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A note on optimal portfolios under regime–switching

Abstract

This paper extends the stochastic dominance rules for normal mixture distributions derived by Levy and Kaplanski (2015). First, the portfolios under consideration are allowed to follow different regime-switching processes. Second, the results are extended from second- to fourth-order stochastic dominance, which is known to be closely related to kurtosis aversion in financial markets and allows to compare mixture distributions with the same overall variance. In particular, when a risk-free asset is available, checking for fourth-order stochastic dominance turns out to amount to a comparison of the regime-specific and overall Sharpe ratios of the portfolios under consideration.

JEL classification: C46; C58; G11

Keywords—portfolio selection; regime–switching; Sharpe ratio; stochastic dominance

1 Introduction

Empirical evidence suggests that the distribution of financial returns displays regime–switching behavior, e.g., between bull and bear market periods (for a review of the literature, see, e.g., Guidolin, 2011; and Ang and Timmermann, 2012). If regime changes are stochastic and returns are Gaussian *within* regimes, then the overall conditional distribution is a mixture of normals.[1] More precisely, assume that

(i) there are k different states (or *regimes*) of the market,

(ii) the (conditional) probability that the market will be in state j is λ_j, $j = 1, \ldots, k$,[2]

(iii) conditional on the market being in state j, the portfolio return under consideration, R, is Gaussian with mean μ_j and variance σ_j^2, with $(\mu_i, \sigma_i^2) \neq (\mu_j, \sigma_j^2)$ for $i \neq j$.[3]

Then the portfolio return R follows a k–component finite normal mixture distribution with density

$$f_R(r) = \sum_{j=1}^{k} \lambda_j \phi(r; \mu_j, \sigma_j^2), \tag{1}$$

where the *component densities* $\phi(r; \mu_j, \sigma_j^2)$ are Gaussian with *component means* μ_j and *component variances* σ_j^2, i.e.,

$$\phi(r; \mu_j, \sigma_j^2) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left\{-\frac{(r-\mu_j)^2}{2\sigma_j^2}\right\}, \quad j = 1, \ldots, k, \tag{2}$$

and the strictly positive regime probabilities or *mixing weights* satisfy $\sum_{j=1}^{k} \lambda_j = 1$.

In applications to financial return data, it is usually found that the market regimes differ mainly in volatility, whereas the component means are rather close in value and their differences are (typically) statistically insignificant (Ang and Timmermann, 2012).

[1] Taboga (2005) relates the literature on regime–switching to that on portfolio selection under ambiguity, pointing out that "[i]t seems natural to interpret the existence of multiple regimes as ambiguity about asset returns: under each regime, returns have a different probability distribution and the investor is not able to identify which one correctly describes returns next period."

[2] Typically the regime probabilities will be (conditionally) time–varying. E.g., a popular approach to modeling persistent regimes is the Markov–switching model (cf. Hamilton, 2008). There is no need to make this explicit in our presentation.

[3] Just as for the regime probabilities, conditional time–variation of the regime–specific means and variances can be accommodated as well, e.g., via a regime–switching GARCH specification (cf. Haas and Paolella, 2012).

Thus, in the following discussion, and as in Levy and Kaplanski (2015), we assume that, in (1),

$$\mu_1 = \cdots = \mu_k =: \mu, \tag{3}$$

which is referred to as a *scale normal mixture*. In this case, the mean and the variance of R are given by

$$\mathrm{E}(R) = \mu, \quad \text{and} \quad \mathrm{Var}(R) = \sum_{j=1}^{k} \lambda_j \sigma_j^2, \tag{4}$$

respectively. For later reference, we denote the k–component scale mixture described by (1)–(3) as

$$R \sim \mathrm{MixN}(\lambda_1, \ldots, \lambda_k, \mu, \sigma_1^2, \ldots, \sigma_k^2).$$

If markets are subject to regime–switching and thus returns follow a mixed normal distribution, then a comparison of portfolios based on the *overall* mean and variance as given by (4) will not be sufficient. Rather, to arrive at unambiguous comparisons of portfolios, the more general theory of stochastic dominance can be employed (cf. Levy, 2006). Levy and Kaplanski (2015) have recently established the stochastic dominance rule of order two (i.e., for all risk averters) for the situation where the mixture distributions involved in the comparison have two components and share the same mixing weights (i.e., are subject to the same regime–switching behavior).

