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## On the Mechanics of Economic Convergence

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## On the Mechanics of Economic Convergence

Thomas M. Steger

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Economics Working Paper Series



Eidgenössische Technische Hochschule Zürich  
Swiss Federal Institute of Technology Zurich

# On the Mechanics of Economic Convergence<sup>\*</sup>

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ETH Zurich

June 2003

In macroeconomic dynamic models the speed at which output converges to its steady state is of outstanding interest. Theoretical investigations usually focus on the asymptotic speed of convergence only. This procedure is, however, unnecessarily restrictive and hides important information. The paper at hand provides a straightforward and simple analytical decomposition of the instantaneous rate of convergence into its economic determinants. In addition, the resulting convergence-accounting formula is applied to analyse the transition process of a general R&D-based endogenous growth model. As a result, the driving forces behind the convergence process are identified.

*Keywords:* Convergence accounting, rate of convergence, decomposition, convergence mechanisms, R&D-based growth

*JEL classification:* O0; O3; O31; O33; O41

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## 1. Introduction

In the context of growth theory and, more generally, macroeconomic dynamic theory the speed at which output converges to its steady state is of outstanding interest. The two most prominent reasons in favour of this proposition are the following. First, the positive and normative implications of the dynamic model under study along the transition path can differ dramatically from those along the balanced growth path (e.g., Jones, 1995). In this respect it is important to notice that the relative importance of transitional vis-à-vis balanced growth dynamics is crucially determined by the speed of convergence.<sup>1</sup> Second, a number of authors have investigated the dynamics of international or interregional income disparities by employing the  $\beta$ -convergence framework (e.g., de la Fuente, 2002). According to this approach, the time span which is required to reduce the initial gap in income once more depends essentially on the rate of convergence.

Theoretical investigations which intend to assess the speed of convergence implied by the dynamic model under study usually focus on the asymptotic rate of convergence. For models with a one-dimensional stable manifold this is given by the unique stable eigenvalue of the underlying linearised dynamic system (e.g., Ortigueira and Santos, 1997). In the case of multi-dimensional stable manifolds, the asymptotic rate of convergence is usually approximated by the smaller, in absolute terms, of the stable eigenvalues (e.g., Eicher and Turnovsky, 2001). The focus on this quantity alone, however, is unnecessarily restrictive and hides important information on the mechanisms behind the convergence process.

The paper at hand comprises two main parts. The first part provides a fairly simple analytical decomposition of the instantaneous rate of convergence. The basic idea follows the pioneering contribution of Solow (1957) who introduces the analytical tool of growth accounting. This fundamental procedure breaks down the growth rate of aggregate output into contributions from the growth of inputs.<sup>2</sup> Within the underlying paper this basic idea is applied to the rate of convergence. The resulting convergence-accounting formula yields a decomposition of the rate of convergence into its economic determinants. The second part of the paper is concerned with an application of the decomposition to analyse the convergence process of a general R&D-based endogenous growth model. This multi-sector growth model is clearly appropriate to demonstrate the usefulness of the convergence-accounting formula

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<sup>1</sup> The second determinant consists in the frequency and severity of macroeconomic shocks (e.g., Ben-David and Papell, 1995).

<sup>2</sup> See Barro (1999) for a discussion of modern versions of the growth-accounting framework.

since there are different convergence mechanisms at work. In addition, numerical methods are employed in order to detect the relative importance of the different mechanisms behind the convergence process.

Finally, it should be noticed explicitly that the focus here is on the transition process, whereas the majority of contributions to growth theory have analysed the properties of the balanced growth path. The underlying perspective is best described by the following claim formulated by Temple (2003, Section 6): *Ultimately, all that a long-run equilibrium of a model denotes is its final resting point, perhaps very distant in the future. We know very little about this destination, and should be paying more attention to the journey.*

## 2. A simple analytical decomposition of the rate of convergence

This section is concerned with the development of a straightforward and simple analytical decomposition formula for the rate of convergence (ROC). On this occasion, we focus on the speed at which final output converges towards its balanced growth path (BGP). Moreover, since the state of an economy can be summarised by the level of final output, the focus is on the ROC of final output, which is labelled the overall ROC.

The subsequent decomposition comprises two main steps. The first relies on the well-known growth accounting formula and yields an exact relation. In contrast, the second step requires to focus on the linearised dynamic system and, therefore, yields a local approximation of the ROC. Despite this fact, however, the two-step decomposition proposed in this paper allows a deeper insight into the ultimate causes of the convergence process than the usual approximations.

