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Abstract

In general scheduling a sports league is a difficult combinatorial optimization problem. We study some variants of round robin tournaments and analyze the relationship with the planar three index assignment problem. The complexity of scheduling a round robin tournaments is settled by a reduction from the planar three index assignment problem.

Furthermore, integer programming models are introduced. We pick up a popular idea and decompose the overall problem in order to obtain two subproblems which can be solved sequentially. The latter subproblem can be represented as a planar three index assignment problem which makes corresponding solution techniques amenable to sports league scheduling.

Keywords: Sports league scheduling, round robin tournaments, first-break-then-schedule, three index assignment

1 Introduction

Sport league scheduling covers a huge variety of different problems arising in practice. In literature both temporally-relaxed and temporally-constrained problems are studied. The former contains a number of time periods which is larger than the minimum number of periods necessary to schedule all matches while in the latter case exactly the minimum number of periods required to ensure a feasible schedule is given.

The focus of this paper is on round robin tournaments (RRT), where scheduling is temporally constrained. We consider a set T of n teams. If n is odd we can add a dummy team and, hence, we can assume, without loss of generality, that n is even. In a r-RRT each team plays r times against each other, either at home or away. Each team has to play at least $\lfloor \frac{r}{2} \rfloor$ times at home against each other team. Obviously, this implies that no team can play more than $\lceil \frac{r}{2} \rceil$ times at home and the resulting schedule is somehow balanced with respect to the venues of the matches. Furthermore, a team $i \in T$ has to play exactly once in each period and, hence, we have a set P of $r(n-1)$ periods altogether.

We consider the case $r = 1$, that is, single round robin tournaments (SRRT), an instance of which is provided in table 1, and the case $r = 2$, that is double round robin tournaments (DRRT). In DRRT each team has to play once at home and once away against each other team; see table 2.

period	1	2	3	4	5
match 1	1-2	5-6	3-4	4-5	5-1
match 2	5-3	1-4	2-5	3-1	4-2
match 3	4-6	2-3	1-6	2-6	3-6

Table 1: SRRT for $n = 6$

A special case of DRRT is the mirrored DRRT. Here a match between teams $i, j \in T$ takes place in period t at i's home if and only if a match between teams i and j takes place in period $((t+n-1) \bmod (2n-2))$ at j's home, that is, the tournament is divided into 2 rounds; see table 3. A mirrored r-RRT is defined analogously having r rounds.

In the literature a variety of approaches for scheduling RRTs have been published most of which are based on graph models; see, e.g., de Werra [4, 6] and Drexl and Knust [7]. A schedule can

period	1	2	3	4	5	6	7	8	9	10
match 1	1-2	6-5	5-6	4-5	5-1	5-4	3-4	2-1	1-5	4-3
match 2	5-3	4-1	1-4	3-1	4-2	1-3	2-5	3-5	2-4	5-2
match 3	4-6	3-2	2-3	2-6	3-6	6-2	1-6	6-4	6-3	6-1

Table 2: DRRT for $n = 6$

period	1.1	1.2	1.3	1.4	1.5	2.1	2.2	2.3	2.4	2.5
match 1	1-2	5-6	3-4	4-5	5-1	2-1	6-5	4-3	5-4	1-5
match 2	5-3	1-4	2-5	3-1	4-2	3-5	4-1	5-2	1-3	2-4
match 3	4-6	2-3	1-6	2-6	3-6	6-4	3-2	6-1	6-2	6-3

Table 3: Mirrored DRRT for $n = 6$

be represented as an edge-coloring of the complete graph consisting of n vertices with $n-1$ colors. An edge-coloring of a graph is a coloring of the edges such that adjacent edges have different colors. Each vertex corresponds to a team and each edge represents a match between its incident teams. Consequently, edges having the same color represent matches taking place in the same period. If we orient each edge we obtain a complete schedule containing the information about the period and the venue of a match.

A variety of constraints can be added to the generic RRT constraints mentioned above. A team playing at home (away) in two consecutive periods is said to have a break in the latter period. Often the number of breaks should be minimized. De Werra [5] shows that a league cannot be scheduled having less than $n-2$ breaks. Aiming at the minimum number of breaks implies that the numbers of breaks for the teams is different. For fairness reasons it is often required that the number of breaks is equally balanced for all the teams.

Bartsch [2] and Bartsch et al. [3] compile a bunch of constraints which have to be assured when scheduling the professional soccer leagues of Austria or Germany. In particular, organizational, attractiveness, and fairness constraints are considered to be important. Organizational constraints cover a set of rules which have to guarantee that all the matches can be scheduled according to the regulations. Attractiveness constraints focus on what stadium visitors, television spectators and the players expect from the sequence of matches (that is, a varied, eventful, and exciting season). Finally, fairness constraints have to guarantee that no team is handicapped or favored in comparison with the competitors.

Russel [22] studies carry-over effects in RRT: Two teams $i, j \in T$ playing against each team $k \in T' \subseteq T \backslash \{i, j\}, |T'| > 1$, in the same order is unpleasant and, therefore, usually should be avoided.