These assumptions are reasonable in many applications. E.g., a typical finding in the literature is that two regimes are appropriate to describe the distribution of stock returns (e.g., Guidolin and Timmermann, 2006; Guidolin, 2011). On the other hand, Guidolin and Timmermann (2006) find that three regimes are required for bond returns. More generally, the coherence between the regimes of different asset classes or geographical areas may be less than perfect (e.g., Bae et al., 2014; and Case et al., 2014). Thus, the portfolio returns of funds following different investment strategies, leading to investments in different markets or asset classes, can be characterized by regime–switching processes which are not perfectly synchronized, or even have different numbers of regimes. We therefore extend the stochastic dominance criteria of Levy and Kaplanski (2015) to the situation where the portfolios under consideration may be governed by different regime–switching processes.

The second extension is that, in addition to second–order stochastic dominance, the (weaker) criterion for fourth–order stochastic dominance is considered, which applies

to utility functions u with $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$, and $u'''' \leq 0$. The positive and negative sign, respectively, of u''' and u'''' are typically referred to as representing *prudence* and *temperance* in the literature on decision making under uncertainty. Both plausibility and evidence point towards the presence of both of these attitudes in the typical investor's behavior (for discussion and references, see Eeckhoudt, 2012; and Eeckhoudt and Schlesinger, 2013).

From the viewpoint of the problem considered herein, i.e., deriving stochastic dominance relations for normal mixture distributions, an advantage of extending the scope to fourth–order dominance is that it enables us to compare distributions with the same overall variance, as given in (4). For example, it is now possible to compare a k–component normal mixture with a single–component Gaussian distribution with the same variance. Moreover, when a risk–free asset is available, it turns out that fourth–order stochastic dominance requires specific relations between the regime–specific and overall Sharpe ratios (expected excess return per unit of standard deviation) of the portfolios involved in the comparison.

Despite their more general nature, the derivation of the results is also somewhat simplified, making use of the concavity of two–parameter (mean and standard deviation/variance) preference functions derived under normality.

2 Stochastic dominance criteria for normal mixture distributions

The stochastic dominance rule of order n identifies the preferred distributions for an investor with a utility function $u \in \mathcal{U}_n$, where

$$\mathcal{U}_n = \{u : (-1)^i u^{(i)}(x) \leq 0, \quad \forall x \in \mathbb{R}, \quad i = 1, \ldots, n\},$$

where $u^{(i)}(x)$ is the ith derivative of u at x. We are interested in the case when the two distributions involved are scale normal mixtures as defined by (1)–(3). As in Levy and Kaplanski (2015), we proceed in two steps. In the first step, it is assumed that the two portfolios to be compared have the same expected return (Section 2.1). We drop this assumption in the second step, where a risk–free asset is introduced (Section 2.2).

2.1 Situation without a risk–free asset and equal means

In case of normally distributed returns, $R \sim \mathrm{N}(\mu, \sigma^2)$, expected utility can be expressed as a function V of μ (mean) and σ (standard deviation),[4]

$$\begin{aligned} V(\mu, \sigma) &= \int_{-\infty}^{\infty} u(x)\phi(x; \mu, \sigma^2)dx \\ &= \int_{-\infty}^{\infty} u(\sigma z + \mu)\phi(z)dz, \end{aligned} \tag{5}$$

where $\phi(z)$ is the standard normal density. We may also define

$$W(\mu, \sigma^2) := V(\mu, \sigma), \tag{6}$$

which expresses expected utility as a function W of mean and variance rather than mean and standard deviation. In case of a normal mixture, expected utility is given by a probability–weighted linear combination of μ/σ preference functions of the form (5) or (6). The derivation of the second– and fourth–order stochastic dominance criteria for normal mixtures makes use of the fact that (cf. Chipman, 1973)[5]

$$V \text{ is concave in } \sigma \text{ if } u \in \mathcal{U}_2, \text{ and } W \text{ is concave in } \sigma^2 \text{ if } u \in \mathcal{U}_4. \tag{7}$$

We compare two random variables R_1 and R_2, where R_1 is k–component mixed normal,

$$R_1 \sim \mathrm{MixN}(\lambda_1, \ldots, \lambda_k, \mu, \sigma_1^2, \ldots, \sigma_k^2), \tag{8}$$

and R_2 is m–component mixed normal,

$$R_2 \sim \mathrm{MixN}(\omega_1, \ldots, \omega_m, \mu, \theta_1^2, \ldots, \theta_m^2). \tag{9}$$

Recall the convention that $\sigma_i^2 \neq \sigma_j^2$ and $\theta_i^2 \neq \theta_j^2$ for $i \neq j$. Otherwise, with equal component means, two of the components in the respective mixture would be identical and could be merged. Also note that it may be the case that, e.g., $k = 1$, i.e., one of the distributions under consideration may be Gaussian.

[4] Technical conditions for this and the following relations to hold for the utility function (such as growth conditions to guarantee that expected utility exists finite) can be found in Schneeweiß (1966, Ch. 4) and Chipman (1973). They are satisfied for any even half–reasonable utility function.