The speed at which final output converges to its BGP can be measured by the instantaneous ROC  $\psi_y(t) := -\frac{\dot{y}(t) - \dot{\tilde{y}}(t)}{y(t) - \tilde{y}(t)}$ , where  $y(t)$  is (final) output at time  $t$ .<sup>3</sup> A tilde above a variable denotes its value along the BGP and a dot its rate of change during a small period of time, i.e.,  $\dot{y}(t) := \frac{dy(t)}{dt}$ . Provided that the dynamic system under study is formulated

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<sup>3</sup> In the following “final output” is often simply denoted as “output” provided that no ambiguity results.

in stationary variables, the preceding definition can be simplified to read  $\psi_y := -\frac{\dot{y}}{y-\tilde{y}}$ .<sup>4</sup> A positive (negative) ROC indicates an economy which converges to (diverges from) its BGP.

As usual, output is considered to result from a number of input factors (denoted by  $x_i$ ) according to the production technology  $y = f(x_i)$  with  $i \in \{1, 2, \dots, n\}$ . In order to analyse the ROC of output we, therefore, have to focus on the dynamic system governing the evolution of the input factors over time:

$$\dot{x}_i = g_i(x_i, c_u) \quad (1)$$

$$\dot{c}_u = h_u(x_i, c_u), \quad (2)$$

where  $x_i$  with  $i \in \{1, 2, \dots, n\}$  denotes a set of state variables,  $c_u$  with  $u \in \{1, 2, \dots, m\}$  denotes a set of control or costate variables and  $g_i(\cdot)$  and  $h_u(\cdot)$  are the respective flow functions. The dynamic system (1) and (2) is assumed to possess a unique stationary solution defined by  $g_i(x_i, c_u) = 0$  and  $h_u(x_i, c_u) = 0$ , which is labelled as  $\{\tilde{x}_i, \tilde{c}_u\}$ . Linearising (1) and (2) yields the Jacobian matrix ( $J$ ) of the system. The eigenvalues result from the solution to the characteristic equation  $|\tilde{J} - \lambda I| = 0$ , where  $\tilde{J}$  is the Jacobian matrix evaluated at  $\{\tilde{x}_i, \tilde{c}_u\}$ . It is further assumed that this characteristic equation yields  $n$  eigenvalues with negative real part, denoted as  $\lambda_j$  with  $j \in \{1, 2, \dots, n\}$ , and  $m$  eigenvalues with positive real part. Since the number of jump variables equals the number of unstable eigenvalues ( $m$ ) and the dimension of the state space equals the dimension of the stable manifold ( $n$ ), the equilibrium  $\{\tilde{x}_i, \tilde{c}_u\}$  is saddle-point stable and indeterminacy can be ruled out.

The main result of this paper can be stated as follows. The instantaneous ROC of final output given by  $y = f(x_i)$  with  $i \in \{1, 2, \dots, n\}$  along the (linear) stable manifold of dimension  $n$  can be decomposed by the subsequent convergence-accounting formula:

$$\psi_y \cong \sum_{i=1}^n \sum_{j=1}^n \sigma_i b_i a_{i,j} \lambda_j \quad (3)$$

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<sup>4</sup> To simplify notation the time index is omitted.

with

$$\sigma_i := \frac{\partial f(\cdot)}{\partial x_i} \frac{x_i}{f(\cdot)}$$

$$b_i := \left(1 - \frac{\tilde{x}_i}{x_i}\right) \bigg/ \left(1 - \frac{\tilde{y}}{y}\right)$$

$$a_{i,j} := \frac{B_j v_{i,j} e^{\lambda_j t}}{\sum_{j=1}^n B_j v_{i,j} e^{\lambda_j t}},$$

where  $B_j$  denote arbitrary constants of integration and  $v_{i,j}$  the elements of the eigenvector  $j$  associated with the stable eigenvalue  $\lambda_j$ . In order to grasp the structure of the decomposition formula it should be noticed that  $i$  indexes the state variables and  $j$  indexes the stable eigenvalues in the double sum on the RHS of equation (3).

