from Michigan Consumer Survey data set. Shaded area denote NBER dated recessions.

We further transform the model into the conventional state space form for econometric analysis. Notice that there are several approaches to specify the state space form. We specify the output observation equation taking the form of output growth, instead of output level itself. That is, we take the first difference of the GDP data. The key reason is the following. [Watson \(1986\)](#) has proved that if one tries to estimate  $y_t^n$  (a unit root process with drift) and  $y_t^g$  (a stationary process) from the raw output  $y_t$  data, one cannot uniquely pin down the trend and the cycles unless assuming either zero correlation or perfect correlation between the innovation of these two processes. However, [Morley, Nelson, and Zivot \(2003\)](#) shows that if the goal is only to back out the trend and the cycle without directly estimating the correlation, one does not have identification issues. Therefore, one can first difference the data and correlation can be backed out by the estimated trend and cycle.

We follow the second approach by differencing data and set up the state equations.

Intuitively, we only need to keep track of output gaps and past inflation as the state variables. This is because the natural output levels can be directly calculated once we know output gaps (note: realized output levels are directly observed) and unemployment gaps can be backed out by the Okun’s law; past inflation is needed in the Phillips curve. Specifically, we use (1)-(6), (8), and (9) to express the system. The “state equations” are

$$y_t^g = \rho_1 y_{t-1}^g + \rho_2 y_{t-2}^g + \varepsilon_t^g,$$

$$\pi_t = \theta_0 y_t^g + \theta_1 y_{t-1}^g + \theta_2 \pi_{t-1} + \varepsilon_t^\pi.$$

When observations multiplied by 400 or 100 to adjust the data to annualized increase, we express the “Measurement equations” as:

(1) Real output growth (% , annualized)

$$100 \{ \log(Y_t) - \log(Y_{t-1}) \} = 400 \{ (\rho_1 - 1) y_{t-1}^g + \rho_2 y_{t-2}^g + \varepsilon_t^g + \varepsilon_t^n + \mu_y \},$$

where again  $Y_t$  is the real GDP.

(2) Unemployment growth (%)

$$u_t - u_{t-1} = 100 \{ \varepsilon_t^u + \eta_0 (y_t^g - y_{t-1}^g) + \eta_1 (y_{t-1}^g - y_{t-2}^g) \}.$$

(3) Inflation (% , annualized)

$$400 \{ \log(P_t) - \log(P_{t-1}) \} = 400 (\pi_t + \mu_\pi).$$

(4) Inflation expectation (% , annualized)

$$\Pi_t^{e4} = 100 \left\{ \xi_0^e y_t^g + \xi_1^e y_{t-1}^g + \frac{(1 - \theta_2^5) \theta_2}{1 - \theta_2} \pi_t + \varepsilon_t^e + 4\mu_\pi^e + 4\mu_\pi \right\}.$$

where  $\mu_\pi^e$  is a constant term representing the sample survey systematic difference from the model, and  $\varepsilon_t^e \sim N(0, \sigma_e^2)$  is the measurement errors.  $\mu_\pi^e$  could come from various sources. For instance, the model use the “representative” consumer’s inflation expectation and consumers pay attention to every product. The surveyed respondents, however, may only pay attention to only a subset of consumer products whose inflation might not perfectly correlated with all-product inflation.

The “state equations” and “measurement equations” can be rewritten as the state space

canonical form, after which we use Kalman filter and Bayesian methods to estimate the parameters and corresponding unobserved components. Define the state and the noise as

$$\mathbf{s}_t \stackrel{\text{def}}{=} [y_t^g, y_{t-1}^g, y_{t-2}^g, \pi_t]^\top, \quad \varepsilon_t \stackrel{\text{def}}{=} [\varepsilon_t^g, 0, 0, \varepsilon_t^\pi]^\top.$$

Then, the state transition is

$$\mathbf{s}_t = \mathbf{A}\mathbf{s}_{t-1} + \varepsilon_t, \quad (11)$$

where  $\mathbf{A}$  and the variance-covariance matrix  $\Omega$  of  $\varepsilon_t$  is

$$\mathbf{A} = \begin{bmatrix} 1 & & & -\theta_0^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & \\ & 1 \\ \theta_1 & & \theta_2 \end{bmatrix},$$

$$\Omega = \begin{bmatrix} 1 & & & -\theta_0^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \sigma_g^2 & & \sigma_{g\pi} \\ & 0 & \\ & & 0 \\ \sigma_{\pi g} & & \sigma_\pi^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -\theta_0^{-1} & & & 1 \end{bmatrix}.$$

Define the observations and observation errors are

$$\mathbf{y}_t \stackrel{\text{def}}{=} \begin{bmatrix} 100 \{\log(Y_t) - \log(Y_{t-1})\} \\ u_t - u_{t-1} \\ 400 \{\log(P_t) - \log(P_{t-1})\} \\ \Pi_t^{e4} \end{bmatrix}, \quad \nu_t \stackrel{\text{def}}{=} [400\varepsilon_t^n, 100\varepsilon_t^u, 0, 100\varepsilon_t^e]^\top.$$

Those are linked to the state

$$\mathbf{y}_t = \mathbf{H}\mathbf{s}_t + \mathbf{B} + \nu_t, \quad (12)$$

where  $\mathbf{H}$ ,  $\mathbf{B}$ , and the variance-covariance matrix  $\Sigma$  of  $\nu_t$  are:

$$\mathbf{H} = \begin{bmatrix} 400 & -400 & & \\ 100\eta_0 & 100(\eta_1 - \eta_0) & -100\eta_1 & \\ & & & 400 \\ 100\xi_0^e & 100\xi_1^e & & \frac{100(1-\theta_2^5)\theta_2}{1-\theta_2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 400\mu_y \\ 0 \\ 400\mu_\pi \\ 400\mu_\pi + 400\mu_\pi^e \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} 1.6 \times 10^5 \sigma_n^2 & 4 \times 10^4 \sigma_{un} & & & & \\ 4 \times 10^4 \sigma_{nu} & 10^4 \sigma_u^2 & & & & \\ & & & & & \\ & & & & 0 & \\ & & & & & \sigma_e^2 \end{bmatrix}.$$