[5] With subscripts denoting partial derivatives, this follows for V from $V_{\sigma\sigma} = \int z^2 u''(\sigma z + \mu)\phi(z)dz$. For W, Chipman (1973) makes use of the fundamental result that W satisfies the "heat equation" $W_{\sigma^2} = \frac{1}{2}W_{\mu\mu}$, which can be seen from the normal density in (2) satisfying $\phi_{\mu\mu}(x; \mu, \sigma^2)/2 = \phi_{\sigma^2}(x; \mu, \sigma^2) = \phi(x; \mu, \sigma^2)[(x-\mu)^2 - \sigma^2]/(2\sigma^4)$, and differentiation under the integral sign. Thus $W_{\sigma^2\sigma^2} = \frac{1}{2}(W_{\mu\mu})_{\sigma^2} = \frac{1}{2}(W_{\sigma^2})_{\mu\mu} = \frac{1}{4}W_{\mu\mu\mu\mu} = \frac{1}{4}\int u''''(\sigma z + \mu)\phi(z)dz$.

In this situation, a sufficient condition for second–order stochastic dominance, which generalizes Theorem 1 of Levy and Kaplanski (2015), is as follows.

Proposition 1. *Random variable R_1 in (8) second–order stochastically dominates random variable R_2 in (9) if*

(i) it is possible to label the mixture components such that

$$\theta_1 < \cdots < \theta_\ell \leq \sigma_1 < \cdots < \sigma_k \leq \theta_{\ell+1} < \cdots < \theta_m, \tag{10}$$

and

(ii)

$$\sum_{j=1}^{k} \lambda_j \sigma_j \leq \sum_{i=1}^{m} \omega_i \theta_i. \tag{11}$$

Condition (11) is also necessary. It is also necessary that

$$\max\{\sigma_1, \ldots, \sigma_m\} \leq \max\{\theta_1, \ldots, \theta_m\}. \tag{12}$$

Remark 1. *It is clear that Proposition 1 does not cover all situations of stochastic dominance among scale normal mixture distributions. E.g., if $m = k = 2$, $\sigma_1^2 < \theta_1^2 < \sigma_2^2 < \theta_2^2$ (which is not covered by condition (10)), and $\lambda_1 \geq \omega_1$ (and hence $\lambda_2 = 1 - \lambda_1 \leq \omega_2 = 1 - \omega_1$), then R_1 dominates R_2, but this case is trivial. As pointed out by Levy and Kaplanski (2015), in the two–regime setting, "the interesting non–trivial case is the case where [R_1] has a lower volatility than [R_2] in one regime, but a higher volatility in the other regime", which is the situation covered by condition (10).*

Proof. (i) For *sufficiency* of (10) and (11), note that, since $u \in \mathcal{U}_2$ implies that V, as defined in (5), is concave in the standard deviation,

$$\begin{aligned} \mathrm{E}(u(R_2)) &= \sum_{i=1}^{m} \omega_i V(\mu, \theta_i) \\ &= \sum_{i=1}^{\ell} \omega_i V(\mu, \theta_i) + \sum_{i=\ell+1}^{m} \omega_i V(\mu, \theta_i) \\ &\leq \overline{\omega} V(\mu, \overline{\theta}_1) + (1 - \overline{\omega}) V(\mu, \overline{\theta}_2), \end{aligned}$$

where

$$\overline{\omega} = \omega_1 + \cdots + \omega_\ell, \quad \overline{\theta}_1 = \frac{1}{\overline{\omega}} \sum_{i=1}^{\ell} \omega_i \theta_i, \quad \text{and} \quad \overline{\theta}_2 = \frac{1}{1 - \overline{\omega}} \sum_{i=\ell+1}^{m} \omega_i \theta_i.$$

Thus, random variable R_2 in (9) is second–order stochastically dominated by the two–regime normal mixture variable

$$\tilde{R}_2 \sim \text{MixN}(\overline{\omega}, 1-\overline{\omega}, \mu, \overline{\theta}_1^2, \overline{\theta}_2^2),$$

which shares with R_2 the properties (10) and (11), i.e.,

$$\overline{\theta}_1 \leq \sigma_1 < \cdots < \sigma_k \leq \overline{\theta}_2, \tag{13}$$

and

$$\sum_{j=1}^{k} \lambda_j \sigma_j \leq \overline{\omega}\overline{\theta}_1 + (1-\overline{\omega})\overline{\theta}_2 = \sum_{i=1}^{m} \omega_i \theta_i. \tag{14}$$

