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**Research Report**

## US Health and Aggregate Fluctuations: Technical Appendix

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# US Health and Aggregate Fluctuations: Technical Appendix

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## 1 Household Optimization Problem

The household has to solve a stochastic optimal control problem, with market and non-market consumption and hours as controls, and home and market capital as endogenous state variables. Set up the Lagrangian

$$\begin{aligned} \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{ & \ln C_t + \psi \ln G_t + \theta \ln(1 - H_t^g - BG^{-\xi} - H_t^w) \\ & + \lambda_t [(1 - \tau^l)w_t H_t^w + (1 - \tau^k)r_t K_t - T_t - K_{t+1} + (1 - \delta^k)K_t - C_t] \\ & + \mu_t [Z_t (H_t^g)^\phi + (1 - \delta^g)G_t - G_{t+1}] \} \end{aligned}$$

This is a concave programming problem, so the FOCs, together with the additional, boundary ("transversality") conditions for private physical capital and government bonds are both necessary and sufficient for an optimum.

To derive the FOCs, first we take the derivative of the Lagrangian w.r.t  $C_t^h$  (holding all other variables unchanged) and set it to 0, i.e.  $\mathcal{L}_{C_t^h} = 0$ . That will result in the following expression

$$L_{C_t} = 0 \rightarrow \frac{1}{C_t} = \lambda_t \tag{1}$$

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This optimality condition equates marginal utility of consumption to the marginal utility of wealth.

Now take the derivative of the Lagrangian w.r.t  $K_{t+1}$  (holding all other variables unchanged) and set it to 0, i.e.  $L_{K_{t+1}} = 0$ . That will result in the following expression

$$L_{K_{t+1}} = 0 \rightarrow \lambda_t^1 = \beta E_t \lambda_{t+1}^1 \left[ (1 - \tau^k) \alpha \frac{Y_{t+1}}{K_{t+1}} + (1 - \delta^k) \right] \quad (2)$$

This is the Euler equation, which determines how consumption is allocated across periods.

Now take the derivative of the Lagrangian w.r.t  $H_t^w$ , i.e.  $L_{H_t^w} = 0$  (holding all other variables unchanged). This will result in the following expression

$$L_{H_t^w} = 0 \rightarrow \frac{\theta}{1 - H_t^g - H_t^s - H_t^w} = \lambda_t (1 - \tau^l) (1 - \alpha) \frac{Y_t}{H_t^w} = \frac{1}{C_t} (1 - \tau^l) (1 - \alpha) \frac{Y_t}{H_t^w} \quad (3)$$

Now take the derivative of the Lagrangian w.r.t  $H_t^g$ , i.e.  $L_{H_t^g} = 0$  (holding all other variables unchanged). This will result in the following expression

$$L_{H_t^g} = 0 \rightarrow \frac{\theta}{1 - H_t^g - H_t^s - H_t^w} = \mu_t \phi Z_t (H_t^g)^{\phi-1} \quad (4)$$

or

$$\mu_t = \frac{\frac{1}{C_t} (1 - \tau^l) (1 - \alpha) \frac{Y_t}{H_t^w}}{\phi Z_t (H_t^g)^{\phi-1}}$$

Now take the derivative of the Lagrangian w.r.t  $G_t^h$ , i.e.  $L_{G_t^h} = 0$  (holding all other variables unchanged). This will result in the following expression

$$L_{G_{t+1}} = 0 \rightarrow \beta \left\{ \frac{\psi}{G_{t+1}} + \frac{\theta B \xi G_{t+1}^{-\xi-1}}{1 - H_{t+1}^g - H_{t+1}^s - H_{t+1}^w} + \mu_{t+1} (1 - \delta^g) \right\} = \mu_t \quad (5)$$

Note that

$$\frac{\theta B \xi G_{t+1}^{-\xi-1}}{1 - H_{t+1}^g - H_{t+1}^s - H_{t+1}^w} = B \xi G_{t+1}^{-\xi-1} \mu_{t+1} \phi Z_t (H_{t+1}^g)^{\phi-1}$$

Then

$$\beta \left\{ \frac{\psi}{G_{t+1}} + B \xi G_{t+1}^{-\xi-1} \mu_{t+1} \phi Z_t (H_{t+1}^g)^{\phi-1} + \mu_{t+1} (1 - \delta^g) \right\} = \mu_t \quad (6)$$

This FOC shows the optimality condition for inter-temporal allocation of health: the household equates the benefits and costs. The discounted benefit has three parts: First, a higher

health level tomorrow brings higher utility (the direct effect of health). Second, better health means less sick time, hence indirectly more time to work and consumption. Thirdly, higher health means higher undepreciated health level and thus less replenishment is needed to get back to the old level. The cost is that a larger replenishment was done in the previous period.

