

Biørn, Erik

**Working Paper**

## Revisiting, from a Frischian point of view, the relationship between elasticities of intratemporal and intertemporal substitution

Memorandum, No. 04/2017

**Provided in Cooperation with:**

Department of Economics, University of Oslo

*Suggested Citation:* Biørn, Erik (2017) : Revisiting, from a Frischian point of view, the relationship between elasticities of intratemporal and intertemporal substitution, Memorandum, No. 04/2017, University of Oslo, Department of Economics, Oslo

This Version is available at:

<https://hdl.handle.net/10419/165966>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# MEMORANDUM

No 04/2017

April 2017

**Revisiting, from a Frischian point of view,  
the relationship between elasticities of  
intratemporal and intertemporal substitution**

The seal of the University of Oslo is a circular emblem. It features a central figure of a woman in classical attire, holding a lyre. The text 'UNIVERSITAS OSLOENSIS' is inscribed around the top inner edge of the circle, and 'MDCCCXXXIII' is at the bottom. The seal is rendered in a light gray tone.

**Erik Biørn**

ISSN: 0809-8786

---

Department of Economics  
University of Oslo

This series is published by the  
**University of Oslo**  
**Department of Economics**

P. O.Box 1095 Blindern  
N-0317 OSLO Norway  
Telephone: + 47 22855127  
Fax: + 47 22855035  
Internet: <http://www.sv.uio.no/econ>  
e-mail: [econdep@econ.uio.no](mailto:econdep@econ.uio.no)

In co-operation with  
**The Frisch Centre for Economic  
Research**

Gaustadalleén 21  
N-0371 OSLO Norway  
Telephone: +47 22 95 88 20  
Fax: +47 22 95 88 25  
Internet: <http://www.frisch.uio.no>  
e-mail: [frisch@frisch.uio.no](mailto:frisch@frisch.uio.no)

### **Last 10 Memoranda**

---

No 03/17	Jon Vislie Resource Extraction and Uncertain Tipping Points
No 02/17	Wiji Arulampalam, Michael P. Devereux and Federica Liberini Taxes and the Location of Targets
No 01/17	Erik Biørn Identification and Method of Moments Estimation in Polynomial Measurement Error Models
No 19/16	Erik Biørn <i>Panel data estimators and aggregation</i>
No 18/16	Olav Bjerkholt <i>Wassily Leontief and the discovery of the input-output approach</i>
No 17/16	Øystein Kravdal <i>New Evidence about effects of reproductive variables on child mortality in sub-Saharan Africa</i>
No 16/16	Moti Michaeli and Daniel Spiro <i>The dynamics of revolutions</i>
No 15/16	Geir B. Asheim, Mark Voorneveld and Jörgen W. Weibull <i>Epistemically robust strategy subsets</i>
No 14/16	Torbjørn Hanson <i>Estimating output mix effectiveness: A scenario approach</i>
No 13/16	Halvor Mehlum and Kalle Moene <i>Unequal power and the dynamics of rivalry</i>

---

Previous issues of the memo-series are available in a PDF® format at:  
<http://www.sv.uio.no/econ/english/research/unpublished-works/working-papers/>

REVISITING, FROM A FRISCHIAN POINT OF VIEW,  
THE RELATIONSHIP BETWEEN ELASTICITIES OF  
INTRATEMPORAL AND INTERTEMPORAL SUBSTITUTION

ERIK BIØRN

Department of Economics, University of Oslo,  
P.O. Box 1095 Blindern, 0317 Oslo, Norway

E-mail: erik.biorn@econ.uio.no

ABSTRACT: How are substitution in the spatial and in the temporal sense connected? Can estimates based on data with spatial variation be transmitted into values appropriate for exploring temporal variation, and vice versa? This paper, building on, *inter alia*, Frisch (1959), attempts to give some clarifications and provides elements of a synthesis. The income elasticity of the marginal utility of income, whose inverse is often called the ‘Frisch parameter’, and its multi-period counterpart, the wealth elasticity of the marginal utility of wealth, are key concepts. Three cases with additive preferences are discussed, and in that context the role of the (generalized) ‘Frisch parameter’ as an overall measure of substitution across commodities and over time is reconsidered.

KEYWORDS: Frisch elasticity. Frisch demand. Intertemporal substitution. Additive preferences. Homotheticity. Marginal utility of income. Marginal utility of wealth

JEL CLASSIFICATION: D11, D12, D91

ACKNOWLEDGEMENTS: Helpful comments and suggestions from Olav Bjerkholt are gratefully acknowledged.

# 1 INTRODUCTION

Substitution is in common language ‘the act of putting one thing or person in the place of another’. In consumer demand the term refers to substitution in space, replacing different goods with each other at the same time, *intratemporal substitution*, and to exchanging items of the same good (or of different goods) between periods, *intertemporal substitution*. How is substitution in the spatial sense related to substitution in the temporal sense? Can estimates obtained from data with spatial variation be applied in exploring temporal variation and vice versa? The answers to these questions are important on the one hand because elasticities of intertemporal substitution are crucial parameters in numerous macro models, with implications to, *inter alia*, the saving-consumption decision, on the other hand because within-period commodity substitution and systems of demand and substitution elasticities are core elements in numerous micro-economic and -econometric studies and policy analyses. It would be convenient if the two concepts coincide, making estimates from cross-section and time related studies potentially interchangeable.

Because of its importance for policy analysis it is not surprising that there is a substantial literature dealing with the intra-inter-temporal substitution distinction; see Kim (1993), McLaughlin (1995), Blundell (1999), Browning (2005), and Pakoš (2011), for examples, and Thimme (2016) for a literature review related to intertemporal substitution. However, as the model and assumptions, *e.g.*, with respect to parametrization, differ and are not always fully spelt out, it may be difficult to see the similarities and differences in approaches and conclusions. This paper is an attempt to give some clarifications and provide elements of a synthesis, without contrasting different *parametric* specifications. The seminal article on demand elasticities in a static, ‘non-parametric’ context, Frisch (1959), building on Frisch (1932, 1936), is a starting point, in addition to Houthakker (1960) and Theil (1975).

The paper proceeds as follows: Sections 2–4 deal with the static, one-period model, Sections 5–7 with multi-period models. Section 2 gives some basics for the static model, Section 3 surveys implied Frischian results on substitution. The resulting decompositions of the rates of increase of consumption involve rates of increase of price indexes, based on mean budget shares (the Divisia price index), and on marginal budget shares (the Frisch price index). Cases with homothetic and with additively separable utility functions are considered, the latter in Section 4. The measures of substitution motivated by this theory, often used in ‘calibration’ exercises, are strongly related to assumptions made about the underlying preferences both when it comes to (non-)additivity and to (non-)homotheticity. For the multi-period models, basics are given in Section 5. Generalized results on substitution, covering both the spatial and the temporal dimension, are considered in Section 6. Special cases imposing additive separability of the utility function over periods, across commodities and over both dimensions, and important implications, are considered in Section 7. Aggregate elasticities and some of their relationships

are considered in Section 8, while Section 9 brings cases with more restrictive additive specifications of the utility functions, which makes the model better suited to applications, in focus. Conclusions are summarized in Section 10.

## 2 ONE-PERIOD MODEL: BASICS

Let the quantity vector of the  $n$  commodities and the vector of their prices for a static, one-period model be given by  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{p} = (p_1, \dots, p_n)$ . The utility function and the marginal utilities are

$$(2.1) \quad U = U(\mathbf{x}),$$

$$(2.2) \quad u_i = u_i(\mathbf{x}) \equiv \partial U(\mathbf{x}) / \partial x_i, \quad i = 1, \dots, n.$$

Denoting the vector of marginal utilities as  $\mathbf{u} = (u_1, \dots, u_n)$ , the system (2.2) implies a system of *inverse marginal utility functions* of the form

$$(2.3) \quad x_i = h_i(\mathbf{u}), \quad i = 1, \dots, n,$$

see Frisch (1959), Eqs. (24)–(25). The  $(n \times n)$ -matrix whose element  $(i, j)$  is  $\partial h_i(\mathbf{u}) / \partial u_j$  is the inverse of the Hessian matrix of  $U(\mathbf{x})$ , *i.e.*, the  $(n \times n)$ -matrix whose element  $(i, j)$  is  $\partial u_i(\mathbf{x}) / \partial x_j$ . Following Frisch, we define<sup>1</sup>

$$u_{ij} \equiv \frac{\partial u_i(\mathbf{x})}{\partial x_j} \frac{x_j}{u_i(\mathbf{x})},$$

$$\phi_{ij} \equiv \frac{\partial h_i(\mathbf{u})}{\partial u_j} \frac{u_j}{h_i(\mathbf{u})}, \quad i, j = 1, \dots, n.$$

The first-order conditions for maximization of  $U$  subject to

$$(2.4) \quad \sum_{i=1}^n p_i x_i \equiv \mathbf{p}\mathbf{x}' = y,$$

given total expenditure (income)  $y$ , with Lagrange multiplier  $\omega$ , which can be interpreted as the marginal utility of income at optimum, or the conditions for minimization of  $y$  given  $U$ , with Lagrange multiplier  $\omega^{-1}$ , are<sup>2</sup>

$$(2.5) \quad u_i(\mathbf{x}) = \omega p_i, \quad i = 1, \dots, n.$$

From (2.5) and Young's theorem it follows that

$$(2.6) \quad p_i x_i u_{ij} = p_j x_j u_{ji},$$

$$(2.7) \quad p_i x_i \phi_{ij} = p_j x_j \phi_{ji}, \quad i, j = 1, \dots, n.$$

---

<sup>1</sup>Frisch (1959), see Eqs. (26)–(27), denotes  $u_{ij}$  and  $\phi_{ij}$  ( $x_{ij}$  in his notation) as the ‘utility accelerations’ and the ‘want elasticities’, respectively.

<sup>2</sup>Browning, Deaton and Irish (1985, pp. 507–508) give a third interpretation of the optimizing conditions, as obtained by maximization of the consumer's ‘profit function’, interpreting  $\omega^{-1}$  as the price (marginal cost) of utility.

The ordinary (Marshall), compensated (Hicks), and marginal income-utility constant (Frisch) demand functions, and the marginal utility of income function have the respective forms:<sup>3</sup>

$$(2.8) \quad x_i = f_i(\mathbf{p}, y),$$

$$(2.9) \quad x_i = g_i(\mathbf{p}, U),$$

$$(2.10) \quad x_i = h_i(\omega \mathbf{p}), \quad i = 1, \dots, n,$$

$$(2.11) \quad \omega = \omega(\mathbf{p}, y),$$

to which correspond Engel, Cournot, Slutsky and Frisch demand elasticities:

$$\begin{aligned} E_i &\equiv \frac{\partial f_i(\mathbf{p}, y)}{\partial y} \frac{y}{f_i(\mathbf{p}, y)}, & i = 1, \dots, n, \\ e_{ij} &\equiv \frac{\partial f_i(\mathbf{p}, y)}{\partial p_j} \frac{p_j}{f_i(\mathbf{p}, y)}, \\ \varepsilon_{ij} &\equiv \frac{\partial g_i(\mathbf{p}, U)}{\partial p_j} \frac{p_j}{g_i(\mathbf{p}, U)}, \\ \phi_{ij} &\equiv \frac{\partial h_i(\omega \mathbf{p})}{\partial p_j} \frac{p_j}{h_i(\omega \mathbf{p})}, & i, j = 1, \dots, n, \end{aligned}$$

and ‘marginal utility flexibilities’ (Frisch’s terminology) of income and prices:

$$\begin{aligned} \omega_y &\equiv \frac{\partial \omega(\mathbf{p}, y)}{\partial y} \frac{y}{\omega(\mathbf{p}, y)}, \\ \omega_j &\equiv \frac{\partial \omega(\mathbf{p}, y)}{\partial p_j} \frac{p_j}{\omega(\mathbf{p}, y)}, & j = 1, \dots, n. \end{aligned}$$

The first flexibility or its inverse, sometimes denoted as the *Frisch parameter*, often plays a core role in modeling substitution. The latter is also used extensively by Frisch (1959) in exploring relationships between the classical elasticities, the Frischian flexibilities and the ‘want elasticities’  $\phi_{ij}$ , which clarify the argument.

The homogeneity of (2.8) in  $(\mathbf{p}, y)$ , of (2.9) in  $\mathbf{p}$ , and of (2.11) in  $(\mathbf{p}, y)$ , of degrees, 0, 0, and  $-1$ , respectively, imply the well-known relationships:

$$(2.12) \quad \sum_{j=1}^n e_{ij} + E_i = 0,$$

$$(2.13) \quad \sum_{j=1}^n \varepsilon_{ij} = 0, \quad i = 1, \dots, n,$$

$$(2.14) \quad \sum_{j=1}^n \omega_j + \omega_y = -1.$$

The budget shares and the Hicks-Allen<sup>4</sup> and Frisch elasticities of substitution,

$$(2.15) \quad \alpha_i \equiv \frac{p_i x_i}{\mathbf{p} \mathbf{x}'},$$

$$(2.16) \quad \sigma_{ij} \equiv \frac{\varepsilon_{ij}}{\alpha_j},$$

$$(2.17) \quad \psi_{ij} \equiv \frac{\phi_{ij}}{\alpha_j},$$

---

<sup>3</sup>Reminder: (2.8) is obtained by solving for  $\mathbf{x}$  from (2.4) and (2.5) for given  $\mathbf{p}$  and  $y$ , (2.9) is obtained by solving for  $\mathbf{x}$  from (2.1) and (2.5) for given  $\mathbf{p}$  and  $U$ ; and (2.10) is obtained by solving for  $\mathbf{x}$  from (2.5) for given  $\omega$  and  $\mathbf{p}$ . The marginal utility of income constant demand functions, introduced in Frisch (1959, p.184), are labeled *Frisch demand* functions by Browning, Deaton and Irish (1985, p. 505), a term used by several authors thereafter.

<sup>4</sup>Blackorby and Russell (1981, 1989) call it the Allen elasticity of substitution; see Allen (1938, Secs. 13.7–13.8).

satisfy

$$(2.18) \quad \sigma_{ij} = \sigma_{ji} \iff \alpha_i \varepsilon_{ij} = \alpha_j \varepsilon_{ji},$$

$$(2.19) \quad \psi_{ij} = \psi_{ji} \iff \alpha_i \phi_{ij} = \alpha_j \phi_{ji},$$

denoted as Slutsky-symmetry and Frisch-symmetry, respectively:<sup>5</sup> Each set then has  $\frac{1}{2}n(n-1)$  distinct cross elasticities and  $n$  direct elasticities.