The preceding proposition can be easily proved as follows. The starting point is the definition of the ROC of output  $\psi_y := -\frac{\dot{y}}{y - \tilde{y}}$ . Notice that we focus on the stationary dynamic system, i.e., we assume that the model is transformed into normalised (or scale-adjusted) variables.<sup>5</sup> At first, the RHS of this definition is tautologically extended to  $\psi_y = -\frac{\dot{y}/y}{1 - \tilde{y}/y}$ . In the next step, the growth rate of  $y$  is expressed by the well-known growth accounting equation which yields:

$$\psi_y = -\frac{\sum_{i=1}^n \sigma_i \frac{\dot{x}_i}{x_i}}{1 - \frac{\tilde{y}}{y}} \tag{4}$$

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<sup>5</sup> For the neo-classical growth model with (exogenous) technical progress this would require to express the dynamic system in units of effective labour.



with  $\sigma_i$  denoting the elasticity of input factor  $x_i$  in final output production, as defined in equation (3). At this stage it should be noted that there is a systematic relationship between a variable's growth rate and its ROC given by  $\frac{\dot{q}}{q} = -\frac{\dot{q}}{q-\tilde{q}} \left( -\frac{q-\tilde{q}}{q} \right) = -\psi_q \left( 1 - \frac{\tilde{q}}{q} \right)$ . Using this relation we can substitute the growth rate of the input factors  $x_i$  in equation (4) by the respective modified ROC which yields:

$$\psi_y = \sum_{i=1}^n \sigma_i b_i \psi_{x_i}, \quad (5)$$

where  $b_i$  is defined as in equation (3). Equation (5) represents the first step of the decomposition and is labelled Decomposition 1. This relation is already quite instructive since it decomposes the instantaneous ROC of output into the ROC of the input factors ( $\psi_{x_i}$ ) multiplied by appropriate weights respectively. This decomposition is still exact since it does not require any (linear) approximation.

If we are ready to focus on the linear stable manifold as an approximation of the non-linear stable manifold we can take the second step of the decomposition. The instantaneous ROC of  $x_i$  may then be expressed as follows:

$$\psi_{x_i} \cong \sum_{j=1}^n a_{i,j} \lambda_j, \quad (6)$$

where  $a_{i,j}$  is again defined as in equation (3). Equation (6) is labelled Decomposition 2. Substituting  $\psi_{x_i}$  in equation (5) by the RHS of equation (6) leads to the initially proposed convergence-accounting formula equation (3).

Finally, it should be noted that the asymptotic ROC of output is given by:

$$\lim_{t \rightarrow \infty} \psi_y := \tilde{\psi}_y \cong -\lambda_1, \quad (7)$$

where  $\lambda_1$  denotes the smallest, in absolute terms, of the stable eigenvalues. Equation (7) implies two kinds of approximations. First, by focusing on the linear solution the global convergence implications are approximated by the local convergence behaviour.<sup>6</sup> Second, in the case of multi-dimensional stable manifolds equation (7) describes the asymptotic ROC approximated by the smaller, in absolute terms, of the stable eigenvalues. Apart from these approximation considerations it should be clear that the determinants behind the (instantaneous) ROC should be investigated carefully if we want to enhance our understanding of out-of-steady-state dynamics.

### 3. Interpretation of the convergence-accounting formula

The convergence-accounting formula shown in equation (3) can be used to disentangle the driving forces behind the convergence process. Moreover, in quantitative applications (theoretical and empirical) the relative importance of the different convergence mechanisms can be identified. In order to give a clear economic interpretation of these mechanisms, the different components of the decomposition are explained in turn.

#### **Elasticities in final output production**

Let us start with the elasticities of the input factors  $x_i$  in the production of  $y$  denoted by  $\sigma_i$ .<sup>7</sup> This determinant points to the importance of the underlying final output technology for the overall ROC. To fully understand the role of this component we need two pieces of information. First, it is well known from growth accounting that the contribution of an input factor to the growth rate of output increases with the respective elasticity of production. Second, as already shown, the relation between a variable's rate of growth and its ROC is given by  $\frac{\dot{q}}{q} = \psi_q \left( \frac{\tilde{q}}{q} - 1 \right)$ . This implies that an increase in the growth rate is accompanied by a rise in the ROC provided that the proportional distance from the steady state is held constant. Taking both arguments together shows that growth in  $x_i$  along the transition path contributes

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<sup>6</sup> Based on a human-capital growth model, Ortigueira and Santos (1997) show that the local rate of convergence provides a valid estimate for the global convergence behaviour. On the other hand, using a linear model with subsistence consumption Steger (2000) shows that the local rate of convergence is not a good estimate for the global convergence behaviour.