## 2 Representative Firm's Problem

The profit function is maximized when the derivatives of that function are set to zero. Therefore, the optimal amount of capital - holding the level of technology  $A_t$  and labor input  $H_t^w$  constant - is determined by setting the derivative of the profit function with respect to  $K_t$  equal to zero. This derivative is

$$\alpha A_t (H_t^w)^{1-\alpha} K_t^{\alpha-1} - r_t = 0 \quad (7)$$

where  $\alpha A_t (H_t^w)^{1-\alpha} K_t^{\alpha-1}$  is the marginal product of capital because it expresses how much output will increase if capital increases by one unit. The economic interpretation of this First-Order Condition (FOC) is that in equilibrium, firms will rent capital up to the point where the benefit of renting an additional unit of capital, which is the marginal product of capital, equals the rental cost, i.e the interest rate.

$$r_t = \alpha A_t (H_t^w)^{1-\alpha} K_t^{\alpha-1} \quad (8)$$

Now, multiply by  $K_t$  and rearrange terms. This gives the following relationship:

$$\alpha A_t (H_t^w)^{1-\alpha} K_t^{\alpha-1} K_t = r_t K_t \quad \text{or} \quad \alpha Y_t = r_t K_t \quad (9)$$

because

$$\alpha A_t (H_t^w)^{1-\alpha} K_t^{\alpha-1} K_t = \alpha A_t (H_t^w)^{1-\alpha} K_t^\alpha = \alpha Y_t \quad (10)$$

To derive firms' optimal labor demand, set the derivative of the profit function with respect to the labor input equal to zero, holding technology and capital constant:

$$(1 - \alpha) A_t (H_t^w)^{-\alpha} K_t^\alpha - w_t = 0 \quad \text{or} \quad w_t = (1 - \alpha) A_t (H_t^w)^{-\alpha} K_t^\alpha \quad (11)$$

In equilibrium, firms will hire labor up to the point where the benefit of hiring an additional hour of labor services, which is the marginal product of labor, equals the cost, i.e the hourly

wage rate.

Now multiply both sides of the equation by  $H_t^w$  and rearrange terms to yield

$$(1 - \alpha)A_t(H_t^w)^{-\alpha}K_t^\alpha H_t^w = w_t H_t^w \quad \text{or} \quad (1 - \alpha)Y_t = w_t H_t^w \quad (12)$$

Next, it will be shown that in equilibrium, economic profits are zero. Using the results above one can obtain

$$\Pi_t = Y_t - r_t K_t - w_t H_t^w = Y_t - \alpha Y_t - (1 - \alpha)Y_t = 0 \quad (13)$$

Indeed, in equilibrium, economic profits are zero.

### 3 Steady-State Computation

Shutting down all stochasticity, we obtain

$$1 = \beta[(1 - \tau^k)\alpha\frac{y}{k} + 1 - \delta^k]$$

$$\frac{1}{c} = \lambda$$

$$\frac{\theta}{1 - h^g - Bg^{-\xi} - h^w} = \frac{1}{c}(1 - \tau^l)(1 - \alpha)\frac{y}{h^w}$$

$$\beta\left\{\frac{\psi}{g} + B\xi g^{-\xi-1}\mu\phi(h^g)^{\phi-1} + \mu(1 - \delta^g)\right\} = \mu$$

Collecting all  $\mu$  terms to the right we obtain

$$\beta\frac{\psi}{g} = \mu[1 - \beta B\xi g^{-\xi-1}\phi(h^g)^{\phi-1} - \beta(1 - \delta^g)]$$

$$\mu = \frac{\beta\psi/g}{1 - \beta B\xi g^{-\xi-1}\phi(h^g)^{\phi-1} - \beta(1 - \delta^g)}$$

$$c + \delta^k k = k^\alpha (h^w)^{1-\alpha}$$

Outline of the steady-state computation procedure: First, from the Euler equation, we can obtain the capital-to-output ratio  $k/y$ , or capital-to-hours ratio, since  $y/k = (k/h^w)^{\alpha-1}$ . Then we impose that in steady-state  $h^w = 1/3$ , and obtain steady-state capital stock  $k$ , investment  $i^k$  and output  $y$ . From market clearing, we obtain steady-state consumption  $c$ . Next, using that steady-state exercise hours  $h^g = 0.02$  and the law of motion of health, we can compute steady-state health status  $g = (1/\delta^g)(h^g)^\phi$ . Setting steady-state sick days  $h^s = 0.02$  in the relationship  $h^s = Bg^{-\xi}$ , we obtain  $B = h^s g^\xi$ . Finally, we can find  $\theta$  from the marginal rate of substitution between consumption and hours worked.