### 3 ONE-PERIOD MODEL: FRISCHIAN RESULTS ON SUBSTITUTION

Frisch (1959) demonstrates several important results on demand elasticities and commodity substitution. We recapitulate some of them and give useful extensions.

From (2.5) it follows, by taking elasticity with respect to  $y$  and to  $p_j$ , letting  $\delta_{ij}=1$  for  $i=j$  and  $=0$  for  $i \neq j$ , that

$$(3.1) \quad \sum_{j=1}^n u_{ij} E_j = \omega_y, \quad i = 1, \dots, n,$$

$$(3.2) \quad \sum_{k=1}^n u_{ik} e_{kj} = \omega_j + \delta_{ij}, \quad i, j = 1, \dots, n.$$

Solving (3.1) as one equation system in  $(E_1, \dots, E_n)$  and (3.2) as  $n$  equation systems in  $(e_{1j}, \dots, e_{nj})$  ( $j = 1, \dots, n$ ), utilizing (2.7) and the relationships between the derivatives of (2.2) and (2.3), we obtain

$$(3.3) \quad E_i = \omega_y \sum_{k=1}^n \phi_{ik}, \quad i = 1, \dots, n,$$

$$(3.4) \quad e_{ij} = \phi_{ij} + \omega_j \sum_{k=1}^n \phi_{ik} = \phi_{ij} + \frac{\omega_j}{\omega_y} E_i, \quad i, j = 1, \dots, n.$$

Equations (3.1)–(3.4) correspond to Frisch (1959), Eqs. (42)–(45). From (3.3), (3.4), (2.16) and the Slutsky-equation<sup>6</sup>

$$(3.5) \quad e_{ij} = \varepsilon_{ij} - \alpha_j E_i \equiv \alpha_j (\sigma_{ij} - E_i), \quad i, j = 1, \dots, n,$$

it follows that

$$(3.6) \quad \varepsilon_{ij} = \phi_{ij} + (\omega_j + \alpha_j \omega_y) \sum_{k=1}^n \phi_{ik} = \phi_{ij} + \left( \frac{\omega_j}{\omega_y} + \alpha_j \right) E_i, \quad i, j = 1, \dots, n.$$

Furthermore, (2.4) implies:<sup>7</sup>

$$(3.7) \quad \sum_{i=1}^n \alpha_i = 1,$$

$$(3.8) \quad \sum_{i=1}^n \alpha_i E_i \equiv \sum_{i=1}^n \gamma_i = 1,$$

$$(3.9) \quad \sum_{i=1}^n \alpha_i e_{ij} = -\alpha_j, \quad j = 1, \dots, n,$$

where  $\gamma_i = \partial(p_i x_i) / \partial y \equiv \alpha_i E_i$  is the marginal budget share of commodity  $i$ . Corresponding expressions for the Slutsky- and Frisch-elasticities are:<sup>8</sup>

<sup>5</sup>Reminder: Let  $C(\mathbf{p}, U) \equiv \sum_i p_i g_i(\mathbf{p}, U)$  be the cost function, and  $C_i$  and  $C_{ij}$  its derivatives with respect to  $p_i$  and  $(p_i, p_j)$ . Since Young's theorem and Shephard's Lemma imply  $C_{ij} = C_{ji}$  and  $g_i(\mathbf{p}, U) = C_i$ , (2.18) follows because  $\alpha_j = p_j C_j / C$  and  $\varepsilon_{ij} = C_{ij} p_j / C_i \implies \sigma_{ij} = \varepsilon_{ij} / \alpha_j = C_{ij} C / (C_i C_j) = \sigma_{ji}$ , while (2.19) is implied by (2.7). See also Uzawa (1962), Blackorby and Russell (1981), Eq. (1), and Blackorby and Russell (1989), Eq. (3).

<sup>6</sup>On the origin of the Slutsky equation, its history and its dissemination, see Slutsky (1915), Barnett (2007), and Bjerkholt (2015).

<sup>7</sup>Reminder: (3.7) is implied by (2.4) and (2.15), (3.8) and (3.9) follow by taking the elasticity of  $\sum_i p_i f_i(\mathbf{p}, y) = y$  with respect to  $y$  and  $p_j$ , respectively.

<sup>8</sup>Reminder: (3.10) follows from (2.18) and (2.13); (3.11) follows from (3.6) and (3.10).



$$(3.10) \quad \sum_{i=1}^n \alpha_i \varepsilon_{ij} = 0,$$

$$(3.11) \quad \sum_{i=1}^n \alpha_i \phi_{ij} = - \left( \frac{\omega_j}{\omega_y} + \alpha_j \right), \quad j = 1, \dots, n.$$

From (2.19), (3.3) and (3.11) it follows that

$$(3.12) \quad \omega_j + \alpha_j \omega_y = -\alpha_j E_j \equiv -\gamma_j,$$

$$(3.13) \quad \sum_{i=1}^n \alpha_i \phi_{ij} = \alpha_j \frac{E_j}{\omega_y}, \quad j = 1, \dots, n,$$

(3.12) corresponding to Frisch (1959), Eq. (50). Expressions connecting  $(e_{it}, \varepsilon_{ij})$  to  $(\phi_{ij}, \omega_y, \alpha_j, E_j)$  follow by inserting (3.12) in (3.4) and (3.6), which gives, see Frisch (1959), Eqs. (52)–(53):<sup>9</sup>

$$(3.14) \quad e_{ij} = \phi_{ij} - \alpha_j E_i \left( \frac{E_j}{\omega_y} + 1 \right),$$

$$(3.15) \quad \varepsilon_{ij} = \phi_{ij} - \alpha_j \frac{E_j E_i}{\omega_y} \iff \sigma_{ij} = \psi_{ij} - \frac{E_i E_j}{\omega_y}, \quad i, j = 1, \dots, n.$$

These equations are basic to the following argument.<sup>10</sup>

Following Theil (1975, Section 1.4) and Houthakker (1960, footnote 8),<sup>11</sup> the Hicks-Allen elasticity  $\sigma_{ij}$  is the *total substitution effect* and its components,  $\psi_{ij}$  and

$$(3.16) \quad S_{ij} \equiv \frac{E_i E_j}{\omega_y},$$

are the *specific and general substitution effects*.<sup>12</sup> Then (3.14) and (3.15) read

$$(3.17) \quad e_{ij} = \phi_{ij} - \alpha_j (S_{ij} + E_i) \equiv \alpha_j (\psi_{ij} - S_{ij} - E_i),$$

$$(3.18) \quad \varepsilon_{ij} = \phi_{ij} - \alpha_j S_{ij} \iff \sigma_{ij} = \psi_{ij} - S_{ij}.$$

We will denote  $S_{ij}$  as the *Houthakker-Theil general (HTG) substitution elasticity*. To summarize:

**Proposition 1:**

1. The HTG elasticities are connected to  $\omega_y^{-1}$  by  $S_{ij} = E_i E_j \omega_y^{-1}$ , which under homotheticity ( $E_i = 1$ ) simply gives  $S_{ij} = \omega_y^{-1}$  for all  $i$  and  $j$ .
2. No one-to-one relationships in general exist between the Frisch elasticities and  $\omega_y^{-1}$ , for given  $E_i$ . From (3.3) is seen that  $\phi_{i1}, \dots, \phi_{in}$  are related to  $\omega_y^{-1}$  via  $\sum_{j=1}^n \phi_{ij} = E_i \omega_y^{-1}$ , which because of (3.8) implies

$$(i) \quad \sum_{i=1}^n \sum_{j=1}^n \alpha_i \phi_{ij} = \omega_y^{-1} \text{ in general,}$$

$$(ii) \quad \sum_{j=1}^n \phi_{ij} = \omega_y^{-1} \text{ for all } i, \text{ under homotheticity.}$$

<sup>9</sup>Equations corresponding to (3.13) and (3.4), using income and price derivatives instead of elasticities, can be found in *e.g.* Theil (1975), Eqs. (3.9) and (3.14), in matrix notation, the last in scalar notation as Eq. (4.1).

<sup>10</sup>Versions of (3.15) have been frequently used in the literature as a kind of a ‘vehicle’ in discussing of (intra- and/or intertemporal) substitution in consumer demand; see *e.g.*, Theil (1975), Eq. (4.3), Browning, Deaton and Irish (1985), Eq. (1.16), McLaughlin (1995), Eq. (3), and Blundell (1999), Eq. (1.1).

<sup>11</sup>See also Barten (1964, Section 2.1).

<sup>12</sup>Theil (1975), in Eqs. (4.3), (4.5), and (4.7), uses derivatives instead of elasticities.

As is well known, however, homothetic preferences is an extreme case, inconsistent with well-established empirical regularities in consumer behaviour, *e.g.*, Engel's Law.

From (3.7), (3.8) and (3.13)–(3.16), it follows that

$$(3.19) \quad \sum_{i=1}^n \sum_{j=1}^n \alpha_i e_{ij} = -1,$$

$$(3.20) \quad \sum_{i=1}^n \sum_{j=1}^n \alpha_i \varepsilon_{ij} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{ij} = 0,$$

$$(3.21) \quad \sum_{i=1}^n \sum_{j=1}^n \alpha_i \phi_{ij} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \psi_{ij} = \omega_y^{-1},$$

$$(3.22) \quad \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j S_{ij} = \omega_y^{-1},$$

where (3.21) corresponds to Frisch (1959), Eq. (47). Hence, the *overall* budget-share-weighted Hicks-Allen, Frisch and HTG elasticities are 0,  $\omega_y^{-1}$ , and  $\omega_y^{-1}$ , respectively.

Arguably, (3.20)–(3.22) do not strictly measure substitution, since an appropriate measure should *only include cross-commodity terms* ( $i \neq j$ ). Now

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j \sigma_{ij} &\equiv - \sum_{i=1}^n \alpha_i^2 \sigma_{ii}, \\ \sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j \psi_{ij} &\equiv \omega_y^{-1} - \sum_{i=1}^n \alpha_i^2 \psi_{ii}, \\ \sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j S_{ij} &\equiv \omega_y^{-1} - \sum_{i=1}^n \alpha_i^2 S_{ii}, \end{aligned}$$

so that (2.15)–(2.17) imply:

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \alpha_i (\varepsilon_{ij} - \alpha_j \varepsilon_{jj}) &\equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{ij} - \sum_{j=1}^n \alpha_j^2 \sigma_{jj} \\ &\equiv \sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j \sigma_{ij}, \\ \sum_{i=1}^n \sum_{j=1}^n \alpha_i (\phi_{ij} - \alpha_j \phi_{jj}) &\equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \psi_{ij} - \sum_{j=1}^n \alpha_j^2 \psi_{jj} \\ &\equiv \sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j \psi_{ij}, \\ \sum_{i=1}^n \sum_{j=1}^n \alpha_i [\alpha_j (S_{ij} - \alpha_j S_{jj})] &\equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j S_{ij} - \sum_{j=1}^n \alpha_j^2 S_{jj} \\ &\equiv \sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j S_{ij}. \end{aligned}$$

Hence, since  $\sum_{i=1}^n \alpha_i^2 \sigma_{ii} \equiv \sum_{i=1}^n (\psi_{ii} \alpha_i^2 - \omega_y^{-1} \gamma_i^2)$  and  $\sum_{i=1}^n \alpha_i^2 S_{ii} \equiv \omega_y^{-1} \sum_{i=1}^n \gamma_i^2$ , it follows from (3.20)–(3.22), with ‘modified substitution elasticities’ defined as

$$(3.23) \quad \sigma_{ij}^* \equiv \sigma_{ij} - \alpha_j \sigma_{jj},$$

$$(3.24) \quad \psi_{ij}^* \equiv \psi_{ij} - \alpha_j \psi_{jj},$$

$$(3.25) \quad S_{ij}^* \equiv S_{ij} - \alpha_j S_{jj},$$

that

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{ij}^* &\equiv - \sum_{i=1}^n \alpha_i^2 \sigma_{ii} \equiv \sum_{i=1}^n (\gamma_i^2 \omega_y^{-1} - \alpha_i^2 \psi_{ii}), \\ \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \psi_{ij}^* &\equiv \omega_y^{-1} - \sum_{i=1}^n \alpha_i^2 \psi_{ii}, \\ \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j S_{ij}^* &\equiv \omega_y^{-1} - \sum_{i=1}^n \alpha_i^2 S_{ii} \equiv (1 - \sum_{i=1}^n \gamma_i^2) \omega_y^{-1}. \end{aligned}$$

Equation (3.23) is related to the transformation which takes Slutsky and Allen elasticities into ‘Morishima elasticities’, defined as  $m_{ji} = \varepsilon_{ij} - \varepsilon_{jj} \equiv \alpha_j (\sigma_{ij} - \sigma_{jj})$ ; see Blackorby and Russell (1981), Eqs. (1)–(4), Blackorby and Russell (1989), Eq. (8), and McFadden (1963). While  $m_{ji}$  ‘nets out’ the direct effect  $\sigma_{jj}$  by a simple deduction,  $\sigma_{ij}^*$  weights this effect by the budget share  $\alpha_j$ . The severe limitations of

(Hicks-)Allen elasticities of substitution (AES)  $\sigma_{ij}$  ( $i \neq j$ ) for  $n > 2$ , pointed out by Blackorby and Russell (1981, Sections 1–2; 1989), should, however, be recalled:

“...while the AES reduces to the original Hicksian concept in the two-dimensional case, in general it preserves none of the salient properties of the Hicksian notion....the Allen elasticity of substitution (i) is not a measure of the “ease” of substitution, or curvature of the isoquant, (ii) provides no information about relative factor shares, and (iii) cannot be interpreted as a (logarithmic) derivative of a quantity ratio with respect to a price ratio (or the marginal rate of substitution). As a quantitative measure, it has no meaning; as a qualitative measure, it adds no information to that contained in the (constant output) cross-price elasticity.....the elasticity-of-substitution concept, as originally conceived by Hicks, has nothing to do with the substitute/complement taxonomy”. [Blackorby and Russell (1989, pp. 882–884)].