<sup>7</sup> These are exogenous constants in the Cobb-Douglas case and a function of the input vector in the CES case.

stronger to growth in  $y$ , the larger is  $\sigma_i$ . Moreover, a higher growth rate of  $y$  translates, *ceteris paribus*, into a higher ROC of  $y$ .

It is clearly instructive to notice that the decomposition conducted here can be equally applied to explicit multi-sectoral models. Let us assume that there is still a single final output good which can be used for consumption or investment. Final output is produced by employing a row of input stocks (e.g., physical capital, technology and human capital). In addition, it is quite natural to assume that these inputs are themselves produced by employing the same array of inputs. The (intensive) production function of final output may then be expressed as  $y = f(\theta_i x_i)$  with  $\theta_i$  ( $0 \leq \theta_i \leq 1$ ) denoting intersectoral allocation variables, which give the respective share of resource  $x_i$  allocated to final output production.<sup>8</sup> In this case, equation (5) has to be extended to include the ROC of the allocation variables ( $\psi_{\theta_i}$ ) in addition to the ROC of the input factors ( $\psi_{x_i}$ ):

$$\psi_y = \sum_{i=1}^n \sigma_i b_i \psi_{x_i} + \sum_{i=1}^n \sigma_i d_i \psi_{\theta_i} \quad (8)$$

with

$$d_i := \left(1 - \frac{\tilde{\theta}_i}{\theta_i}\right) \bigg/ \left(1 - \frac{\tilde{y}}{y}\right)$$

The proposed decomposition, therefore, indicates that two fundamental convergence mechanisms should be distinguished. On the one hand, a shock which pushes the economy out of its steady state induces an accumulation (or decumulation) of reproducible resources (the accumulation-decumulation mechanism). On the other hand, the intersectoral reallocation of resources represents a second convergence mechanism (the resource-reallocation mechanism). This mechanism is captured and can be quantified by the terms  $\sigma_i d_i \psi_{\theta_i}$  in equation (8).<sup>9</sup>

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<sup>8</sup> To clarify the corresponding empirical concepts: Gross output value is given by  $y + \sum \dot{x}_i$ , while gross value added amounts simply to  $y$ .

<sup>9</sup> Formally, both mechanisms contribute directly to convergence of output and should, therefore, be considered as equally important in this respect. From an economic perspective, however, the

## Distance from the steady state

The second component of the decomposition formula [equation (3)] is the proportional distance of  $x_i$  from its steady state in relation to the proportional distance of  $y$  from its steady state as expressed by  $b_i := \left(1 - \frac{\tilde{x}_i}{x_i}\right) / \left(1 - \frac{\tilde{y}}{y}\right)$ . This “distance from the steady state” is mainly determined by the specific shock under consideration; to be precise, this holds true for the initial distance from the steady state. The interpretation of this component is as follows. The absolute movement of  $x_i$  towards its steady state increases, holding the ROC of  $x_i$  constant, with the proportional distance of  $x_i$  from its steady state, as expressed by the nominator of  $b_i$ . Moreover, the contribution of this absolute movement to the ROC of  $y$  is larger, the smaller the average proportional gap of all input factors. The average proportional gap of all input factors is expressed by the denominator of  $b_i$ .

## Direction of the deviation from the steady state

To understand the meaning of the weights  $a_{i,j} := \frac{B_j v_{i,j} e^{\lambda_j t}}{\sum_{j=1}^n B_j v_{i,j} e^{\lambda_j t}}$  let us consider a simple

example with two input factors, i.e.,  $y = f(x_1, x_2)$ . Assume further that the stable manifold is of dimension two. The stable eigenvalues are denoted by  $\lambda_2 < \lambda_1 < 0$ . The solution to the underlying (linearised) dynamic system for  $x_1$  and  $x_2$  is of the following shape:  $x_1 = B_1 v_{1,1} e^{\lambda_1 t} + B_2 v_{1,2} e^{\lambda_2 t} + \tilde{x}_1$  and  $x_2 = B_1 v_{2,1} e^{\lambda_1 t} + B_2 v_{2,2} e^{\lambda_2 t} + \tilde{x}_2$ . The instantaneous ROC of  $x_1$ , say, can hence be expressed as  $\psi_{x_1} \cong -(a_{1,1} \lambda_1 + a_{1,2} \lambda_2)$  with  $a_{1,1} = \frac{B_1 v_{1,1} e^{\lambda_1 t}}{B_1 v_{1,1} e^{\lambda_1 t} + B_2 v_{1,2} e^{\lambda_2 t}}$