## 4 Per capita stationary DCE

Since the model in stationary and per capita terms by definition, there is no need to transform the optimality conditions, i.e  $X_t^h = X_t = x_t$ . We obtain the following system of equations that describe the DCE.

$$y_t = a_t k_t^\alpha (h_t^w)^{1-\alpha} \quad (14)$$

$$y_t = c_t + k_{t+1} - (1 - \delta^k)k_t \quad (15)$$

$$\frac{1}{c_t} = \lambda_t \quad (16)$$

$$\lambda_t = \beta E_t \lambda_{t+1} \left[ (1 - \tau^k) \alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta^k \right] \quad (17)$$

$$\frac{\theta}{1 - h_t^g - h_t^s - h_t^w} = \lambda_t (1 - \tau^l) (1 - \alpha) \frac{y_t}{h_t^w} \quad (18)$$

$$\frac{\theta}{1 - h_t^g - h_t^s - h_t^w} = \mu_t \phi z_t (h_t^g)^{\phi-1} \quad (19)$$

$$\beta \left\{ \frac{\psi}{g_{t+1}} + B \xi g_{t+1}^{-\xi-1} \mu_{t+1} \phi z_t (h_{t+1}^g)^{\phi-1} + \mu_{t+1} (1 - \delta^g) \right\} = \mu_t \quad (20)$$

$$k_{t+1} = i_t^k + (1 - \delta^k)k_t \quad (21)$$

$$g_{t+1} = i_t^g + (1 - \delta^g)g_t \quad (22)$$

$$h_t^s = Bg_t^{-\xi} \quad (23)$$

$$i_t^g = z_t(h_t^g)^\phi \quad (24)$$

Therefore, the DCE is summarized by Equations (22)-(32) in the paths of the following 11 variables  $\{y_t, c_t, g_t, i_t^k, i_t^g, k_t, h_t^w, h_t^g, h_t^s, \lambda_t, \mu_t\}_{t=0}^\infty$  given the paths of government transfers  $\{t_t\}$  and the exogenously set stationary processes,  $\{a_t, z_t\}_{t=0}^\infty$ , whose motion was determined in the previous subsection.

## 5 Log-linearized system

Since there is no closed-form general solution for the model, the model is approximated by log-linearizing the stationary DCE equations, where  $\hat{x}_t = \ln x_t - \ln x$ . The log-linearized system of model equations is as below<sup>1</sup>

$$k\hat{k}_{t+1} = y\hat{y}_t - c\hat{c}_t + (1 - \delta^k)k\hat{k}_t \quad (25)$$

$$0 = -\hat{y}_t + \hat{a}_t + \alpha\hat{k}_t + (1 - \alpha)\hat{h}_t^w \quad (26)$$

$$-\hat{c}_t = \hat{\lambda}_t \quad (27)$$

$$\hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + \frac{\beta(1 - \tau^k)\alpha y}{k} E_t\hat{y}_{t+1} - \frac{\beta(1 - \tau^k)\alpha y}{k} E_t\hat{k}_{t+1} \quad (28)$$

$$\hat{h}_t^s = -\xi\hat{g}_t \quad (29)$$

$$\frac{h^w}{(1 - h_t^w - h_t^s - h_t^w)}\hat{h}_t^w + \frac{h^s}{(1 - h_t^w - h_t^s - h_t^w)}\hat{h}_t^s \quad (30)$$

$$+ \frac{h^g}{(1 - h_t^w - h_t^s - h_t^w)}\hat{h}_t^g = \hat{\lambda}_t + \hat{y}_t - \hat{h}_t^w \quad (31)$$

$$\hat{\mu}_t + \hat{z}_t + \phi\hat{h}_t^g = \hat{\lambda}_t + \hat{y}_t - \hat{h}_t^w \quad (32)$$

$$\begin{aligned} & -\left[\frac{\beta\psi}{g\mu} + \beta\xi(\xi + 1)g^{-\xi-1}z\phi(h^g)^{\phi-1}\right]\hat{g}_{t+1} + \beta[\xi g^{-\xi-1}z\phi(h^g)^{\phi-1} + (1 - \delta^g)]\hat{\mu}_{t+1} \\ & + \beta\xi g^{-\xi-1}z\phi(h^g)^{\phi-1}\hat{z}_{t+1} - \beta\xi g^{-\xi-1}z\phi(\phi - 1)(h^g)^{\phi-1}\hat{h}_{t+1}^g = \hat{\mu}_t \end{aligned} \quad (33)$$

