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Working Paper

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SSE/EFI Working Paper Series in Economics and Finance, No. 748

Provided in Cooperation with:

EFI - The Economic Research Institute, Stockholm School of Economics

Suggested Citation: Jonsson, Adam; Voorneveld, Mark (2014) : The limit of discounted utilitarianism, SSE/EFI Working Paper Series in Economics and Finance, No. 748, Stockholm School of Economics, The Economic Research Institute (EFI), Stockholm

This Version is available at:

<https://hdl.handle.net/10419/122141>

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THE LIMIT OF DISCOUNTED UTILITARIANISM

ADAM JONSSON AND MARK VOORNEVELD

ABSTRACT. This paper presents an infinite-horizon version of intergenerational utilitarianism that is both satisfactorily complete and consistent. By studying discounted utilitarianism as the discount factor tends to one, we obtain a welfare criterion — limit-discounted utilitarianism — that combines efficiency and the equal treatment of generations with analytical tractability and a high degree of comparability. We show that limit-discounted utilitarianism satisfies a number of consistency properties; in particular, it provides (i) an intuitive link between preferences over infinite-horizon streams and large, but finite-horizon truncations, and (ii) a complete view of the consequences of delaying streams with well-defined finite averages. The latter is formulated through a principle of compensation. Through this compensation principle, limit-discounted utilitarianism gives a coherent view on the consequences of delaying welfare which is compatible with stationarity. Limit-discounted utilitarianism is characterized on a large domain of infinite-horizon utility streams.

SSE/EFI WORKING PAPER SERIES IN ECONOMICS AND FINANCE, NO. 748

Date: March 21, 2014.

Key words and phrases. Intergenerational equity; aggregating infinite utility streams; generalized utilitarianism.

The second author acknowledges financial support by the Wallander-Hedelius Foundation under grant P2010-0094:1.

1. INTRODUCTION

Utilitarianism is the normative theory which says that the right action among a set of alternatives is the one that produces the most good. The accompanying notion of maximizing aggregate utility is easy to make precise for finitely many agents, but becomes problematic in comparing intergenerational welfare distributions over an infinite number of future generations. In this context, the alternatives are specified by infinite utility streams $u = (u_1, u_2, \dots)$ that typically do not have a well-defined finite sum $\sum_{n=1}^{\infty} u_n$. In the theory of economic growth, such an infinite stream of utilities is often evaluated by the discounted utilitarian criterion. When that criterion is allowed to rank intergenerational welfare distributions, ethical problems arise. The discounting of future generations' utilities was criticized already by Pigou [24]. It was famously rejected by Ramsey [25], whose well known moral position was that any desire to do so must be caused by a "weakness of the imagination". Nowadays it is standard practice in welfare economics to combine the *Strong Pareto* (**SP**) axiom with an anonymity condition to ensure that all generations are treated equally, where *Finite Anonymity* (**FA**) is a widely accepted minimum requirement.¹ The construction of criteria that meet the minimum requirement is however associated with complications, the full extent of which has only been understood recently. First, it is not possible to define a function over the set of utility streams whose induced order satisfies **FA** and **SP**. (Diamond's [12] original version of this result was obtained under an additional continuity assumption on the function in question; the general impossibility theorem is due to Basu and Mitra [7].) Second, although it is in principle possible to define an ordering that respects both axioms if the condition of numerical representability is dropped (cf. Svensson [28]), Lauwers [23] and Zame [30] have shown that the construction of such an ordering necessitates some form of the Axiom of Choice. Stated more dramatically, every explicit description of preferences that satisfies **SP** and **FA** is incomplete; see [30, Theorem 4] and [23, p. 33].

The above restrictions motivate the study of incomplete preference descriptions. A pre-order or *social welfare relation* (SWR) is a binary relation that is reflexive and transitive, but not necessarily complete. Practically all authors who have proposed SWRs take on the infinite horizon problem with a large undiscounted problem. Indeed, for the Suppes-Sen grading principle of Suppes [27] and Svensson [28], for the classical overtaking and catching-up criteria of von Weizsäcker [29] and Gale [15], for the utilitarian SWR of Basu and Mitra [9], and many others, a necessary condition for declaring $u = (u_1, u_2, \dots)$ at least as good as $v = (v_1, v_2, \dots)$ is that $\sum_{i=1}^n (u_i - v_i) \geq 0$ holds for all large n . For such criteria it is not difficult to find noncomparable pairs of utility streams and a social context that would force us to consider them. Suppose, for instance, that $u = (u_1, u_2, \dots)$ is a periodic stream of utilities, and that we wish to compare u with its *postponement* $(0, u_1, u_2, \dots)$ or, more generally, with (c, u_1, u_2, \dots) .

How should delaying the stream of utilities one generation be viewed by society if the first generation is compensated by c ?

¹Axioms **SP** and **FA** are stated formally in Section 3.

That the classic criteria are unable to compare general periodic streams and fail to confirm the equivalence of $(1, 0, 1, 0, \dots)$ and $(0, 1, 0, 1, \dots)$ have led many authors to impose strengthened axioms of anonymity. Two examples of such *extended* anonymity axioms that are compatible with **SP** are Lauwers's [22] *Fixed-Step Anonymity* (see e.g. Fleurbaey and Michel [14], Asheim and Banerjee [2], Sakai [26], Kamaga and Kojima [19]) and the *Strong Relative Anonymity* axiom in Asheim et al [3]. As exemplified by these conditions, imposing invariance under infinite permutations is a risky business, and may come at the cost of Koopmans's [20] *Stationarity* axiom. Our objective is to address some fundamental questions related to that problem from a utilitarian perspective and propose an alternative approach. Rather than letting the finite horizon tend to infinity, we study the discounted criterion as the discount factor tends to 1. In the limit, all generations are treated equally.

The idea of discounting with discount factors that tend to one is not new. A theory of discount rate asymptotics for stochastic dynamic optimization problems has been developed by Dutta [13]. Recently, the relevance of the idea for intergenerational equity was defended by Basu and Mitra [9, pp. 360-361] in a "robustness check" of their welfare criterion.² In this paper, we formalize such a robustness requirement and refer to it as *Discounting Consistency*. When this property is linked with utilitarianism for summable streams, it defines a pre-order that combines equal treatment of generations and a high degree of comparability. We call it the *limit-discounted utilitarian* (LDU) criterion. Together with the LDU criterion there is a natural generalization of the concept of aggregate utility for infinite streams. While this concept is defined through classic tools from analysis, it paves the way to a principle that sheds new light on the ethical consequences of delaying welfare. The *Compensation Principle* says that it is defensible to postpone an infinite stream if and only if the first generation is compensated by the average utility over subsequent generations.³ For postponements of constant streams, this conclusion is a direct consequence of the *Strong Pareto* axiom. For delays of general periodic streams, the aforementioned criteria yield no answers at all, and criteria satisfying extended anonymity lead to inconsistencies that include the violation of *Stationarity*; cf. Section 7. Through the *Compensation Principle*, limit-discounted utilitarianism gives a coherent view on the consequences of delaying welfare which is compatible with *Stationarity*.

The outline of the paper is as follows: Section 2 contains definitions and notation. Section 3 gives an informal description of the problem of extending finite utilitarianism without compromising coherence properties that have generally been described as desirable in the literature. Section 4 presents the limit-discounted utilitarian criterion and establishes some of its basic properties, including its ability to compare all pairs of periodic streams. Section 5 is entirely devoted to a discussion of the effects of delaying welfare and the derivation of the *Compensation Principle*. Section 6 collects further properties of limit-discounted utilitarianism and gives an axiomatic description of the LDU criterion. Section 7 examines the trade-off between our criterion and criteria satisfying extended axioms of anonymity. Section 8 concludes.

²They attribute the idea behind the robustness check to Jörgen Weibull.

³The general formulation and interpretation of this principle is given in Section 5.