and  $a_{1,2} = \frac{B_2 v_{1,2} e^{\lambda_2 t}}{B_1 v_{1,1} e^{\lambda_1 t} + B_2 v_{1,2} e^{\lambda_2 t}}$ . Since  $\lambda_2 < \lambda_1 < 0$ , it follows that  $\lim_{t \rightarrow \infty} a_{1,1} = 1$  and  $\lim_{t \rightarrow \infty} a_{1,2} = 0$ . This implies, not surprisingly, that the asymptotic ROC (valid for  $x_1$ ,  $x_2$  and  $y$ ) is given by  $\lim_{t \rightarrow \infty} \psi_{x_1} \cong -\lambda_1$ . Moreover, the arbitrary constants of integration ( $B_1$  and  $B_2$ ) are crucially determined by the system’s initial deviation from its steady state.<sup>10</sup> The

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resource-reallocation mechanism is of instrumental character with respect to the accumulation or decumulation of resources. For details on this point see Section 4.4.

<sup>10</sup> In addition, the arbitrary constants of integration depend on the complete set of eigenvectors of the system.

initial deviations  $\{x_i(0) - \tilde{x}_i, c_u(0) - \tilde{c}_u\}$  result from the specific shock under consideration, which gives rise to the process of convergence.

The weights  $a_{i,j}$  account for the fact that it is not only the magnitude of the initial deviation from the steady state that matters. In addition, the direction of this initial deviation turns out to be of major importance. More specifically, if the shock under consideration moves the economy primarily in the direction given by the eigenvector associated with the larger, in absolute terms, of the stable eigenvalues, then the subsequent convergence process is faster and vice versa. This points to the fact, not considered in the literature so far, that the type of the specific shock under consideration might be crucial for the speed at which an economy converges to its steady state.

### **Stable eigenvalues**

Finally, the ROC depends, of course, on the stable eigenvalues  $\lambda_j$ . These quantities are the ultimate measures of the speed at which an exponential decay process proceeds. In addition, it should be observed that the eigenvalues are themselves endogenously determined by preferences and technologies (e.g., Barro and Sala-i-Martin, 1992).

## **4. Application to a general R&D-based model of endogenous growth**

In order to demonstrate the usefulness of the decomposition described above, the convergence-accounting formula is applied to investigate the convergence process of a multi-sector growth model. On this occasion, a fairly general R&D-based endogenous growth model is employed as an example.

### **4.1. The basic structure of the model**

The market equilibrium of a general R&D-based endogenous growth model of the increasing-variety type is considered. The model is general in the sense that each factor of production (labour, capital and technological knowledge) is productive in each sector (final output, intermediate goods and R&D).<sup>11</sup>

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<sup>11</sup> Specific models which are included in this class comprise the first-generation of R&D-based growth models like the original Romer (1990) model, the non-scale models of Jones (1995) and Eicher and Turnovsky (1999b, 2001).

The production side consists of three sectors. First, the final output sector produces a homogeneous good that can be used for consumption or investment purposes. Second, the intermediate goods sector produces differentiated intermediate goods that serve as inputs in the production of final output. Third, the R&D sector searches for ideas (designs), which are the technical prerequisite to produce new intermediate goods. Households choose their level of consumption and inelastically supply one unit of labour at every point in time.

The state variables are the stock of physical capital ( $K$ ) and the number of designs ( $A$ ). The model comprises three choice variables, namely the level of consumption ( $C$ ), the share of labour ( $\theta$ ) and the share of capital ( $\phi$ ) devoted to the production of final output.<sup>12</sup> Finally, since we have three distinct types of goods, there are three prices. Final output serves as the numeraire, its price is set equal to unity. The price of intermediate goods is denoted by  $p$  and the price of designs by  $v$ , respectively. The dimension of the dynamic system can be reduced by eliminating the price of intermediate goods ( $p$ ).