<sup>1</sup>Detailed derivations in Appendix 12.4.1-12.4.16

$$\hat{k}_{t+1} = \delta^k \hat{i}_t + (1 - \delta^k) \hat{k}_t \quad (34)$$

$$\hat{g}_{t+1} = \delta^g \hat{i}_t^g + (1 - \delta^g) \hat{g}_t \quad (35)$$

$$\hat{i}_t^g = \hat{z}_t + \phi \hat{h}_t^g \quad (36)$$

$$\hat{a}_{t+1} = \rho_a \hat{a}_t + \epsilon_{t+1}^a \quad (37)$$

$$\hat{z}_{t+1} = \rho_z \hat{z}_t + \epsilon_{t+1}^z \quad (38)$$

## 6 Auto- and Cross-correlation Functions

The main emphasis in this subsection is on the ACFs and CCFs of labor market variables. In particular, close attention is paid to cyclical properties of public and private wage rates and hours. To establish 95% confidence intervals for the theoretical ACFs and CCFs, as in Gregory and Smith (1991), the simulated time series are used to obtain 1000 ACFs and CCFs. The mean ACFs and CCFs are computed by averaging across simulations, as well as the corresponding standard error across simulations. Those moments allow for the lower and upper bounds for the ACFs confidence intervals to be estimated. The empirical ACFs and CCFs are then plotted, together with the theoretical ones. If empirical ACFs lie within the confidence region, this means that the calibrated model fits data well.

Empirical ACFs and CCFs were generated from a Vector Auto-Regressive (VAR) process of order 1. Since ACFs and CCFs are robust to identifying restrictions (Canova (2007), Ch.7), the VAR(1) was left unrestricted. However, not all the variables in  $\hat{\mathbf{x}}_t$  are observable. As proposed by Kim and Pagan (1994), a VAR was estimated using  $\tilde{\mathbf{x}}_t$ , model variables on which data was available, where

$$\tilde{\mathbf{x}}_t = \left[ y_t \quad c_t \quad i_t \quad h_t^w \quad w_t^p \right]'. \quad (39)$$

## 7 Log-linearization

### 7.1 Linearized Market Clearing

$$c_t + k_{t+1} - (1 - \delta)k_t = y_t \quad (40)$$



Take natural logarithms from both sides to obtain

$$\ln[c_t + k_{t+1} - (1 - \delta)k_t] = \ln(y_t) \quad (41)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln[c_t + k_{t+1} - (1 - \delta)k_t]}{dt} = d \ln(y_t) \quad (42)$$

Simplify to obtain

$$\left[ \frac{1}{c_t + k_{t+1} - (1 - \delta)k_t} \right] \left[ \frac{dc_t}{dt} \frac{c}{c} + \frac{dk_{t+1}}{dt} \frac{k}{k} - (1 - \delta) \frac{dk_t}{dt} \frac{k}{k} \right] = \frac{dy_t}{dt} \frac{1}{y} \quad (43)$$

Pass to log-deviations to obtain

$$\frac{1}{y} [\hat{c}_t c + \hat{k}_{t+1} k - (1 - \delta) \hat{k}_t k] = \hat{y}_t \quad (44)$$

Multiply both sides to obtain

$$\hat{c}_t c + \hat{k}_{t+1} k - (1 - \delta) \hat{k}_t k = y \hat{y}_t \quad (45)$$

Rearrange terms to obtain

$$k \hat{k}_{t+1} = y \hat{y}_t - c \hat{c}_t + (1 - \delta) k \hat{k}_t \quad (46)$$

## 7.2 Linearized Production Function

$$y_t = a_t (k_t)^\alpha (h_t^w)^{1-\alpha} \quad (47)$$

Take natural logarithms to obtain

$$\ln y_t = \ln a_t + \alpha \ln k_t + (1 - \alpha) \ln h_t^w \quad (48)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln y_t}{dt} = \frac{d \ln a_t}{dt} + \alpha \frac{d \ln k_t}{dt} + (1 - \alpha) \frac{d \ln h_t^w}{dt} \quad (49)$$

Simplify to obtain

$$\frac{1}{y} \frac{dy_t}{dt} = \frac{1}{a} \frac{da_t}{dt} + \frac{\alpha}{k} \frac{dk_t}{dt} + \frac{(1 - \alpha)}{h^w} \frac{dh_t^w}{dt} \quad (50)$$

Pass to log-deviations to obtain

$$0 = -\hat{y}_t + \hat{a}_t + \alpha \hat{k}_t + (1 - \alpha) \hat{h}_t^w \quad (51)$$