We show that R_1 second–order dominates $\tilde{R}_2$. First note that, from (14),

$$1 - \overline{\omega} \geq \frac{\sum_{j=1}^{k} \lambda_j \sigma_j - \overline{\theta}_1}{\overline{\theta}_2 - \overline{\theta}_1}.$$

Hence, since $V(\mu, \overline{\theta}_2) < V(\mu, \overline{\theta}_1)$, and by (7), i.e., concavity,

$$\begin{aligned}
\text{E}(u(\tilde{R}_2)) &= \overline{\omega} V(\mu, \overline{\theta}_1) + (1-\overline{\omega}) V(\mu, \overline{\theta}_2) \\
&= V(\mu, \overline{\theta}_1) + (1-\overline{\omega})(V(\mu, \overline{\theta}_2) - V(\mu, \overline{\theta}_1)) \\
&\leq V(\mu, \overline{\theta}_1) + \frac{\sum_{j=1}^{k} \lambda_j \sigma_j - \overline{\theta}_1}{\overline{\theta}_2 - \overline{\theta}_1} (V(\mu, \overline{\theta}_2) - V(\mu, \overline{\theta}_1)) \\
&= \sum_{j=1}^{k} \lambda_j \left\{ V(\mu, \overline{\theta}_1) + \frac{V(\mu, \overline{\theta}_2) - V(\mu, \overline{\theta}_1)}{\overline{\theta}_2 - \overline{\theta}_1} (\sigma_j - \overline{\theta}_1) \right\} \\
&\leq \sum_{j=1}^{k} \lambda_j V(\mu, \sigma_j) = \text{E}(u(R_1)).
\end{aligned}$$

(ii) For *necessity* of (11), we use the general result that a random variable X with distribution function (cdf) F second–order dominates a random variable Y with distribution G if and only if (cf. Levy, 2006)

$$\Delta(t) = \int_{-\infty}^{t} [G(x) - F(x)] dx \geq 0 \quad \forall t. \tag{15}$$

To use this in our context, we note that the "cumulative cdf" $\int_{-\infty}^{t} F(x)dx$ can be expressed in terms of the first–order lower partial moment (cf. Bawa, 1975; and Ingersoll, 1987, p. 98),

$$\int_{-\infty}^{t} F(x)dx = \int_{-\infty}^{t} (t-x) f(x) dx. \tag{16}$$

Now assume $X \sim \mathrm{N}(\mu, \sigma^2)$, i.e., $f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$, and $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, where $\phi(z) = (2\pi)^{-1/2}e^{-z^2/2}$ and $\Phi(z) = \int_{-\infty}^{z}\phi(\xi)d\xi$ are the standard normal pdf and cdf, respectively. Then we get

$$\begin{aligned}\int_{-\infty}^{t}\Phi\left(\frac{x-\mu}{\sigma}\right)dx &\overset{(16)}{=} \int_{-\infty}^{t}(t-x)\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)dx \\ &= \int_{-\infty}^{(t-\mu)/\sigma}(t-\mu-\sigma z)\phi(z)dz \\ &= (t-\mu)\Phi\left(\frac{t-\mu}{\sigma}\right)+\sigma\phi\left(\frac{t-\mu}{\sigma}\right),\end{aligned}$$

where the substitution $z = (x-\mu)/\sigma$ was made in the second line, and the third line follows from $\phi'(z) = -z\phi(z)$, i.e., $\int z\phi(z)dz = -\phi(z)$.

Therefore, when substituting R_1 in (8) for X and R_2 in (9) for Y, (15) evaluated at $t = \mu$ is given by

$$\Delta(\mu) = \frac{1}{\sqrt{2\pi}}\left(\sum_{i=1}^{m}\omega_i\theta_i - \sum_{j=1}^{k}\lambda_j\sigma_j\right),$$

and the necessity of (11) follows.

To see the necessity of (12), consider the exponential utility function, $u(x) = -e^{-cx}$, where $c > 0$ is the coefficient of constant absolute risk aversion. Clearly $u \in \mathcal{U}_2$. In this case, expected utility under mixture (8) is

$$\mathrm{E}(u(R_1)) = -e^{-c\mu}\sum_{j=1}^{k}\lambda_j e^{c^2\sigma_j^2/2}. \tag{17}$$

As $c \to \infty$, mixture (9) will be preferred over mixture (8) if $\max_j\{\sigma_j\} > \max_i\{\theta_i\}$, thus (12) is necessary.[6]

∎

The essence of Proposition 1 is contained in Figure 1, which depicts the situation for $k = m = 2$. The dash–dotted line is $V(\mu, \sigma)$ defined in (5), which is decreasing and concave in σ. Expected utility of random variable R_1 in (8) is located somewhere on

[6] Intuitively, the behavior under extreme risk aversion ($c \to \infty$) is completely determined by the investor's desire to select the distribution with the least unfavorable worst–case scenario, as represented by the most turbulent regime. But note that this interpretation should be taken *cum grano salis*, since part of the result for $c \to \infty$ may be due to the unlimited liability implied by the normal mixture distribution and the explosion of $u(x) = -e^{-cx}$ towards $-\infty$ as $x \to -\infty$. See also the discussion in Footnote 10 of Levy and Kaplanski (2015).