2. PRELIMINARIES

2.1. Formal setting. $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of positive integers, \mathbb{R} the set of real numbers, $\mathcal{U} = \{u \in \mathbb{R}^{\mathbb{N}} : \sup_{n \in \mathbb{N}} |u_n| < +\infty\}$ is the set of bounded⁴ utility streams $u = (u_1, u_2, \dots)$, where u_n represents total welfare of generation n . Our axioms shall be indexed by subsets of $\mathcal{U} \times \mathcal{U}$ that will first arise when we consider the sets of streams that are eventually zero and summable, respectively:

$$\begin{aligned}\mathcal{U}_0 &= \{u \in \mathcal{U} : u_n = 0 \text{ for all but finitely many } n \in \mathbb{N}\}, \\ \mathcal{U}_{\text{fin}} &= \{u \in \mathcal{U} : \sum_{n=1}^{\infty} u_n \text{ is convergent}\}.\end{aligned}$$

For $n \in \mathbb{N}$, $c = (c_1, \dots, c_n) \in \mathbb{R}^n$, $a \in \mathbb{R}$ and $u \in \mathcal{U}$, we sometimes write

$$\begin{aligned}\underline{a}_n &= (\overbrace{a, \dots, a}^{n \text{ times}}), \quad \underline{a} = (a, a, \dots), \quad au = (au_1, au_2, \dots), \\ (c_1, \dots, c_n)^\infty &= (c_1, \dots, c_n, c_1, \dots, c_n, c_1, \dots), \\ (c, u) &= (c_1, \dots, c_n, u_1, u_2, \dots), \\ u_{[n]} &= (u_1, \dots, u_n, \underline{0}), \quad u^{[n]} = (u_{n+1}, u_{n+2}, \dots).\end{aligned}$$

For $j \in \mathbb{N}$, let $e_j \in \mathcal{U}$ be the stream with j -th entry equal to one and all other entries equal to zero. A *block* is a subset B or $B(i, j)$ of \mathbb{N} of the type $\{i, i+1, \dots, j\}$, $i < j$. The length of the block $B = B(i, j)$ is defined $|B| = j - i$. Given $u \in \mathcal{U}$ and $B = B(i, j)$, let $u_{[B]} = \sum_{k=i}^j u_k e_k$. For $u, v \in \mathcal{U}$, write $u \geq v$ if $u_n \geq v_n$ for all n , and $u > v$ if $u \geq v$ and $u \neq v$. We say that u *dominates* v when $u \geq v$ holds. For $u \in \mathcal{U}$ and $c \in \mathbb{R}$, we call $(0, u)$ the *postponement* of u and refer to (c, u) as the postponement of u with *compensation*⁵ c .

2.2. Permutations. A permutation is a one-to-one map from \mathbb{N} onto \mathbb{N} . The set of all permutations is denoted Π . We say that $\pi \in \Pi$ *finite* if there is an $N \in \mathbb{N}$ with $\pi(n) = n$ for all $n \geq N$, and *cyclic* if π^n (the composition of π with itself n times) is the identity for some $n \in \mathbb{N}$. A utility stream u' is a (finite) permutation of utility stream u if there is a (finite) permutation π with $u' = \pi(u) := (u_{\pi(1)}, u_{\pi(2)}, \dots)$.

2.3. Average utility, total utility and extended total utility. Given $u \in \mathcal{U}$, we denote the *average* utility

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n u_i \tag{1}$$

by \bar{u} if the limit (1) exists. (The limit is necessarily finite since u is bounded.) The *total utility* is defined

$$\sigma(u) = \sum_{n=1}^{\infty} u_n \tag{2}$$

⁴This domain is chosen for notational convenience, as some statements are more easily read if we are able consider sums and scalar multiples on a common space. Our discussion would not be affected if \mathcal{U} were replaced by the infinite Cartesian product of the unit interval $[0, 1]$.

⁵Koopmans et al. [21] refer to (c, u) as the postponement of u with “insertion” c .

if the series converges or if the sum equals $+\infty$ or $-\infty$. For $\delta \in (0, 1)$, the *discounted* total utility⁶ $\sigma_\delta: \mathcal{U} \rightarrow \mathbb{R}$ is given by

$$\sigma_\delta(u) = \sum_{n=1}^{\infty} \delta^{n-1} u_n. \quad (3)$$

The *generalized* or *extended* total utility is defined

$$\sigma^*(u) = \lim_{\delta \rightarrow 1^-} \sigma_\delta(u) \quad (4)$$

when the limit exists, again allowing the values $\pm\infty$.

2.4. Binary relations. A *pre-order* or *social welfare relation* (SWR) is a reflexive and transitive binary relation \succsim on \mathcal{U} ; $u \succsim v$ means that u is considered at least as good as v by society, $u \sim v$ means that $u \succsim v$ and $v \succsim u$, $u \succ v$ means that $u \succsim v$ holds but $v \succsim u$ does not. We call u and v (\succsim -)comparable if we have either $u \succsim v$ or $v \succsim u$, or both. The SWR \succsim is an *extension* of the SWR \succsim' if for all $u, v \in \mathcal{U}$: (i) $u \succsim' v$ implies $u \succsim v$, and (ii) $u \succ' v$ implies $u \succ v$. If \succsim extends \succsim' , we call \succsim' *weaker* than \succsim .

3. THE PRINCIPLE OF TOTAL UTILITY

All welfare criteria that we shall discuss satisfy

$$\text{for all } u, v \in \mathcal{U} \text{ with } u - v \in \mathcal{U}_0: \quad u \succsim v \text{ if and only if } \sigma(u - v) \geq 0. \quad (5)$$

We refer to this property as *finite utilitarianism*. Our basic aim is to construct a SWR that satisfies intuitively appealing generalizations of finite utilitarianism in such a way that we are sensitive to the interests of each generation and treat everyone the same. Formally, we call a SWR *ethical*, an ESWR for short, if it satisfies:

Strong Pareto (SP): For all $u, v \in \mathcal{U}$, if $u > v$, then $u \succ v$,

Finite Anonymity (FA): For all $u, v \in \mathcal{U}$, if there are $i, j \in \mathbb{N}$ with $u_i = v_j, u_j = v_i$, and $u_k = v_k$ for all other $k \in \mathbb{N} \setminus \{i, j\}$, then $u \sim v$.

Apart from these widely accepted minimum requirements, obvious generalizations of finite utilitarianism would be to require⁷

$$u \succsim v \text{ for all } u, v \in \mathcal{U}_{\text{fin}} \text{ with } \sigma(u) \geq \sigma(v), \quad (6)$$

or, more generally, that

$$u \succsim v \text{ if } u - v \in \mathcal{U}_{\text{fin}} \text{ and } \sigma(u - v) \geq 0. \quad (7)$$

In addition, we shall require a correspondence between the large but finite problem and the infinite horizon problem (to be formalized later), as well as Koopmans's [20] condition for dynamic consistency:

Stationarity: For all $u, v \in \mathcal{U}$ and $c \in \mathbb{R}$, $u \succsim v$ if and only if $(c, u) \succsim (c, v)$.

⁶Many authors refer to (3) as the discounted utilitarian criterion.

⁷Banerjee [5, p. 329] explicitly expresses the desirability of (6).

Unlike known criteria, the social welfare relation that is introduced in this paper combines the properties above with a high degree of comparability,⁸ allowing explicit evaluation of pairs of utility streams as long as we are not deliberately searching for pathological examples.⁹ To illustrate these differences we revisit two examples from the literature where the utilitarian and overtaking ESWRs do not give broadly accepted conclusions. As in [9] and [28], we denote them by \succsim_U and \succsim_W , respectively:

$$u \succsim_U v \Leftrightarrow \exists N \in \mathbb{N} \text{ with } \sum_{i=1}^N (u_i - v_i) \geq 0 \text{ and } u_n - v_n \geq 0 \text{ for all } n > N,$$

$$u \succsim_W v \Leftrightarrow \exists N \in \mathbb{N} \text{ with } \sum_{i=1}^n (u_i - v_i) \geq 0 \text{ for all } n \geq N.$$

The utilitarian SWR is a natural generalization of finite utilitarianism to pairs of utility streams where one dominates the other beyond some finite horizon. Remarkably, this degree of comparability turns out to be sufficient to characterize optima in several models of economic growth.¹⁰ But as we indicated in the introduction, social contexts that call for more comparability arise quite naturally.

Example 1. Consider the periodic sequence $(1, 0, 1, 0, \dots)$ and its postponement with compensation c , i.e., the streams

$$u = (1, 0, 1, \dots) \quad \text{and} \quad v = (c, 1, 0, 1, \dots). \quad (8)$$

They are clearly not \succsim_U -comparable for any real c , and the overtaking SWR compares u and v only for certain values of c . Since

$$\sum_{i=1}^n (u_i - v_i) = \begin{cases} 1 - c & \text{if } n \text{ is odd,} \\ -c & \text{if } n \text{ is even,} \end{cases}$$

u and v are not \succsim_W -comparable if $c \in (0, 1)$. If $c \leq 0$, then $u \succ_W v$, where $v \succ_W u$ if $c \geq 1$. In particular, $(1, 0)^\infty \succ_W (0, 1)^\infty$.