### 7.3 Linearized FOC consumption

$$\frac{1}{c_t} = \lambda_t \quad (52)$$

Take natural logarithms from both sides to obtain

$$\ln \frac{1}{c_t} = \ln \lambda_t \quad (53)$$

$$-\ln c_t = \ln \lambda_t \quad (54)$$

Totally differentiate with respect to time to obtain

$$-\frac{d \ln c_t}{dt} = \frac{d \ln \lambda_t}{dt} \quad (55)$$

$$-\frac{dc_t}{c_t} = \frac{d\lambda_t}{\lambda_t} \quad (56)$$

Pass to log-deviations to obtain

$$-\hat{c}_t = \hat{\lambda}_t \quad (57)$$

### 7.4 Linearized no-arbitrage condition for capital

$$\lambda_t = \beta E_t \lambda_{t+1} \left[ (1 - \tau^k) r_{t+1} + 1 - \delta^k \right] \quad (58)$$

Substitute out the real interest rate with the marginal product of capital

$$\lambda_t = \beta E_t \left[ \lambda_{t+1} (1 - \tau^k) \alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta^k \right] \quad (59)$$

Take natural logarithms from both sides to obtain

$$\ln \lambda_t = \ln E_t \left[ \lambda_{t+1} (1 - \tau^k) \alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta^k \right] \quad (60)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln \lambda_t}{dt} = \frac{d \ln E_t \left[ \lambda_{t+1} (1 - \tau^k) \alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta^k \right]}{dt} \quad (61)$$

$$\frac{1}{\lambda} \frac{d\lambda_t}{dt} = E_t \left\{ \frac{1}{\lambda (1 - \tau^k) \alpha \frac{y}{k} + 1 - \delta^k} \left[ (1 - \tau^k) \alpha \frac{y}{k} + 1 - \delta^k \right] \frac{d\lambda_{t+1}}{dt} \frac{\lambda}{\lambda} \right. \quad (62)$$

$$\left. + \frac{\lambda (1 - \tau^k) \alpha}{k} \frac{dy_{t+1}}{dt} \frac{y}{y} - \left[ \frac{\lambda (1 - \tau^k) \alpha y}{k^2} \right] \frac{dk_{t+1}^p}{dt} \frac{k}{k} \right\} \quad (63)$$

Pass to log-deviations to obtain

$$\hat{\lambda}_t = E_t \left\{ \hat{\lambda}_{t+1} + \frac{(1 - \tau^k)\alpha y}{\left[ (1 - \tau^k)\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta^k \right] k} \hat{y}_{t+1} - \frac{\alpha y}{\left[ (1 - \tau^k)\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta^k \right] k} \hat{k}_{t+1} \right\} \quad (64)$$

Observe that

$$(1 - \tau^k)\alpha \frac{y}{k} + 1 - \delta^k = 1/\beta \quad (65)$$

Plug the above expression back to obtain

$$\hat{\lambda}_t = E_t \left[ \hat{\lambda}_{t+1} + \frac{\beta(1 - \tau^k)\alpha y}{k} \hat{y}_{t+1} - \frac{\beta(1 - \tau^k)\alpha y}{k} \hat{k}_{t+1} \right] \quad (66)$$

Apply the expectations operator to each term to obtain

$$\hat{\lambda}_t = E_t \hat{\lambda}_{t+1} + \frac{\beta(1 - \tau^k)\alpha y}{k} E_t \hat{y}_{t+1} - \frac{\beta(1 - \tau^k)\alpha y}{k} E_t \hat{k}_{t+1} \quad (67)$$

## 7.5 Linearized Sick Days

$$h_t^s = B g_t^{-\xi} \quad (68)$$

Take natural logarithms from both sides to obtain

$$\ln h_t^s = \ln B g_t^{-\xi} \quad (69)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln h_t^s}{dt} = -\xi \frac{d \ln g_t}{dt} \quad (70)$$

Simplify to obtain

$$\frac{dh_t^s}{dt} \frac{1}{h^s} = -\xi \frac{dg_t}{dt} \frac{1}{g} \quad (71)$$

Pass to log-deviations to obtain

$$\hat{h}_t^s = -\xi \hat{g}_t \quad (72)$$

## 7.6 Linearized FOC hours

$$\theta(1 - h_t^w - h_t^s - h_t^w)^{-1} = \lambda_t(1 - \tau^l)\alpha \frac{y_t}{h_t^w} \quad (73)$$

Take natural logarithms from both sides to obtain

$$\ln \theta(1 - h_t^w - h_t^s - h_t^w)^{-1} = \ln \lambda_t(1 - \tau^l)\alpha \frac{y_t}{h_t^w} \quad (74)$$

Simplify to obtain

$$-\ln(1 - h_t^w - h_t^s - h_t^g) = \ln \lambda_t + \ln y_t - \ln h_t^w \quad (75)$$