The deep parameters to be estimated (with some calibrated) are

$$(\beta, \alpha, \kappa, \mu_y, \mu_\pi, \mu_\pi^e, \rho_1, \rho_2, \eta_0, \eta_1)$$

together with the standard deviation of the shocks and correlation of the shocks

$$(\sigma_n, \sigma_g, \sigma_u, \sigma_\pi, \sigma_e, \rho_{ng}, \rho_{nu}, \rho_{n\pi}, \rho_{gu}, \rho_{g\pi}, \rho_{u\pi}),$$

where the subscripts indicate the correlation between two variables. For example,  $\rho_{ng} = \sigma_{ng} \sigma_n^{-1} \sigma_g^{-1}$  is the correlation between  $\varepsilon_t^n$  and  $\varepsilon_t^g$ .

(11) and (12) form a system that can be handled via the Kalman filter. We derive the likelihood conditioning seeing all the observations, given parameters. Then we proceed to Bayesian estimation and use Markov Chain Monte Carlo (MCMC) with a random walk Metropolis-Hasting to calculate the posterior and produce the estimates of unobserved components. More detailed Bayesian methods for structural macro model can be found e.g. in [An and Schorfheide \(2007\)](#).

## 4 Results

We launch the Bayesian structural estimation in this section. The estimation exploits survey data and compare with situation without survey data. The estimated NAIRU and unemployment gap are quite different in these two cases.

### 4.1 Priors

We keep  $\beta$  and  $\kappa$  at a calibrated value, since data we use are not very informative on that.  $\beta$  is directly linked to interest rate. The inverse of  $\beta$  should be equal to the real interest rate. Historical real interest rate can be approximated by around 4% annually (see [Mehra and Prescott \(1985\)](#)) which translates into  $\beta = 0.99$  in our quarterly frequency.  $\kappa$  is the sum of intertemporal elasticity of substitution and disutility from labor supply. Notice that intertemporal elasticity of substitution is equal to risk-aversion which is usually set to 1 or



2. The disutility from labor supply is usually 3 implying a saving rate of wages around 33%. We set  $\kappa = 4$ .

Table 1: Priors

Parameters	Prior Distribution	Prior Mean	Prior Standard Deviation
$\zeta$	gamma	0.5000	0.3000
$\mu_y$	normal	0.0076	0.0020
$\mu_\pi$	normal	0.0076	0.0020
$\rho_1$	normal	1.3500	0.0500
$\rho_2$	normal	-0.5000	0.0500
$\eta_0$	normal	-0.4000	0.1000
$\eta_1$	normal	0.0000	0.2000
$\sigma_n$	inverse gamma	0.0085	0.0010
$\sigma_g$	inverse gamma	0.0020	0.0010
$\sigma_\pi$	inverse gamma	0.0076	0.0010
$\sigma_u$	inverse gamma	0.0016	0.0050
$\sigma_e$	inverse gamma	0.0020	0.0010
$\rho_{ng}$	normal	0.0000	0.0025
$\rho_{n\pi}$	normal	0.0000	0.0025
$\rho_{nu}$	normal	0.0000	0.0025
$\rho_{g\pi}$	normal	0.0000	0.0025
$\rho_{gu}$	normal	0.0000	0.0025
$\rho_{u\pi}$	normal	0.0000	0.0025

Notes:  $\alpha = 0.75$ ,  $\beta = 0.99$ , and  $\kappa = 4$  are calibrated

Next, we illustrate the parameters' prior choices.  $\alpha$  measures how frequent the average price adjustment can be in the economy. Existing micro studies and macro estimation (see e.g. [Gali and Gertler \(1999\)](#)) all point to about 3 quarters to 6 quarters adjustment and we thus center  $\alpha$  around 0.75 with a Normal prior i.e., there will be an average  $1/(1 - \alpha) = 4$  quarters adjustment periods.  $\zeta$  measures the indexation degree of price adjustment to past inflation. It thus control inflation persistence. We center it around 0.5, choose a Gamma prior, give it more weight on values smaller than 0.5 such that it is unlikely that inflation will be very persistent.

In a balanced growth economy, inflation will tend to grow at the same rate of real GDP growth. For the mean GDP growth rate, we center  $\mu_y$  around 0.0076. After being annualized, this is the average growth rate of U.S. real GDP in our sample (3.04%). As a result, we center the trend inflation  $\mu_\pi$  around 0.0076. Notice that both of real GDP growth and inflation are positive in the long-run. We thus choose a Gamma prior for both  $\mu_y$  and  $\mu_\pi$ . To get  $\mu_\pi^e$ , we use the difference between average (annualized) inflation and one-year ahead inflation expectation, which is 0.73%. Then, we set  $\mu_\pi^e$  as 0.0018, observe that  $\rho_1$  and  $\rho_2$  are coefficients of the AR(2) process of output gap, which is transitory component in real GDP.

Previous studies which use only GDP data, such as [Morley, Nelson, and Zivot \(2003\)](#), show that  $\rho_1$  is close to 1.35 and  $\rho_2$  is close to  $-0.5$  for quarterly frequency. Hence, the prior is centered around 1.35 for  $\rho_1$  and  $-0.5$  for  $\rho_2$ . Since we do not have additional information on these two parameters, we use Normal priors for both  $\rho_1$  and  $\rho_2$ .