The question of whether $(1, 0)^\infty$ and $(0, 1)^\infty$ should be considered equivalent is of fundamental importance in welfare economics, where some authors interpret any attempt to distinguish them as a violation of equal treatment. We return to this interpretation several times below. A seemingly related property of \succsim_W that has been discussed in the literature is the ‘‘time-preference’’ by which it ranks summable streams (see e.g. [9, p. 361] and [22, p. 225]). We illustrate with two examples.

Example 2. If $u = (1, 0, 0, \dots)$, $v = (1/2, 1/4, 1/8, \dots)$, then $\sum_{i=1}^\infty u_i = \sum_{i=1}^\infty v_i = 1$. But $\sum_{i=1}^n (u_i - v_i) = 2^{-n} > 0$ for every n , so $u \succ_W v$. For another example, take $u \in \mathcal{U}_{\text{fin}}$, where each $u_n > 0$, and let $v = (0, u)$. We again have $\sum_{i=1}^\infty u_i = \sum_{i=1}^\infty v_i$, yet \succsim_W ranks u above v since $\sum_{i=1}^n (u_i - v_i) = u_n > 0$ for all n .

⁸See, in particular, Propositions 1, 2, and 3.

⁹Although every SWR has a complete extension [28] and extensions of ESWRs are ethical, *Stationarity* is not automatically inherited of extensions of a stationary SWR. So apart from explicit evaluation, increased comparability is not worth much without such consistency requirements.

¹⁰Basu and Mitra [9] consider a model of economic growth without discounting and show that their criterion is able to characterize optimal paths. Asheim and Tungodden [4] obtain similar results for economies that satisfy a productivity condition.

The overtaking criterion thus violates (6) since $u \succ_W (0, u)$ may hold even though u is summable, and it has the property that $(1, 0)^\infty$ is ranked above $(0, 1)^\infty$. Let us point out something obvious but important: the existence of such ESWRs does *not* imply that all SWRs that prefer $(1, 0)^\infty$ to $(0, 1)^\infty$ have some type of disrespect for equal treatment. In fact, it will become apparent that in comparing u with $(0, u)$, we need to distinguish whether u is summable or not. In the second case, we cannot expect $u \sim (0, u)$ to hold for all u . For example, if $u' = (1, 1, 1, \dots)$ and \succsim is any SWR satisfying **SP**, then

$$u' = (1, 1, 1, \dots) \succ (0, 1, 1, 1, \dots) = (0, u'). \quad (9)$$

Here $u' - (0, u') = (1, 0, 0, \dots)$ is summable with $\sigma(u' - (0, u')) = 1$, so the more general utilitarian principle (7) still applies. We state this principle formally for future reference:

The Principle of Total Utility: for all $u, v \in \mathcal{U}$ with $u - v \in \mathcal{U}_{\text{fin}}$: $u \succsim v$ if and only if $\sigma(u - v) \geq 0$.

It is not hard to define a modification of \succsim_W that satisfies the *Principle of Total Utility*. For example, set¹¹

$$u \succsim_{W'} v \quad \Leftrightarrow \quad \liminf_{N \rightarrow \infty} \sum_{n=1}^N (u_n - v_n) \geq 0.$$

This modification, however, does not lead to a considerable increase in comparability. In particular, the utility streams (8) remain incomparable for every $c \in (0, 1)$.

4. A NEW WELFARE CRITERION: THE LIMIT-DISCOUNTED UTILITARIAN SWR

The *limit-discounted utilitarian* (LDU) criterion that we propose and study in this paper is defined:

$$u \succsim_{\text{LDU}} v \quad \Leftrightarrow \quad \liminf_{\delta \rightarrow 1^-} \sum_{n=1}^{\infty} \delta^{n-1} (u_n - v_n) \geq 0. \quad (10)$$

It defines an ethical SWR that satisfies many familiar and desirable properties (see Theorem 2). Two streams u and v are \succsim_{LDU} -comparable unless $\liminf_{\delta \rightarrow 1^-} \sigma_\delta(u - v) < 0 < \limsup_{\delta \rightarrow 1^-} \sigma_\delta(u - v)$; for the latter, see Example 3.

The extended total utility $\sigma^*(u) = \lim_{\delta \rightarrow 1^-} \sum_{n=1}^{\infty} \delta^{n-1} u_n$ is the Abel sum of the series $\sigma(u) = \sum_{n=1}^{\infty} u_n$. By Abel's theorem, $\sigma^*(u) = \sigma(u)$ if the series is convergent. This immediate gives:

Proposition 1. *The LDU criterion satisfies the Principle of Total Utility.*

Moreover, there are interesting pairs that $\succsim_{W'}$ cannot compare for which the limit $\sigma^*(u - v)$ exists:

Proposition 2. *Suppose that $u, v \in \mathcal{U}$ are periodic, or — more generally — that $u - v$ is periodic beyond a finite horizon. Then u and v are \succsim_{LDU} -comparable.*

¹¹This is one version of the catching-up criterion of Gale [15].

Proof. We show that $\sigma^*(u-v)$ is defined if $d := u-v$ is eventually periodic, letting k, p be such that $d_n = d_{n+p}$ for $n \geq k$. If $k = 1$, then

$$\sigma_\delta(d) = \sum_{n=0}^{\infty} (d_1 + d_2\delta + \dots + d_p\delta^{p-1})\delta^{np} = \frac{\sum_{i=1}^p d_i\delta^{i-1}}{1-\delta^p}. \quad (11)$$

We have $\sum_{i=1}^p d_i\delta^{i-1} \rightarrow \sum_{i=1}^p d_i$ as $\delta \rightarrow 1^-$, so $\sigma_\delta(d) \rightarrow +\infty$ or $-\infty$ if $\sum_{i=1}^p d_i \neq 0$. If $\sum_{i=1}^p d_i = 0$, then (11) is of the form $(0/0)$ as $\delta \rightarrow 1^-$ and l'Hôpital's rule yields

$$\sigma^*(d) = -\frac{\sum_{i=1}^p d_i(i-1)}{p}. \quad (12)$$

If $k \geq 2$, then $\sigma^*(d) = +\infty$ or $-\infty$ when $\sum_{i=1}^p d_i \neq 0$, and $\sigma^*(d) = \sum_{n=1}^{k-1} d_n - \sum_{i=1}^p d_i(i-1)/p$ if $\sum_{i=1}^p d_i = 0$. \square

For the streams $u = (1, 0, 1, 0, \dots)$ and $v = (c, 1, 0, 1, 0, \dots)$ in Example 1, we have $d = u - v = (1 - c, -1, 1, \dots)$ and a direct calculation gives

$$\sigma^*(d) = \lim_{\delta \rightarrow 1^-} \sum_{n=1}^{\infty} (-\delta)^{n-1} - c = \lim_{\delta \rightarrow 1^-} \frac{1}{1+\delta} - c = 1/2 - c.$$

The limit-discounted utilitarian criterion thus ranks $(1, 0)^\infty$ above $(0, 1)^\infty$.

Two frequently cited reasons for why $(0, 1)^\infty$ and $(1, 0)^\infty$ should be considered equivalent — against the LDU criterion's prediction — are that (i) the first stream is “just” the postponement of the second, and (ii) they can be obtained from each other by a permutation. The relevance of infinite permutations for intergenerational equity will be discussed briefly in Section 7. But let us already now emphasize the well known fact that if we insist on *Stationarity* and *Strong Pareto* and decide to compare $(0, 1)^\infty$ and $(1, 0)^\infty$ (as opposed to refusing to compare them), *then we must rank $(1, 0)^\infty$ above $(0, 1)^\infty$* ; see Proposition 4 below. In other words, every Paretian SWR that is indifferent between $(0, 1)^\infty$ and $(1, 0)^\infty$ violates the *Stationarity* axiom. That (i) leads to a similar inconsistency will be seen in the next section, where we pay special attention to the ethical consequences of postponing infinite streams of utility.