Totally differentiate with respect to time to obtain

$$-\frac{-\frac{dh_t^g}{dt} - \frac{dh_t^s}{dt} - \frac{dh_t^w}{dt}}{(1 - h_t^w - h_t^s - h_t^g)} = \frac{d \ln \lambda_t}{dt} + \frac{d \ln y_t}{dt} - \frac{d \ln h_t^w}{dt} \quad (76)$$

$$\frac{\frac{dh_t^g}{dt} + \frac{dh_t^s}{dt} + \frac{dh_t^w}{dt}}{(1 - h_t^w - h_t^s - h_t^g)} = \frac{d \ln \lambda_t}{dt} + \frac{d \ln y_t}{dt} - \frac{d \ln h_t^w}{dt} \quad (77)$$

$$\frac{\frac{dh_t^g}{dt} \frac{h_t^g}{h_t^g} + \frac{dh_t^s}{dt} \frac{h_t^s}{h_t^s} + \frac{dh_t^w}{dt} \frac{h_t^w}{h_t^w}}{(1 - h_t^w - h_t^s - h_t^g)} = \hat{\lambda}_t + \hat{y}_t - \hat{h}_t^w \quad (78)$$

Pass to log-deviations to obtain

$$\begin{aligned} & \frac{h_t^w}{(1 - h_t^w - h_t^s - h_t^g)} \hat{h}_t^w + \frac{h_t^s}{(1 - h_t^w - h_t^s - h_t^g)} \hat{h}_t^s \\ & + \frac{h_t^g}{(1 - h_t^w - h_t^s - h_t^g)} \hat{h}_t^g = \hat{\lambda}_t + \hat{y}_t - \hat{h}_t^w \end{aligned} \quad (79)$$

## 7.7 MPL equalization

$$\mu_t \phi z_t (h_t^g)^{\phi-1} = \lambda_t (1 - \tau^l) \alpha \frac{y_t}{h_t^w} \quad (80)$$

Take natural logarithms from both sides to obtain

$$\ln \mu_t z_t \phi (h_t^g)^{\phi-1} = \ln \lambda_t (1 - \tau^l) \alpha \frac{y_t}{h_t^w} \quad (81)$$

Simplify to obtain

$$\ln \mu_t + \ln z_t + (\phi - 1) \ln h_t^g = \ln \lambda_t + \ln y_t - \ln h_t^w \quad (82)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln \mu_t}{dt} + \frac{d \ln z_t}{dt} + (\phi - 1) \frac{d \ln h_t^g}{dt} = \frac{d \ln \lambda_t}{dt} + \frac{d \ln y_t}{dt} - \frac{d \ln h_t^w}{dt} \quad (83)$$

$$\frac{d \mu_t}{dt} \frac{1}{\mu} + \frac{d z_t}{dt} \frac{1}{z} + (\phi - 1) \frac{d h_t^g}{dt} \frac{1}{h_t^g} = \frac{d \lambda_t}{dt} \frac{1}{\lambda} + \frac{d y_t}{dt} \frac{1}{y} - \frac{d h_t^w}{dt} \frac{1}{h_t^w} \quad (84)$$

Pass to log-deviations to obtain

$$\hat{\mu}_t + \hat{z}_t + (\phi - 1) \hat{h}_t^g = \hat{\lambda}_t + \hat{y}_t - \hat{h}_t^w \quad (85)$$

## 7.8 Health status dynamic equation

$$\beta \left\{ \frac{\psi}{g_{t+1}} + B\xi g_{t+1}^{-\xi-1} \mu_{t+1} \phi z_t (h_{t+1}^g)^{\phi-1} + \mu_{t+1} (1 - \delta^g) \right\} = \mu_t \quad (86)$$

Take natural logarithms from both sides to obtain

$$\ln \beta \left\{ \frac{\psi}{g_{t+1}} + B\xi g_{t+1}^{-\xi-1} \mu_{t+1} \phi z_t (h_{t+1}^g)^{\phi-1} + \mu_{t+1} (1 - \delta^g) \right\} = \ln \mu_t \quad (87)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln \left\{ \frac{\beta \psi}{g_{t+1}} + \beta \mu_{t+1} [B\xi g_{t+1}^{-\xi-1} \phi z_t (h_{t+1}^g)^{\phi-1} + \mu_{t+1} (1 - \delta^g)] \right\}}{dt} = \frac{d \ln \mu_t^2}{dt} \quad (88)$$