$\eta_0$  and  $\eta_1$  measure the sensitivity of Okun’s Law. Empirical studies such as [Prachowny \(1993\)](#) show that 1% increase of unemployment rate tend to reduce current output gap by about 2.5% to 3%. Therefore, we center Normal priors of  $\eta_0$  around  $-0.4$  and  $\eta_1$  around 0.

Now, we turn to parameters related exogenous shocks. Using the whole sample from 1960Q1 to 2014Q3, the standard deviation of (annualized) output and (annualized) inflation fluctuation are 3.4% and 3.05% while the standard deviation of unemployment rate fluctuation is around 1.6%. Therefore, we center the prior of  $\sigma_n$ ,  $\sigma_\pi$ , and  $\sigma_u$  around 0.85%, 0.76%, and 1.6% respectively. We further center the prior of  $\sigma_g$  around 0.20% to allow more weights on output fluctuation that is not due to nominal frictions i.e.,  $\sigma_n > \sigma_g$  which was found in previous trend and cycle studies for GDP. Since  $\varepsilon_e$  measures the difference between inflation and inflation expectation,  $\sigma_e$  is chosen to center around the standard deviation of the gap between inflation and inflation expectation.

Finally, for the correlation of shocks, having no precise information, we center the correlations around zero. We use Normal priors with relatively large standard deviations, 0.25. [Table 1](#) summarizes all parameters’ prior information.

## 4.2 Estimation Results

After estimation, the posterior means, together with 5% and 95% Bayesian intervals are summarized in [Table 2](#). For the sake of space, we do not report posterior modes and the significance of the modes, as the modes are similar to the mean values reported.

The indexation to past inflation is smaller than 0.5 implying a small degree of inflation persistence. The sum of AR(2) coefficients  $\rho_1$  and  $\rho_2$  is about 0.90, and the output gaps are relatively persistent. From posterior means of  $\eta_0$ , we know that a 1% drop of output will imply 0.56% increase of unemployment rate, similar to previous estimation of Okun’s Law; in addition, a 1% percent lower output gap today increases tomorrow’s unemployment rate by 0.18%. Notice that the standard deviation of  $\eta_1$ ’s posterior is large. The posterior mode of  $\eta_1$  is 0.1643 and is not significant (with  $t$  statistics being 0.9637), implying less persistent effects from output on unemployment.

None of the mode of shocks’ correlation is significant at 5% level, except the correlation between output trend and unemployment  $\rho_{nu} < 0$ . These facts could be confirmed by the

Table 2: Results from Metropolis-Hastings

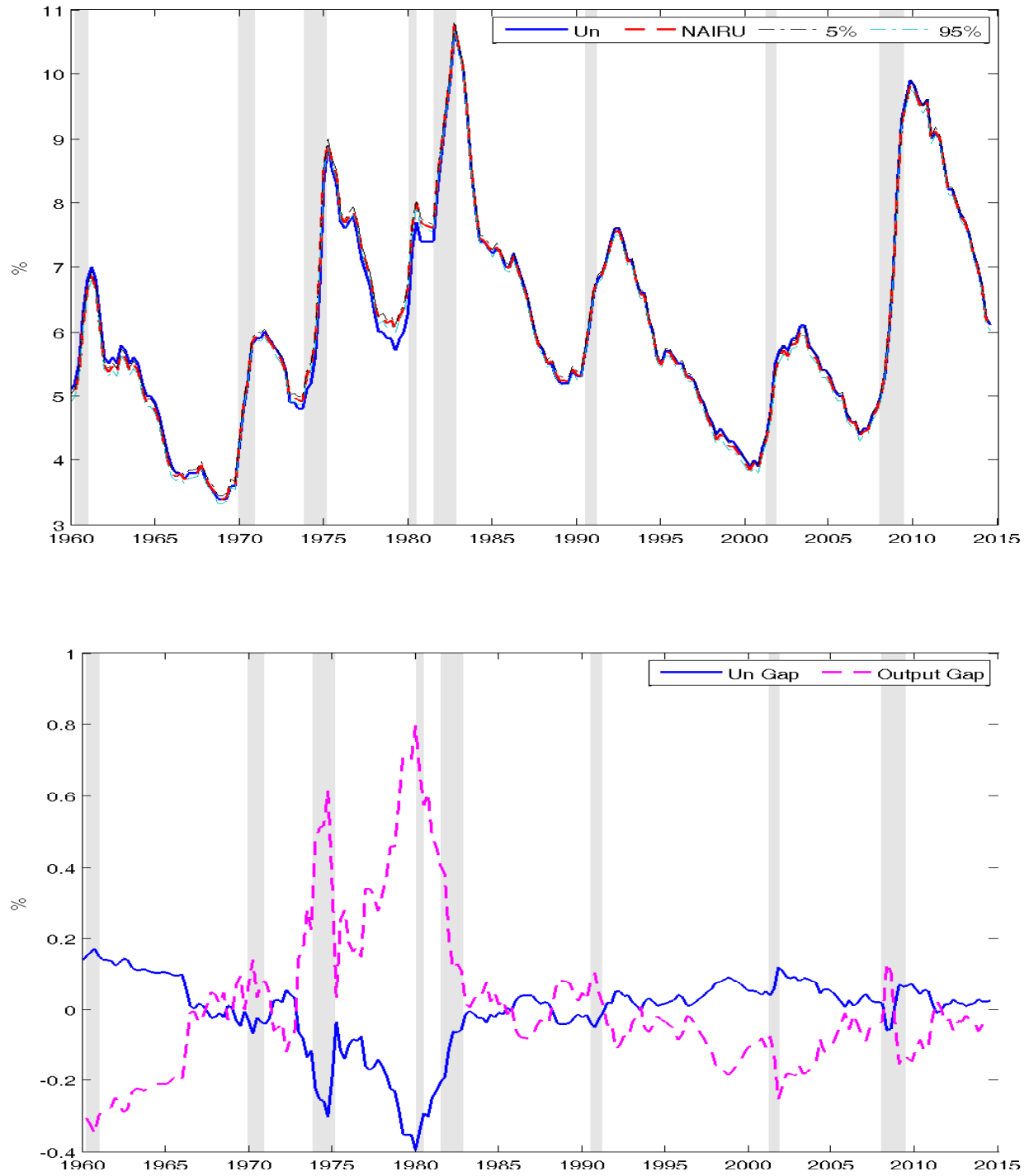
	Prior	Prior mean	Prior s.d.	Posterior mean	Posterior s.d.	5%	95%
$\zeta$	gamma	0.500	0.3000	0.1540	0.0624	0.0565	0.2603
$\mu_y$	norm	0.008	0.0020	0.0077	0.0004	0.0070	0.0083
$\mu_\pi$	norm	0.008	0.0020	0.0093	0.0007	0.0082	0.0104
$\rho_1$	norm	1.400	0.1000	1.1764	0.0615	1.0798	1.2837
$\rho_2$	norm	-0.500	0.1000	-0.2763	0.0567	-0.3780	-0.1940
$\eta_0$	norm	-0.400	0.1000	-0.4374	0.0996	-0.5978	-0.2656
$\eta_1$	norm	0.000	0.2000	-0.1642	0.1844	-0.4364	0.1711
$\sigma_g$	invg	0.002	0.0010	0.0007	0.0001	0.0006	0.0008
$\sigma_n$	invg	0.009	0.0010	0.0084	0.0004	0.0077	0.0090
$\sigma_\pi$	invg	0.008	0.0010	0.0045	0.0002	0.0042	0.0049
$\sigma_u$	invg	0.002	0.0050	0.0033	0.0002	0.0030	0.0036
$\sigma_e$	invg	0.002	0.0010	0.0017	0.0007	0.0006	0.0028
$\rho_{ng}$	norm	-0.400	0.2500	-0.0850	0.0734	-0.2179	0.0260
$\rho_{nu}$	norm	0.000	0.2500	-0.6050	0.0429	-0.6696	-0.5287
$\rho_{n\pi}$	norm	0.000	0.2500	0.1206	0.0751	0.0033	0.2486
$\rho_{gu}$	norm	0.000	0.2500	-0.1099	0.0724	-0.2386	0.0018
$\rho_{g\pi}$	norm	0.000	0.2500	0.1267	0.0953	-0.0444	0.2618
$\rho_{u\pi}$	norm	0.000	0.2500	-0.0231	0.0721	-0.1407	0.0904