Although finding a feasible schedule might already be difficult it is often desired to find a "good one". The travelling tournament problem (see, e.g., Easton et al. [8]) is the problem to find the RRT inducing least travel cost. A further objective is to maximize the teams' preferences to play home or away in specific periods.

In previous research, a couple of different techniques are employed to tackle the scheduling of sports leagues. Nemhauser and Trick [21], for example, use integer programming while Henz [13] intercedes constraint programming. Moreover, Easton et al. [9] and Trick [24] propose to combine constraint and integer programming. Furthermore, much effort has been spent on neighborhood search; see, e.g., Anagnostopoulos et al. [1], Hamiez and Hao [12] and Henz [14].

The outline of this paper is as follows: In section 2 we define the problems considered in this paper and we present integer programming formulations. In section 3 we provide a proof of the complexity of SRRT. Section 4 further examines the relationship between our scheduling problem and the planar three index assignment problem. We propose a scheduling approach based on a well-known decomposition scheme comprising two subproblems. We show how to represent the second subproblem as planar three index assignment problem. The last section gives a summary and an outlook to future research.

2 Problem description

In this section we describe a single round robin tournament, a round robin tournament with an arbitrary number of r rounds, and the planar three index assignment problem.

2.1 Single round robin tournaments

Given a set T of teams and a set P of periods with $|P| = |T| - 1$, each triple $(i, j, p) \in T \times T \times P, i \neq j$, represents a match of team i against team j at i's home in period p. For each such triple a cost $c_{i,j,p}$ is known. A feasible solution of the minimum cost SRRT problem corresponds to a set of $\frac{|T|(|T|-1)}{2}$ triples such that (i) for each pair $(i, j) \in T \times T, i \neq j$, exactly one triple of the form (i, j, p) or (j, i, p) with $p \in P$ is chosen (implying that each pair of teams has to meet) and such that (ii) for each pair $(i, p) \in T \times P$ exactly one triple of the form (i, j, p) or (j, i, p) with $j \in T \backslash \{i\}$ is chosen (implying that each team plays exactly once in each period). The problem is to find a feasible solution having the minimum sum of chosen triples' cost.
An integer programming formulation of this problem is as follows. We use a representation of the minimum cost SRRT presented by Trick [24] employing $n(n-1)^2$ binary variables and $\frac{3n(n-1)}{2}$ constraints. In this formulation $x_{i,j,p}$ equals 1 if team i plays at home against team j in period p, and it is 0 otherwise.

SRRT-formulation

$$\min \sum_{i \in T} \sum_{j \in T \backslash \{i\}} \sum_{p \in P} c_{i,j,p} x_{i,j,p} \tag{1}$$

$$\text{s.t.} \sum_{p \in P} (x_{i,j,p} + x_{j,i,p}) = 1 \qquad \forall\, i, j \in T, i < j \tag{2}$$

$$\sum_{j \in T \backslash \{i\}} (x_{i,j,p} + x_{j,i,p}) = 1 \qquad \forall\, i \in T, p \in P \tag{3}$$

$$x_{i,j,p} \in \{0, 1\} \qquad \forall\, i, j \in T, i \neq j, p \in P \tag{4}$$

The objective function (1) minimizes the total cost of chosen matches. Constraints (2) ensure that each team plays each other team exactly once. Constraints (3) assure that each team plays exactly once in each period. Constraints (2) are called 1-factor constraints and constraints (3) are denoted all-different constraints in the literature; see for example Trick [24].
In the objective (1) we used the abstract term "cost". In order to emphasize the practical relevance we summarize some aspects of what the "cost" $c_{i,j,p}$ of real-world tournaments might cover:

- Teams usually have preferences for playing at home at certain rounds, a fact which can easily be expressed through $c_{i,j,p}$.

- Since a major objective of the organizers of a tournament is to maximize attendance we can represent the economic value of the estimated attendance by $c_{i,j,p}$ and tackle the maximization version of the SRRT.

- Often, a stadium is owned by some public agency and teams do have to pay a fee for each match taking place in that particular stadium. This fee can be represented by $c_{i,j,p}$.

- In terms of more complex models the SRRT might be used as a subproblem, e.g., within a Lagrangean relaxation or a column generation framework. Then, $c_{i,j,p}$ is used to cover dual information also.

- A special case of SRRT arises when the costs are restricted to $\{0,1\}$. Then $c_{i,j,p} = 1$ denotes that team i cannot play team j in team i's home venue in period p, whereas $c_{i,j,p} = 0$ denotes that this is possible. What we are interested in is to determine whether a feasible schedule, that is, a zero-cost schedule, exists or not.

We refer to the case when the costs are restricted to $\{0,1\}$ as Availability Constrained SRRT, or AC-SRRT for short. The AC-SRRT is of relevance because of two reasons:

- First, stadiums may not always be available (leading to $c_{i,j,p} = 1$ for all j if team i's home venue is not available in period p), or team i can, in some period p, only play in stadiums not too far away from its home base. Moreover, if team i does not want to play away in period p then $c_{j,i,p}$ is set to 1 for all j.