## 4.2. The model framework

The reduced form of the general R&D-based endogenous growth model consists in the following dynamic system (an appendix available upon request provides the details of the model).

$$\dot{k} = y - c - \delta_K k - \beta_K n k \quad (9)$$

$$\dot{a} = j - \beta_A n a \quad (10)$$

$$\dot{c} = \frac{c}{\gamma} [r - \delta_K - \rho - (1 - \gamma)n] - \beta_K n c \quad (11)$$

$$\dot{v}_a = v_a [r - \delta_K - (\beta_K - \beta_A)n] - \pi \quad (12)$$

$$\sigma_L \frac{y}{\theta} = v_a \frac{\eta_L^p j}{1 - \theta} \quad (13)$$

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<sup>12</sup> It might appear confusing at first glance that there are three sectors but only two intersectoral allocation variables. However, it should be noticed that the final output sector and the intermediate goods sector use essentially the same technology. Moreover, the consumption rate determines the allocation of resources to the production of consumption goods and intermediate goods (i.e., capital goods).

$$\sigma_K \frac{y}{\phi} = v_a \frac{\eta_K^p j}{1-\phi} \quad (14)$$

with

$$r := \frac{\sigma_K^2 y}{\phi k} \quad \text{and} \quad \pi := \frac{(1-\sigma_K)\sigma_K y}{\phi a}$$

**TABLE 1**

The notation is explained in Table 1. The dynamic system displayed above is formulated in scale-adjusted variables, which are defined as follows:  $y := Y/L^{\beta_K}$ ,  $k := K/L^{\beta_K}$ ,  $c := C/L^{\beta_K}$ ,  $a := A/L^{\beta_A}$ ,  $j := J/L^{\beta_A}$ ,  $v_a := v/L^{\beta_K-\beta_A}$ . As a consequence, the (unique) BGP corresponds to the (unique) stationary solution of the above system [resulting from  $\hat{k} = \hat{a} = \hat{c} = \hat{v}_a = 0$  together with (13) and (14)]. The balanced growth rates are given by

$$\hat{Y} = \hat{K} = \hat{C} = \beta_K n \quad \text{and} \quad \hat{A} = \beta_A n \quad \text{with} \quad \beta_K := \frac{\sigma_L(1-\eta_A) + \eta_L \sigma_A}{(1-\eta_A)(1-\sigma_K) - \eta_K \sigma_A} \quad \text{and}$$

$$\beta_A := \frac{\eta_L(1-\sigma_K) + \eta_K \sigma_L}{(1-\eta_A)(1-\sigma_K) - \eta_K \sigma_A}. \quad \text{Due to the existence of duplication externalities in R\&D, the}$$

social elasticities of the private inputs ( $\eta_L$  and  $\eta_K$ ) are a composite of private elasticities ( $\eta_L^p$  and  $\eta_K^p$ ) and external elasticities ( $\eta_L^e$  and  $\eta_K^e$ ), i.e.,  $\eta_L := \eta_L^p + \eta_L^e$  and  $\eta_K := \eta_K^p + \eta_K^e$ .

Equations (9) and (10) show the equations of motion of physical and technological capital. Equation (11) is the Keynes-Ramsey rule of optimal consumption. The dynamics of the price of designs is given by equation (12). Finally, equations (13) and (14) are the conditions for an efficient intersectoral allocation of labour and physical capital.

### 4.3. Parameterisation and calibration

The production functions of the final-output and R&D sector are assumed to be of the following shape (the variables are expressed in original form, not in scale-adjusted form).

$$Y = F(.) = \alpha_F A^{\sigma_A} (\theta L)^{\sigma_L} (\phi K)^{\sigma_K} \quad (15)$$

with  $\alpha_F, \sigma_A, \sigma_L, \sigma_K > 0$ ;  $\sigma_L + \sigma_K = 1$

$$\dot{A} = J(.) = \alpha_J A^{\eta_A} [(1-\theta)L]^{\eta_L} [(1-\phi)K]^{\eta_K} \quad (16)$$

with

$$\alpha_J, \eta_A, \eta_L, \eta_K > 0; 0 < \eta_L + \eta_K \leq 1; (\eta_L^p + \eta_K^p = 1, -1 < \eta_L^e + \eta_K^e \leq 0)$$

Equation (15) shows the final-output technology and equation (16) displays the R&D technology, where  $\dot{A}$  represents output of the R&D sector. From (15) and (16) together with the definition of scale-adjusted variables as well as  $\beta_K$  and  $\beta_A$  one can derive the production functions in scale-adjusted variables  $y = \alpha_F a^{\sigma_A} \theta^{\sigma_L} (\phi k)^{\sigma_K}$  and  $j = \alpha_J a^{\eta_A} (1-\theta)^{\eta_L} [(1-\phi)k]^{\eta_K}$ .