Simplify to obtain

$$\frac{\frac{d \left\{ \frac{\beta \psi}{g_{t+1}} + \beta \mu_{t+1} [B\xi g_{t+1}^{-\xi-1} \phi z_t (h_{t+1}^g)^{\phi-1} + \mu_{t+1} (1 - \delta^g)] \right\}}{dt}}{\frac{\beta \psi}{g} + \beta \mu [B\xi g^{-\xi-1} \phi z (h^g)^{\phi-1} + \mu (1 - \delta^g)]} = \frac{d \ln \mu_t}{dt} \quad (89)$$

Note that

$$\frac{\beta \psi}{g} + \beta \mu [B\xi g^{-\xi-1} \phi z (h^g)^{\phi-1} + \mu (1 - \delta^g)] = \mu \quad (90)$$

Then

$$\frac{1}{\mu} \frac{d}{dt} \left\{ \frac{\beta \psi}{g_{t+1}} + \beta \mu_{t+1} [B\xi g_{t+1}^{-\xi-1} \phi z_t (h_{t+1}^g)^{\phi-1} + \mu_{t+1} (1 - \delta^g)] \right\} = \frac{d \ln \mu_t}{dt} \quad (91)$$

$$\begin{aligned} & \frac{1}{\mu} \left\{ -\frac{\beta \psi}{g} \frac{dg_{t+1}}{dt} \frac{1}{g} + \beta \left[ \xi g^{-\xi-1} z \phi (h^g)^{\phi-1} + (1 - \delta^g) \right] \frac{d\mu_{t+1}}{dt} \frac{\mu}{\mu} \right. \\ & \quad \left. + \beta \mu \xi g^{-\xi-1} z \phi (h^g)^{\phi-1} \frac{dz_{t+1}}{dt} \frac{1}{z} - \beta \mu \xi (\xi + 1) \right. \\ & \left. g^{-\xi-1} z \phi (h^g)^{\phi-1} \frac{dg_{t+1}}{dt} \frac{1}{g} - \beta \mu \xi g^{-\xi-1} z \phi (\phi - 1) (h^g)^{\phi-1} \frac{dh_{t+1}^g}{dt} \frac{1}{h^g} \right\} = \frac{d\mu_t}{dt} \frac{1}{\mu} \end{aligned} \quad (92)$$

Pass to log-deviations to obtain

$$\begin{aligned} & -\frac{\beta \psi}{g\mu} \hat{g}_{t+1} + \beta \left[ \xi g^{-\xi-1} z \phi (h^g)^{\phi-1} + (1 - \delta^g) \right] \hat{\mu}_{t+1} + \beta \xi g^{-\xi-1} z \phi (h^g)^{\phi-1} \hat{z}_{t+1} \\ & - \beta \xi (\xi + 1) g^{-\xi-1} z \phi (h^g)^{\phi-1} \hat{g}_{t+1} - \beta \xi g^{-\xi-1} z \phi (\phi - 1) (h^g)^{\phi-1} \hat{h}_{t+1}^g = \hat{\mu}_t \end{aligned} \quad (93)$$

Rearrange terms to obtain

$$\begin{aligned} & - \left[ \frac{\beta \psi}{g\mu} + \beta \xi (\xi + 1) g^{-\xi-1} z \phi (h^g)^{\phi-1} \right] \hat{g}_{t+1} \\ & + \beta \left[ \xi g^{-\xi-1} z \phi (h^g)^{\phi-1} + (1 - \delta^g) \right] \hat{\mu}_{t+1} \\ & + \beta \xi g^{-\xi-1} z \phi (h^g)^{\phi-1} \hat{z}_{t+1} - \beta \xi g^{-\xi-1} z \phi (\phi - 1) (h^g)^{\phi-1} \hat{h}_{t+1}^g = \hat{\mu}_t \end{aligned} \quad (94)$$

## 7.9 Linearized Private Capital Accumulation

$$k_{t+1} = i_t^k + (1 - \delta^k)k_t \quad (95)$$

Take natural logarithms from both sides to obtain

$$\ln k_{t+1} = \ln(i_t^k + (1 - \delta^k)k_t) \quad (96)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln k_{t+1}}{dt} = \frac{1}{i_t^k + (1 - \delta^k)k_t} \frac{d(i_t^k + (1 - \delta^k)k_t)}{dt} \quad (97)$$

Observe that since  $i^k = \delta^k k$ , it follows that  $i^k + (1 - \delta^k)k = \delta^k k + (1 - \delta^k)k = k$ . Then

$$\frac{dk_{t+1}}{dt} \frac{1}{k} = \frac{1}{k} \frac{di_t^k}{dt} \frac{i^k}{i^k} + \frac{k}{i^k + (1 - \delta^k)k_t} dt \frac{k}{k} \quad (98)$$

Pass to log-deviations to obtain

$$\hat{k}_{t+1} = \frac{\delta^k k}{k} \hat{i}_t^k + \frac{(1 - \delta^k)k}{k} \hat{k}_t \quad (99)$$