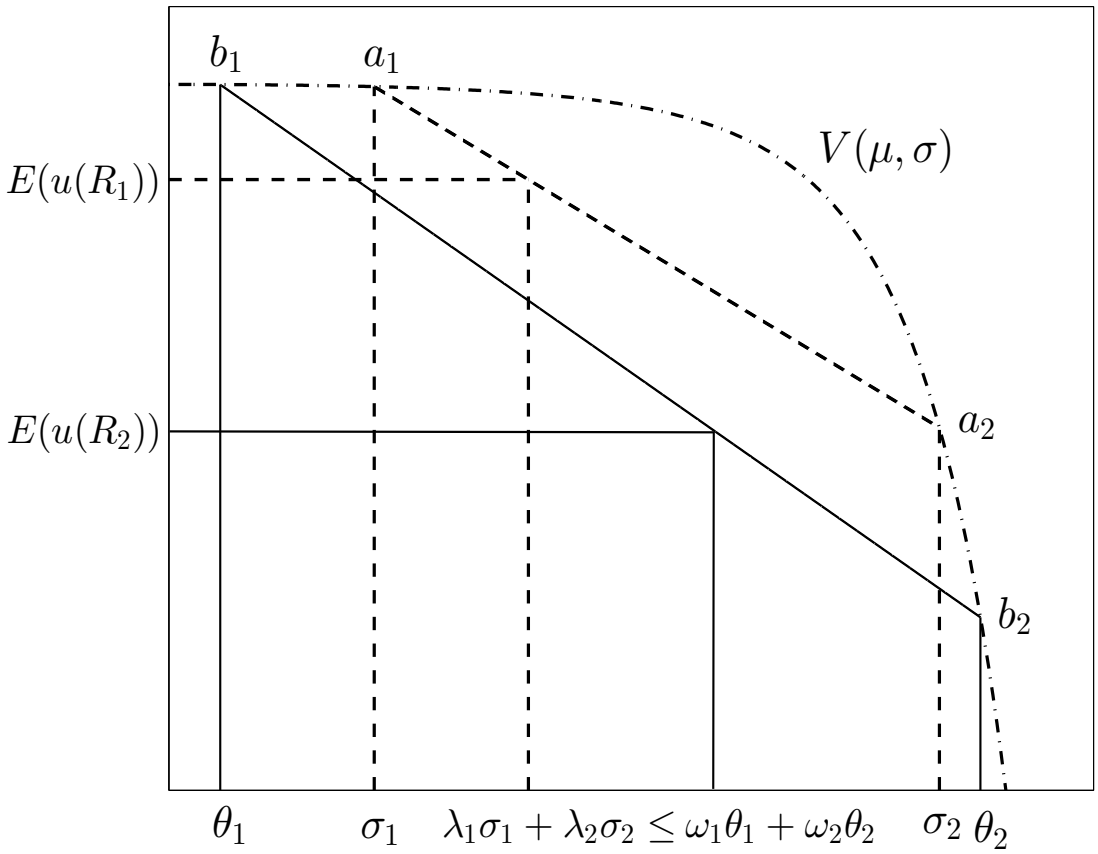

Figure 1: Illustration of Proposition 1 for $k = m = 2$.

the dashed line between a_1 and a_2, and expected utility of random variable R_2 in (9) is somewhere on the solid line between b_1 and b_2. Moreover, since $\lambda_1\sigma_1+\lambda_2\sigma_2 \leq \omega_1\theta_1+\omega_2\theta_2$, the former is to the left of the latter, and thus $\mathrm{E}(u(R_1)) \geq \mathrm{E}(u(R_2))$.