Since  $\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j = 1$ , overall Hicks-Allen, Frisch, and HTG *cross*-elasticities can be defined as the following means of the  $n^2$  values of  $\sigma_{ij}^*$ ,  $\psi_{ij}^*$ ,  $S_{ij}^*$ :

$$(3.26) \quad \bar{\sigma}^* = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{ij}^* \equiv - \sum_{i=1}^n \alpha_i^2 \sigma_{ii} \equiv \sum_{i=1}^n (\gamma_i^2 \omega_y^{-1} - \alpha_i^2 \psi_{ii}),$$

$$(3.27) \quad \bar{\psi}^* = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \psi_{ij}^* \equiv \omega_y^{-1} - \sum_{i=1}^n \alpha_i^2 \psi_{ii},$$

$$(3.28) \quad \bar{S}^* = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j S_{ij}^* \equiv \omega_y^{-1} - \sum_{i=1}^n \alpha_i^2 S_{ii} \equiv (1 - \sum_{i=1}^n \gamma_i^2) \omega_y^{-1}.$$

Alternative measures are the means of the  $n(n-1)$  values of  $\sigma_{ij}$ ,  $\psi_{ij}$ ,  $S_{ij}$  ( $i \neq j$ ), after renormalization, using  $\sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j \equiv 1 - \sum_{i=1}^n \alpha_i^2$ :

$$(3.29) \quad \bar{\sigma} = \frac{\sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j \sigma_{ij}}{\sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j} \equiv \frac{\sum_{i=1}^n (\gamma_i^2 \omega_y^{-1} - \alpha_i^2 \psi_{ii})}{1 - \sum_{i=1}^n \alpha_i^2},$$

$$(3.30) \quad \bar{\psi} = \frac{\sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j \psi_{ij}}{\sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j} \equiv \omega_y^{-1} + \frac{\sum_{i=1}^n \alpha_i^2 (\omega_y^{-1} - \psi_{ii})}{1 - \sum_{i=1}^n \alpha_i^2},$$

$$(3.31) \quad \bar{S} = \frac{\sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j S_{ij}}{\sum_{i=1}^n \sum_{j=1, \neq i}^n \alpha_i \alpha_j} \equiv \omega_y^{-1} + \frac{\sum_{i=1}^n (\alpha_i^2 - \gamma_i^2) \omega_y^{-1}}{1 - \sum_{i=1}^n \alpha_i^2}.$$

The aggregate Houthakker-Theil decompositions corresponding to (3.18) thus read:

$$\begin{aligned} \bar{\sigma}^* &= \bar{\psi}^* - \bar{S}^*, \\ \bar{\sigma} &= \bar{\psi} - \bar{S}. \end{aligned}$$

To summarize:

**Proposition 2:**

1. While any  $\sigma_{ij}$  depends on  $\omega_y^{-1}$ , the latter is no aggregate indicator of Hicks-Allen substitution. We have:  $\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{ij} = 0$ , while  $\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \psi_{ij} = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j S_{ij} = \omega_y^{-1}$ . The overall cross-substitution elasticities satisfy (3.26)–(3.31).
2. If  $n$  is large,  $\sum_i \alpha_i^2$ ,  $\sum_i \gamma_i^2$  are small, and  $\psi_{ii}$  is finite, then  $(\bar{\psi}^*, \bar{\psi}) \approx \omega_y^{-1}$ ,  $(\bar{S}^*, \bar{S}) \approx \omega_y^{-1}$ , and  $(\bar{\sigma}^*, \bar{\sigma}) = (\bar{\psi}^* - \bar{S}^*, \bar{\psi} - \bar{S}) \approx 0$ .

*Example:* The stylized case with homotheticity and equal budget shares:  $\alpha_i = \frac{1}{n}$ ,  $E_i = 1 \implies \sum \alpha_i^2 = \sum \gamma_i^2 = \frac{1}{n}$ , letting  $\tilde{\psi} = \frac{1}{n} \sum \psi_{ii}$ , gives

$$\begin{aligned}\bar{\sigma}^* &= \frac{1}{n}(\omega_y^{-1} - \tilde{\psi}), & \bar{\sigma} &= \frac{1}{n-1}(\omega_y^{-1} - \tilde{\psi}), \\ \bar{\psi}^* &= \omega_y^{-1} - \frac{1}{n}\tilde{\psi}, & \bar{\psi} &= \frac{n}{n-1}(\omega_y^{-1} - \frac{1}{n}\tilde{\psi}), \\ \bar{S}^* &= \frac{n-1}{n}\omega_y^{-1}, & \bar{S} &= \omega_y^{-1}.\end{aligned}$$

Interesting byproducts of these results, useful for doing *comparative statics*, follow by inserting for  $e_{ij}$  from (3.5) and (3.15) in the general expression for the *rate of increase of  $x_i$*  induced by arbitrary changes in income and prices. We first get

$$\frac{dx_i}{x_i} = E_i \frac{dy}{y} + \sum_{j=1}^n e_{ij} \frac{dp_j}{p_j} \equiv E_i \left( \frac{dy}{y} - \sum_{j=1}^n \alpha_j \frac{dp_j}{p_j} \right) + \sum_{j=1}^n \varepsilon_{ij} \frac{dp_j}{p_j},$$

and next

$$\begin{aligned}(3.32) \quad \frac{dx_i}{x_i} &= E_i \left( \frac{dy}{y} - \sum_{j=1}^n \alpha_j \frac{dp_j}{p_j} \right) + \sum_{j=1}^n \left( \phi_{ij} - \alpha_j \frac{E_i E_j}{\omega_y} \right) \frac{dp_j}{p_j} \\ &\equiv E_i \left( \frac{dy}{y} - \frac{dP_\alpha}{P_\alpha} \right) + \sum_{j=1}^n \phi_{ij} \frac{dp_j}{p_j} - \frac{E_i}{\omega_y} \frac{dP_\gamma}{P_\gamma},\end{aligned}$$

where  $dP_\alpha/P_\alpha$  and  $dP_\gamma/P_\gamma$  are the changes in price indexes which weight commodity prices by, respectively, average and marginal budget shares:

$$\frac{dP_\alpha}{P_\alpha} = \sum_{j=1}^n \alpha_j \frac{dp_j}{p_j}, \quad \frac{dP_\gamma}{P_\gamma} = \sum_{j=1}^n \gamma_j \frac{dp_j}{p_j}.$$

The former is (the rate on increase of) the *Divisia price index*, the latter is often denoted as (the rate of increase of) the *Frisch price index*, after Frisch (1932; pp. 74–82) and Frisch (1936).<sup>13</sup> Using (3.13), we finally rewrite (3.32) as

$$\begin{aligned}(3.33) \quad \frac{dx_i}{x_i} &= E_i \left( \frac{dy}{y} - \frac{dP_\alpha}{P_\alpha} \right) + \sum_{j=1}^n \phi_{ij} \left( \frac{dp_j}{p_j} - \frac{dP_\alpha}{P_\alpha} \right) + \frac{E_i}{\omega_y} \left( \frac{dP_\alpha}{P_\alpha} - \frac{dP_\gamma}{P_\gamma} \right) \\ &\equiv E_i \left( \frac{dy}{y} - \frac{dP_\alpha}{P_\alpha} \right) + \phi_{ii} \left( \frac{dp_i}{p_i} - \frac{dP_\alpha}{P_\alpha} \right) \\ &\quad + \sum_{j=1, \neq i}^n \phi_{ij} \left( \frac{dp_j}{p_j} - \frac{dP_\alpha}{P_\alpha} \right) + \frac{E_i}{\omega_y} \left( \frac{dP_\alpha}{P_\alpha} - \frac{dP_\gamma}{P_\gamma} \right),\end{aligned}$$

which decomposes  $dx_i/x_i$  into a real income effect,  $n$  real price effects, one direct and  $n-1$  cross-effects, and a ‘correction’ term reflecting the difference between the Divisia and the Frisch price indexes:

$$\frac{dP_\alpha}{P_\alpha} - \frac{dP_\gamma}{P_\gamma} \equiv \sum_j \alpha_j (1 - E_j) \frac{dp_j}{p_j}.$$

Equivalently, using  $\sum_{j=1}^n \phi_{ij} = E_i \omega_y^{-1}$  and rearranging, we get

<sup>13</sup>See also Theil (1975; Sections 5.1 and 5.2) and Barnett and Serletis (2008).

$$(3.34) \quad \frac{dx_i}{x_i} = E_i \left( \frac{dy}{y} - \frac{dP_\alpha}{P_\alpha} \right) + \frac{E_i}{\omega_y} \left( \frac{dp_i}{p_i} - \frac{dP_\gamma}{P_\gamma} \right) + \sum_{j=1}^n \phi_{ij} \left( \frac{dp_j}{p_j} - \frac{dp_i}{p_i} \right).$$

This expression decomposes  $dx_i/x_i$  into a real income effect (based on the Divisia price index), a real price effect (based on the Frisch price index) and  $n-1$  cross-price effects. Both the real price effects and the cross-price effects involve  $\omega_y^{-1}$ , the latter via  $\sum_{j=1}^n \phi_{ij} = E_i \omega_y^{-1}$ . Imposing homotheticity, and thus eliminating ‘scale effects’, the expression simplifies to

$$\frac{dx_i}{x_i} = \left( \frac{dy}{y} - \frac{dP_\alpha}{P_\alpha} \right) + \omega_y^{-1} \left( \frac{dp_i}{p_i} - \frac{dP_\alpha}{P_\alpha} \right) + \sum_{j=1}^n \phi_{ij} \left( \frac{dp_j}{p_j} - \frac{dp_i}{p_i} \right),$$

which shows succinctly the roles of  $\omega_y^{-1}$  and the  $\phi_{ij}$ s in describing substitution.

#### 4 ONE-PERIOD MODEL AND SUBSTITUTION: ADDITIVE UTILITY

Additively separable utility, ‘want independence’ in the terminology of Frisch, is often assumed in empirical demand analysis and makes the Hessian matrix of  $U(\mathbf{x})$  diagonal. On the ordinality-cardinality distinction and the imposition of additive utility to save parameters, he says:

“By making assumptions about want independence, we introduce concepts that are not invariant under a general monotonic transformation of the utility indicator. I am not very concerned about this.....By introspection we are frequently in a position to say something fairly definite and plausible about the non-invariant concepts, for instance about want-independence.....From assumptions as to want-independence there follow very definite conclusions about certain observable demand phenomena. Assumptions of this sort can therefore be classified as refutable hypotheses..... I shall therefore not plead guilty of heresy even if I do work with choice-theory concepts that are not invariant under a general monotonic transformation of the utility indicator. At any subsequent step one can, if need be, verify whether any derived expression is invariant or not.” [Frisch (1959, pp.177–178)]

If (2.1) has this form, *i.e.*,

$$(4.1) \quad U = U(\mathbf{x}) = \sum_{i=1}^n U_i(x_i),$$

(2.2), (2.3), and (2.5) simplify to

$$(4.2) \quad u_i = u_i(x_i) \equiv dU_i(x_i)/dx_i,$$

$$(4.3) \quad x_i = h_i(u_i) = u_i^{-1}(u_i),$$

$$(4.4) \quad u_i(x_i) = \omega p_i, \quad i = 1, \dots, n,$$

and  $u_{ij} = \delta_{ij} u_{ii}$ ,  $\phi_{ij} = \delta_{ij} \phi_{ii}$  and  $\phi_{ii} = 1/u_{ii}$ . In the Houthakker-Theil usage, *all ‘specific cross-substitution’ is eliminated, i.e.*, Frischian cross-substitution vanishes. Then (3.1)–(3.4) simplify to

$$(4.5) \quad u_{ii} E_i = \omega_y \quad \iff \quad E_i = \omega_y \phi_{ii}, \quad i = 1, \dots, n,$$

$$(4.6) \quad u_{ii} e_{ij} = \omega_j + \delta_{ij} \quad \iff \quad e_{ij} = (\omega_j + \delta_{ij}) \phi_{ii}, \quad i, j = 1, \dots, n,$$

while

$$(4.7) \quad \phi_{ij} = \frac{\delta_{ij} E_i}{\omega_y} \iff \psi_{ij} = \frac{\delta_{ij} E_i}{\alpha_j \omega_y}, \quad i, j = 1, \dots, n,$$

so that (3.14) and (3.15) simplify to

$$(4.8) \quad e_{ij} = \frac{E_i}{\omega_y} [\delta_{ij} - \alpha_j E_j] - \alpha_j E_i,$$

$$(4.9) \quad \varepsilon_{ij} = \frac{E_i}{\omega_y} [\delta_{ij} - \alpha_j E_j] \iff \sigma_{ij} = \frac{E_i}{\omega_y} \left[ \frac{\delta_{ij}}{\alpha_j} - E_j \right].$$

Further, (3.21) simplifies to

$$(4.10) \quad \sum_{i=1}^n \alpha_i \phi_{ii} \equiv \sum_{i=1}^n \alpha_i^2 \psi_{ii} = \omega_y^{-1},$$

and since  $\psi_{ii} = E_i \omega_y^{-1} / \alpha_i$ , (3.26) and (3.29) simplify to

$$(4.11) \quad \bar{\sigma}^* = -(1 - \sum_{i=1}^n \gamma_i^2) \omega_y^{-1},$$

$$(4.12) \quad \bar{\sigma} = -\frac{1 - \sum_{i=1}^n \gamma_i^2}{1 - \sum_{i=1}^n \alpha_i^2} \omega_y^{-1}.$$

The results are summarized in:

**Proposition 3:**

1. *Additive utility, implying  $\phi_{ij} = 0$  ( $i \neq j$ ), gives a one-to-one relationship between the direct Frisch elasticities and  $\omega_y^{-1}$ :  $\phi_{ii} = E_i \omega_y^{-1}$ . Hence,*
  - (i)  $\sigma_{ij} = -E_i E_j \omega_y^{-1}$  ( $i \neq j$ ) under additivity,
  - (ii)  $\sum_{i=1}^n \alpha_i \phi_{ii} = \omega_y^{-1}$  under additivity,
  - (iii)  $\phi_{ii} = \omega_y^{-1}$ ,  $\sigma_{ij} = -\omega_y^{-1}$  ( $i \neq j$ ) under additivity and homotheticity.
2. *Since then the aggregate Frisch cross-elasticities equal  $(\bar{\psi}^*, \bar{\psi}) = 0$ , their aggregate Hicks-Allen counterparts become  $(\bar{\sigma}^*, \bar{\sigma}) = (\bar{\psi}^* \bar{S}^*, \bar{\psi} \bar{S}) \approx -\omega_y^{-1}$ .*