5. THE COMPENSATION PRINCIPLE

While the *Principle of Total Utility* requires $u \sim (0, u)$ to hold if u is summable, the extent to which this indifference can and should persist for delays of general utility streams has not been rigorously investigated. Specifically, although the impatience implications of axiom sets that properly include **SP** have been studied intensively in the literature stemming from Koopmans [20] and Koopmans et al. [21], we have not been able to find a study of time-preference for SWRs that are merely assumed to satisfy the strong Pareto axiom¹². As we have already noted, this axiom *alone* makes $(1, 1, 1, \dots)$ superior to its postponement $(0, 1, 1, \dots)$. Regardless of whether this observation gives new information on the topic, the fact remains that a number of researchers fail to appreciate that time-preference of the type $u \succ (0, u)$ is already inherent in the very general concept of a Paretian SWR. We have found examples in the literature where the preference of $(1, 1, 1, \dots)$ to $(0, 1, 1, \dots)$ is described as a deficiency of overtaking, where some authors use this

¹²The systematic study of time-preference was initiated by Koopmans [20]. For recent developments and an up-to-date list of references, see e.g. Banerjee and Dubey [6].

claim to argue that ranking $(1, 0, 1, \dots)$ above $(0, 1, 0, \dots)$ displays disrespect for equal treatment of the same kind¹³. Given that all Paretian SWRs rank $(1, 1, \dots)$ above $(0, 1, 1, \dots)$, speaking of this ranking as a deficiency of any particular ESWR can only be described as misleading. For postponements of $u' := (1, 0)^\infty$, the situation is undecided. Since $u' - (0, 0, u') = (1, 0, 0, \dots)$, two postponements cause a total utility loss of one unit. In terms of extended total utility, *each* postponement incurs a loss of one half — see (15). As is well known, it is *possible* to rank $(0, u') \sim u'$ without violating **SP** (see Section 7). But if we do, because *Strong Pareto* obliges us to rank $(0, 0, u')$ *below* u' , we will be forced to contradict ourselves if we justify ranking $(0, u') \sim u'$ on the grounds that the first sequence is the postponement of the second. That is, if $u' := (1, 0)^\infty$ and $(0, u') \sim u'$, then $u'' := (0, 1)^\infty$ satisfies $u'' \succ (0, u'')$ by transitivity and *Strong Pareto*.

The discussion above shows that for every ethical SWR, certain streams u have the property that (c, u) is inferior to u if the compensation $c = 0$. Let us consider the question of how a principle of compensation can be combined with *The Principle of Total Utility*. If (7) is satisfied, compensating the first generation by c is equivalent to distributing c units of utility over any finite or infinite number of generations.¹⁴ In this context, it is *humanity* that must be compensated, so the natural candidate for the amount of compensation for one delay is the average (1), assuming that it is defined. Since $\bar{u} = 0$ if u is summable, this compensation is consistent with the *Principle of Total Utility* that requires equivalence $u \sim (0, u)$ for streams $u \in \mathcal{U}_{\text{fin}}$.

The Compensation Principle¹⁵ *It is ethically acceptable to postpone an infinite utility stream with well-defined long-term average if the first generation is compensated by the average utility over subsequent generations.*

Formally,

$$u \sim (\bar{u}, u), \quad (13)$$

provided that \bar{u} is defined.

Theorem 1. *For $u \in \mathcal{U}$, if \bar{u} is defined, then*

$$(\bar{u}, u) \sim_{\text{LDU}} u. \quad (14)$$

The proof of Theorem 1 uses the relationship between the Abel and Cesàro methods of summation. For this relationship, see e.g. Hardy [16, Chapter V].

Proof. It suffices to prove that with $d := (c, u) - u = (c - u_1, u_1 - u_2, u_2 - u_3, \dots)$,

$$\sigma^*(d) = c - \bar{u}. \quad (15)$$

For this it is enough to show that the series $\sum_{n=1}^{\infty} d_n$ is Cesàro-summable to $c - \bar{u}$, i.e., that

$$\frac{s_1 + \dots + s_n}{n} \rightarrow c - \bar{u} \text{ as } n \rightarrow \infty, \quad (16)$$

¹³See for example Heal [17, p. 1115]

¹⁴Formally, if $u \sim (c, u)$ and $v \in \mathcal{U}$ has sum $\sigma(v) = c$, then $v \sim (c, \underline{0})$ and $u \sim (c, u) = (0, u) + (c, \underline{0}) \sim (0, u) + v$ by the *Principle of Total Utility*.

¹⁵Arrow [1] uses the same term to refer to a different method of collective choice.

where $s_n = \sum_{i=1}^n d_i$, $n \geq 1$; cf. [16, Theorem 55]. Here $\sum_{i=1}^n d_i = c - u_n$, $n \geq 1$, so

$$\frac{s_1 + \dots + s_n}{n} = c - \frac{u_1 + \dots + u_n}{n}. \quad (17)$$

Since \bar{u} exists by assumption, the right hand side of (17) tends to $c - \bar{u}$ as $n \rightarrow \infty$. \square

Every ESWR \succsim satisfying the *Compensation Principle* ranks $(1, 0)^\infty$ above $(0, 1)^\infty$.¹⁶ $(1, 0)^\infty$ has average $1/2$, so the *Compensation Principle* and *Strong Pareto* give

$$(1, 0)^\infty \sim (1/2, (1, 0)^\infty) \succ (0, (1, 0)^\infty) = (0, 1)^\infty.$$

That the overtaking criterion \succsim_W ranks these two streams the same way, however, is unrelated to the *Compensation Principle*: Example 1 showed that $(c, (1, 0)^\infty)$ and $(1, 0)^\infty$ are not \succsim_W -comparable if $c \in (0, 1)$.

6. PROPERTIES OF LDU AND CHARACTERIZATION

We established that limit-discounted utilitarianism satisfies the *Principle of Total Utility* in Proposition 1 and the *Compensation Principle* in Theorem 1. The purpose of this section is to study a few more properties of limit-discounted utilitarianism. Most properties are summarized in Theorem 2. Theorem 3 characterizes limit-discounted utilitarianism to be the only ethical SWR on a large domain to satisfy a selection of these properties, plus a robustness check relating the ordering between streams u and v to those of discounted streams with discount factors near one.

6.1. Basic axioms.

Finite Unit Comparability (FC): For all $u, v \in \mathcal{U}$, if $u \succsim v$, then $u + \alpha \succsim v + \alpha$ for every $\alpha \in \mathcal{U}_0$.

Horizon Consistency (HC): For all $u, v \in \mathcal{U}$, if there exists $N \in \mathbb{N}$ with $u_{[n]} \succ v_{[n]}$ for all $n \geq N$, then $u \succsim v$.

ε -Continuity (εC): For all $u, v \in \mathcal{U}$, if $u + (\varepsilon, \mathbf{0}) \succsim v$ for all $\varepsilon > 0$, then $u \succsim v$.

The ε -Continuity axiom appears to be original to the present paper. **FC** and **HC** on the other hand are relaxations of standard axioms from the literature. *Finite Unit Comparability* is a weaker form of the *Partial Unit Comparability* axiom from [9]. *Horizon Consistency* captures the idea formulated in Brock [10, p. 929] that “decisions on infinite programs are consistent with decisions on finite programs of length n if n is large enough”, but **HC** is weaker than Brock’s third axiom, and of the related “weak consistency” axiom in [9, p. 359]: it allows weak preference rather than demanding that $u \succ v$ hold if $u_{[n]} \succ v_{[n]}$ for large n .

It should be noted that these stronger versions of **HC** (those that require $u \succ v$ if $u_{[n]} \succ v_{[n]}$ holds for large n) induce time-preference of the type discussed in Example 2: by considering the streams in that example, it is not hard to verify that all SWRs satisfying finite utilitarianism (5) and the strengthened version of *Horizon Consistency* violate the *Principle of Total Utility*. In contrast, *Horizon Consistency* is compatible with (7), and its restriction to $\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}}$ is a necessary condition (see Remark 2).

¹⁶Proposition 4 gives a different motivation for this ranking.

All properties that we discuss can be restricted to an arbitrary subset \mathcal{V} of $\mathcal{U} \times \mathcal{U}$ by merely requiring that the condition in question hold when $(u, v) \in \mathcal{V}$, instead of for all $u, v \in \mathcal{U}$. The axioms that we obtain in this way will be denoted $\mathbf{UC}(\mathcal{V})$, $\mathbf{HC}(\mathcal{V})$, etc. When $\mathcal{V} = \mathcal{U} \times \mathcal{U}$, the index set will be suppressed.