- Second and more important, the AC-SRRT will serve as a means to formally state the complexity of SRRT in section 3.

2.2 Round robin tournaments with r rounds

In this section we study the case of $r = 2$, that is double round robin tournaments (DRRT), and the general case denoted as r-RRT where r can take on arbitrary values,
DRRTs are quite common in practice, and can be described as follows. Given are a set T of teams and a set P of periods with $2(|T| - 1) = |P|$. Costs $c_{i,j,p}$ for each triple $(i,j,p) \in (T \times T \times P), i \neq j$, are known. A feasible solution to the minimum cost DRRT problem corresponds to $|T|(|T| - 1)$ triples such that for each pair $(i,j) \in T \times T, i \neq j$, exactly one triple of the form (i,j,p) with $p \in P$ is chosen. Furthermore, for each pair $(i,p) \in T \times P$ exactly one triple of the form (i,j,p) or (j,i,p) with $j \in T \backslash \{i\}$ has to be chosen. Our formulation of the minimum cost DRRT problem uses $2n(n-1)^2$ binary variables and it has $2n(n-1)$ constraints.
Again, the objective function (5) minimizes cost of chosen matches. Constraint (6) ensures two matches between i and j exactly one of which takes place at one opponents home. Constraint (7) assures that each team plays exactly once per period.
Notice that DRRT covers tournaments which do not consist of two rounds being SRRTs. Tournament problems having mirrored rounds can be represented by an SRRT as outlined in section 2.1.
This concept can be generalized to the minimum cost r-RRT with a set of T teams, a set of P periods with $r(|T| - 1) = |P|$, and cost for each triple $(i,j,p) \in (T \times T \times P), i \neq j$.

DRRT-formulation

$$\min \sum_{i \in T} \sum_{j \in T \setminus \{i\}} \sum_{p \in P} c_{i,j,p} x_{i,j,p} \tag{5}$$

$$\begin{aligned}
\text{s.t.} \sum_{p \in P} x_{i,j,p} &= 1 && \forall i,j \in T : i \neq j && (6)\\
\sum_{j \in T \setminus \{i\}} (x_{i,j,p} + x_{j,i,p}) &= 1 && \forall i \in T, p \in P && (7)\\
x_{i,j,p} &\in \{0,1\} && \forall\, i,j \in T, i \neq j, p \in P && (8)
\end{aligned}$$

r-RRT-formulation

$$\min \sum_{i \in T} \sum_{j \in T \setminus \{i\}} \sum_{p \in P} c_{i,j,p} x_{i,j,p} \tag{9}$$

$$\begin{aligned}
\text{s.t.} \sum_{p \in P} x_{i,j,p} &\geq \lfloor \frac{r}{2} \rfloor && \forall i,j \in T : i \neq j && (10)\\
\sum_{p \in P} (x_{i,j,p} + x_{j,i,p}) &= r && \forall i,j \in T : i < j && (11)\\
\sum_{j \in T \setminus \{i\}} (x_{i,j,p} + x_{j,i,p}) &= 1 && \forall i \in T, p \in P && (12)\\
x_{i,j,p} &\in \{0,1\} && \forall\, i,j \in T, i \neq j, p \in P && (13)
\end{aligned}$$

A feasible solution is a set of $\frac{r|T|(|T|-1)}{2}$ triples such that for each pair $(i,j) \in T \times T, i \neq j$, at least $\lfloor \frac{r}{2} \rfloor$ triples of the form (i,j,p) with $p \in P$ are chosen and such that for each pair $(i,p) \in T \times P$ exactly one triple of the form (i,j,p) or (j,i,p) with $j \in T \setminus \{i\}$ is selected as well. The formulation (9) to (13) uses $rn(n-1)^2$ binary variables and it has $(r+\frac{3}{2})n(n-1)$ constraints.
In case r is even, constraints (10) and (11) can be simplified to:

$$\sum_{p \in P} x_{i,j,p} = \frac{r}{2} \qquad \forall i,j \in T : i \neq j \tag{14}$$

A special case of the minimum cost r-RRT, the minimum cost mirrored r-RRT, is confined to r-RRT's having all teams $i \in T$ play against its opponents in the same order in each round. Obviously, we can reduce a minimum cost mirrored r-RRT to a minimum cost SRRT where $c_{i,j,p}$ represents the overall cost of corresponding matches in further rounds.

2.3 Planar three index assignment problem

As we will see later the so-called planar three index assignment problem (TIAP) is intimately related to round robin tournaments. The TIAP can be described by recalling that three m-sets I, J, K are given, as well as a cost $d_{i,j,k}$ for each triple $(i,j,k) \in I \times J \times K$. The goal is to find m^2 triples such that each pair in $(I \times J) \cup (I \times K) \cup (J \times K)$ is present exactly once. We give here a formulation of TIAP as an integer program according to, e.g., Spieksma [23] using

m^3 binary variables and $3m^2$ constraints. In this formulation $x_{i,j,k}$ equals 1 if triple (i,j,k) is chosen, and it is 0 otherwise.