Remark 2. *Proposition 1 extends Theorem 1 of Levy and Kaplanski (2015), who consider the situation $k = m = 2$, $\lambda_1 = \omega_1$, and show that, if the second regime is the high–volatility regime for both R_1 and R_2, then $\sigma_2 \leq \theta_2$ and $\lambda_1\sigma_1+(1-\lambda_1)\sigma_2 \leq \lambda_1\theta_1+(1-\lambda_1)\theta_2$ are necessary and sufficient for R_1 to second–order dominate R_2. Without the restriction $\lambda_1 = \omega_1$, however, it is required to consider the entire pattern of the regime–specific standard deviations. That is, in general, the necessary conditions (11) and (12), i.e., smaller average standard deviation and less extreme worst–case scenario, are not sufficient for second–order stochastic dominance. To illustrate, let the utility function be exponential, i.e., $u(x) = -e^{-cx}$, $c > 0$, so that expected utility is given by (17). Suppose*

that both R_1 in (8) and R_2 in (9) are characterized by two components ($k = m = 2$), and that $\lambda_1 = 0.8$, $\sigma_1 = 0.1$, $\sigma_2 = 1$, $\omega_1 = 0.9$, $\theta_1 = 0.5$, and $\theta_2 = 1$, so that (11) and (12) are valid, but (10) is violated. Since the worst–case scenario is the same under R_1 and R_2 but more likely under R_1, investors characterized by sufficiently large c ($c > 1.88$) will prefer R_2 over R_1. If we decrease σ_2 from 1 to 0.95, then both investors with sufficiently small and investors with sufficiently large c will prefer R_1, whereas those with c roughly between 2.6 and 3.6 prefer R_2. For the latter, the higher probability of the high–volatility regime under R_1 outweighs its only slight reduction in turbulence relative to the high–volatility regime under R_2.

A potential drawback of condition (11) is that it cannot be interpreted in terms of the overall standard deviation of the return distribution. In particular, the linear combination of the regime–specific standard deviations $\sigma_1, \ldots, \sigma_k$ of the scale mixture (8) is not the standard deviation of R_1.[7] Rather, inspection of (4) reveals that such a relation applies to the regime–specific *variances*, with the overall standard deviation being $\sqrt{\mathrm{Var}(R_1)} = \sqrt{\sum_j \lambda_j \sigma_j^2}$. Extending the scope of analysis to fourth–order stochastic dominance allows us to replace requirement (19) by a weaker condition on the variances of the variables under consideration. This result is stated next.

Proposition 2. *Random variable R_1 in (8) fourth–order stochastically dominates random variable R_2 in (9) if*

(i) it is possible to label the mixture components such that

$$\theta_1^2 < \cdots < \theta_\ell^2 \leq \sigma_1^2 < \cdots < \sigma_k^2 \leq \theta_{\ell+1}^2 < \cdots < \theta_m^2, \tag{18}$$

and

(ii)

$$\mathrm{Var}(R_1) = \sum_{j=1}^{k} \lambda_j \sigma_j^2 \leq \sum_{i=1}^{m} \omega_i \theta_i^2 = \mathrm{Var}(R_2). \tag{19}$$

Condition (19) is also necessary. It is also necessary that

$$\max\{\sigma_1^2, \ldots, \sigma_m^2\} \leq \max\{\theta_1^2, \ldots, \theta_m^2\}. \tag{20}$$

[7] Condition (11) can be expressed as $\mathrm{E}\,|R_1 - \mu| \leq \mathrm{E}\,|R_2 - \mu|$.

Remark 3. *Note that conditions (18) and (20) are just (10) and (12), rewritten in terms of variances rather than standard deviations. The new part is (19), which weakens (11) and requires us to compare the (overall) variances of the mixture distributions involved.*

Proof. The largest part of the proof of Proposition 2 is a reformulation of that of Proposition 1, with standard deviations replaced by variances, and V replaced by W defined in (6). It remains to show the necessity of (19). To see this, consider the utility function $u(x) = -e^{-cx}$, $c > 0$, which is in $\mathcal{U}_4$. In this case,

$$\begin{aligned} \mathrm{E}(u(R_1)) &= -e^{-c\mu} \sum_{j=1}^{k} \lambda_j e^{c^2\sigma_j^2/2} \\ &= -e^{-c\mu} \sum_{j=1}^{k} \lambda_j \left(1 + \frac{c^2\sigma_j^2}{2} + O(c^4)\right) \\ &= -e^{-c\mu} \left(1 + \frac{c^2}{2} \operatorname{Var}(R_1) + O(c^4)\right), \quad \text{as } c \to 0. \end{aligned}$$

Thus, for small enough risk aversion, c, the distribution with the smaller variance will be preferred. ∎

Remark 4. *Since the exponential utility function satisfies* $\operatorname{sgn} u^{(n)} = (-1)^{n+1}$*, the proof of Proposition 2 makes clear that condition (19) cannot be further weakened by considering stochastic dominance of orders higher than the fourth.*