Table 2 shows the set of parameters which underlies the numerical investigations. In general, those parameters which have close real-world counterparts are specified according to empirical estimates. The remaining parameters are chosen to lie within (theoretically) plausible ranges such that the growth rate of per capita output and per capita technology result in empirically plausible values. The set of parameters shown in Table 2 is very similar to those used in previous exercises (e.g., Lucas 1988; Ortigueira and Santos, 1997; Jones and Williams, 2000; Eicher and Turnovsky, 2001).

**TABLE 2**

With respect to the empirical plausibility of the underlying set of parameters several aspects are worth being noted. First, following Eicher and Turnovsky (2001, p. 100) both sectors are characterised by mildly increasing returns to scale in all three factors of production:  $\sigma_A + \sigma_L + \sigma_K = 1.20$  and  $\eta_A + \eta_L + \eta_K = 1.24$ . Moreover, Jones and Williams (2000, p. 74) argue that the social elasticity of the private inputs in R&D ( $\eta_L + \eta_K$ ) should lie within the range of 0.5 and 1; the baseline set of parameters implies  $\eta_L + \eta_K = 0.7$ .

Finally, and most importantly, the underlying set of parameters results in a balanced growth rate of final output given by  $\hat{Y} \cong 0.025$ . In addition, the implied growth rate of total factor productivity (TFP) amounts to  $\sigma_A \hat{A} \cong 0.006$ . These values are roughly in line with the empirical picture on growth in industrialised countries. The average growth rate of output in the G7 economies for the period 1980 to 2000 amounts to 2.7 %. Moreover, available



evidence on the growth rate of TFP in the G7 (1980 to 2000) yields values of about 0.9 % (Colecchia and Schreyer, 2002).<sup>13</sup>

## 4.4. Results

### 4.4.1. Basic quantitative implications and description of the transition process

The general R&D-based growth model parameterised and calibrated as described in the preceding section yields surprisingly plausible quantitative implications. Table 3 shows the growth rates  $(\hat{Y}, \sigma_A \hat{A})$ , key economic ratios  $(\widetilde{Y/K}, \widetilde{C/Y})$  and intersectoral allocation variables  $(\tilde{\theta}, \tilde{\phi})$  along the BGP as well as the asymptotic ROC of per capita output  $(\tilde{\psi}_{Y/L})$ .<sup>14</sup>

The ROC of per capita output amounts to 0.0084 implying a half-life of about 80 years. According to this result, transitional dynamics are of major importance. However, this value gives the asymptotic ROC. The instantaneous ROC of (scale-adjusted) output  $(\psi_y)$  is clearly variable along the transition as displayed in Figure 1. Since the instantaneous ROC falls along the transition, the average ROC is higher than the asymptotic ROC.

**TABLE 3**

**FIGURE 1**

Before decomposing the ROC of output into its economic determinants, the underlying transition process is described concisely. Figure 2 (a) shows the trajectory in  $(k, a)$ -plane. The dotted lines display the  $\dot{k} = 0$ -locus and the  $\dot{a} = 0$ -locus with the first one having the higher slope. Both state variables ( $k$  and  $a$ ) converge monotonically. Notice that an increase in scale-adjusted capital, for example, means that capital (or capital per capita) grows at a rate which is higher than the balanced growth rate.

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<sup>13</sup> Moreover, Jones and Williams (2000) report that the average growth rate of TFP for the U.S. from 1948 to 1997 amounted to 1.2 %. The fact that the implied growth rate of TFP is below the empirical values of 0.9 % and 1.2 % does not represent a problem. The reason lies in the fact that empirical estimates of TFP growth usually overestimate the contribution of technical progress since TFP also increases as a result of efficiency changes and economies of scale.

<sup>14</sup> The link between the ROC of scale-adjusted output  $(y := Y/L^{\beta_k})$  and per capita output  $(Y/L)$  is given by  $\tilde{\psi}_{Y/L} = \tilde{\psi}_y - (\beta_k - 1)n$ .