TIAP-formulation

$$\min \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} d_{i,j,k}x_{i,j,k} \tag{15}$$

$$\begin{aligned}
\text{s.t.} \sum_{k\in K} x_{i,j,k} &= 1 \qquad \forall\, i\in I, j\in J && (16)\\
\sum_{j\in J} x_{i,j,k} &= 1 \qquad \forall\, i\in I, k\in K && (17)\\
\sum_{i\in I} x_{i,j,k} &= 1 \qquad \forall\, j\in J, k\in K && (18)\\
x_{i,j,k} &\in \{0,1\} \quad \forall\, i\in I, j\in J, k\in K && (19)
\end{aligned}$$

The objective function (15) sums up the chosen triples' cost. Constraints (16), (17), and (18) force each pair in $(I\times J)\cup(I\times K)\cup(J\times K)$ to be contained in the chosen triples.

3 Complexity

To the best of our knowledge, the complexity of SRRT has not been formally stated in the literature. First we prove that AC-SRRT is NP-complete by giving a reduction from TIAP.

Theorem 1 *The AC-SRRT is NP-complete.*

Proof. We prove the theorem by presenting a reduction from TIAP to AC-SRRT. TIAP is proved to be NP-complete in Frieze [11].
The decision version of TIAP can be described as follows:

Input: Three m-sets I, J, K, and a set $A \subseteq I\times J\times K$.

Question: Does there exist a set of m^2 triples $M\subseteq A$ such that each pair $(i,j)\in(I\times J)$, $(i,k)\in(I\times K)$, and $(j,k)\in(J\times K)$ is contained exactly once in a triple from M?
We assume, without loss of generality, that m is even. Given an instance of TIAP, we now build the instance of AC-SRRT as follows. There are $2m$ teams, so we have $|T| = n = 2m$ (and of course $|P| = 2m-1$). Further, we set

$$c_{i,m+j,p} = \begin{cases} 0 & \text{for each triple } (i,j,p)\in A,\\ 1 & \text{for each triple } (i,j,p)\notin A, \end{cases}$$

and

$$c_{i,j,p} = \begin{cases} 1 & \text{for } i = 1,\ldots,m, j = 1,\ldots,m, i \neq j, p = 1,\ldots,m, \\ 1 & \text{for } i = m+1,\ldots,2m, j = m+1,\ldots,2m, i \neq j, p = 1,\ldots,m, \\ 1 & \text{for } i = m+1,\ldots,2m, j = 1,\ldots,m, p = 1,\ldots,m, \\ 1 & \text{for } i = 1,\ldots,m, j = m+1,\ldots,2m, p = m+1,\ldots,2m-1, \\ 0 & \text{for } i = 1,\ldots,m, j = 1,\ldots,m, i \neq j, p = m+1,\ldots,2m-1, \\ 0 & \text{for } i = m+1,\ldots,2m, j = m+1,\ldots,2m, i \neq j, p = m+1,\ldots,2m-1, \\ 1 & \text{for } i = m+1,\ldots,2m, j = 1,\ldots,m, p = m+1,\ldots,2m-1. \end{cases}$$

This completes the description of the instance of AC-SRRT.

A yes-answer to the TIAP instance corresponds to a feasible solution to AC-SRRT in the following way. First, the triples (i,j,k) that constitute the solution of TIAP give rise to the following partial solution of AC-SRRT: Team i plays team $m+j$ in period k in team i's home venue. Since in this way we use only triples from A, we have ensured that each match between a team i with $1 \leq i \leq m$, and a team j with $m+1 \leq j \leq 2m$ is scheduled with zero cost. Second, to schedule the remaining matches, let us first deal with the matches between different teams i and j with $1 \leq i,j \leq m$. Observe that we must assign these matches to periods $m+1,\ldots,2m-1$ in order to have a zero-cost solution. Assigning these matches to $m-1$ periods can be seen as edge-coloring a complete graph (recall that an edge-coloring of a graph is a coloring of the edges such that adjacent edges have different colors). Indeed, the graph that results when there is a vertex for each of the first m teams, and an edge for each match to be played is complete. It is well-known (see Mendelsohn and Rosa [18]) that, in case m is even – as we assumed – $(m-1)$ colors suffice to edge color a complete graph on m nodes. The resulting coloring gives us a feasible assignment of matches to periods (edges with the same color correspond to matches played in the same period). In this way each period receives $\frac{m}{2}$ matches, each with zero cost. By using the same procedure for different teams i and j with $m+1 \leq i,j \leq 2m$, we find an assignment of the corresponding matches to periods $m+1,\ldots,2m-1$. Total cost of these matches equals zero. Hence, we have found a feasible solution to AC-SRRT.

Finally, if a zero-cost solution to AC-SRRT instance exists, it is not difficult to show that TIAP admits a solution. Indeed, let us focus on the matches between teams i and j with $1 \leq i \leq m$ and $m+1 \leq j \leq 2m$. From the construction it is clear that the existence of a zero-cost solution implies that team j never plays at its home venue against team i since this costs 1. Hence, the assignment of matches of team i against team j to periods p, $p = 1,\ldots,m$ (which must exist since we assumed a zero-cost solution to AC-SRRT exists), gives us the solution to TIAP. □

Theorem 1 implies that the minimum cost SRRT problem is NP-hard.