*Example:* The stylized case with additivity, homotheticity and equal budget shares:  $\alpha_i = \frac{1}{n}$ ,  $E_i = 1 \implies \sum \alpha_i^2 = \sum \gamma_i^2 = \frac{1}{n}$ ,  $\psi_{ii} = \bar{\psi} = n \omega_y$ , gives

$$\begin{aligned} \bar{\sigma}^* &= -\frac{n-1}{n} \omega_y^{-1}, & \bar{\sigma} &= -\omega_y^{-1}, \\ \bar{\psi}^* &= 0, & \bar{\psi} &= 0, \\ \bar{S}^* &= \frac{n-1}{n} \omega_y^{-1}, & \bar{S} &= \omega_y^{-1}. \end{aligned}$$

Imposing additivity, *i.e.*,  $\phi_{ij} = \delta_{ij} \phi_{ii}$ , (3.34) is simplified to

$$(4.13) \quad \frac{dx_i}{x_i} = E_i \left( \frac{dy}{y} - \frac{dP_\alpha}{P_\alpha} \right) + \frac{E_i}{\omega_y} \left( \frac{dp_i}{p_i} - \frac{dP_\gamma}{P_\gamma} \right),$$

which says that the real price and the real income elasticities are proportional across  $i$ , with factor of proportionality  $\omega_y^{-1}$ . If also homotheticity prevails, we get simply

$$\frac{dx_i}{x_i} = \frac{dy}{y} - \frac{dP_\alpha}{P_\alpha} + \omega_y^{-1} \left( \frac{dp_i}{p_i} - \frac{dP_\alpha}{P_\alpha} \right),$$

which again elucidates the role of  $\omega_y^{-1}$  in describing substitution.<sup>14</sup> The changed interpretation of  $(\bar{\sigma}^*, \bar{\sigma})$  when additivity is imposed, is reflected in the disappearance of the summation term in the expression for  $dx_i/x_i$ .

<sup>14</sup>In the (highly degenerate) case where also  $\omega_y = -1$  (exemplified by the Cobb-Douglas utility function), the latter equation simply gives  $dx_i/x_i = dy/y - dp_i/p_i$  for all  $i$  (implying price and income invariant budget shares).

## 5 MULTI-PERIOD-COMMODITY MODEL. BASICS

The second part of the paper will be concerned with combined intra-intertemporal substitution, formalized as an extension of the model in Section 2 to a  $T$ -period,  $n$ -commodity model, imagining a consumer in period 1, who decides on consumption in this period and makes plans for periods  $2, \dots, T$ . The quantity vector of the  $n$  commodities and the vector of their marginal utilities and prices for period  $t$  are, respectively,  $\mathbf{x}_t = (x_{1t}, \dots, x_{nt})$ ,  $\mathbf{u}_t = (u_{1t}, \dots, u_{nt})$  and  $\mathbf{p}_t = (p_{1t}, \dots, p_{nt})$ . Further, we define  $\bar{\mathbf{p}}_t = \beta_t \mathbf{p}_t$ , where  $\beta_t$  is the discounting factor for period  $t$  ( $\beta_1 = 1$ ) and the composite  $nT$ -vectors  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_T)$ ,  $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_T)$ ,  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_T)$ ,  $\bar{\mathbf{p}} = (\bar{\mathbf{p}}_1, \dots, \bar{\mathbf{p}}_T)$ . The intertemporal utility function, the marginal utilities and their inverses, the counterparts to (2.1)–(2.3), can then be expressed as, respectively:

$$(5.1) \quad U = U(\mathbf{x}_1, \dots, \mathbf{x}_T) \equiv U(\mathbf{x}),$$

$$(5.2) \quad u_{it} = u_{it}(\mathbf{x}) \equiv \partial U(\mathbf{x}) / \partial x_{it},$$

$$(5.3) \quad x_{it} = h_{it}(\mathbf{u}), \quad i = 1, \dots, n; t = 1, \dots, T,$$

giving

$$u_{it,j_s} \equiv \frac{\partial u_{it}(\mathbf{x})}{\partial x_{j_s}} \frac{x_{j_s}}{u_{it}(\mathbf{x})},$$

$$\phi_{it,j_s} \equiv \frac{\partial h_{it}(\mathbf{u})}{\partial u_{j_s}} \frac{u_{j_s}}{h_{it}(\mathbf{u})}, \quad i, j = 1, \dots, n; t, s = 1, \dots, T.$$

The first-order conditions for maximization of  $U$  subject to

$$(5.4) \quad \sum_{t=1}^T \sum_{i=1}^n \bar{p}_{it} x_{it} \equiv \sum_{t=1}^T \bar{\mathbf{p}}_t \mathbf{x}'_t = W,$$

for given wealth  $W$  (the multi-period counterpart to  $y$ ), with Lagrange multiplier  $\Omega$  (the multi-period counterpart to  $\omega$ ), are

$$(5.5) \quad u_{it}(\mathbf{x}) = \Omega \bar{p}_{it}, \quad i = 1, \dots, n; t = 1, \dots, T.$$

It follows from (5.5) and Young's theorem that (2.6) and (2.7) are generalized to

$$(5.6) \quad \bar{p}_{it} x_{it} u_{it,j_s} = \bar{p}_{j_s} x_{j_s} u_{j_s,it},$$

$$(5.7) \quad \bar{p}_{it} x_{it} \phi_{it,j_s} = \bar{p}_{j_s} x_{j_s} \phi_{j_s,it}, \quad i, j = 1, \dots, n; t, s = 1, \dots, T.$$

The multi-period Marshall, Hicks, and Frisch demand functions and the marginal utility of wealth function have the respective forms:

$$(5.8) \quad x_{it} = f_{it}(\bar{\mathbf{p}}, W),$$

$$(5.9) \quad x_{it} = g_{it}(\bar{\mathbf{p}}, U),$$

$$(5.10) \quad x_{it} = h_{it}(\Omega \bar{\mathbf{p}}), \quad i = 1, \dots, n; t = 1, \dots, T,$$

$$(5.11) \quad \Omega = \Omega(\bar{\mathbf{p}}, W),$$

to which correspond Engel, Cournot, Slutsky, and Frisch elasticities:

$$\begin{aligned}
E_{it} &\equiv \frac{\partial f_{it}(\bar{\mathbf{p}}, W)}{\partial W} \frac{W}{f_{it}(\bar{\mathbf{p}}, W)}, & i = 1, \dots, n; t = 1, \dots, T, \\
e_{it,js} &\equiv \frac{\partial f_{it}(\bar{\mathbf{p}}, W)}{\partial \bar{p}_{js}} \frac{\bar{p}_{js}}{f_{it}(\bar{\mathbf{p}}, W)}, \\
\varepsilon_{it,js} &\equiv \frac{\partial g_{it}(\bar{\mathbf{p}}, U)}{\partial \bar{p}_{js}} \frac{\bar{p}_{js}}{g_{it}(\bar{\mathbf{p}}, U)}, \\
\phi_{it,js} &\equiv \frac{\partial h_{it}(\Omega \bar{\mathbf{p}})}{\partial \bar{p}_{js}} \frac{\bar{p}_{js}}{h_{it}(\Omega \bar{\mathbf{p}})}, & i, j = 1, \dots, n; t, s = 1, \dots, T,
\end{aligned}$$

and marginal utility flexibilities of wealth and prices:

$$\begin{aligned}
\Omega_W &\equiv \frac{\partial \Omega(\bar{\mathbf{p}}, W)}{\partial W} \frac{W}{\Omega(\bar{\mathbf{p}}, W)}, \\
\Omega_{js} &\equiv \frac{\partial \Omega(\bar{\mathbf{p}}, W)}{\partial \bar{p}_{js}} \frac{\bar{p}_{js}}{\Omega(\bar{\mathbf{p}}, W)}, & j = 1, \dots, n; s = 1, \dots, T.
\end{aligned}$$

The inverse of the first flexibility frequently occurs in modeling intertemporal substitution. We will denote it as the *generalized Frisch parameter*. The homogeneity of (5.8) in  $(\bar{\mathbf{p}}, W)$ , of (5.9) in  $(\bar{\mathbf{p}}, W)$  and of (5.11) in  $\bar{\mathbf{p}}$ , of degrees 0, 0, and  $-1$ , respectively, imply, as counterparts to (2.12)–(2.14),

$$(5.12) \quad \sum_{j=1}^n \sum_{s=1}^T e_{it,js} + E_{it} = 0,$$

$$(5.13) \quad \sum_{s=1}^T \sum_{j=1}^n \varepsilon_{it,js} = 0, \quad i = 1, \dots, n; t = 1, \dots, T,$$

$$(5.14) \quad \sum_{j=1}^n \sum_{s=1}^T \Omega_{js} + \Omega_W = -1.$$

The multi-period budget shares and the Hicks-Allen and Frisch elasticities of (intra- and intertemporal) substitution become, respectively,

$$(5.15) \quad \alpha_{it} = \frac{\bar{p}_{it} x_{it}}{\bar{\mathbf{p}} \mathbf{x}'},$$

$$(5.16) \quad \sigma_{it,js} \equiv \frac{\varepsilon_{it,js}}{\alpha_{js}},$$

$$(5.17) \quad \psi_{it,js} \equiv \frac{\phi_{it,js}}{\alpha_{js}}, \quad i, j = 1, \dots, n; t, s = 1, \dots, T,$$

while generalized Slutsky-symmetry and Frisch-symmetry are expressed by

$$(5.18) \quad \sigma_{it,js} = \sigma_{js,it} \iff \alpha_{it} \varepsilon_{it,js} = \alpha_{js} \varepsilon_{js,it},$$

$$(5.19) \quad \psi_{it,js} = \psi_{js,it} \iff \alpha_{it} \phi_{it,js} = \alpha_{js} \phi_{js,it}.$$

Each set has  $nT$  direct elasticities and  $\frac{1}{2}nT(nT-1)$  cross-elasticities. As will be explained in Section 7, this number is reduced when restrictions are imposed on the utility function.

## 6 MULTI-PERIOD MODEL: FRISCHIAN RESULTS ON SUBSTITUTION

The multi-period counterparts to (3.1) and (3.2), obtained from (5.5), become

$$(6.1) \quad \sum_{j=1}^n \sum_{s=1}^T u_{it,js} E_{js} = \Omega_W,$$

$$(6.2) \quad \sum_{k=1}^n \sum_{r=1}^T u_{it,kr} e_{kr,js} = \Omega_{js} + \delta_{ij} \delta_{ts}, \quad i, j = 1, \dots, n; t, s = 1, \dots, T.$$



The former gives one equation system in  $(E_{11}, \dots, E_{n1}, \dots, E_{1T}, \dots, E_{nT})$ , the latter gives  $nT$  systems in  $(e_{11,js}, \dots, e_{n1,js}, \dots, e_{1T,js}, \dots, e_{nT,js})$  ( $j=1, \dots, n; s=1, \dots, T$ ). Solving (6.1) and (6.2), utilizing (5.7) and the relationships between the derivatives of (5.2) and (5.3), gives as generalizations of (3.3)–(3.6) and (3.14)–(3.15),

$$(6.3) \quad E_{it} = \Omega_W \sum_{k=1}^n \sum_{s=1}^T \phi_{it,ks}, \quad i = 1, \dots, n; t = 1, \dots, T,$$

$$(6.4) \quad e_{it,js} = \phi_{it,js} + \frac{\Omega_{js}}{\Omega_W} E_{it} = \phi_{it,js} - \alpha_{js} E_{it} \left( \frac{E_{js}}{\Omega_W} + 1 \right),$$

$$(6.5) \quad \varepsilon_{it,js} = \phi_{it,js} + \left( \frac{\Omega_{js}}{\Omega_W} + \alpha_{js} \right) E_{it} = \phi_{it,js} - \alpha_{js} \frac{E_{js} E_{it}}{\Omega_W} \iff$$

$$\sigma_{it,js} = \psi_{it,js} + \left( \frac{\Omega_{js}}{\alpha_{js} \Omega_W} + 1 \right) E_{it} = \psi_{it,js} - \frac{E_{js} E_{it}}{\Omega_W},$$

$$(6.6) \quad e_{it,js} = \varepsilon_{it,js} - \alpha_{js} E_{it} = \alpha_{js} (\sigma_{it,js} - E_{it}), \quad i, j = 1, \dots, n; t, s = 1, \dots, T.$$

The counterparts to (3.7)–(3.13) become

$$(6.7) \quad \sum_{i=1}^n \sum_{t=1}^T \alpha_{it} = 1,$$

$$(6.8) \quad \sum_{i=1}^n \sum_{t=1}^T \alpha_{it} E_{it} \equiv \sum_{i=1}^n \sum_{t=1}^T \gamma_{it} = 1,$$

$$(6.9) \quad \sum_{i=1}^n \sum_{t=1}^T \alpha_{it} e_{it,js} = -\alpha_{js},$$

$$(6.10) \quad \sum_{i=1}^n \sum_{t=1}^T \alpha_{it} \varepsilon_{it,js} = 0,$$

$$(6.11) \quad \sum_{i=1}^n \sum_{t=1}^T \alpha_{it} \phi_{it,js} = - \left( \frac{\Omega_{js}}{\Omega_W} + \alpha_{js} \right) = \alpha_{js} \frac{E_{js}}{\Omega_W},$$

$$(6.12) \quad \Omega_{js} + \alpha_{js} \Omega_W = -\alpha_{js} E_{js} \equiv -\gamma_{js}, \quad j = 1, \dots, n; s = 1, \dots, T.$$

The HTG substitution elasticity between commodity  $i$  in period  $t$  and commodity  $j$  in period  $s$ , *i.e.*, the counterpart to (3.16), is

$$(6.13) \quad S_{it,js} = \frac{E_{it} E_{js}}{\Omega_W},$$

giving as counterparts to (3.17) and (3.18):

$$(6.14) \quad e_{it,js} = \phi_{it,js} - \alpha_{js} (S_{it,js} + E_{it}) \equiv \alpha_{js} (\psi_{it,js} - S_{it,js} - E_{it}),$$

$$(6.15) \quad \varepsilon_{it,js} = \phi_{it,js} - \alpha_{js} S_{it,js} \iff \sigma_{it,js} = \psi_{it,js} - S_{it,js}.$$

To summarize, Proposition 1 is generalized to:

**Proposition 4:**

1. The HTG elasticities are connected to  $\Omega_W^{-1}$  by  $S_{it,js} = E_{it} E_{js} \Omega_W^{-1}$ , which under homotheticity ( $E_{it} = 1$ ) simply gives  $S_{it,js} = \Omega_W^{-1}$  for all  $i, j, t, s$ .
2. No one-to-one relationships in general exist between the generalized Frisch elasticities and  $\Omega_W^{-1}$ , for given  $E_{it}$ . From (6.3) is seen that  $\phi_{it,11}, \dots, \phi_{it,1T}, \dots, \phi_{it,n1}, \dots, \phi_{it,nT}$  are related to  $\Omega_W^{-1}$  via  $\sum_{j=1}^n \sum_{s=1}^T \phi_{it,js} = E_{it} \Omega_W^{-1}$ , which because of (6.8) implies
  - (i)  $\sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \phi_{it,js} = \Omega_W^{-1}$  in general,
  - (ii)  $\sum_{j=1}^n \sum_{s=1}^T \phi_{it,js} = \Omega_W^{-1}$  for all  $(i, t)$ , under homotheticity.