Finally, simplify to obtain

$$\hat{k}_{t+1} = \delta^k \hat{i}_t^k + (1 - \delta^k) \hat{k}_t \quad (100)$$

## 7.10 Linearized Health Accumulation

$$g_{t+1} = i_t^g + (1 - \delta^g)g_t \quad (101)$$

Take natural logarithms from both sides

$$\ln g_{t+1} = \ln(i_t^g + (1 - \delta^g)g_t) \quad (102)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln g_{t+1}}{dt} = \frac{1}{i_t^g + (1 - \delta^g)g_t} \frac{d(i_t^g + (1 - \delta^g)g_t)}{dt} \quad (103)$$

Observe that since  $i^g = \delta^g g$ , it follows that  $i^g + (1 - \delta^g)g = \delta^g g + (1 - \delta^g)g = g$ . Then

$$\frac{dg_{t+1}}{dt} \frac{1}{g} = \frac{1}{g} \frac{di_t^g}{dt} \frac{i^g}{i^g} + \frac{g}{i^g + (1 - \delta^g)g_t} dt \frac{g}{g} \quad (104)$$

Pass to log-deviations to obtain

$$\hat{g}_{t+1} = \frac{\delta^g g}{g} \hat{i}_t^g + \frac{(1 - \delta^g)g}{g} \hat{g}_t \quad (105)$$

Finally, simplify to obtain

$$\hat{g}_{t+1} = \delta^g \hat{i}_t^g + (1 - \delta^g) \hat{g}_t \quad (106)$$

## 7.11 Health investment equation

$$i_t^g = z_t (h_t^g)^\phi \quad (107)$$

Take natural logarithms from both sides to obtain

$$\ln i_t^g = \ln z_t + (\phi) \ln h_t^g \quad (108)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln i_t^g}{dt} = \frac{d \ln z_t}{dt} + \phi \frac{d \ln h_t^g}{dt} \quad (109)$$

Simplify to obtain

$$\frac{di_t^g}{dt} \frac{1}{i^g} = \frac{dz_t}{dt} \frac{1}{z} + \phi \frac{dh_t^g}{dt} \frac{1}{h^g} \quad (110)$$

Pass to log-deviations to obtain

$$\hat{i}_t^g = \hat{z}_t + \phi \hat{h}_t^g \quad (111)$$

## 7.12 Linearized Technology Shock Process

$$\ln a_{t+1} = (1 - \rho^a) \ln a + \rho^a \ln a_t + \epsilon_{t+1}^a \quad (112)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln a_{t+1}}{dt} = (1 - \rho^a) \frac{d \ln a}{dt} + \rho^a \frac{d \ln a_t}{dt} + \frac{d \epsilon_{t+1}^a}{dt} \quad (113)$$

$$\frac{da_{t+1}}{dt} \frac{1}{a} = \rho^a \frac{da_t}{dt} \frac{1}{a} + \epsilon_{t+1}^a \quad (114)$$

where for  $t = 1$   $d\epsilon_{t+1}^a \approx \ln(e^{\epsilon_{t+1}^a}/e^{\epsilon^a}) = \epsilon_{t+1}^a - \epsilon^a = \epsilon_{t+1}^a$  since  $\epsilon^a = 0$

Pass to log-deviations to obtain

$$\hat{a}_{t+1} = \rho^a \hat{a}_t + \epsilon_{t+1}^a \quad (115)$$

## 7.13 Linearized Health Stochastic Process

$$\ln z_{t+1} = (1 - \rho^z) \ln z + \rho^z \ln z_t + \epsilon_{t+1}^z \quad (116)$$

Totally differentiate with respect to time to obtain

$$\frac{d \ln z_{t+1}}{dt} = (1 - \rho^z) \frac{d \ln z}{dt} + \rho^z \frac{d \ln z_t}{dt} + \frac{d \epsilon_{t+1}^z}{dt} \quad (117)$$

Simplify to obtain

$$\frac{dz_{t+1}}{dt} \frac{1}{z} = \rho_z \frac{dz_t}{dt} \frac{1}{z} + \epsilon_{t+1}^z \quad (118)$$

where for  $t = 1$   $d\epsilon_{t+1}^z \approx \ln(e^{\epsilon_{t+1}^z}/e^{\epsilon^z}) = \epsilon_{t+1}^z - \epsilon^z = \epsilon_{t+1}^z$  since  $\epsilon^z = 0$ .

Pass to log-deviations to obtain

$$\hat{z}_{t+1} = \rho_z \hat{z}_t + \epsilon_{t+1}^z \quad (119)$$