posterior standard deviation reported in Table 2. Notice that  $\rho_{nu} < 0$  implies that long-term technology improvement reduces unemployment, even though it may possibly leave workers transit from jobs with old technology to jobs with new technology. However, it may also imply that more populations are out of labor force which could reduce unemployment rate at the same time.

In contrast to previous studies, the innovation of the GDP trend only slightly negatively correlated with the innovation to the GDP cycle ( $\rho_{ng} > 0$ ). Moreover, note that inflation expectation contains useful information of policy effects, since surprise inflation can increase output gap ( $\rho_{g\pi} > 0$ ) and reduce unemployment ( $\rho_{u\pi} < 0$ ). But both the effects are small and not statistically significant, which suggests that monetary policy quickly stables short-run fluctuations. That is, there is not much further room for improvement.

Figure 3 plots the estimated NAIRU, unemployment gaps, and output gaps. Unemployment gaps tend to increase in recessions and output gaps tend to drop. Several distinguished features need to be noticed. (1) The magnitude of unemployment gaps are very small, compared to existing literature. In recessions, the gaps will move almost to zero. (2) The 1975-1982 period exhibits a long lasting negative unemployment gap. This result, reflecting Friedman (1968), is likely due to high inflation policy at that time until Paul Volker committed to a low inflation policy regime. Unemployment catches up with NAIRU in the subsequent 1981-1982 recessions with the accommodating high interest rate. (3) The

Figure 3: Unemployment, NAIRU, and Gaps



Top panel: estimated NAIRU and 5% and 95% confidence intervals together with unemployment rate. Inflation expectation data is used for the estimation. Shaded area denote NBER dated recessions. Bottom panel: estimated unemployment gap and output gap. No inflation expectation data is used for the estimation.

NAIRU's 5% and 95% Bayesian posterior intervals are with 0.5% magnitude, much smaller than previous estimations. This result confirms that we can estimate NAIRU more accurately once we have inflation expectation data, as this information indicates the magnitude of unemployment gaps and output gaps.

In summary, NAIRU is very precisely estimated and traces closely the realized unemployment rate. Given the price rigidities imposed (there is on average 4 quarters price adjustment), monetary policy is probably most effective in dampening frictions due to nominal rigidities. Importantly, such exercise shows how informative inflation expectation is. It contains private agents' belief about underlying economic fundamental and policy making.

### 4.3 Removing Survey Data

We perform a counterfactual experiment by removing the survey data of inflation expectation. That is, there is no observation equation for inflation expectation. Then, we redo the whole exercise by imposing the same priors as before. The posterior results are summarized in Table 3. Again, we focus on mean values of the estimation, as the corresponding mode values are very similar to them.