In the following section we study approaches to sports league scheduling which make use of the fact that round robin tournaments are intimately related to planar three index assignments.

4 Scheduling approach

4.1 Problem decomposition

As outlined above the minimum cost SRRT problem is hard to solve. Therefore, solution approaches usually are based on a decomposition of the problem. A frequently used decom-

position scheme is to separate the problem of finding the venue of a match and the problem of finding a match's date.
In the literature two types of decomposition schemes are predominant:

- **First-break-then-schedule:** First, a decision about the matches' venues is made. Afterwards, the matches are assigned to the periods in which they shall take place.
- **First-schedule-then-break:** First, each match (determined by the pair of teams without a venue) is assigned to a period. Based on this timetable each match's venue is fixed.

  Several topics related to this decomposition scheme are studied in Elf et al. [10], Knust and von Thaden [15] and Miyashiro and Matsui [20].

In the following we analyze the first-break-then-schedule decomposition scheme. The decisions about matches' venues are discussed in section 4.2. Our focus is, however, on the scheduling problem when the venues are already fixed. In particular we provide an encoding of the scheduling problem as a planar TIAP in section 4.3 which paths the way to employ approaches for the latter one.

4.2 Home-away-pattern sets

A home-away-pattern (HAP) for a particular team $i \in T$ is a string of length equal to the number of periods containing A's and H's. Each entry contains the information whether i plays at home (H) or away (A) in the period corresponding to the entry's index. If we assign a HAP to each single team this assignment is called a HAP set. In terms of first-break-then-schedule a HAP set restricts the following scheduling procedure such that two teams having the same entry for a specific period p cannot play against each other in p. If two teams i and j have different entries in period p a match between these teams can be scheduled in p. Of course, this match has to take place at the venue of the team having the H. Such a match will be called feasible in the following. All other matches are infeasible.
HAP sets have been studied recently. Obviously, we cannot choose n HAPs arbitrarily and assign them to the teams. For example consider the trivial case in which two teams $i, j \in T$ have identical HAPs: Apparently, i and j can never play against each other. If we can construct an RRT based on a HAP set we call it feasible. Note that feasibility does not depend on the specific assignment of HAPs to teams but only on the set of HAPs. Several necessary conditions are known for a HAP set to be feasible. An overview is provided in Miyashiro et al. [19]. Furthermore, in [19] a strong necessary condition for the feasibility of the special case where HAP sets have a minimum number of breaks is proposed and conjectured to be sufficient, too. However, neither this condition is known to be sufficient nor a similar strong necessary condition for general HAP sets is known so far.
Examples for feasible and infeasible HAP sets of SRRTs with $n = 6$ teams are given in tables 4 and 5, respectively. In particular, the HAP set in table 4 corresponds to the SRRT given in table 1 if $i - j$ is interpreted as a match between i and j at i's home. Apparently, this HAP set is feasible. To the contrary, there are two obvious reasons for the HAP set in table 5 to be unfeasible: Teams 1 and 2 can never play against each other and period 1 has more teams playing at home than away.
Often, a set of HAPs is given in advance when a sport scheduling problem has to be solved. Otherwise the first step of our scheme is to generate a feasible HAP set. Since sufficient conditions for the feasibility of a set of HAPs are not known Nemhauser et al. [21] propose to

period	1	2	3	4	5
team 1	H	H	H	A	A
team 2	A	H	H	H	A
team 3	A	A	H	H	H
team 4	H	A	A	H	H
team 5	H	H	A	A	H
team 6	A	A	A	A	A

Table 4: Feasible HAP set for $n = 6$

period	1	2	3	4	5
team 1	H	H	H	A	A
team 2	H	H	H	A	A
team 3	A	A	H	H	H
team 4	H	A	A	H	H
team 5	H	H	A	H	H
team 6	A	A	A	A	A

Table 5: Infeasible HAP set for $n = 6$

generate several HAPs considering simple feasibility conditions. In order to limit the solution space they construct mirrored HAPs having an A for period p iff there is an H for period $((p + \frac{|P|}{2}) \bmod |P|)$. This property is used in our construction of a TIAP in order to schedule the matches. Nemhauser et al. [21] state that it is not difficult to find feasible HAP sets although feasibility of a single generated HAP set cannot be guaranteed. Miyashiro et al. [19] use an integer programming formulation to check the feasibility of potential HAP sets.
If there are several feasible HAP sets available a specific HAP set has to be chosen. Furthermore, the HAPs belonging to that particular HAP set have to be assigned to the teams. In order to do so we can construct a linear assignment problem for each HAP set which guarantees that exactly one HAP h is assigned to each team i. To this end, for example, estimated attendance for a home-game at i's venue can be employed in order to define the benefit $b_{h,i}$ for an assignment of h to i.
More precisely, let HAS denote the set of available HAP sets and let HA denote one particular HAP set. Then, given HA we obtain the model formulation (20) to (23) where the variable $a_{h,i} \in \{0, 1\}$ indicates whether HAP h is assigned to each team i ($a_{h,i} = 1$) or not ($a_{h,i} = 0$).