Proposition 2 can be interpreted in the sense that, for a given level of overall variance, temperate investors (i.e., investors with $u \in \mathcal{U}_4$) prefer a more even distribution of the variances across states, so in order to rule out the likelihood of an unpleasant surprise (in the form of extremely turbulent market conditions) once the "veil of ignorance" with regard to the prevailing market regime has been raised.[8] This interpretation is particularly clear–cut when a temperate investor compares any MixN distribution (8) with $k > 1$ with the single–component Gaussian distribution with the same mean and variance. The latter will always be preferred since

$$W\left(\mu, \sum_{j=1}^{k} \lambda_j \sigma_j^2\right) \geq \sum_{j=1}^{k} \lambda_j W(\mu, \sigma_j^2).$$

[8] This interpretation is closely related to the *definition* of temperance suggested in Eeckhoudt and Schlesinger (2006), namely, as a preference for "disaggregating the harms" in the sense that an agent prefers an even distribution of independent zero–mean shocks across states of nature.

That is, for a given overall variance, temperate investors prefer a certain state of the world, with a given variance, over the uncertainty about the prevailing volatility level that comes along with the mixture distribution. Risk aversion is not sufficient for this conclusion, since $\sqrt{\sum_j \lambda_j \sigma_j^2} > \sum_j \lambda_j \sigma_j$, and thus condition (11) is violated.

2.2 Situation with a risk–free asset and different means

Now suppose there is a risk–free asset with return r_f and the risky portfolios under consideration have possibly different expected returns, i.e., (8) and (9) are replaced with

$$R_1 \sim \text{MixN}(\lambda_1, \ldots, \lambda_k, \mu_1, \sigma_1^2, \ldots, \sigma_k^2), \tag{21}$$

and

$$R_2 \sim \text{MixN}(\omega_1, \ldots, \omega_m, \mu_2, \theta_1^2, \ldots, \theta_m^2), \tag{22}$$

respectively, where possibly $\text{E}(R_1) = \mu_1 \neq \mu_2 = \text{E}(R_2)$. Assume that we can lend and borrow any desired amount at the risk–free rate, r_f, and that both risky portfolios (21) and (22) have positive excess returns over the risk–free rate, i.e., $\mu_i > r_f$, $i = 1, 2$.

We can mix both of the risky portfolios (21) and (22) with the risk–free asset to obtain new portfolios with returns $\tilde{R}_1(\alpha_1)$ and $\tilde{R}_2(\alpha_2)$, respectively, where α_i is the weight of the risky portfolio i in the corresponding new (overall) portfolio, $i = 1, 2$. Then $\tilde{R}_i(\alpha_i) = \alpha_i R_i + (1 - \alpha_i) r_f$, $i = 1, 2$, and

$$\tilde{R}_1(\alpha_1) \sim \text{MixN}(\lambda_1, \ldots, \lambda_k, \tilde{\mu}_1(\alpha_1), \alpha_1^2\sigma_1^2, \ldots, \alpha_1^2\sigma_k^2), \tag{23}$$

and

$$\tilde{R}_2(\alpha_2) \sim \text{MixN}(\omega_1, \ldots, \omega_m, \tilde{\mu}_2(\alpha_2), \alpha_2^2\theta_1^2, \ldots, \alpha_2^2\theta_m^2), \tag{24}$$

where the (overall) portfolio return

$$\tilde{\mu}_i(\alpha_i) = \alpha_i \mu_i + (1 - \alpha_i) r_f = r_f + \alpha_i(\mu_i - r_f), \quad i = 1, 2. \tag{25}$$

Now fix an (overall) portfolio mean return μ_p, $\mu_p > r_f$. With unlimited lending and borrowing at the risk–free rate, any μ_p can be achieved by an appropriately selected weight α_i. Solving (25) shows that the suitable weight is

$$\alpha_i^\star = \frac{\mu_p - r_f}{\mu_i - r_f}, \quad i = 1, 2. \tag{26}$$

Now apply the stochastic dominance criteria of Propositions 1 and 2 to random variables $\tilde{R}_1(\alpha_1^\star)$ and $\tilde{R}_2(\alpha_2^\star)$, defined, respectively, by (23) and (24), with α_i given by (26), $i = 1, 2$. It follows from Proposition 1 that $\tilde{R}_1(\alpha_1^\star)$ second–order stochastically dominates $\tilde{R}_2(\alpha_2^\star)$ if we can label the components such that

$$\begin{aligned}\frac{\mu_p - r_f}{\mu_2 - r_f}\theta_1 < \cdots < \frac{\mu_p - r_f}{\mu_2 - r_f}\theta_\ell \\ \leq \frac{\mu_p - r_f}{\mu_1 - r_f}\sigma_1 < \cdots < \frac{\mu_p - r_f}{\mu_1 - r_f}\sigma_k \\ \leq \frac{\mu_p - r_f}{\mu_2 - r_f}\theta_{\ell+1} < \cdots < \frac{\mu_p - r_f}{\mu_2 - r_f}\theta_m,\end{aligned}$$