6.2. Properties of limit-discounted utilitarianism.

Theorem 2. *Limit-discounted utilitarianism, i.e., binary relation \succsim_{LDU} in (10), defines a SWR that satisfies **SP**, **FA**, **Stationarity**, **FC**, **HC**, and $\varepsilon\mathbf{C}$.*

Proof. Rewrite (10) as

$$u \succsim_{\text{LDU}} v \quad \Leftrightarrow \quad \sigma_1(u - v) \geq 0,$$

where $\sigma_1: \mathcal{U} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is defined

$$\sigma_1(u) = \liminf_{\delta \rightarrow 1^-} \sigma_\delta(u). \quad (18)$$

We first verify that \succsim_{LDU} is indeed a SWR: It is reflexive since $\sigma_1(\underline{0}) = 0$. To see that \succsim_{LDU} is transitive, let $u, v, w \in \mathcal{U}$ satisfy $u \succsim_{\text{LDU}} v$ and $v \succsim_{\text{LDU}} w$: $\sigma_1(u - v) \geq 0$ and $\sigma_1(v - w) \geq 0$. For fixed $\delta \in (0, 1)$, $\sigma_\delta(u - w) = \sigma_\delta(u - v) + \sigma_\delta(v - w)$, so $\sigma_1(u - w) \geq \sigma_1(u - v) + \sigma_1(v - w) \geq 0 + 0 = 0$, i.e., $u \succsim_{\text{LDU}} w$.

SP: If $u > v$, then $\sigma_\delta(u - v)$ is a positive, increasing function of $\delta \in (0, 1)$, so its limit $\sigma^*(u - v)$ exists in $(0, \infty]$ and $\sigma^*(v - u) = -\sigma^*(u - v) \in [-\infty, 0)$. Thus $u \succ_{\text{LDU}} v$.

FA: If $u, v \in \mathcal{U}$ have $u_i = v_j, u_j = v_i$, and $u_k = v_k$ for all $k \neq i, j$, then

$$\sigma_\delta(u - v) = \delta^{i-1}(u_i - v_i) + \delta^{j-1}(u_j - v_j) = \delta^{i-1}(u_i - u_j) + \delta^{j-1}(u_j - u_i) \rightarrow 0$$

as $\delta \rightarrow 1^-$. Likewise, $\sigma_1(v - u) = 0$. Together, they give $u \sim_{\text{LDU}} v$.

Stationarity: A consequence of that $\sigma_1(u - v) = \sum_{i=1}^n (u_i - v_i) + \sigma_1(u^{[n]}, v^{[n]})$ for $u, v \in \mathcal{U}$ and $n \in \mathbb{N}$.

FC: This follows from that $(u + \alpha) - (v + \alpha) = u - v$ for all $u, v, \alpha \in \mathcal{U}$.

$\varepsilon\mathbf{C}$: Let $u, v \in \mathcal{U}$ have $u + (\varepsilon, \underline{0}) \succsim_{\text{LDU}} v$ for all $\varepsilon > 0$. Then

$$0 \leq \sigma_1(u + (\varepsilon, \underline{0}) - v) = \varepsilon + \sigma_1(u - v)$$

for each $\varepsilon > 0$, which implies $\sigma_1(u - v) \geq 0$, hence $u \succsim_{\text{LDU}} v$.

HC: Let $u, v \in \mathcal{U}$ and $N \in \mathbb{N}$ be such that $u_{[n]} \succ_{\text{LDU}} v_{[n]}$ for all $n \geq N$. The streams $u_{[n]}$ and $v_{[n]}$ are summable, so this is equivalent to that $s_n := \sum_{i=1}^n (u_i - v_i) > 0$ for all $n \geq N$ by Proposition 1. Summation by parts gives

$$\sum_{i=N+1}^{\infty} \delta^{i-1}(u_i - v_i) = \sum_{i=N+1}^{\infty} \delta^{i-1}(s_i - s_{i-1}) = (1 - \delta) \sum_{i=N+1}^{\infty} \delta^{i-1} s_i - \delta^N s_N,$$

which means (since each $s_n > 0$) that

$$\sum_{i=1}^{\infty} \delta^{i-1}(u_i - v_i) > \sum_{i=1}^N \delta^{i-1}(u_i - v_i) - \delta^N s_N.$$

Therefore,

$$\sigma_1(u - v) = \liminf_{\delta \rightarrow 1^-} \sum_{i=1}^{\infty} \delta^{i-1}(u_i - v_i) \geq \liminf_{\delta \rightarrow 1^-} \left(\sum_{i=1}^N \delta^{i-1}(u_i - v_i) - \delta^N s_N \right) = 0,$$

i.e., $u \succsim_{\text{LDU}} v$. \square

Remark 1. A trivial modification of the proof of **HC** shows that $u \succ_{\text{LDU}} v$ holds if $\sum_{i=1}^n (u_i - v_i) \geq 0$ for all sufficiently large n . So all \succ_W -comparable pairs are \succ_{LDU} -comparable.

From a utilitarian perspective, it is reasonable to rank u above v when $\bar{u} > \bar{v}$. (We do not want $\bar{u} = \bar{v}$ to imply $u \sim v$ since utility changes in any finite number of generations have no effect on (1).) The next results shows that the LDU criterion does discriminate among streams with different averages.¹⁷

Proposition 3. For all $u, v \in \mathcal{U}$ with well-defined averages \bar{u}, \bar{v} , respectively: if $\bar{u} > \bar{v}$, then $u \succ_{\text{LDU}} v$.

Proof. Let $u, v \in \mathcal{U}$ have $\bar{u} > \bar{v}$. Then there are $\varepsilon > 0$ and $N \in \mathbb{N}$ with $\frac{1}{n} \sum_{i=1}^n u_i - \varepsilon > \frac{1}{n} \sum_{i=1}^n v_i$ for all $n \geq N$. Hence, $\sum_{i=1}^n u_i - \varepsilon > \sum_{i=1}^n v_i$ for all $n \geq N$. Equivalently: for all $n \geq N$, summable streams $u_{[n]} - (\varepsilon, \underline{0})$ and $v_{[n]}$ satisfy

$$\sigma_1(u_{[n]} - (\varepsilon, \underline{0}) - v_{[n]}) = \sigma(u_{[n]} - (\varepsilon, \underline{0}) - v_{[n]}) > 0,$$

i.e., $u_{[n]} - (\varepsilon, \underline{0}) \succ_{\text{LDU}} v_{[n]}$. By **HC**, $u - (\varepsilon, \underline{0}) \succ_{\text{LDU}} v$. By **SP**, $u \succ_{\text{LDU}} v$. \square

6.3. A characterization of limit-discounted utilitarianism. As mentioned in the introduction, the idea of checking for consistency with the discounted utilitarian criterion for discount factors close to 1 appears in Basu and Mitra [9], p. 360-363. Their “robustness check” of an ethical SWR \succsim consists of determining whether “the ranking between two alternatives provided by \succsim is preserved for discount factors close to 1” in the sense that $u \succ v$ implies $\sigma_\delta(u - v) > 0$ for all δ sufficiently close to 1. In this vein, the *Discounting Consistency* property that we introduce below formalizes the idea that the problem with non-discounted streams can be approximated by the problem with discounted streams with discount factors close to one:

Discounting Consistency (DC): For all $u, v \in \mathcal{U}$, if there is a $\hat{\delta} \in (0, 1)$ with

$$(u_1, \delta u_2, \delta^2 u_3, \dots, \delta^{i-1} u_i, \dots) \succ (v_1, \delta v_2, \delta^2 v_3, \dots, \delta^{i-1} v_i, \dots) \quad (19)$$

for all $\delta \in (\hat{\delta}, 1)$, then $u \succsim v$.

Note that this property is stated in terms of discounted *streams*, not discounted sums. Since discounted (bounded) streams are summable, the following extension of the key Lemma from [9] establishes the connection between the two.

Lemma 1. If \succsim is an ESWR that satisfies **FC**($\mathcal{U}_0 \times \mathcal{U}_0$), **HC**($\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}}$) and $\varepsilon\mathbf{C}$ ($\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}}$), then

$$\text{for all } u, v \in \mathcal{U}_{\text{fin}}: \quad u \succsim v \text{ if and only if } \sigma(u) \geq \sigma(v). \quad (20)$$

Remark 2. For ethical SWRs, **FC**($\mathcal{U}_0 \times \mathcal{U}_0$), **HC**($\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}}$) and $\varepsilon\mathbf{C}$ ($\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}}$) are also necessary for (20), so these axioms need no further motivation besides The Principle of Total Utility. For the necessity and logical independence of the three axioms, we refer to our companion paper [18].