Assignment$_{HA}$

$$\max Z_{HA} = \sum_{i \in T} \sum_{h \in HA} a_{h,i} b_{h,i} \tag{20}$$

$$\text{s.t.} \sum_{i \in T} a_{h,i} = 1 \qquad \forall\, h \in HA \tag{21}$$

$$\sum_{h \in HA} a_{h,i} = 1 \qquad \forall\, i \in T \tag{22}$$

$$a_{h,i} \in \{0, 1\} \qquad \forall\, i \in T, h \in HA \tag{23}$$

Obviously, the assignment's benefit shall be maximized as stated in (20). The constraints (21), (22), and (23) define a linear assignment problem which can be solved efficiently by, e.g., the Hungarian method. Finally, the overall most suitable HAP set HA' has to be found, that is,

$$HA' = \text{argmax}_{HA \in HAS} Z_{HA} \tag{24}$$

has to be solved. We restrain ourselves from going into details here because the scope of this paper is on the scheduling subproblem.

4.3 Scheduling and three-index-assignment

4.3.1 Double round robin tournaments

Given a HAP set the next step is to decide which match of the DRRT to schedule in which period. Let the minimum cost DRRT problem be defined by the set of teams T, the set of periods P with $|P| = (2|T| - 2)$ and cost $c_{i,j,p}$ for each triple $(i,j,p) \in (T \times T \times P), i \neq j$. In the following we will construct a planar TIAP by defining three sets T'_1, T'_2, and P' and cost $c'_{i',j',p'}$ for all triples $(i',j',p') \in (T'_1 \times T'_2 \times P')$. We set $T'_1 = T'_2 = T$ and interpret T'_1 as teams playing home and T'_2 as teams playing away. Therefore, each team occurs twice in this representation and we refer to team i' in T'_1 and T'_2 as i'_1 and i'_2, respectively. The third set $P' = P$ is associated with periods.

In terms of the TIAP described in section 2.3 we can interpret a tournament as a triple (i',j',p') defining a match between team i' and team j' at i''s home in period p'. Doing so we obtain a tournament where each team i' plays twice per period, once represented by i'_1 and once represented by i'_2 as well. This means each team i' has one match at its home and one match at the opponents venue. Furthermore, each pair of teams (i',j') meets twice during the tournament: Once i'_1 plays at home against j'_2 and once j'_1 plays at home against i'_2. Additionally, each team i' plays once against each other since constraint (16) forces i'_1 to play against i'_2 once. Of course, in terms of real world tournaments this does not make sense. If we neglect, however, the matches of a team against itself we realize immediately that the matches noticed above are exactly the matches a DRRT consists of. All we have to do is to rearrange the periods in which the matches take place and to get rid off the matches of teams against themselves.

In the following we employ a given HAP set from section 4.2 in order to construct a planar TIAP whose solution can be interpreted as a DRRT using knowledge of the HAP set.

Let H_p and A_p be sets containing teams playing at home and away in period p, respectively. Due to the property of mirrored HAPs we have $H_p = A_{p+|T|-1}$ and $A_p = H_{p+|T|-1}$ for $p < |T|$. A match between two teams i and j at i's home and a match between the same two teams at j's home can never both be feasible in period p. Furthermore, a match between two teams i and j at i's home is feasible in period p if and only if a match between i and j at j's home is feasible in period $p + |T| - 1$. Hence, we can employ triples (i',j',p') and (j',i',p') with $p' < |T|$ to represent the matches between i and j in periods p' and $p' + |T| - 1$. Note that we have an "artificial period" $p' = |T|$ in the TIAP since the minimum cost DRRT problem has $2|T| - 2$ periods.

Now we instantiate a planar TIAP by setting its size n' to the number of teams $|T|$ and employing cost $c_{i,j,p}$ of the minimum cost DRRT problem and a big M to define cost $c'_{i',j',p'}$ according to (25) to (30).

$$c'_{i',j',p'} = \begin{cases} M & \text{if } p' = n' \wedge i' \neq j', & (25) \\ -M & \text{if } p' = n' \wedge i' = j', & (26) \\ M & \text{if } p' < n' \wedge i' \in A_{p'} \wedge j \in A_{p'}, & (27) \\ M & \text{if } p' < n' \wedge i' \in H_{p'} \wedge j \in H_{p'}, & (28) \\ c_{i',j',p'} & \text{if } p' < n' \wedge i' \in H_{p'} \wedge j' \in A_{p'}, & (29) \\ c_{i',j',p'+|T|-1} & \text{if } p' < n' \wedge i' \in A_{p'} \wedge j' \in H_{p'}. & (30) \end{cases}$$

(25) and (26) force all "artificial matches" of teams against themselves to take place in the