The *overall* multi-period budget-share-weighted Hicks-Allen, Frisch and HTG elasticities of substitution equal 0,  $\Omega_W^{-1}$ , and  $\Omega_W^{-1}$ , respectively, since the counterparts to (3.19)–(3.22) become

$$(6.16) \quad \sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} e_{it,js} = -1,$$

$$(6.17) \quad \sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \varepsilon_{it,js} \equiv \sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \alpha_{js} \sigma_{it,js} = 0,$$

$$(6.18) \quad \sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \phi_{it,js} \equiv \sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \alpha_{js} \psi_{it,js} = \Omega_W^{-1},$$

$$(6.19) \quad \sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \alpha_{js} S_{it,js} = \Omega_W^{-1}.$$

Direct generalizations of (3.23)–(3.25) are

$$(6.20) \quad \sigma_{it,js}^* \equiv \sigma_{it,js} - \alpha_{js} \sigma_{js,js},$$

$$(6.21) \quad \psi_{it,js}^* \equiv \psi_{it,js} - \alpha_{js} \psi_{js,js},$$

$$(6.22) \quad S_{it,js}^* \equiv S_{it,js} - \alpha_{js} S_{js,js}.$$

Since  $\sum_i \sum_t \sum_j \sum_s \alpha_{it} \alpha_{js} = 1$  and  $\sum_{i,t} \sum_{j,s,(j,s) \neq (i,t)} \alpha_{it} \alpha_{js} = 1 - \sum_i \sum_t \alpha_{it}^2$ , we obtain as generalizations of (3.26)–(3.28) and (3.29)–(3.31), boldface letters symbolizing substitution elasticities aggregated over all commodities and periods jointly:

$$(6.23) \quad \bar{\sigma}^* = \sum_i \sum_t \sum_j \sum_s \alpha_{it} \alpha_{js} \sigma_{it,js}^* \equiv \sum_i \sum_t (\gamma_{it}^2 \Omega_W^{-1} - \alpha_{it}^2 \psi_{it,it}),$$

$$(6.24) \quad \bar{\psi}^* = \sum_i \sum_t \sum_j \sum_s \alpha_{it} \alpha_{js} \psi_{it,js}^* \equiv \Omega_W^{-1} - \sum_i \sum_t \alpha_{it}^2 \psi_{it,it},$$

$$(6.25) \quad \bar{S}^* = \sum_i \sum_t \sum_j \sum_s \alpha_{it} \alpha_{js} S_{it,js}^* \equiv (1 - \sum_i \sum_t \gamma_{it}^2) \Omega_W^{-1},$$

$$(6.26) \quad \bar{\sigma} = \frac{\sum_{i,t} \sum_{j,s,(j,s) \neq (i,t)} \alpha_{it} \alpha_{js} \sigma_{it,js}}{\sum_{i,t} \sum_{j,s,(j,s) \neq (i,t)} \alpha_{it} \alpha_{js}} \equiv \frac{\sum_i \sum_t (\gamma_{it}^2 \Omega_W^{-1} - \alpha_{it}^2 \psi_{it,it})}{1 - \sum_i \sum_t \alpha_{it}^2},$$

$$(6.27) \quad \bar{\psi} = \frac{\sum_{i,t} \sum_{j,s,(j,s) \neq (i,t)} \alpha_{it} \alpha_{js} \psi_{it,js}}{\sum_{i,t} \sum_{j,s,(j,s) \neq (i,t)} \alpha_{it} \alpha_{js}} \equiv \Omega_W^{-1} + \frac{\sum_i \sum_t \alpha_{it}^2 (\Omega_W^{-1} - \psi_{it,it})}{1 - \sum_i \sum_t \alpha_{it}^2},$$

$$(6.28) \quad \bar{S} = \frac{\sum_{i,t} \sum_{j,s,(j,s) \neq (i,t)} \alpha_{it} \alpha_{js} S_{it,js}}{\sum_{i,t} \sum_{j,s,(j,s) \neq (i,t)} \alpha_{it} \alpha_{js}} \equiv \Omega_W^{-1} + \frac{\sum_i \sum_t (\alpha_{it}^2 - \gamma_{it}^2)}{1 - \sum_i \sum_t \alpha_{it}^2} \Omega_W^{-1}.$$

Finally, the Houthakker-Theil decompositions of the aggregate Hicks-Allen cross-elasticities, corresponding to (6.15), are

$$\begin{aligned} \bar{\sigma}^* &= \bar{\psi}^* - \bar{S}^*, \\ \bar{\sigma} &= \bar{\psi} - \bar{S}. \end{aligned}$$

The results can be summarized in the following generalization of Proposition 2:

**Proposition 5:**

1. While any  $\sigma_{it,js}$  depends on  $\Omega_W^{-1}$ ,  $\Omega_W^{-1}$  is no aggregate indicator of Hicks-Allen substitution. We have  $\sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \alpha_{js} \sigma_{it,js} = 0$ , while  $\sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \alpha_{js} \psi_{it,js} = \sum_{i=1}^n \sum_{t=1}^T \sum_{j=1}^n \sum_{s=1}^T \alpha_{it} \alpha_{js} S_{it,js} = \Omega_W^{-1}$ . The overall cross-substitution elasticities satisfy (6.23)–(6.28).
2. If  $nT$  is large,  $\sum_t \sum_i \alpha_{it}^2$ ,  $\sum_t \sum_i \gamma_{it}^2$  are small, and  $\psi_{it,it}$  is finite, then  $(\bar{\psi}^*, \bar{\psi}) \approx \Omega_W^{-1}$ ,  $(\bar{S}^*, \bar{S}) \approx \Omega_W^{-1}$ , and  $(\bar{\sigma}^*, \bar{\sigma}) = (\bar{\psi}^* - \bar{S}^*, \bar{\psi} - \bar{S}) \approx 0$ .

*Example:* The stylized case with homotheticity and equal budget shares:  $\alpha_{it} = \frac{1}{nT}$ ,  $E_{iT} = 1 \implies \sum_i \sum_t \alpha_{it}^2 = \sum_i \sum_t \gamma_{it}^2 = \frac{1}{nT}$ , letting  $\tilde{\psi} = \frac{1}{nT} \sum_i \sum_t \psi_{it,it}$ , gives

$$\begin{aligned}\bar{\sigma}^* &= \frac{1}{nT}(\Omega_W^{-1} - \tilde{\psi}), & \bar{\sigma} &= \frac{1}{nT-1}(\Omega_W^{-1} - \tilde{\psi}), \\ \bar{\psi}^* &= \Omega_W^{-1} - \frac{1}{nT}\tilde{\psi}, & \bar{\psi} &= \frac{nT}{nT-1}(\Omega_W^{-1} - \frac{1}{nT}\tilde{\psi}), \\ \bar{S}^* &= \frac{nT-1}{nT}\Omega_W^{-1}, & \bar{S} &= \Omega_W^{-1}.\end{aligned}$$

## 7 MULTI-PERIOD MODEL AND SUBSTITUTION: ADDITIVE UTILITY

We next consider three specializations of the multi-period model with the utility function (5.1) restricted to be additively separable. The quotation from Frisch in of Section 4, on the ordinality-cardinality distinction and the non-invariance of important measures to general monotonic transformations, is relevant also here. Hoel (1975) elaborates, and illustrates, by a simple parametric model, the interpretation of a counterpart to  $\Omega_W$  in a related context. Browning, conducting an empirical analysis involving both durable and non-durable goods, concludes<sup>15</sup>

“That preferences over purchases are not separable over time (even if one takes a time period such as a year) seems to be well established. What is a good deal less obvious is the source of such intertemporal dependencies. .... (1) Time separability is rejected. (2) Once one allows for intertemporal nonseparabilities, neither the rational nor the myopic variants of the model are rejected by the data. (3) Not accounting for intertemporal dependencies biases considerably the estimates of intratemporal allocation. For example, the categorization of goods as luxuries or necessities is considerably changed by allowing for nonseparabilities over time.” [Browning (1999, pp. 631–632)]

1. First, let  $U(\cdot)$  be *additively separable over periods*, having the form:

$$(7.1) \quad U = U(\mathbf{x}) = \sum_{t=1}^T U_t(\mathbf{x}_t),$$

where  $\mathbf{x}_t = (x_{1t}, \dots, x_{nt})$  and  $\mathbf{u}_t = (u_{1t}, \dots, u_{nt})$ . The Hessian matrix of  $U(\mathbf{x})$  is block-diagonal, the blocks representing periods, and (5.2), (5.3), and (5.5) become (subscript  $P$  signaling that  $u(\cdot)$  and  $h(\cdot)$  have period-specific arguments)<sup>16</sup>

$$(7.2) \quad u_{it} = u_{it,P}(\mathbf{x}_t) \equiv \partial U_t(\mathbf{x}_t) / \partial x_{it},$$

$$(7.3) \quad x_{it} = h_{it,P}(\mathbf{u}_t),$$

$$(7.4) \quad u_{it,P}(\mathbf{x}_t) = \Omega \bar{p}_{it}.$$

Since (7.1) implies  $u_{it,jt} = \phi_{it,jt} = 0$ ,  $t \neq s$ , (6.3) and (6.5) simplify to

$$(7.5) \quad E_{it} = \Omega_W \sum_{k=1}^n \phi_{it,kt},$$

$$(7.6) \quad \varepsilon_{it,jt} = \delta_{ts} \phi_{it,jt} - \alpha_{js} E_{jt} \Omega_W^{-1} \iff \sigma_{it,jt} = \delta_{ts} \psi_{it,jt} - E_{jt} \Omega_W^{-1}.$$

2. Next, let  $U(\cdot)$  be *additively separable over commodities*, having the form:

$$(7.7) \quad U = U(\mathbf{x}) = \sum_{i=1}^n U_{(i)}(\mathbf{x}_{(i)}),$$

<sup>15</sup>See also Crossley and Low (2011).

<sup>16</sup>The  $(n \times n)$ -matrix whose element  $(i, j)$  is  $\partial h_{it,P}(\mathbf{u}_t) / \partial u_{jt}$  is the inverse of the Hessian matrix of  $U_t(\mathbf{x}_t)$ , i.e., the  $(n \times n)$ -matrix whose element  $(i, j)$  is  $\partial u_{it,P}(\mathbf{x}_t) / \partial x_{jt}$ .

where  $\mathbf{x}_{(i)} = (x_{i1}, \dots, x_{iT})$  and  $\mathbf{u}_{(i)} = (u_{i1}, \dots, u_{iT})$ . After a reordering of the elements, the Hessian matrix is block-diagonal, with blocks representing commodities, (5.2), (5.3), and (5.5) become (subscript  $C$  signaling that  $u(\cdot)$  and  $h(\cdot)$  have commodity-specific arguments)<sup>17</sup>

$$(7.8) \quad u_{it} = u_{it.C}(\mathbf{x}_{(i)}) \equiv \partial U_{(i)}(\mathbf{x}_{(i)}) / \partial x_{it},$$

$$(7.9) \quad x_{it} = h_{it.C}(\mathbf{u}_{(i)}),$$

$$(7.10) \quad u_{it.C}(\mathbf{x}_{(i)}) = \Omega \bar{p}_{it}, \quad t = 1, \dots, T.$$

Since (7.1) implies  $u_{it,j_s} = \phi_{it,j_s} = 0$ ,  $i \neq j$ , (6.3) and (6.5) simplify to

$$(7.11) \quad E_{it} = \Omega_W \sum_{s=1}^T \phi_{it,is},$$

$$(7.12) \quad \varepsilon_{it,j_s} = \delta_{ij} \phi_{it,is} - \alpha_{j_s} E_{j_s} E_{it} \Omega_W^{-1} \iff \sigma_{it,j_s} = \delta_{ij} \psi_{it,is} - E_{j_s} E_{it} \Omega_W^{-1}.$$

3. Finally, let  $U(\cdot)$  be *fully separable*, having the form:

$$(7.13) \quad U = U(\mathbf{x}) = \sum_{t=1}^T \sum_{i=1}^n U_{it}(x_{it}).$$

Since this implies  $u_{it,j_s} = \delta_{ij} \delta_{ts} u_{it,it}$ ,  $\phi_{it,j_s} = \delta_{ij} \delta_{ts} \phi_{it,it}$ , where  $\phi_{it,it} = 1/u_{it,it}$ , (6.3) and (6.5) become simply

$$(7.14) \quad E_{it} = \Omega_W \phi_{it,it},$$

$$(7.15) \quad \varepsilon_{it,j_s} = \delta_{ij} \delta_{ts} \phi_{it,it} - \alpha_{j_s} E_{j_s} E_{it} \Omega_W^{-1} = E_{it} \Omega_W^{-1} (\delta_{ij} \delta_{ts} - \alpha_{j_s} E_{j_s}) \iff$$

$$\sigma_{it,j_s} = \delta_{ij} \delta_{ts} \psi_{it,it} - E_{j_s} E_{it} \Omega_W^{-1} = E_{it} \Omega_W^{-1} (\delta_{ij} \delta_{ts} / \alpha_{j_s} - E_{j_s}).$$

The first and third kind of separability are often considered. While the second is more rare, it may deserve attention not only as reference point, but also because it may be used to handle preference rigidities, habit formation, and durable goods.