and

$$\frac{\mu_p - r_f}{\mu_1 - r_f}\sum_{j=1}^{k}\lambda_j\sigma_j \leq \frac{\mu_p - r_f}{\mu_2 - r_f}\sum_{i=1}^{m}\omega_i\theta_i,$$

or, alternatively,

$$\frac{\mu_2 - r_f}{\theta_m} < \cdots < \frac{\mu_2 - r_f}{\theta_{\ell+1}} \leq \frac{\mu_1 - r_f}{\sigma_k} < \cdots < \frac{\mu_1 - r_f}{\sigma_1} \leq \frac{\mu_2 - r_f}{\theta_\ell} < \cdots < \frac{\mu_2 - r_f}{\theta_1}, \quad (27)$$

and

$$\frac{\mu_1 - r_f}{\sum_{j=1}^{k}\lambda_j\sigma_j} \geq \frac{\mu_2 - r_f}{\sum_{i=1}^{m}\omega_i\theta_i}. \quad (28)$$

Following the same logic, Proposition 2 implies that $\tilde{R}_1(\alpha_1^\star)$ fourth–order stochastically dominates $\tilde{R}_2(\alpha_2^\star)$ if we can label the components such that (27) holds and

$$\frac{\mu_1 - r_f}{\sqrt{\mathrm{Var}(R_1)}} = \frac{\mu_1 - r_f}{\sqrt{\sum_{j=1}^{k}\lambda_j\sigma_j^2}} \geq \frac{\mu_2 - r_f}{\sqrt{\sum_{i=1}^{m}\omega_i\theta_i^2}} = \frac{\mu_2 - r_f}{\sqrt{\mathrm{Var}(R_2)}}. \quad (29)$$

Conditions (28) and (29) are also *necessary* for second– and fourth–order stochastic dominance, respectively. Moreover,

$$\min_j\left\{\frac{\mu_1 - r_f}{\sigma_j}\right\} \geq \min_i\left\{\frac{\mu_2 - r_f}{\theta_i}\right\} \quad (30)$$

is also necessary for both.

Note that conditions (27) and (28) do not depend on μ_p, i.e., if they hold, they hold for *any* μ_p. That is, if (27) and (28) hold, then any portfolio combination of R_2 in (22) and the risk–free rate is second–order dominated by an appropriately constructed portfolio consisting of R_1 in (21) and the risk–free rate, i.e., random variable (21) second–order dominates random variable (22). Similarly, if (27) and (29) hold, then random variable (21) fourth–order dominates random variable (22).

The quantities appearing in (27) and (30) are the *regime–specific* Sharpe ratios of the portfolios involved in the comparison, whereas condition (29) says that the *overall* Sharpe ratio of R_2 must not exceed that of R_1 for fourth–order stochastic dominance. The interpretation is the same as that of Proposition 2. To see this, recall that, if Portfolio 1 has a higher Sharpe ratio than Portfolio 2, then by combining Portfolio 1 with the risk–free asset we can achieve any expected return with a lower overall portfolio variance than by combining Portfolio 2 with the risk–free asset (Sharpe, 1966). Thus, with risk–free lending and borrowing, investors preferring a more even distribution of Sharpe ratios over the regimes, for a given overall Sharpe ratio, is equivalent to investors preferring a more even distribution of regime–specific variances, for a given overall variance.

3 Concluding remarks

The stochastic dominance criterion for normal mixture distributions derived by Levy and Kaplanski (2015) has been generalized and extended to be applicable to a broader set of situations. As in Levy and Kaplanski (2015), it has been assumed that regimes are characterized by different volatility levels, whereas expected returns are constant across regimes. This is consistent with empirical evidence, i.e., differences between regime–specific variances are typically highly statistically significant, whereas differences between regime–specific means are not (c.f. Ang and Timmermann, 2012).[9] As a result, the overall return distribution is leptokurtic but symmetric. A source of asymmetry different from regime–specific expected returns is *within–regime* skewness, which can be allowed for, e.g., by replacing the Gaussian in (1) by Azzalini's (1985) *skew–normal* distribution. This naturally leads to the class of finite mixtures of skew–normal distributions (e.g., Bernardi, 2013; and Otiniano et al., 2015), which nests the class of normal mixtures and may be useful when portfolio return distributions display asymmetries. Stochastic dominance criteria for single–component skew–normal distributions have been investigated by Blasi and Scarlatti (2012). It may be worthwhile to consider whether these can be generalized to finite mixtures of skew–normal distributions.

[9] Further empirical evidence is presented in Levy and Kaplanski (2015) for monthly returns of 311 mutual funds.

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