When not using inflation expectation, the estimation results are in sharp contrast. The indexation to past inflation  $\zeta$  is now larger than 0.5 implying a large degree of inflation

Table 3: Results from Metropolis-Hastings (no survey data is used)

	Prior	Prior mean	Prior s.d.	Posterior mean	Posterior s.d.	5%	95%
$\zeta$	gamma	0.500	0.3000	0.726	0.0845	0.5930	0.8691
$\mu_y$	norm	0.008	0.0020	0.008	0.0004	0.0069	0.0083
$\mu_\pi$	norm	0.008	0.0020	0.009	0.0012	0.0069	0.0105
$\rho_1$	norm	1.300	0.1000	1.288	0.0780	1.1708	1.4267
$\rho_2$	norm	-0.500	0.1000	-0.588	0.0663	-0.6998	-0.4830
$\eta_0$	norm	-0.400	0.1000	-0.458	0.1299	-0.6639	-0.2318
$\eta_1$	norm	0.000	0.2000	-0.553	0.2000	-0.8822	-0.2097
$\sigma_g$	invga	0.002	0.0010	0.002	0.0003	0.0010	0.0020
$\sigma_n$	invga	0.009	0.0010	0.008	0.0003	0.0073	0.0085
$\sigma_\pi$	invga	0.008	0.0010	0.006	0.0003	0.0052	0.0062
$\sigma_u$	invga	0.002	0.0050	0.003	0.0002	0.0023	0.0031
$\rho_{ng}$	norm	0.000	0.2500	0.027	0.2165	-0.3256	0.3546
$\rho_{nu}$	norm	0.000	0.2500	-0.581	0.0498	-0.6665	-0.5045
$\rho_{n\pi}$	norm	0.000	0.2500	0.089	0.1068	-0.0727	0.2747
$\rho_{gu}$	norm	0.000	0.2500	-0.119	0.2482	-0.4950	0.2771
$\rho_{g\pi}$	norm	0.000	0.2500	-0.453	0.1472	-0.7269	-0.2423
$\rho_{u\pi}$	norm	0.000	0.2500	0.088	0.1472	-0.1199	0.3413

persistence. The sum of AR(2) coefficients  $\rho_1$  and  $\rho_2$  is about 0.70 such that the output gaps are not very persistent. From posterior means of  $\eta_0$ , we know that a 1% drop of output will increase unemployment by 0.46%, a little larger than the previous estimation; in addition, a 1% lower output gap today increases tomorrow's unemployment rate by 0.59%, which is much larger than the effect estimated from using survey data.

Unlike previous estimation, all the correlations of the shocks are significant. The innovation to the GDP trend still positively correlates with the innovation to the GDP cycle ( $\rho_{ng} > 0$ ), while negatively correlates with innovation to the unemployment ( $\rho_{nu} < 0$ ). This is likely to be the case when there is a positive long-term technology improvement firms tend to hire workers and produce more output. During the transition period, there will be positive output gap, and reduced unemployment will be seen ( $\rho_{gu} < 0$ ). Finally, more production in the economy contribute to a lower inflation pressure i.e.,  $\rho_{g\pi} < 0$ .

Figure 4 shows the estimated NAIRU, output gaps, and unemployment gaps. When the economy enters recessions, unemployment rates quickly catch up and surpass NAIRU. Although again unemployment gaps tend to move up in recessions and output gaps tend to move down in recessions, the estimated gaps are different from previous estimations. For example, the unemployment gap is almost zero after 1980 recessions if one uses the survey data, while the gap is about 1% without using the survey data. In the past recovery from financial crisis, there is a long period (2009-2012) of positive excess unemployment rates. However, starting from early 2013, excess unemployment rates tend to be zero which is similar to the previous estimation.

To summarize, in contrast to the estimation with the survey data, this estimation shows that the sizes of unemployment gaps and output gaps are large, which contribute to the relatively smooth NAIRU curve in Figure 4. Large gaps imply that there are still rooms for policy to stabilize the economy. Notice that the 5% and 95% bounds of the Bayesian credible set is generally within 1% to 1.2%, which is significant larger than that when survey data is used.

#### 4.4 Why Survey Data Contains Information

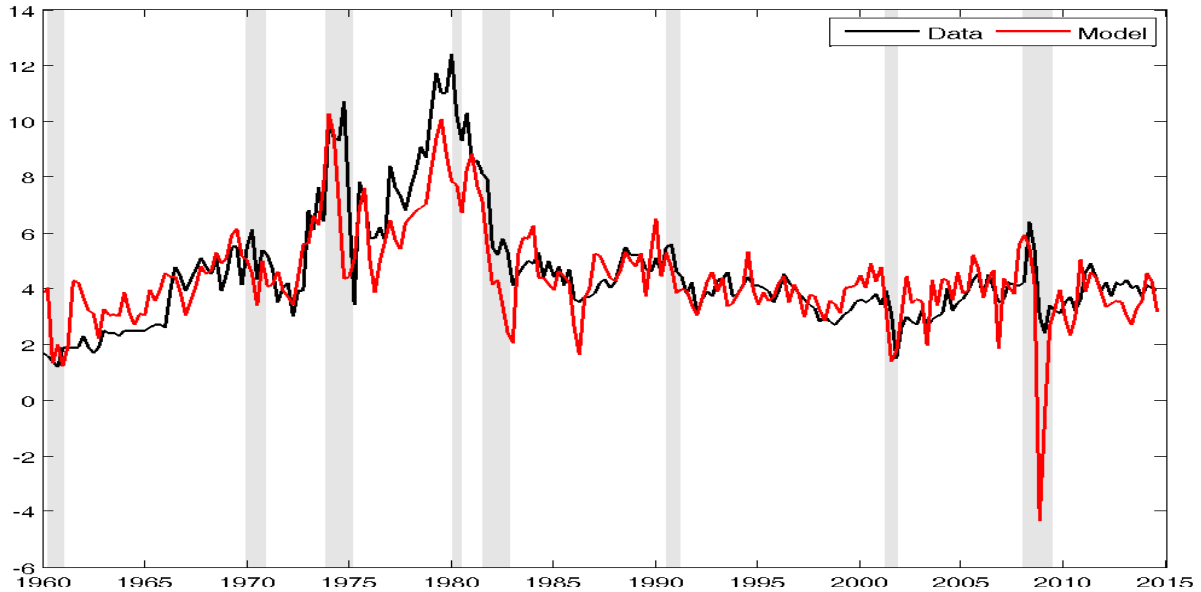
To understand why the survey data contains useful information, it is instructive to compare the model generated inflation expectation in Section 4.3 to the data. The expectation generated from the estimated model without using inflation expectation is  $100 \mathbf{E}_t[\pi_{t+1} + \pi_{t+2} + \pi_{t+3} + \pi_{t+4} + 4\mu_\pi + 4\mu_\pi^e]$  in percentage terms. Not surprisingly, this model generated expectation co-moves with the data counterpart (Figure 5), but the gaps between these two