Briefly, (7.1), (7.7), and (7.13) restrict Frisch-substitution by simplifying (6.18) to, respectively,

$$(7.16) \quad \sum_{i=1}^n \sum_{j=1}^n \sum_{t=1}^T \alpha_{it} \alpha_{jt} \psi_{it,jt} = \Omega_W^{-1},$$

$$(7.17) \quad \sum_{i=1}^n \sum_{t=1}^T \sum_{s=1}^T \alpha_{it} \alpha_{is} \psi_{it,is} = \Omega_W^{-1},$$

$$(7.18) \quad \sum_{i=1}^n \sum_{t=1}^T \alpha_{it}^2 \psi_{it,it} = \Omega_W^{-1}.$$

These equations can be viewed as extensions of Frisch (1959), Eq. (47). Therefore:

- The quadruple sum (6.18) restricts the  $\frac{1}{2}nT(nT-1)$  cross-elasticities  $\psi_{it,j_s} = \psi_{j_s,it}$ .
- The triple sum (7.16) restricts the  $\frac{1}{2}nT(n-1)$  non-zero cross-elasticities  $\psi_{it,jt} = \psi_{jt,it}$ .
- The triple sum (7.17) restricts the  $\frac{1}{2}nT(T-1)$  non-zero cross-elasticities  $\psi_{it,is} = \psi_{is,it}$ .
- The double sum (7.18) restricts no cross-elasticity, as  $\psi_{it,j_s} = 0$ , for  $(j, s) \neq (i, t)$ .

The following generalization of Proposition 3 summarizes the results:

---

<sup>17</sup>The  $(T \times T)$ -matrix whose element  $(t, s)$  is  $\partial h_{it.C}(\mathbf{u}_{(i)}) / \partial u_{is}$  is the inverse of the Hessian matrix of  $U_{(i)}(\mathbf{x}_{(i)})$ , i.e., the  $(T \times T)$ -matrix whose element  $(t, s)$  is  $\partial u_{it.C}(\mathbf{x}_{(i)}) / \partial x_{is}$ .

**Proposition 6:**

1. Fully additive utility,  $\psi_{it,ks}=0, (i,t) \neq (k,s)$ , gives a relationship between the direct Frisch elasticities and  $\Omega_W^{-1}$ :  $\phi_{it,it} = E_{it}\Omega_W^{-1}$ . Hence,
  - (i)  $\sigma_{it,js} = -E_{it}E_{js}\Omega_W^{-1}, (i,t) \neq (j,s)$ , under full additivity,
  - (ii)  $\sum_{i=1}^n \sum_{t=1}^T \alpha_{it}\phi_{it,it} = \Omega_W^{-1}$  under full additivity,
  - (iii)  $\phi_{it,it} = \Omega_W^{-1}, \sigma_{it,js} = -\Omega_W^{-1}, (i,t) \neq (j,s)$ , under full additivity and homotheticity.
2. Since the aggregate Frisch cross-elasticities then equal  $(\bar{\psi}^*, \bar{\psi})=0$ , their Hicks-Allen counterparts become  $(\bar{\sigma}^*, \bar{\sigma}) = (\bar{\psi}^* - \bar{S}^*, \bar{\psi} - \bar{S}) \approx -\Omega_W^{-1}$ .
3. Utility additively separable over periods,  $\psi_{it,js}=0, t \neq s$ , implies

$$\begin{aligned} \sum_{i=1}^n \sum_{t=1}^T \alpha_{it}\psi_{it,jt} &= E_{jt}\Omega_W^{-1}, \\ \sum_{i=1}^n \sum_{j=1}^n \sum_{t=1}^T \alpha_{it}\alpha_{jt}\psi_{it,jt} &= \Omega_W^{-1}. \end{aligned}$$

Utility additively separable across commodities,  $\psi_{it,js}=0, i \neq j$ , implies

$$\begin{aligned} \sum_{i=1}^n \sum_{t=1}^T \alpha_{it}\psi_{it,is} &= E_{is}\Omega_W^{-1}, \\ \sum_{i=1}^n \sum_{t=1}^T \sum_{s=1}^n \alpha_{it}\alpha_{is}\psi_{it,is} &= \Omega_W^{-1}. \end{aligned}$$

*Example:* The stylized case with full additivity, homotheticity, and equal budget shares:  $\alpha_{it} = \frac{1}{nT}, E_{iT} = 1 \implies \sum_i \sum_t \alpha_{it}^2 = \sum_i \sum_t \gamma_{it}^2 = \frac{1}{nT}, \psi_{it,it} = \tilde{\psi} = nT\Omega_W^{-1}$ , is characterized by the following aggregate cross-elasticities:

$$\begin{aligned} \bar{\sigma}^* &= -\frac{nT-1}{nT}\Omega_W^{-1}, & \bar{\sigma} &= -\Omega_W^{-1}, \\ \bar{\psi}^* &= 0, & \bar{\psi} &= 0, \\ \bar{S}^* &= \frac{nT-1}{nT}\Omega_W^{-1}, & \bar{S} &= \Omega_W^{-1}. \end{aligned}$$

## 8 SUBSTITUTION: AGGREGATE IMPLICATIONS

The results in Sections 5 through 7 have aggregate implications relevant to the questions posed in the introduction. In this section we will show that aggregate analogues to the relationships between ‘two-dimensional’ elasticities are easily obtained. Let  $\alpha_{\cdot t} \equiv \sum_{i=1}^n \alpha_{it}$  and  $\alpha_{i\cdot} \equiv \sum_{t=1}^T \alpha_{it}$ , and write the period-conditional (within-period) and commodity-conditional (within-commodity) budget shares as

$$(8.1) \quad \alpha_{i|t} \equiv \alpha_{it}/\alpha_{\cdot t}, \quad \sum_{i=1}^n \alpha_{i|t} = 1, \quad t = 1, \dots, T,$$

$$(8.2) \quad \tilde{\alpha}_{t|i} \equiv \alpha_{it}/\alpha_{i\cdot}, \quad \sum_{t=1}^T \tilde{\alpha}_{t|i} = 1, \quad i = 1, \dots, n.$$

Average, budget-share-weighted Engel elasticities for period  $t$  and for commodity  $i$  are defined as, respectively,

$$(8.3) \quad E_{\cdot t} \equiv \sum_{i=1}^n \alpha_{i|t} E_{it}, \quad \sum_{t=1}^T \alpha_{\cdot t} E_{\cdot t} = 1,$$

$$(8.4) \quad E_{i\cdot} \equiv \sum_{t=1}^T \tilde{\alpha}_{t|i} E_{it}, \quad \sum_{i=1}^n \alpha_{i\cdot} E_{i\cdot} = 1,$$

and average, double budget-share-weighted Hicks-Allen, Frisch, and HTG elasticities for periods  $(t, s)$  and for commodities  $(i, j)$  as, respectively,<sup>18</sup>

<sup>18</sup>On the use of cost-shares in aggregation of elasticities of substitution in production function and cost functions contexts, and its relation to ‘Hicks’ Aggregation Theorem’, see Diewert (1974) and Miyagiwa and Papageorgiou (2007).

$$(8.5) \quad \sigma_{\cdot t, \cdot s} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_{i|t} \alpha_{j|s} \sigma_{it, js},$$

$$(8.6) \quad \psi_{\cdot t, \cdot s} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_{i|t} \alpha_{j|s} \psi_{it, js},$$

$$(8.7) \quad S_{\cdot t, \cdot s} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_{i|t} \alpha_{j|s} S_{it, js},$$

and

$$(8.8) \quad \sigma_{i \cdot, j \cdot} \equiv \sum_{t=1}^T \sum_{s=1}^T \tilde{\alpha}_{t|i} \tilde{\alpha}_{s|j} \sigma_{it, js},$$

$$(8.9) \quad \psi_{i \cdot, j \cdot} \equiv \sum_{t=1}^T \sum_{s=1}^T \tilde{\alpha}_{t|i} \tilde{\alpha}_{s|j} \psi_{it, js},$$

$$(8.10) \quad S_{i \cdot, j \cdot} \equiv \sum_{t=1}^T \sum_{s=1}^T \tilde{\alpha}_{t|i} \tilde{\alpha}_{s|j} S_{it, js}.$$

From (6.5), (6.13), (6.17)–(6.19), and (8.1)–(8.10) it follows that equations similar to (3.16) and (3.18) connect these aggregate elasticities:

$$(8.11) \quad S_{\cdot t, \cdot s} = E_{\cdot t} E_{\cdot s} \Omega_W^{-1},$$

$$(8.12) \quad \sigma_{\cdot t, \cdot s} = \psi_{\cdot t, \cdot s} - S_{\cdot t, \cdot s}, \quad t, s = 1, \dots, T,$$

$$(8.13) \quad S_{i \cdot, j \cdot} = E_{i \cdot} E_{j \cdot} \Omega_W^{-1},$$

$$(8.14) \quad \sigma_{i \cdot, j \cdot} = \psi_{i \cdot, j \cdot} - S_{i \cdot, j \cdot}, \quad i, j = 1, \dots, n,$$

while aggregate counterparts to (3.20)–(3.22) become

$$(8.15) \quad \sum_{t=1}^T \sum_{s=1}^T \alpha_{\cdot t} \alpha_{\cdot s} \sigma_{\cdot t, \cdot s} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{i \cdot, j \cdot} = 0,$$

$$(8.16) \quad \sum_{t=1}^T \sum_{s=1}^T \alpha_{\cdot t} \alpha_{\cdot s} \psi_{\cdot t, \cdot s} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \psi_{i \cdot, j \cdot} = \Omega_W^{-1},$$

$$(8.17) \quad \sum_{t=1}^T \sum_{s=1}^T \alpha_{\cdot t} \alpha_{\cdot s} S_{\cdot t, \cdot s} \equiv \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j S_{i \cdot, j \cdot} = \Omega_W^{-1}.$$

With additive utility, the expressions for the aggregate Hicks-Allen elasticities, (8.12) and (8.14), using (7.6), (7.12), and (7.15), simplify to, respectively,

$$(8.18) \quad \sigma_{\cdot t, \cdot s} = \delta_{ts} \psi_{\cdot t, \cdot t} - S_{\cdot t, \cdot s}, \quad \psi_{\cdot t, \cdot t} = \begin{cases} \sum_{i=1}^n \sum_{j=1}^n \alpha_{i|t} \alpha_{j|t} \psi_{it, jt}, & \text{under period add.}, \\ \sum_{i=1}^n \alpha_{i|t}^2 \psi_{it, it}, & \text{under full add.}, \end{cases}$$

$$(8.19) \quad \sigma_{i \cdot, j \cdot} = \delta_{ij} \psi_{i \cdot, i \cdot} - S_{i \cdot, j \cdot}, \quad \psi_{i \cdot, i \cdot} = \begin{cases} \sum_{t=1}^T \sum_{s=1}^T \tilde{\alpha}_{t|i} \tilde{\alpha}_{s|i} \psi_{it, is}, & \text{under comm. add.}, \\ \sum_{t=1}^T \tilde{\alpha}_{t|i}^2 \psi_{it, it}, & \text{under full add.} \end{cases}$$

To conclude, the aggregate elasticities defined, by using as weights period-conditional and commodity-conditional budget shares as in (8.3)–(8.10), satisfy:

**A.** Simple analogues exist (i) between the above two-dimensional relationships (6.13) and (6.15) and the temporally and spatially aggregated relationships, and (ii) between the uni-dimensional relationships (3.16) and (3.18) and the commodity-specific, temporally aggregated relationships.

**B.** The inverse Frisch parameter,  $\Omega_W^{-1}$ , has a core role in these aggregate relationships as it has in the disaggregate ones.

## 9 MORE PARSIMONIOUS MULTI-PERIOD MODELS

To preserve economy in parametrization, it is rarely convenient, in empirical applications, to let the sub-utility functions  $U_t$  and  $U_{(i)}$  be unrestricted. As an extension

of Section 7, we in this section impose restrictions on the within-period or within-commodity relative marginal utilities which result in what we will call *models with restricted additive separability*. We specifically reconsider, under three sets of restrictions, the Frischian flexibilities and the Hicks-Allen substitution elasticities and their components.