¹⁷This result is generalized in [18], where the conclusion is derived axiomatically.

Proof. We first show that if the ESWR \succsim satisfies $\mathbf{FC}(\mathcal{U}_0 \times \mathcal{U}_0)$, then

$$\text{for all } u \in \mathcal{U}_0 : \quad u \sim (\sigma(u), \underline{0}). \quad (21)$$

Note that

$$ae_1 - ae_j \sim \underline{0} \text{ for all } j \in \mathbb{N}, a \in \mathbb{R}. \quad (22)$$

If $j = 1$, this follows from reflexivity, so let $j > 1$. By \mathbf{FA} ,

$$\frac{1}{2}ae_1 - \frac{1}{2}ae_j \sim -\frac{1}{2}ae_1 + \frac{1}{2}ae_j.$$

By $\mathbf{FC}(\mathcal{U}_0 \times \mathcal{U}_0)$ with $\alpha = \frac{1}{2}ae_1 - \frac{1}{2}ae_j$, this yields (22). Given $u \in \mathcal{U}_0$, let $m(u) = \min\{n \in \mathbb{N} : u_k = 0 \text{ for all } k > n\}$. If $m(u) = 1$, $u = (\sigma(u), \underline{0})$, and $u \sim (\sigma(u), \underline{0})$ follows from reflexivity. If $m(u) = n > 1$, (22) gives $u_n e_1 - u_n e_n \sim \underline{0}$, so by $\mathbf{FC}(\mathcal{U}_0 \times \mathcal{U}_0)$, adding u to both sides gives

$$(u_1 + u_n, u_2, \dots, u_{n-1}, \underline{0}) \sim u.$$

The left vector has at most $n - 1$ nonzero coordinates. So repeating this with $u_{n-1}e_1 - u_{n-1}e_{n-1} \sim \underline{0}, \dots, u_2e_1 - u_2e_2 \sim \underline{0}$, each time changing a (possibly) nonzero coordinate to zero and adding it to coordinate 1, we find $u \sim (\sigma(u), \underline{0})$.

By reflexivity of \succsim if $\sigma(u) = \sigma(v)$ and the Pareto axiom \mathbf{SP} otherwise, (21) gives:

$$\text{for all } (u, v) \in \mathcal{U}_0 \times \mathcal{U}_0 : \quad u \succsim v \Leftrightarrow (\sigma(u), \underline{0}) \succsim (\sigma(v), \underline{0}) \Leftrightarrow \sigma(u) \geq \sigma(v). \quad (23)$$

To show that (20) holds if \succsim in addition satisfies $\mathbf{HC}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$ and $\mathbf{eC}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$, take $u, v \in \mathcal{U}_{\text{fin}}$. Firstly, if $\sum_{n=1}^{\infty} u_n \geq \sum_{n=1}^{\infty} v_n$, let $\varepsilon > 0$. By convergence of the partial sums, there is an $N \in \mathbb{N}$ with $\sum_{i=1}^n u_i + \frac{\varepsilon}{2} > \sum_{i=1}^n v_i$ for all $n \geq N$. Equivalently: for all $n \geq N$, streams $u_{[n]} + (\varepsilon/2, \underline{0})$ and $v_{[n]}$ in \mathcal{U}_0 satisfy

$$\sigma(u_{[n]} + (\varepsilon/2, \underline{0})) = \sum_{i=1}^n u_i + \frac{\varepsilon}{2} > \sum_{i=1}^n v_i = \sigma(v_{[n]}).$$

By (23), $u_{[n]} + (\varepsilon/2, \underline{0}) \succ v_{[n]}$ for all $n \geq N$. By $\mathbf{HC}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$, $u + (\varepsilon/2, \underline{0}) \succ v$. By \mathbf{SP} , $u + (\varepsilon, \underline{0}) \succ v$. Since $\varepsilon > 0$ was arbitrary, $\mathbf{eC}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$ gives $u \succsim v$.

Secondly, if $u \succsim v$, we show that $\sum_{n=1}^{\infty} u_n \geq \sum_{n=1}^{\infty} v_n$. If, to the contrary, $\sum_{n=1}^{\infty} u_n < \sum_{n=1}^{\infty} v_n$, then $\sum_{n=1}^{\infty} v_n - \varepsilon \geq \sum_{n=1}^{\infty} u_n$ for $\varepsilon > 0$ sufficiently small. Applying the previous case to summable $v - (\varepsilon, \underline{0})$ and u gives $v - (\varepsilon, \underline{0}) \succ u$. By \mathbf{SP} , $v \succ u$, contradicting $u \succsim v$. \square

We characterize the LDU criterion on the set $\mathcal{D} \subset \mathcal{U} \times \mathcal{U}$ of pairs (u, v) for which $\sigma^*(u - v)$ is defined. That is,

$$\mathcal{D} := \{(u, v) \in \mathcal{U} \times \mathcal{U} : \lim_{\delta \rightarrow 1^-} \sigma_{\delta}(u - v) \text{ exists in } \mathbb{R} \cup \{-\infty, +\infty\}\}.$$

In the following example we construct a pair (u, v) that is not in \mathcal{D} by “diluting” $(1, -1, 1, \dots)$; cf. [16, Section 3.9].

Example 3. Let $d_i = (-1)^n$ if $i = 2^n$ for some integer $n \geq 0$ and $d_i = 0$ otherwise. Then $\sigma_{\delta}(d)$ oscillates as $\delta \rightarrow 1^-$ [16, p. 59-60], so $u = d - (r, \underline{0})$ and $v = \underline{0}$ are not \succsim_{LDU} -comparable if $r \in (\liminf_{\delta \rightarrow 1^-} \sigma(d), \limsup_{\delta \rightarrow 1^-} \sigma(d))$.

Another way of constructing pairs that are not \succsim_{LDU} -comparable would be to take u, v such that $\sum_{n=1}^N (u_n - v_n)$ oscillates between $-\infty$ and $+\infty$ as $N \rightarrow +\infty$. However, our domain is sufficiently large to contain all streams that we earlier proved to

be comparable (see Propositions 1, 2,3 and Theorem 1); we discuss possible variants in Section 6.4.

Theorem 3. *Let \succsim be an ESWR on \mathcal{U} satisfying $\mathbf{FC}(\mathcal{U}_0 \times \mathcal{U}_0)$, $\mathbf{HC}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$, $\varepsilon\mathbf{C}(\mathcal{D})$ and $\mathbf{DC}(\mathcal{D})$. Then \succsim coincides with \succsim_{LDU} on \mathcal{D} :*

$$\text{for all } (u, v) \in \mathcal{D} : \quad u \succsim_{\text{LDU}} v \quad \Leftrightarrow \quad u \succsim v. \quad (24)$$

Conversely, if (24) holds, then \succsim satisfies $\mathbf{FC}(\mathcal{U}_0 \times \mathcal{U}_0)$, $\mathbf{HC}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$, $\varepsilon\mathbf{C}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$ and $\mathbf{DC}(\mathcal{D})$.

Proof. By Theorem 2, \succsim_{LDU} is an ESWR satisfying \mathbf{FC} , \mathbf{HC} , $\varepsilon\mathbf{C}$ and consequently their restrictions to any subset. To show that \succsim_{LDU} satisfies \mathbf{DC} (and hence $\mathbf{DC}(\mathcal{D})$), let $u, v \in \mathcal{U}$ and $\hat{\delta} \in (0, 1)$ be such that (19) holds for all $\delta \in (\hat{\delta}, 1)$. The streams in (19) are summable, which by Lemma 1 means that $\sum_{i=1}^{\infty} \delta^{i-1}(u_i - v_i) > 0$ for all $\delta \in (\hat{\delta}, 1)$. Hence, $\liminf_{\delta \rightarrow 1^-} \sum_{i=1}^{\infty} \delta^{i-1}(u_i - v_i) \geq 0$, i.e., $u \succsim_{\text{LDU}} v$.

To see that any ESWR with the stated properties coincides with \succsim_{LDU} on \mathcal{D} , let \succsim be any ESWR that satisfies them. In particular, \succsim satisfies \mathbf{SP} and \mathbf{FA} . NOTATION: For $u \in \mathcal{U}$ and $\delta \in (0, 1)$, write $\delta^{\mathbb{Z}^+}u := (u_1, \delta u_2, \delta^2 u_3, \dots)$, which is summable since u is bounded.