Figure 4: Unemployment and NAIRU



Top panel: estimated NAIRU and 5% and 95% confidence intervals together with unemployment rate. No inflation expectation data is used for the estimation. Shaded area denote NBER dated recessions. Bottom panel: estimated unemployment gap and output gap. No inflation expectation data is used for the estimation. Shaded area denote NBER dated recessions.

Figure 5: Model Generated Inflation Expectation and the Data



Note: Estimated one year ahead inflation expectation and inflation expectation in the data. No inflation expectation data is used for the estimation. Shaded area denote NBER dated recessions.

start to increase in recessions and persist for a while.

Such discrepancy will be important information to determine the size of unemployment and output gaps, which explain why the results are very different under these two estimation strategies. Conceptually, if the black line is identical with the red line, the estimation in Section 4.3 is the same as the estimation in Section 4.2. Therefore, all the different findings in the estimation are due to the gaps in these two lines, which provides us extra information in estimating output gaps and unemployment gaps.

More specifically, the discrepancy between these two lines suggest how effective is the monetary policy. This result shows that monetary policy are fairly effective in eliminating nominal frictions, only if we use the inflation expectation data.

Focus for example on the 2008 recession periods. Without using the survey data, the model implies the inflation forecast to be -4% while the survey data says inflation drops but will still be a little above 2%. This is possibly due to agent's belief on the strong policy reaction to the great recession. As a result, unemployment and output gaps are much smaller in the estimation in Section 4.2, as we take into account people's belief on current and future policy response.



## 4.5 Out-of-Sample Forecast

It is important to compare the forecasting performance of the two estimation exercises. In order to check this, we estimate the model using data from 1960Q1 to 2006Q1, and then perform an out-of-sample recursive scheme. Considering a forecast horizon of 1 Q ahead. Notice that there are large disturbances during the 2008 financial crisis. Hence we do not perform the forecast for 2008Q3 to 2009Q2. We are then left with 30 forecasts for output growth, unemployment growth, and inflation.

Let us compare the mean squared forecast error (MSFE) of each observable in the estimation with the survey data to that of an estimation without the survey data. The forecast takes into account both uncertainty about parameters and uncertainty about future shocks. The following table summarize the result.

Table 4: Forecasting Performance

	Using Survey Data	No Survey Data
GDP Growth	1.05	1 (normalized)
Unemployment Growth	0.93	1 (normalized)
Inflation	0.98	1 (normalized)

Notes: Mean squared forecast error comparison for different variables. The mean squared errors when no survey data is used are normalized to 1

This exercise supports the estimation of using the inflation expectation data by putting it as observations, at least for inflation and unemployment. GDP forecast with inflation expectation data does slightly worse. The survey data facilitates the estimation to extract information of the underlying states. Not only we obtain different estimation results of the NAIRU curve (together with different output gaps), we also gain prediction power by incorporating the survey data into the model.

Intuitively, private agents know more information than us who are performing econometric exercises. While the information is generally hidden, we still can extract some information by looking at their forecast of future variables (which in our case is inflation expectation). Importantly, we do not model monetary policy explicitly, but the inflation expectation can indicate private agents' belief on the direction of monetary policy as well as economic fundamental shocks.

## 4.6 Discussion

We highlight the need to incorporate using data of inflation expectation in estimating NAIRU. The key motivation is that the inflation expectation affects unemployment dynamics

and the information contained in survey data can guide us in estimating the underlying economy. Several caveats are worth to be discussed here.

We illustrate one way to incorporating survey data by expressing it as a noisy observation of the function of underlying states. However, since not all forecast performance is better with using survey data in our proposed method, we suspect further methods of incorporating survey data and together with macro models are needed. Perhaps, a time-varying weight should be given to survey data as observations of underlying states. This aspect is left for future research.

Inflation expectation is informative about the underlying economy. Given the degree of price rigidities, one can infer that monetary policy is effective in dampening frictions arising from nominal rigidities. However, adding observed interest rate policy might change the estimation. Further work is needed to show if NAIRU continues to track unemployment closely when policy might be constrained to some extent.

We model NAIRU as a unit root process for ease of computing likelihood and posteriors. U.S. data of unemployment rate is very persistent and that is why many economists prefer the unit-root process setup (see e.g., [Gordon \(1997\)](#)). Nevertheless, there are always bounds for unemployment rate and NAIRU. To model the NAIRU process more accurately, one may need a process that is close to unit root but always bounded. One of possible way of modeling might be a mean-reverting unit root process: when unemployment rate goes very high, it will tend to go down very soon. This set-up has been analyzed for exchange rate dynamics in [Nicolau \(2002\)](#). We do not choose this process because our focus is on using inflation expectation data and we simplify the filtering problem. One can choose a more complicated process and use non-linear filtering technique.

As in many studies, we treat boom time and recessions time symmetrically. As pointed out by recent business cycle research, asymmetry might be an important features of business cycle and especially for labor market. A different way of modeling the job creation and destruction will possibly alter the estimation. However, even we consider the asymmetry, estimation with inflation expectation data could still suggest relative small gaps. We suspect this because the model generated inflation expectation (from the estimation when no survey data is used) has larger deviation from the data in recessions ([Figure 5](#)).