"artificial period". (27) and (28) forbid those matches which can take place neither in p' nor in $(p' + |T| - 1)$ since both opponents play either at home or away. (29) defines cost for matches which can take place at i''s venue in period p' since $i' \in H_{p'}$. (30) determines cost for matches at i''s home in period $(p' + |T| - 1)$ if $i' \in A_{p'}$ and, therefore, $i' \in H_{p'+|T|-1}$.
After solving this planar TIAP we have a solution x' associated with cost $C'(x') = \sum_{i' \in T_1'} \sum_{j' \in T_2'} \sum_{p' \in P} c'_{i',j',p'} x_{i',j',p'}$. We can construct a solution x for the minimum cost DRRT problem from x' as follows:

(a) ignore all chosen triples (i', j', p') having $p' = n'$,

(b) add all chosen triples (i', j', p') having $p' < n'$ and $i' \in H_{p'}$ to x,

(c) for all chosen triples (i', j', p') having $p' < n'$ and $i' \in A_{p'}$ add triple $(i', j', p' + |T| - 1)$ to x.

Note that due to the high cost associated with infeasible matches only feasible matches will be chosen if possible. Otherwise, if an infeasible match is chosen, the given HAP set is not feasible.
The solution x forms a DRRT since (6) and (7) hold. Obviously, (6) is fulfilled through (16): Each team plays at home against each other team. Since we have all "artificial matches" covered by triples $(i', j', |T|)$ the others are contained in triples (i', j', p') with $p' < |T|$. Furthermore, for each $p' < |T|$ there are $\frac{|T|}{2}$ teams of $T_1' \cap H_{p'}$ playing against $\frac{|T|}{2}$ teams of $T_2' \cap A_{p'}$ (due to the fact that infeasible matches are not chosen). Since $H_{p'} \cap A_{p'} = \emptyset$ each team participates in exactly one match in all periods $p \leq |T| - 1$. Considering periods $|T| \leq p \leq (2|T| - 2)$ is straightforward.
The cost of x is given by $C(x) = C'(x') + |T|M$ since chosen triples in x' contain $|T|$ "artificial matches" having cost $-M$ and $|T|(|T| - 1)$ matches having original cost.

4.3.2 Example

In order to illustrate the mechanism we provide an example in the following: We consider $|T| = 4$ teams, costs for periods 1 to 6 as given in tables 6 to 11, and a HAP set as defined in table 12.
Applying (25) to (30) yields cost for the TIAP shown in tables 13 to 16. The cost are singular so the transformation can be followed easily. Attached to each $c'_{i',j',p'}$ we indicate the rule it is constructed with. Note that the costs contained in exactly two out of three cells in tables 6 to 11 are irrelevant in tables 13 to 16 due to the restrictions given by the HAP set.
The optimal solution consists of 16 chosen triples and has associated cost of $438 - 4M$. The variables' values are provided in tables 17 to 20 ("–" means 0). We interpret the triples according to rules (a) to (c) and obtain the DRRT shown in table 21 having cost 438. Again, we indicate the rule according to which the match is interpreted. Note that the solution is not a mirrored DRRT while it obeys the mirrored HAP set given in table 12.

4.3.3 Some further aspects

Obviously, the procedure presented in section 4.3.1 is suitable for mirrored HAP sets only. However, some preprocessing makes the procedure suitable to some other problems as well.
Let HA_p be a string containing all teams' HAP sets entries for period p and let $\overline{HA}_p$ be the complement of HA_p where team i has an A in $\overline{HA}_p$ if and only if it has an H in HA_p. Note that in mirrored HAP sets $HA_p = \overline{HA}_{p+|T|-1}$ holds for each $p \leq |T| - 1$, a fact which is

	1	2	3	4
1	–	01	02	03
2	04	–	05	06
3	07	08	–	09
4	10	11	12	–

Table 6: $c_{i,j,1}$

	1	2	3	4
1	–	13	14	15
2	16	–	17	18
3	19	20	–	21
4	22	23	24	–

Table 7: $c_{i,j,2}$

	1	2	3	4
1	–	25	26	27
2	28	–	29	30
3	31	32	–	33
4	34	35	36	–

Table 8: $c_{i,j,3}$

	1	2	3	4
1	–	61	62	63
2	64	–	65	66
3	67	68	–	69
4	70	71	72	–

Table 9: $c_{i,j,4}$

	1	2	3	4
1	–	49	50	51
2	52	–	53	54
3	55	56	–	57
4	58	59	60	–

Table 10: $c_{i,j,5}$

	1	2	3	4
1	–	37	38	39
2	40	–	41	42
3	43	44	–	45
4	46	47	48	–

Table 11: $c_{i,j,6}$

period	1	2	3	4	5	6
team 1	H	A	H	A	H	A
team 2	A	H	A	H	A	H
team 3	H	H	H	A	A	A
team 4	A	A	A	H	H	H