1. *Frischian flexibilities: restrictive period-separability.* Let in (7.1)  $U_t(\mathbf{x}_t) = \rho_t V_P(\mathbf{x}_t)$ , where  $V_P(\cdot)$  is a common ‘one-period, time invariant, multi-commodity utility index’ and  $\rho_t$  a  $t$ -specific factor of proportionality of marginal utilities – a discounting factor for period  $t$  utility, when sticking to the cardinalistic interpretation. The Lagrange function for maximization of  $U = \sum_{t=1}^T \rho_t V_P(\mathbf{x}_t)$  subject to (5.4), written as  $\sum_{i=1}^n p_{it} x_{it} = y_t$  ( $t = 1, \dots, T$ ),  $\sum_{t=1}^T \beta_t y_t = W$ , is

$$(9.1) \quad \mathcal{L} = \sum_{t=1}^T \rho_t V_P(\mathbf{x}_t) - \sum_{t=1}^T \lambda_t (\sum_{i=1}^n p_{it} x_{it} - y_t) - \Omega (\sum_{t=1}^T \beta_t y_t - W).$$

Since in optimum  $\lambda_t = \Omega \beta_t$ , (7.4) implies

$$(9.2) \quad \rho_t v_{it.P}(\mathbf{x}_t) \equiv \lambda_t p_{it} = \Omega \beta_t p_{it} \iff v_{it.P}(\mathbf{x}_t) = \omega_{Pt} p_{it},$$

where  $v_{it.P}(\mathbf{x}_t) \equiv \partial V_P(\mathbf{x}_t) / \partial x_{it}$  and  $\omega_{Pt} = \lambda_t / \rho_t = \Omega \beta_t / \rho_t$ . The latter is the Lagrange multiplier for the period- $t$ -problem conditional on  $y_t$ :

$$(9.3) \quad \begin{aligned} & \max V_P(\mathbf{x}_t) \text{ subject to } \sum_{i=1}^n p_{it} x_{it} = y_t, \\ & \mathcal{L}_{Pt} = V_P(\mathbf{x}_t) - \omega_{Pt} (\sum_{i=1}^n p_{it} x_{it} - y_t). \end{aligned}$$

This mimics the problem for a myopic consumer in (say) period  $t=1$ . Since

$$(9.4) \quad \ln(\omega_{Pt}) = \ln(\Omega) + \ln(\beta_t) - \ln(\rho_t),$$

it follows, provided that  $\beta_t$  and  $\rho_t$  do not respond to changes in the  $y_s$ , that

$$d \ln(\omega_{Pt}) = d \ln(\Omega) = \Omega_W \sum_{s=1}^T \alpha_{ys} d \ln(y_s),$$

where  $\alpha_{yt} = \partial \ln(W) / \partial \ln(\beta_t y_t) = \beta_t y_t / W$ , and hence,

$$(9.5) \quad d \ln(\omega_{Pt}) / d \ln(\beta_t y_t) = \Omega_W \sum_{s=1}^T \alpha_{ys} d \ln(y_s) / d \ln(y_t),$$

so that

$$(9.6) \quad \frac{d \ln(\omega_{Pt})}{d \ln(y_t)} \begin{matrix} > \\ < \end{matrix} \Omega_W \iff \sum_{s=1}^T \alpha_{ys} \frac{d \ln(y_s)}{d \ln(y_t)} \begin{matrix} > \\ < \end{matrix} 1.$$

Of relevance here is the Hicksian *elasticity of expectation*, Hicks (1939, Chapter XVI).<sup>19</sup> Several authors using parametric models and calibration exercises for teaching or policy application purposes, appear to inadvertently set (the counterpart to) this elasticity to unity. An interpretation is that if  $d \ln(y_s) / d \ln(y_t) = 1$  for all  $s, t$ , there is proportionality between  $\omega_{Pt}$  and  $\Omega$  over time, so that

$$d \ln(\omega_{Pt}) / d \ln(\beta_t y_t) = \Omega_W.$$

<sup>19</sup>See also Arrow and Nerlove (1958), Lachman (1945) and Ozga (1975).

Commenting on price expectations in a similar full separability case (the remark is equally relevant for income expectations and under full separability), Blundell, for example, remarks, without expanding the argument in detail:

“For intertemporal allocations, the approach of Frisch has turned out to be extremely attractive. Additive separability over time is commonly assumed and can easily be extended to account for uncertainty. Moreover, the price elasticities from the Frisch demands are identical with the good-specific intertemporal substitution elasticities..... where we relax the perfect-foresight assumption and allow for uncertainty, the Frisch elasticities have to be interpreted as price responses along an anticipated price path. As such, they do not necessarily correspond to the intertemporal elasticities needed for policy analysis. Policy changes often are unexpected and therefore involve an unanticipated income or wealth effect.” [Blundell (1999, pp. 148-149)]

Equation (9.5) connects the Frischian flexibility of the marginal utility of period- $t$ -income with the marginal utility of wealth flexibility  $\Omega_W$ , and the elasticities of expectation of period income  $y$ , say  $d \ln(y_s)/d \ln(y_1)$ ,  $s = 2, 3, \dots, T$ .

**2. Frischian flexibilities: restrictive commodity-separability.** Let in (7.7)  $U_{(i)}(\mathbf{x}_{(i)}) = \mu_i V_C(\mathbf{x}_{(i)})$ , where  $V_C(\cdot)$  is a ‘commodity-invariant, multi-period utility index’ and  $\mu_i$  an  $i$ -specific factor of proportionality of marginal utilities – interpreted as a ‘utility weight’ for commodity  $i$ . The Lagrange function for maximization of  $U = \sum_{i=1}^n \mu_i V_C(\mathbf{x}_{(i)})$  subject to (5.4), written as  $\sum_{t=1}^T \bar{p}_{it} x_{it} = w_i$  ( $i = 1, \dots, n$ ),  $\sum_{i=1}^n w_i = W$ , where  $w_i$  can be interpreted as a (discounted) ‘commodity- $i$ -expenditure’, is

$$(9.7) \quad \mathcal{L} = \sum_{i=1}^n \mu_i V_C(\mathbf{x}_{(i)}) - \sum_{i=1}^n \chi_i (\sum_{t=1}^T \bar{p}_{it} x_{it} - w_i) - \Omega (\sum_{i=1}^n w_i - W).$$

Since in optimum  $\chi_i = \Omega$ , (7.10) implies

$$(9.8) \quad \mu_i v_{it.C}(\mathbf{x}_{(i)}) = \chi_i \bar{p}_{it} = \Omega \bar{p}_{it} \iff v_{it.C}(\mathbf{x}_{(i)}) = \omega_{C_i} \bar{p}_{it},$$

where  $v_{it.C}(\mathbf{x}_{(i)}) \equiv \partial V_C(\mathbf{x}_{(i)})/\partial x_{it}$  and  $\omega_{C_i} = \chi_i/\mu_i = \Omega/\mu_i$ . The latter is the Lagrange multiplier for the commodity  $i$  problem, conditional on  $w_i$ :

$$(9.9) \quad \begin{aligned} & \max V_C(\mathbf{x}_{(i)}) \text{ subject to } \sum_{t=1}^T \bar{p}_{it} x_{it} = w_i, \\ & \mathcal{L}_{C_i} = V_C(\mathbf{x}_{(i)}) - \omega_{C_i} (\sum_{t=1}^T \bar{p}_{it} x_{it} - w_i). \end{aligned}$$

This mimics the problem for a consumer who, in period  $t=1$ , consider allocation of the discounted sum  $w_i$  on commodity  $i$ , say, foods, housing, or transportation, up to period  $T$ .<sup>20</sup> Since

$$(9.10) \quad \ln(\omega_{C_i}) = \ln(\Omega) - \ln(\mu_i),$$

it follows, provided that  $\mu_i$  does not respond to changes in the  $w$ s, that

$$d \ln(\omega_{C_i}) = d \ln(\Omega) = \Omega_W \sum_{k=1}^N \alpha_{wk} d \ln(w_k),$$

where  $\alpha_{wi} = \partial \ln(W)/\partial \ln(w_i) = w_i/W$ , and hence,

$$(9.11) \quad d \ln(\omega_{C_i})/d \ln(w_i) = \Omega_W \sum_{k=1}^n \alpha_{wk} d \ln(w_k)/d \ln(w_i),$$

<sup>20</sup> A special case is the frequently used one-commodity dynamic model, with total consumption as one bundle.



so that

$$(9.12) \quad \frac{d \ln(\omega_{C_i})}{d \ln(w_i)} \begin{matrix} > \\ \equiv \\ < \end{matrix} \Omega_W \iff \sum_{k=1}^n \alpha_{wk} \frac{d \ln(w_k)}{d \ln(w_i)} \begin{matrix} > \\ \equiv \\ < \end{matrix} 1.$$

If all  $d \ln(w_k)/d \ln(w_i) = 1$ , implying proportionality between  $\omega_{C_i}$  and  $\Omega$  across commodities – which, in view the commodities’ different ‘scale properties’, reflected in the utility function being non-homothetic, seems less realistic than proportionality between  $\omega_{P_t}$  and  $\Omega$  over time – we would in particular have

$$d \ln(\omega_{C_i})/d \ln(w_i) = \Omega_W.$$

Equation (9.11) connects the Frischian flexibility of the marginal utility of commodity- $i$  expenditure with the marginal utility of wealth flexibility  $\Omega_W$ , and the relative changes in commodity expenditures, say  $d \ln(w_k)/d \ln(w_1)$ ,  $k = 2, 3, \dots, n$ , which are the formal counterparts to the income elasticities of expectation.

**3. Frischian flexibilities: restrictive full separability.** Let in (7.13)  $U_{it}(x_{it}) = \mu_i \rho_t V(x_{it})$ , where  $\rho_t$  and  $\mu_i$  are utility weights for, respectively, period  $t$  and commodity  $i$  and  $V(\cdot)$  is a prototype single-commodity utility element. The Lagrange function for maximization of  $U = \sum_{i=1}^n \sum_{t=1}^T \rho_t \mu_i V(x_{it})$  subject to (5.4) is

$$(9.13) \quad \mathcal{L} = \begin{cases} \sum_{t=1}^T \rho_t \sum_{i=1}^n \mu_i V(x_{it}) - \sum_{t=1}^T \lambda_t (\sum_{i=1}^n p_{it} x_{it} - y_t) - \Omega (\sum_{t=1}^T \beta_t y_t - W), \\ \sum_{i=1}^n \mu_i \sum_{t=1}^T \rho_t V(x_{it}) - \sum_{i=1}^n \chi_i (\sum_{t=1}^T \bar{p}_{it} x_{it} - w_i) - \Omega (\sum_{i=1}^n w_i - W), \end{cases}$$

and (9.2) and (9.8), letting  $v'_{it}(x_{it}) \equiv dV(x_{it})/dx_{it}$ , specialize to

$$\begin{aligned} \rho_t v'_{it}(x_{it}) = \lambda_t p_{it} &\iff v'_{it}(x_{it}) = \omega_{P_t} p_{it} \quad (t = 1, \dots, T) &\iff \\ \mu_i v'_{it}(x_{it}) = \chi_i \bar{p}_{it} &\iff v'_{it}(x_{it}) = \omega_{C_i} \bar{p}_{it} \quad (i = 1, \dots, n), \end{aligned}$$

where  $\omega_{P_t} = \lambda_t/\rho_t \equiv \Omega \beta_t/\rho_t$ , and  $\omega_{C_i} = \chi_i/\mu_i \equiv \Omega/\mu_i$ . The latter are the Lagrange multipliers for, respectively,

$$\begin{aligned} \max \sum_{i=1}^n \mu_i V(x_{it}) \text{ subject to } \sum_{i=1}^n p_{it} x_{it} = y_t, \\ \mathcal{L}_{P_t} = \sum_{i=1}^n \mu_i V(x_{it}) - \omega_{P_t} (\sum_{i=1}^n p_{it} x_{it} - y_t), \\ \max \sum_{t=1}^T \rho_t V(x_{it}) \text{ subject to } \sum_{t=1}^T \bar{p}_{it} x_{it} = w_i, \\ \mathcal{L}_{C_i} = \sum_{t=1}^T \rho_t V(x_{it}) - \omega_{C_i} (\sum_{t=1}^T \bar{p}_{it} x_{it} - w_i). \end{aligned}$$

Since (9.4) and (9.10) are still satisfied, these Lagrange multipliers are related by

$$\ln(\omega_{P_t}) + \ln(\rho_t) - \ln(\beta_t) = \ln(\omega_{C_i}) + \ln(\mu_i) = \ln(\Omega).$$

Then, (9.5) and (9.11) summarize, for the cases with restrictive additive separability, the connection between the generalized Frisch parameter (the flexibility of the marginal utility of wealth), the period-specific flexibility of the marginal utility of income, and the commodity-specific flexibility of the marginal utility of expenditure.

4. *Hicks-Allen decomposition: restrictive period-separability.* Conditional on period expenditure  $y_t$ , irrespective of  $\mathbf{p}_s$  and  $y_s$  ( $s \neq t$ ), we have in precise notation, in analogy with (2.8), (2.10), (2.11), and (3.3), for the period- $t$ -problem (9.3),

$$\begin{aligned} x_{i|Pt} &= f_{i|Pt}(\mathbf{p}_t, y_t), & E_{i|Pt} &\equiv \frac{\partial f_{i|Pt}(\mathbf{p}_t, y_t)}{\partial y_t} \frac{y_t}{f_{i|Pt}(\mathbf{p}_t, y_t)}, & i &= 1, \dots, n, \\ x_{i|Pt} &= h_{i|Pt}(\omega_{Pt}\mathbf{p}_t), & \phi_{ij|Pt} &\equiv \frac{\partial h_{i|Pt}(\omega_{Pt}\mathbf{p}_t)}{\partial p_{jt}} \frac{p_{jt}}{h_{i|Pt}(\omega_{Pt}\mathbf{p}_t)}, & i, j &= 1, \dots, n, \\ \omega_{Pt} &= \omega_{Pt}(\mathbf{p}_t, y_t), & \omega_{y|Pt} &\equiv \frac{\partial \omega_{Pt}(\mathbf{p}_t, y_t)}{\partial y_t} \frac{y_t}{\omega_{Pt}(\mathbf{p}_t, y_t)}, \end{aligned}$$

where

$$E_{i|Pt} = \omega_{y|Pt} \sum_{k=1}^n \phi_{ik|Pt}.$$

This gives Hicks-Allen commodity, period- $t$ -conditional substitution elasticities,

$$(9.14) \quad \sigma_{ij|Pt} = \psi_{ij|Pt} - S_{ij|Pt},$$

the Frischian and general commodity substitution elasticities being, respectively,

$$\psi_{ij|Pt} = \frac{\phi_{ij|Pt}}{\alpha_{j|t}}, \quad S_{ij|Pt} = \frac{E_{i|Pt} E_{j|Pt}}{\omega_{y|Pt}}, \quad i, j = 1, \dots, n.$$

If *commodity-separability for period  $t$*  also prevails, then  $\phi_{ij|Pt} = \delta_{ij} \phi_{ii|Pt} = \delta_{ij} E_{i|Pt} \omega_{y|Pt}^{-1}$ .