## 5 Conclusion

We illustrate how forward looking inflation expectation data could be used consistently in a dynamic macro context. Such treatment avoids endogeneity issues in estimating the natural rate of unemployment, or non-accelerating-inflation rate of unemployment (NAIRU), and further gives more information to estimate the structural model more accurately.

We view this paper as an effort to utilize survey data to estimate NAIRU in a model consistent way. Further work can be carried out by using more sophisticated models for estimating private agents' learning behaviors. We believe our results will be useful for many policy debates as expectation data offers more (accurate) information in assessing whether government policy will be effective. The key message here is that if unemployment is close to NAIRU, further round of monetary stimulus might not be useful. To loosen labor market regulations might be the appropriate solution.

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# Appendix

## Details of the Phillips Curve

There are intermediate goods firms and final goods firms. Assume that intermediate firms are identical at the beginning, but produce differentiated products for their pricing history. In addition, there is a competitive final goods market. Each intermediate firm  $i \in [0, 1]$  produces output  $Y_{it}$  by paying labor wages and fixed costs. No capital is needed in the production or capital is assumed to be fixed.  $Y_{it}$  will be assembled into final goods  $Y_t$  (which is the GDP)

$$Y_t = \left( \int Y_{it}^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}},$$

where  $\epsilon$  is the elasticity of substitution among products from consumers' perspectives. For simplicity, the substitution is the same among different goods.

Let  $P_t$  be the nominal price of final goods and  $P_{it}$  be the nominal price of intermediate good  $i$ . A final goods firm maximizes per-period profits by solving

$$\max_{Y_{it}} \{P_t Y_t - \int P_{it} Y_{it} di\},$$

taking as given  $P_t$  and  $P_{it}$ . The demand for each individual goods and the aggregate price level  $P_t$  can be written as

$$Y_{it} = \left( \frac{P_{it}}{P_t} \right)^{-\epsilon} Y_t, \quad P_t = \left\{ \int (P_{it})^{1-\epsilon} di \right\}^{\frac{1}{1-\epsilon}}. \quad (13)$$

Naturally, the (gross) inflation rate at time  $t$  is:

$$\Pi_t = P_t / P_{t-1}.$$

Each intermediate firm can change price with a probability  $1 - \alpha$  where  $\alpha \in (0, 1)$ . For example, if  $\alpha = 0.75$  for a quarterly model, prices are fixed on average for  $1/(1 - \alpha) = 4$  quarters, or a year. With a probability  $\alpha$ , it must keep its price unchanged, except for adjustment indexed to past inflation and trend inflation.  $\alpha$  therefore represents the price rigidities. That is, if firm  $i$  cannot adjust its price, the price  $P_{it}$  will be

$$P_{it} = P_{it-1} \bar{\Pi}^{1-\zeta} \Pi_{t-1}^\zeta.$$

where  $\zeta$  measures the elasticity of the indexation to past inflation and  $\bar{\Pi}$  is the steady state (trend) inflation.

Denote  $P_{it}^*$  as the optimal price that can be adjusted firm  $i$  at time  $t$ . For simplicity, we look for a symmetric equilibrium in which the firms who can optimally reset price adjust to the same price  $P_{it}^*$ , so that aggregate price can be written as

$$P_t = \left\{ (1 - \alpha) (P_t^*)^{1-\epsilon} + \alpha (\bar{\Pi}^{1-\zeta} \Pi_{t-1}^\zeta P_{t-1})^{1-\epsilon} \right\}^{1/(1-\epsilon)}$$

Following the convention, we denote the variable with a bar and without time subscript as the

deterministic steady state and lower case variable as the percentage deviation from its steady state level. For example,  $\pi_t = \log(\Pi_t) - \log(\bar{\Pi})$ . Notice that  $P_t$  and  $P_{it}$  will grow in the steady state with positive inflation but  $\tilde{P}_t = P_{it}/P_t$  will not such that  $\tilde{P}_t$  is stationary. Now, dividing  $P_t$  on both side of the above aggregate price equation, rearranging, and collecting terms, we obtain

$$\tilde{P}_t = \frac{\alpha \bar{\Pi}^{(1-\epsilon)(\zeta-1)}}{1 - \alpha \bar{\Pi}^{(1-\epsilon)(\zeta-1)}} (\pi_t - \zeta \pi_{t-1}) \quad (14)$$

Now consider how to solve  $P_{it}^*$ . The goal of the firm is to pick a price that maximizes discounted total profits of each period, given it cannot adjust optimally later. The firm will not consider those scenarios when it can adjust price optimally, since it will solve a similar problem again. The maximization problem can be written as

$$\max_{P_{it}} \mathbf{E}_t \sum_{s=0}^{\infty} \{\beta\alpha\}^s \left[ \frac{P_{it}}{P_{t+s}} Y_{it+s} - MC_{t+s} (Y_{it+s} + \text{fixed Costs}) \right]$$

where  $MC_{t+s}$  is the marginal cost of producing  $Y_{it+s}$ . Using the demand curve from (13), firm  $i$  effectively solves

$$\max_{P_{it}} \mathbf{E}_t \sum_{s=0}^{\infty} (\beta\alpha)^s \left\{ \left( \frac{P_{it} \Pi_{t-1, t+s-1}}{P_{t+s}} \right)^{1-\epsilon} Y_{it+s} - MC_{t+s} \left( \frac{P_{it} \Pi_{t-1, t+s-1}}{P_{t+s}} \right)^{-\epsilon} (Y_{it+s} + \text{fixed costs}) \right\}$$

where

$$\Pi_{t-1, t+s-1} = \bar{\Pi}^{s(1-\zeta)} \Pi_t^\zeta \Pi_{t+1}^\zeta \dots \Pi_{t+s-1}^\zeta = \bar{\Pi}^{s(1-\zeta)} \left( \frac{P_{t+s-1}}{P_{t-1}} \right)^\zeta$$

and  $\Pi_{t-1, t-1} = 1$ .