To show that for all $(u, v) \in \mathcal{D}$, $u \succsim_{\text{LDU}} v$ if and only if $u \succsim v$, it suffices to prove that for all $(u, v) \in \mathcal{D}$, $u \succ_{\text{LDU}} v$ implies $u \succ v$, and that $u \succ_{\text{LDU}} v$ implies $u \succsim v$.

Let $(u, v) \in \mathcal{D}$ satisfy $u \succ_{\text{LDU}} v$. Since $\lim_{\delta \rightarrow 1^-} \sigma_{\delta}(u - v)$ exists, we must have

$$\lim_{\delta \rightarrow 1^-} \sigma_{\delta}(u - v) \geq 0 \quad \text{but} \quad \lim_{\delta \rightarrow 1^-} \sigma_{\delta}(v - u) = - \lim_{\delta \rightarrow 1^-} \sigma_{\delta}(u - v) \not\geq 0,$$

so $\lim_{\delta \rightarrow 1^-} \sigma_{\delta}(u - v) > 0$. Take $\varepsilon > 0$ and $\hat{\delta} \in (0, 1)$ with $\sigma_{\delta}(u - v) > \varepsilon$ for all $\delta \in (\hat{\delta}, 1)$. By Lemma 1, we then have that

$$\delta^{\mathbb{Z}^+}(u - (\varepsilon, \underline{0})) \succ \delta^{\mathbb{Z}^+}v \quad \text{for every } \delta \in (\hat{\delta}, 1),$$

since all streams lie in \mathcal{U}_{fin} . By $\mathbf{DC}(\mathcal{D})$, $u - (\varepsilon, \underline{0}) \succsim v$. By \mathbf{SP} , $u \succ v$.

Let $(u, v) \in \mathcal{D}$ satisfy $u \succsim_{\text{LDU}} v$. By \mathbf{SP} , $u + (\varepsilon, \underline{0}) \succ_{\text{LDU}} v$ for all $\varepsilon > 0$. Since $(u + (\varepsilon, \underline{0}), v) \in \mathcal{D}$, the previous step gives $u + (\varepsilon, \underline{0}) \succ v$ for all $\varepsilon > 0$. By $\varepsilon\mathbf{C}(\mathcal{D})$, $u \succsim v$.

Finally, let (24) hold: \succsim coincides with \succsim_{LDU} on $\mathcal{D} \supset \mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}} \supset \mathcal{U}_0 \times \mathcal{U}_0$ and therefore inherits the four properties of \succsim_{LDU} . \square

6.4. Alternative explicit and axiomatic descriptions. A variant of our limit-discounted criterion is given by

$$u \succsim_{\text{LDU}^*} v \quad \Leftrightarrow \quad \sigma^*(u - v) \geq 0. \quad (25)$$

Since $\liminf_{\delta \rightarrow 1^-} \sigma_{\delta}(u - v) = \sigma^*(u - v)$ if the limit exists, \succsim_{LDU} is an extension of \succsim_{LDU^*} . The SWR \succsim_{LDU^*} satisfies all properties in Theorem 2 except \mathbf{HC} , which can be seen by defining $u = (1, d_1, d_2, \dots)$, where d is as in Example 3. With $v = \underline{0}$, we have $s_n = \sum_{i=1}^n (u_i - v_i) \geq 1$ for all n , yet u and v are not \succsim_{LDU^*} -comparable. On \mathcal{D} the SWRs coincide, so Theorem 3 remains valid with \succsim_{LDU^*} in the role of \succsim_{LDU} . Since u and v are \succsim_{LDU^*} -comparable precisely when $(u, v) \in \mathcal{D}$, that theorem gives:

Theorem 4. *An ESWR on \mathcal{U} is an extension of \succsim_{LDU^*} if and only if it satisfies $\mathbf{FC}(\mathcal{U}_0 \times \mathcal{U}_0)$, $\mathbf{HC}(\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}})$, $\varepsilon\mathbf{C}(\mathcal{D})$ and $\mathbf{DC}(\mathcal{D})$.*

On $\mathcal{U} \times \mathcal{U}$, a modification of the *Discounting Consistency* axiom is needed.¹⁸ Given the axiom’s convoluted nature, we opted instead for a sharp characterization on a large subdomain (Theorem 3).

7. EXTENDED ANONYMITY VERSUS EXTENDED TOTAL UTILITY

The LDU criterion combines a high degree of comparability¹⁹ with *Strong Pareto*, *Finite Anonymity*, *Stationarity*, *Horizon Consistency* and the *Total Utility Principle*. In addition, it has provided us with a coherent view on delays of infinite streams with well-defined averages. Some authors may object that \succsim_{LDU} ranks $(1, 0)^\infty$ above $(0, 1)^\infty$ by referring to the so-called extended anonymity axioms that these streams have inspired. *Finite Anonymity* means that $u \sim \pi(u)$ holds for all $u \in \mathcal{U}$ and all finite $\pi \in \Pi$, so one way to define variants of this condition is to impose invariance under some set $\mathcal{Q} \subset \Pi$ of “permissible” transformations where the finite permutations are properly contained. This can be done in two ways:

Q-Anonymity (QA) For all $u \in \mathcal{U}$ and all $\pi \in \mathcal{Q}$, $u \sim \pi(u)$,

Relative Q-Anonymity (RQA) For all $u, v \in \mathcal{U}$ and all $\pi \in \mathcal{Q}$, $u \succsim v$ if and only if $\pi(u) \succsim \pi(v)$.

The existence of Pareto-compatible *Q-Anonymity* axioms that imply $(1, 0)^\infty \sim (0, 1)^\infty$ was shown by Lauwers [22]. For *Relative Anonymity*, even the strongest form (called *Strong Relative Anonymity* in [3]) can be combined with **SP**.

At first sight it might appear that stronger invariance conditions should lead to “more equity”. However, a rigorous account of the connection between infinite permutations and intergenerational justice is still lacking.²⁰ As for the consequences of imposing these axioms, they are partially summarized in Proposition 4 below. Roughly, we either end up with non-stationary preferences — for *Q-Anonymity*²¹ — or loose contact with the finite problem. In fact, finite utilitarianism (5) and **RPA** cannot be combined with the following property, which is a weaker requirement than **HC**.²²

¹⁸To state it, say that u δ -dominates v if there is a $\hat{\delta} \in (0, 1)$ such that (19) holds for all $\delta \in (0, \hat{\delta})$, and that u weakly δ -dominates v if $u + (\varepsilon, 0)$ δ -dominates v for every $\varepsilon > 0$. Axioms **εC** and **DC** together imply that for all $u, v \in \mathcal{U}$, if u weakly δ -dominates v , then $u \succsim v$. To characterize extensions of \succsim_{LDU} on the full domain, we must also require $u \succ v$ to hold if v does not weakly dominate u .

Strong Discounting Consistency (SDC): For all $u, v \in \mathcal{U}$, if u weakly δ -dominates v , then $u \succsim v$. If u weakly δ -dominates v but not vice versa, then $u \succ v$.

An ESWR is an extension of \succsim_{LDU} if and only if it satisfies **FC**($\mathcal{U}_0 \times \mathcal{U}_0$), **HC**($\mathcal{U}_{\text{fin}} \times \mathcal{U}_{\text{fin}}$) and **SδC**; the proof is available upon request. Axiom **εC** axiom is no longer present, but this comes at the expense of a property that is less easily interpreted.

¹⁹See, for instance, Propositions 1, 2, 3, and Remark 1.

²⁰Basu and Mitra [8] have posed and completely settled the question of when **QA** can be combined with **SP** by showing that there is a SWR that satisfies both axioms if and only if \mathcal{Q} is a group of cyclic permutations. But they do not justify this class of invariance conditions and clearly state that their investigation should be viewed as a logical consistency-check only.

²¹Though we only prove this for nonrelative anonymity axioms that imply $(1, 0)^\infty \sim (0, 1)^\infty$, Demichelis et al [11] have shown that *Stationarity* places severe restrictions on our possibilities to strengthen **FA** via *Q-Anonymity*.

²²Recall, if necessary, the notation in Section 2.1.

The Correspondence Principle: For $u, v \in \mathcal{U}$, if there is an $N \in \mathbb{N}$ such that $u_{[B]} \succ v_{[B]}$ for every block B with $|B| \geq N$, then $u \succsim v$.