5. *Hicks-Allen decomposition: restrictive commodity-separability.* For the analogous commodity- $i$ -problem (9.9), we have in precise notation, with  $\bar{\mathbf{p}}_{(i)} = (\bar{p}_{i1}, \bar{p}_{i2}, \dots, \bar{p}_{iT}) \equiv (p_{i1}, \beta_2 p_{i2}, \dots, \beta_T p_{iT})$ , conditional on the commodity expenditure  $w_i$ , irrespective of  $\bar{\mathbf{p}}_{(k)}$  and  $w_k$  ( $k \neq i$ ),

$$\begin{aligned} x_{t|Ci} &= f_{t|Ci}(\bar{\mathbf{p}}_{(i)}, w_i), & E_{t|Ci} &\equiv \frac{\partial f_{t|Ci}(\bar{\mathbf{p}}_{(i)}, w_i)}{\partial w_i} \frac{w_i}{f_{t|Ci}(\bar{\mathbf{p}}_{(i)}, w_i)}, & t &= 1, \dots, T, \\ x_{t|Ci} &= h_{t|Ci}(\omega_{Ci}\bar{\mathbf{p}}_{(i)}), & \phi_{ts|Ci} &\equiv \frac{\partial h_{t|Ci}(\omega_{Ci}\bar{\mathbf{p}}_{(i)})}{\partial \bar{p}_{is}} \frac{\bar{p}_{is}}{h_{t|Ci}(\omega_{Ci}\bar{\mathbf{p}}_{(i)})}, & t, s &= 1, \dots, T, \\ \omega_{Ci} &= \omega_{Ci}(\bar{\mathbf{p}}_{(i)}, w_i), & \omega_{w|Ci} &\equiv \frac{\partial \omega_{Ci}(\bar{\mathbf{p}}_{(i)}, w_i)}{\partial w_i} \frac{w_i}{\omega_{Ci}(\bar{\mathbf{p}}_{(i)}, w_i)}, \end{aligned}$$

where

$$E_{t|Ci} = \omega_{w|Ci} \sum_{s=1}^T \phi_{ts|Ci}.$$

This gives Hicks-Allen temporal, commodity- $i$ -conditional substitution elasticities,

$$(9.15) \quad \sigma_{ts|Ci} = \psi_{ts|Ci} - S_{ts|Ci},$$

the Frischian and general temporal substitution elasticities being, respectively,

$$\psi_{ts|Ci} = \frac{\phi_{ts|Ci}}{\tilde{\alpha}_{s|i}}, \quad S_{ts|Ci} = \frac{E_{t|Ci} E_{s|Ci}}{\omega_{w|Ci}}, \quad t, s = 1, \dots, T.$$

If *period-separability for commodity  $i$*  also prevails, then  $\phi_{ts|Ci} = \delta_{ts} \phi_{tt|Ci} = \delta_{ts} E_{t|Ci} \omega_{w|Ci}^{-1}$ .

We can then conclude:

**a.** Full period-separability and commodity-separability for period  $t$  – irrespective of  $y_s, \mathbf{p}_s$  ( $s \neq t$ ) and irrespective of whether commodity-separability prevails for periods  $s \neq t$  – implies

$$(9.16) \quad \sigma_{ij|Pt} = \frac{E_{i|Pt}}{\omega_{y|Pt}} \left( \frac{\delta_{ij}}{\alpha_{j|Pt}} - E_{j|Pt} \right), \quad i, j = 1, \dots, n.$$

**b.** Full commodity-separability and period-separability for commodity  $i$  – irrespective of  $w_k, \bar{\mathbf{p}}_{(k)}$  ( $k \neq i$ ) and irrespective of whether period-separability prevails for commodities  $k \neq i$  – implies

$$(9.17) \quad \sigma_{ts|Ci} = \frac{E_{t|Ci}}{\omega_{w|Ci}} \left( \frac{\delta_{ts}}{\alpha_{s|Ci}} - E_{s|Ci} \right), \quad t, s = 1, \dots, T.$$

**c.** Full period- and commodity-separability implies that jointly,

$$(9.18) \quad \begin{aligned} \sigma_{ij|Pt} &= \frac{E_{i|Pt}}{\omega_{y|Pt}} \left( \frac{\delta_{ij}}{\alpha_{j|Pt}} - E_{j|Pt} \right), \quad i, j = 1, \dots, n; \quad t = 1, \dots, T, \\ \sigma_{ts|Ci} &= \frac{E_{t|Ci}}{\omega_{w|Ci}} \left( \frac{\delta_{ts}}{\alpha_{s|Ci}} - E_{s|Ci} \right), \quad t, s = 1, \dots, T; \quad i = 1, \dots, n. \end{aligned}$$

Contrasting (9.16)–(9.18) with (9.5) and (9.11) – recalling all assumptions about the utility function and the consumer’s optimization problem on which these two sets of relationships rely – raises the empirically relevant question: when and how can estimates of ‘conditional’ Frischian flexibilities like  $\omega_{y|Pt}$  and  $\omega_{w|Ci}$  be translated into estimates of  $\Omega_W$ ? Elements of an answer have been given in the previous sections. In the final section, we give some concluding remarks.

## 10 DISCUSSION AND CONCLUSION

In focus of this paper has been substitution in the spatial and in the temporal sense and the connection between them. Frisch (1959), discussing, *inter alia*, intra-temporal commodity substitution, proposed, and demonstrated the usefulness of systems of demand functions conditional on the marginal utility of income, later labeled ‘Frisch demand functions’. Such functions and their multi-period counterparts, systems of dynamic demand functions conditional on the marginal utility of wealth, have been a starting point for the present paper. A question which comes to the forefront when discussing such systems is to what extent substitution elasticities obtained from data with spatial variation, across commodities, agree with substitution elasticities obtained by exploring data with temporal variation, across time periods, for single commodities or for aggregates of them. More precisely, and stated in econometric terms: when can estimates representing spatial substitution be interchanged with estimates relevant for temporal substitution, and vice versa?

Are there *ceteris paribus* clauses or issues related to autonomy and invariance that should be considered?

In the paper, decompositions of the ‘total’ Hicks-Allen substitution, as following from Slutsky type of substitution in classical demand analysis, into two components: ‘Frisch substitution’, also called ‘specific substitution’, and ‘general substitution’, in the Houthakker-Theil sense, has been repeatedly exploited. The former refers to the ‘genuine substitution’ that follows ‘directly’ from the form of the utility indicator function, and in Frisch’s terminology represented by ‘want elasticities’. The latter refers to substitution following from the commodities’ ‘competition within the budget’. While the former may be, and in practice often is, eliminated for certain commodity combinations through parameter-saving restrictions, the latter can never be made to disappear (although it usually does not involve specific parameters). Of particular importance is the fact that the inverse of the ‘flexibility of the marginal utility of income’ (strictly a cardinal concept), often labeled the ‘Frisch parameter’, is a core element in mathematical expressions of both kinds of substitution. Specific substitution has a different meaning relative to the ‘Frisch parameter’ under additive separability than in less restrictive cases, which should affect procedures for its estimation both in parametric and non-parametric contexts. In the latter role, however, the ‘Frisch parameter’ does not presume a cardinalistic interpretation of utility, which is probably a primary reason for its popularity.

Ideally, though depending on purpose, scale effects, represented by Engel elasticities, and substitution effects, represented by Hicks-Allen elasticities and their components, should be allowed for jointly in modeling. For some applications it may be worthwhile to ‘isolate’ one of them. The literature dealing with parametric specifications give numerous examples as specific parameter restrictions, see *e.g.*, Crossley and Low (2011, Section 2), Pakoš (2011, Section 2 and Appendix A) and Thimme (2016, pp. 228 and 239), for a few recent examples. Sometimes, parametric utility functions, or utility elements, are restricted to be additively separable and/or homothetic. Additive separability, in multi-period, multi-commodity contexts, restricts Frisch cross-elasticities, some of them or all, to be zero, between commodities, between periods or both.<sup>21</sup> Full homotheticity restricts all Engel elasticities to equal one. The latter is, in a multi-commodity context, a quite unrealistic implication which, with reference to, *e.g.*, ‘Engel’s Law’ has been rejected in much empirical work. The relevance of imposing homotheticity on the dynamic utility function in a one-commodity context – neglecting any ‘scale effects’, say over a consumer’s life-cycle, by making all Engel elasticities equal to unity – is still a contested issue, notably when durable goods are involved.

We finish by briefly stating some more specific conclusions: *First*, depending on the commodity grouping, specifications with additive separability across commodi-

---

<sup>21</sup>Crossley and Low (2011, Corollary 1) for example shows that if (i) the intertemporal utility function is additive in period-specific ‘felicity functions’, corresponding to (7.1), (ii) the latter have a certain PIGL or PIGLOG form and (iii) the within period preferences are nonhomothetic, then the elasticity of substitution is identified by the shape of the Engel functions.

ties and non-homotheticity over periods for each commodity are worth considering. *Second*, other forms of restrictive block-additive separability over either time-periods or commodities gives parameter-saving specifications also worth considering. *Third*, when extending a static model to a dynamic one and attempting to ‘translate’ commodity-related elasticities of substitution into temporal elasticities, the Hicksian elasticity of expectation emerges as an interesting concept. *Fourth*, omission of Hicks-Allen or Frisch direct substitution elasticities may be relevant in defining aggregate substitution elasticities. The elasticities obtained are somewhat related to Morishima elasticities. *Fifth*, an appropriate weighting by budget shares is essential to construct from the commodity-temporal level expressions interpretable as aggregate Engel and substitution elasticities. *Sixth*, the Frisch parameter in the income and in the wealth sense cannot, in general, be interchanged. In particular cases, *e.g.*, with additive separability of the utility function over periods and commodities, the translation raises few problems. Otherwise, giving attention to *ceteris paribus* clauses and autonomy (in the econometric sense) is highly recommendable.

#### REFERENCES

- Allen, R.G.D. (1938): *Mathematical Analysis for Economists*. London: Macmillan.
- Arrow, K.J. and Nerlove, M. (1958): A Note on Expectations and Stability. *Econometrica* **26**, 297–305.
- Barnett, V. (2007): The Contextual Sources of Slutsky’s Effect: 1915, 1927, and after. *Journal of the History of Economic Thought* **29**, 403–416.
- Barnett, W.A. and Serletis, A. (2008): Consumer Preferences and Demand Systems. *Journal of Econometrics* **147**, 210–224.
- Barten, A.P. (1964): Consumer Demand Functions under Conditions of Almost Additive Preferences. *Econometrica* **32**, 1–38.
- Bjerkholt, O. (2015): Eugen E. Slutsky’s 1915 Essay: Found But Never Lost. Unpublished paper, August 5, 2015.
- Blackorby, C. and Russell, R.R. (1981): The Morishima Elasticity of Substitution; Symmetry, Constancy, Separability, and its Relationship to the Hicks and Allen Elasticities. *Review of Economic Studies* **48**, 147–158.
- Blackorby, C. and Russell, R.R. (1989): Will the Real Elasticity of Substitution Please Stand Up? (A Comparison of the Allen/Uzawa and Morishima Elasticities.) *American Economic Review* **79**, 882–888.
- Blundell, R. (1999): Consumer Demand and Intertemporal Allocations: Engel, Slutsky and Frisch. Chapter 5 in: Strøm, S. (ed.): *Econometrics and Economic Theory in the 20th Century. The Ragnar Frisch Centennial Symposium*. Cambridge: Cambridge University Press.
- Browning, M. (1999): A Simple Nonadditive Preference Structure for Models of Household Behavior over Time. *Journal of Political Economy* **99**, 607–637.
- Browning, M. (2005): A Working Paper From April 1985: Which Demand Elasticities Do We Know and Which Do We Need to Know for Policy Analysis? *Research in Economics* **59**, 293–320.
- Browning, M., Deaton, A. and Irish, M. (1985): A Profitable Approach to Labor Supply and Commodity Demands over the Life-Cycle. *Econometrica* **53**, 503–544.
- Crossley, T.F. and Low, H.W. (2011): Is the Elasticity of Intertemporal Substitution Constant? *Journal of the European Economic Association* **9**, 87–105.

- Diewert, E. (1974): A Note on Aggregation and Elasticities of Substitution. *Canadian Journal of Economics* **7**, 12–20.
- Frisch, R. (1932): *New Methods of Measuring Marginal Utility*. Tübingen: Mohr (Paul Siebeck).
- Frisch, R. (1936): Annual Survey of General Economic Theory: The Problem of Index Numbers. *Econometrica* **4**, 1–38.
- Frisch, R. (1959): A Complete Scheme for Computing All Direct and Cross Demand Elasticities in a Model with Many Sectors. *Econometrica* **27**, 177–196.
- Hicks J. (1939): *Value and Capital*. Oxford: Clarendon Press
- Hoel, M. (1975): A Note on the Estimation of the Elasticity of the Marginal Utility of Consumption. *European Economic Review* **6**, 411–415.
- Houthakker, H.S. (1960): Additive Preferences. *Econometrica* **28**, 244–257.
- Kim, H.Y. (1993): Frisch Demand Functions and Intertemporal Substitution in Consumption. *Journal of Money, Credit and Banking* **25**, 445–454.
- Lachman, L.M. (1945): A Note on the Elasticity of Expectations. *Economica* **12**, 248–253.
- McFadden, D. (1963): Constant Elasticity of Substitution Production Functions. *Review of Economic Studies* **30**, 73–83.
- McLaughlin, K.J. (1995): Intertemporal Substitution and  $\lambda$ -Constant Comparative Statics. *Journal of Monetary Economics* **35**, 193–213.
- Miyagiwa, K. and Papageorgiou, C. (2007): Endogenous Aggregate Elasticity of Substitution. *Journal of Economic Dynamics and Control* **31**, 2899–2919.
- Ozga, S.A. (1975): *Expectations in Economic Theory*. London: Aldline Transactions.
- Pakoš, M. (2011): Estimating Intertemporal and Intratemporal Substitutions When Both Income and Substitution Effects are Present: The Role of Durable Goods. *Journal of Business & Economic Statistics* **29**, 439–454.
- Slutsky, E.E. (1915): Sulla Teoria Del Bilancio Del Consumatore. *Giornale Degli Economisti e Rivista di Statistica* **51**, 1–26. (English translation by Olga Ragusa (2012): On the Theory of the Budget of the Consumer. *Giornale degli Economisti e Annali di Economia, Nuova Serie* **71**, 173–200.)
- Theil, H. (1975): *Theory and Measurement of Consumer Demand. Vol. 1*. Amsterdam: Elsevier.
- Thimme, J. (2017): Intertemporal Substitution in Consumption: A Literature Review. *Journal of Economic Surveys* **31**, 226–257.
- Uzawa, H. (1962): Production Functions with Constant Elasticities of Substitution. *Review of Economic Studies* **29**, 291–299.