Firms' optimal price setting behaviors after log-linearization leads to the New Keynesian Phillips curve (NKPC)

$$\pi_t = \lambda mc_t + \gamma_f \beta \mathbf{E}_t [\pi_{t+1}] + \gamma_b \pi_{t-1}, \quad (15)$$

Note  $mc_t$  is the percentage deviation of marginal costs of producing output from the trend,  $\pi_t = \log(\Pi_t) - \log(\bar{\Pi})$  is the percentage deviation of inflation from the trend, and the definition of  $\lambda$ ,  $\gamma_f$ , and  $\gamma_b$  is

$$\lambda = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+\zeta\beta)}, \quad \gamma_f = \frac{\beta}{1+\zeta\beta}, \quad \gamma_b = \frac{\zeta}{1+\zeta\beta}.$$

In the NKPC, there are both forward-looking and backward-looking terms, since some firms adjust prices according to previous inflation and trend inflation. Under certain conditions (Walsh (2010)), marginal costs are proportional to the output gap

$$mc_t = \kappa y_t^g = (\sigma + \eta) y_t^g,$$

where  $\sigma$  is the elasticity of consumers' intertemporal substitution between today's consumption goods and tomorrow's consumption goods while  $\eta$  is the disutility from unit labor supply.  $\kappa$ , the sum of these two, is thus the output gap elasticity of marginal cost.

Finally, we also add exogenous shocks (denoted as  $\varepsilon_t^\pi \sim N(0, \sigma_\pi^2)$ ) to inflation such as monetary policy shocks or supply shocks (e.g., exogenous oil price movements) that are outside of our model.

Then, the NKPC can be written as

$$\pi_t = \gamma_f \mathbb{E}_t[\pi_{t+1}] + \gamma_b \pi_{t-1} + \lambda \kappa y_t^g + \varepsilon_t^\pi, . \quad (16)$$

## Inflation Expectation

One can explicitly show the matrix as

$$(I - A_1)^{-1} = \begin{bmatrix} 1 & \rho_2 \\ 1 & 1 - \rho_1 \end{bmatrix} \frac{1}{1 - \rho_1 - \rho_2}$$

$$(I - \theta_2^{-1} A_1)^{-1} = \begin{bmatrix} 1 & \rho_2/\theta_2 \\ 1/\theta_2 & -\rho_1/\theta_2 + 1 \end{bmatrix} \frac{1}{1 - \rho_1/\theta_2 - \rho_2/\theta_2^2}$$

$$I - \theta_2^{-k} A_1^k = \begin{bmatrix} 1 - \theta_2^{-k}(\xi_1^{k+1} - \xi_2^{k+1}) & -\xi_2^{k+1}\xi_1 + \xi_1^{k+1}\xi_2 \\ \xi_2^k - \xi_1^k & 1 + \theta_2^{-k}\xi_1^k\xi_2 - \theta_2^{-k}\xi_2^k\xi_1 \end{bmatrix} \frac{1}{\xi_1 - \xi_2}$$

where  $\xi_1 = \frac{\rho_1}{2} + \left(\frac{\rho_1^2}{4} + \rho_2\right)^{\frac{1}{2}}$  and  $\xi_2 = \frac{\rho_1}{2} - \left(\frac{\rho_1^2}{4} + \rho_2\right)^{\frac{1}{2}}$ .

## Computing $\gamma_0$ and $\gamma_1$

For convenience, we repeat

$$\gamma_0 \gamma_1 = \gamma_f, \quad \gamma_0 + \gamma_1 \gamma_b = 1$$

Then  $\gamma_1$  is the root of the function of

$$f(x) = x^2 - \frac{1}{\gamma_b} x + \frac{\gamma_f}{\gamma_b}.$$

Notice that  $f(0) > 0$ ,  $f(x)$  is symmetric with respect to a vertical line  $x = \frac{1}{2\gamma_b} > 0$ , and  $f(1) = 1 - \frac{1-\gamma_f}{\gamma_b} \leq 0$ . Thus, only the smaller root is smaller than 1 which we assign to  $\gamma_1$

$$\gamma_1 = \frac{1 - \sqrt{1 - 4\gamma_f\gamma_b}}{2\gamma_b} < 1$$

and subsequently  $\gamma_0 = \frac{1 + \sqrt{1 - 4\gamma_f\gamma_b}}{2}$ .

## Proof of the Proposition

We will show  $\gamma_f + \gamma_b \neq 1$ . Suppose not and this implies that  $\theta_2 = \gamma_b/\gamma_0 = 1$  because  $\gamma_0 = \gamma_b$ . To see this, notice that  $\gamma_0 = \frac{1 + \sqrt{1 - 4\gamma_f\gamma_b}}{2}$  and we have

$$(2\gamma_0 - 1)^2 = 1 - 4\gamma_f\gamma_b$$



Using  $\gamma_f + \gamma_b = 1$ , then

$$4\gamma_0^2 - 4\gamma_0 + 1 = 4\gamma_b^2 - 4\gamma_b + 1$$

We know that  $\gamma_0 \neq 0$  and  $\gamma_b \neq 0$ , then  $\gamma_0 = \gamma_b$  so that  $\theta_2 = 1$ . Therefore,

$$\pi_t = \theta_0 y_t^g + \theta_1 y_{t-1}^g + \pi_{t-1} + \varepsilon_t^\pi$$

and inflation is not stable, a contradiction.

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