Table 12: Mirrored HAP set

essential for the procedure. We define a balanced HAP set as a HAP set where each pattern contained in HA_p with $p \leq 2|T| - 2$ exists exactly as often as its complement. An example for a balanced HAP set is given in table 22.
If a given HAP set is not mirrored but balanced we construct an automorphism $a(*)$ on the set of periods such that $a(P)$ induces a mirrored HAP set. Considering the HAP set in table 22 $a(P)$ can be defined as follows:

$$a(P) : (1,2,3,4,5,6) \rightarrow (1,2,5,3,4,6)$$

Now we are able to apply the procedure of section 4.3.1. Afterwards we apply $a^{-1}(P)$ to "reorder" the periods.
Besides the minimum cost DRRT problem considered in section 4.3.1 we can employ techniques developed for TIAPs for several other problems. If we consider for example SRRTs we can construct a planar TIAP having the additional constraint

$$x'_{i',j',p'} = x'_{j',i',p'}.$$

Note that in terms of section 4.3.1 this corresponds to a DRRT having mirrored rounds which is equivalent to a SRRT. By additionally doubling P' we can cover DRRT without mirrored HAPs.
Although we do not have a standard TIAP anymore we can easily modify basic ideas of branching rules or heuristics (see Magos and Miliotis [17] and Magos [16], respectively) for a standard planar TIAP in order to fit to this structure.

	1	2	3	4
1	$M^{(28)}$	$01^{(29)}$	$M^{(28)}$	$03^{(29)}$
2	$64^{(30)}$	$M^{(27)}$	$65^{(30)}$	$M^{(27)}$
3	$M^{(28)}$	$08^{(29)}$	$M^{(28)}$	$09^{(29)}$
4	$70^{(30)}$	$M^{(27)}$	$72^{(30)}$	$M^{(27)}$

Table 13: $c'_{i',j',1}$

	1	2	3	4
1	$M^{(27)}$	$49^{(30)}$	$50^{(30)}$	$M^{(27)}$
2	$16^{(29)}$	$M^{(28)}$	$M^{(28)}$	$18^{(29)}$
3	$19^{(29)}$	$M^{(28)}$	$M^{(28)}$	$21^{(29)}$
4	$M^{(27)}$	$59^{(30)}$	$60^{(30)}$	$M^{(27)}$

Table 14: $c'_{i',j',2}$

	1	2	3	4
1	$M^{(28)}$	$25^{(29)}$	$M^{(28)}$	$27^{(29)}$
2	$40^{(30)}$	$M^{(27)}$	$41^{(30)}$	$M^{(27)}$
3	$M^{(28)}$	$32^{(29)}$	$M^{(28)}$	$33^{(29)}$
4	$46^{(30)}$	$M^{(27)}$	$48^{(30)}$	$M^{(27)}$

Table 15: $c'_{i',j',3}$

	1	2	3	4
1	$-M^{(26)}$	$M^{(25)}$	$M^{(25)}$	$M^{(25)}$
2	$M^{(25)}$	$-M^{(26)}$	$M^{(25)}$	$M^{(25)}$
3	$M^{(25)}$	$M^{(25)}$	$-M^{(26)}$	$M^{(25)}$
4	$M^{(25)}$	$M^{(25)}$	$M^{(25)}$	$-M^{(26)}$

Table 16: $c'_{i',j',4}$

	1	2	3	4
1	–	1	–	–
2	–	–	1	–
3	–	–	–	1
4	1	–	–	–

Table 17: $x'_{i',j',1}$

	1	2	3	4
1	–	–	1	–
2	–	–	–	1
3	1	–	–	–
4	–	1	–	–

Table 18: $x'_{i',j',2}$

	1	2	3	4
1	–	–	–	1
2	1	–	–	–
3	–	1	–	–
4	–	–	1	–

Table 19: $x'_{i',j',3}$

	1	2	3	4
1	1	–	–	–
2	–	1	–	–
3	–	–	1	–
4	–	–	–	1

Table 20: $x'_{i',j',4}$

period	1	2	3	4	5	6
match 1	1-2$^{(b)}$	3-1$^{(b)}$	1-4$^{(b)}$	4-1$^{(c)}$	1-3$^{(c)}$	2-1$^{(c)}$
match 2	3-4$^{(b)}$	2-4$^{(b)}$	3-2$^{(b)}$	2-3$^{(c)}$	4-2$^{(c)}$	4-3$^{(c)}$

Table 21: Example's solution

5 Summary and future work

In this paper we have examined the close relationship between sports league scheduling and planar three index assignment problems. In particular we have proven that single round robin tournament comprises an NP-hard optimization problem. Moreover, we have shown that three index assignment problems play a vital role in first-break-then-schedule decomposition schemes when the venues of matches have already been fixed.

The results exhibited in this paper open up several avenues for future research in sports league scheduling, to mention a few: First, the development of tailored exact and heuristic algorithms addressing the special structure of the three index assignment problems arising in this context. Second, embedding such algorithms in (sequential) first-break-then-schedule decomposition schemes. Third, the formulation of simultaneous optimization models covering break requirements, too. Last but not least, the development of, e.g., Lagrangean relaxation based approaches in order to get tight lower and upper bounds for these optimization models.

period	1	2	3	4	5	6
team 1	A	A	H	H	A	H
team 2	H	H	A	A	H	A
team 3	A	H	H	A	A	H
team 4	H	A	A	H	H	A

Table 22: Balanced HAP set

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