We do not find such inconsistencies particularly surprising considering that the imposed restrictions have no clear ethical or economic content. But our reasons for proposing the alternative that we have presented are entirely practical, namely that it gives us coherence in terms of *Stationarity* and correspondence between the large, finite-horizon problem and the infinite-horizon problem.

Proposition 4. *Let \succsim be an ESWR on \mathcal{U} .*

- (a) *If \succsim satisfies Stationarity and $(1, 0)^\infty$ and $(0, 1)^\infty$ are \succsim -comparable, then $(1, 0)^\infty \succ (0, 1)^\infty$.*
- (b) *If \succsim satisfies finite utilitarianism (5) and Strong Relative Anonymity **RIIA**, then \succsim violates the Correspondence Principle.*

The proof of (b) shows in a rather striking way that Paretian extensions of finite utilitarianism in general cannot discriminate between streams with distinct averages. For a more detailed description of the comparability restrictions implied by **RIIA**, the reader is referred to [3, Section 7].

Proof. (a): Suppose $(1, 0)^\infty \succ (0, 1)^\infty$ does not hold. If the two streams are comparable, this means that $(0, 1)^\infty \succsim (1, 0)^\infty$. By **SP**, $(1, 0)^\infty \succ (0, (0, 1)^\infty)$. By transitivity, $(0, 1)^\infty \succ (0, (0, 1)^\infty)$. By *Stationarity*, omitting the initial zero in both streams, we have $(1, 0)^\infty \succ (0, 1)^\infty$, a contradiction.

(b): Suppose for contradiction that ESWR \succsim satisfies finite utilitarianism and **RIIA**. Define $u = (0, \underline{1}_{100})^\infty$, $v = (1, \underline{0}_{100})^\infty$, and let $w = v + (1, \underline{0})$. For every block B with $|B| \geq 4$, $u_{[B]}$ and $w_{[B]}$ are in \mathcal{U}_0 and satisfy $\sigma(u_{[B]}) > \sigma(w_{[B]})$, so by finite utilitarianism: $u_{[B]} \succ w_{[B]}$. Therefore, $u \succsim w$ by the *Correspondence Principle* and $u \succ v$ by **SP**.

Since u and v have entries in $\{0, 1\}$ and each has infinitely many zero entries and infinitely many one entries, there is a $\pi \in \Pi$ such that $u = \pi(v)$. For instance, let π map the index of k -th zero entry of the first stream to the index of the k -th zero entry of the second stream, and likewise for the one entries. We chose u and v to be ‘symmetric’ in $\{0, 1\}^\mathbb{N}$, in the sense that $|u_i - v_i| = 1$ for all $i \in \mathbb{N}$. Hence, by construction, $v = \pi(u)$. But then **RIIA** together with $u \succsim v$ imply $v = \pi(u) \succsim \pi(v) = u$, a contradiction. \square

8. CONCLUDING REMARKS

The definition of an infinite horizon version of utilitarianism is complicated by the fact that no complete ordering can be represented by a social welfare function or otherwise be described explicitly. In many previous studies it has been suggested that extended notions of anonymity are required to represent intergenerational justice in this context. The present paper departs from this tradition and proposes a replacement for these notions through an extended concept of aggregate utility. We introduced the extended total utility

$$\sigma^*(u) := \lim_{\delta \rightarrow 1^-} \sum_{n=1}^{\infty} \delta^{n-1} u_n \quad (26)$$

and defined a generalization of utilitarianism for summable streams by declaring $u = (u_1, u_2, \dots)$ at least as good as $v = (v_1, v_2, \dots)$ whenever $\sigma^*(u - v)$ is well-defined and nonnegative. That limit-discounted utilitarianism combines the equal treatment of generations with the analytical tractability of discounted utilitarianism allowed us to provide, among other things, a coherent view of the ethical consequences of delaying infinite streams of utility.

REFERENCES

- [1] K. Arrow. *Social Choice and Individual Values*. Wiley, New York, 1963. 2nd edition.
- [2] G. Asheim and K. Banerjee. Fixed-step anonymous overtaking and catching-up. *International Journal of Economic Theory*, 6:147–163, 2010.
- [3] G. Asheim, C. d’Aspremont, and K. Banerjee. Generalized time-invariant overtaking. *Journal of Mathematical Economics*, 38:519–533, 2010.
- [4] B. Ashheim and Tungodden. Justifying sustainability. *J. Environ. Econ. Manage.*, 41:252–268, 2001.
- [5] K. Banerjee. On the extension of the utilitarian and Suppes-Sen social welfare relations to infinite utility streams. *Social Choice and Welfare*, 27:327–339, 2006.
- [6] K. Banerjee and R. S. Dubey. Impatience for weakly Paretian orders: Existence and genericity. *Journal of Mathematical Economics*, 49:134–140, 2013.
- [7] K. Basu and T. Mitra. Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian. *Econometrica*, 71:1557–1563, 2003.
- [8] K. Basu and T. Mitra. *On the existence of Paretian social welfare quasi orderings for infinite utility streams with extended anonymity*. in: J. Roemer, K. Suzumura (Eds.), *Intergenerational Equity and Sustainability*, 2007.
- [9] K. Basu and T. Mitra. Utilitarianism for infinite utility streams: A new welfare criterion and its axiomatic characterization. *Journal of Economic Theory*, 133:350–373, 2007.
- [10] W. A. Brock. An axiomatic basis for the Ramsey-Weizsäcker overtaking criterion. *Econometrica*, 38:927–929, 1970.
- [11] S. Demichelis, T. Mitra, and G. Sorger. *Intergenerational equity and stationarity*. Vienna Univ. Dep. of Economics Online Resources, 2010.
- [12] P. Diamond. The evaluation of infinite utility streams. *Econometrica*, 33:170–177, 1965.
- [13] P. Dutta. What do discounted optima converge to? A theory of discount rate asymptotics in economic models. *Journal of Economic Theory*, 55:64–94, 1991.
- [14] M. Fleurbaey and P. Michel. Intertemporal equity and the extension of the Ramsey criterion. *Journal of Mathematical Economics*, 39:777–802, 2003.
- [15] D. Gale. On optimal development in a multi-sector economy. *Review of Economic Studies*, 34:1–18, 1967.
- [16] G. H. Hardy. *Divergent series*. Éditions Jacques Gabay, Sceaux, 1992. With a preface by J. E. Littlewood and a note by L. S. Bosanquet, Reprint of the revised (1963) edition.
- [17] G. Heal. *Intertemporal welfare economics and the environment*. in: *Handbook of Economics 20* (Eds.: K.-G. Mäler and J.R. Vincent), Elsevier, 2005.
- [18] A. Jonsson and M. Voorneveld. Ethical social welfare relations and utilitarianism for infinite utility streams with finite sums and averages. SSE/EFI Working Paper Series in Economics and Finance, no. 747, 2014.
- [19] K. Kamaga and T. Kojima. On the leximin and utilitarian overtaking criteria with extended anonymity. *Social choice welfare*, 35:277–392, 2010.
- [20] T. Koopmans. Stationary ordinal utility and impatience. *Econometrica*, 28:287–309, 1960.
- [21] T. Koopmans, M. Diamond, and R. Williamsson. Stationary utility and time perspective. *Econometrica*, 32:82–100, 1964.
- [22] L. Lauwers. Infinite utility: Insisting on strong monotonicity. *Australasian Journal of Philosophy*, 75:222–233, 1997.
- [23] L. Lauwers. Ordering infinite utility streams comes at the cost of a non-Ramsey set. *Journal of Mathematical Economics*, 46:32–37, 2010.
- [24] A. Pigou. *The Economics of Welfare*. Macmillan, London, 1920.
- [25] F. P. Ramsey. A mathematical theory of savings. *Economic Journal*, 38:543–559, 1928.

- [26] T. Sakai. Intergenerational equity and an explicit construction of welfare criteria. *Social Choice Welfare*, 35:393–414, 2010.
- [27] P. Suppes. Some formal models of grading principles. *Synthese*, 6:284–306, 1966.
- [28] L.-G. Svensson. Equity among generations. *Econometrica*, 48:1251–1256, 1980.
- [29] C. C. von Weizsäcker. Existence of optimal programs of accumulation for an infinite time horizon. *Review of Economic Studies*, 32:85–104, 1965.
- [30] W. Zame. Can utilitarianism be operationalized? *Theoretical Economics*, 2:187–202, 2010.

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