The economic intuition behind the transition process can be sketched as follows. The source of this adjustment is a permanent increase in the final-output technology parameter ( $\alpha_F$ ) from 0.5 to 1. This sudden increase leads to a massive rise in final output and, therefore, in the accumulation of capital. Furthermore, the price of innovations in terms of final output increases as displayed in Figure 2 (b); notice the initial jump.<sup>15</sup> This change in relative prices in turn induces a reallocation of resources from the final-output sector to the R&D sector as shown in Figure 2 (c) and (d). As more and more resources are reallocated to the R&D sector and output of this sector increases, the price of innovations reaches a maximum and eventually starts to decline. This price movement reverses the resource reallocation process, i.e., the intersectoral allocation variables start to increase. In the long run, the allocation variables return to their initial balanced-growth equilibrium levels; this result is due to the general non-scale character of the underlying growth model.

## FIGURE 2

### 4.4.2. Decomposing the rate of convergence

Let us now turn to the decomposition of the overall ROC. Following the convergence-accounting formula developed in Section 2 two steps of the decomposition are distinguished.

#### **Decomposition 1**

For the multi-sectoral growth model under study Decomposition 1 should be expressed as  $\psi_y = \sum_{i=1}^n \sigma_i b_i \psi_{x_i} + \sum_{i=1}^n \sigma_i d_i \psi_{\theta_i}$ . This decomposition is illustrated by Figure 3. The instantaneous ROC of (scale-adjusted) output is decomposed into a linear combination of the instantaneous ROC of the input factors ( $\psi_k, \psi_a$ ) and the allocation variables ( $\psi_\theta, \psi_\phi$ ).<sup>16</sup> All ROC approach their common long run value, i.e.,  $\tilde{\psi}_k = \tilde{\psi}_a = \tilde{\psi}_\theta = \tilde{\psi}_\phi = -\lambda_1 = 0.017$ , where

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<sup>15</sup> Two mechanisms are behind this price movement: First, the increase in  $\alpha_F$  reduces marginal costs of final-output production. A negatively sloped demand curve for final output implies a falling price of final output goods. Second, since productivity of technology in final output production rises, the price of technology increases.

<sup>16</sup> The term “scale-adjusted” is omitted in the following provided that no ambiguity arises.

$\lim_{t \rightarrow \infty} \psi_k := \tilde{\psi}_k$  [Figure 3 (b), (d), (f) and (h)].<sup>17</sup> The time path of the weights ( $\sigma_K b_k$ ,  $\sigma_A b_a$ ,  $\sigma_L d_l$  and  $\sigma_K d_k$ ) are shown in the left column of Figure 3 [plot (a), (c), (e) and (g)].

From Figure 3 (b) and (d) it is obvious that, at any instant of time, capital converges much faster than technology. This is in line with the adjustment process shown in Figure 2 (a), which displays a convex curve. The observed pattern of adjustment may hence be described as follows. The convergence process is mainly driven by a strengthened capital accumulation due to a favourable technology shock and is supported by a temporary intensification of R&D activities.

To fully understand this pattern of adjustment two points should be noticed. First, the shock under study determines the initial conditions relative to the new steady state. Due to the permanent increase in  $\alpha_F$  the long-run level of both capital and technology rises. This follows immediately from the equations of motion (9) and (10). The rise in  $\alpha_F$  increases output in (9). To realise a long-run equilibrium,  $k$  must also increase until the balanced-growth condition  $\dot{k} = 0$  is once more satisfied. Since  $k$  is also productive in the R&D sector, an increase in  $k$  in turn rises R&D output ( $j$ ).<sup>18</sup> Equation (10) requires  $a$  to increase until  $\dot{a} = 0$  is equally satisfied. Therefore, the shock under consideration directly influences the equation of motion of  $k$  and indirectly influences the equation of motion of  $a$ . As a result, the (proportional) long-run increase in  $k$  exceeds the (proportional) long-run increase in  $a$  [Figure 2 (a)].

To identify the second determinant of this adjustment pattern notice that the dynamic system under consideration, parameterised and calibrated as described in Section 4.3., exhibits two negative eigenvalues denoted as  $\lambda_2 < \lambda_1 < 0$ . It follows that all trajectories must approach the steady state from the direction given by the eigenvector associated with the smaller, in absolute terms, of the stable eigenvalues. For the area below the steady state, the slope of this direction lies between the slope of the  $\dot{a} = 0$ -locus and the  $\dot{k} = 0$ -locus [Figure 2 (a)].

The time paths of the ROC of the intersectoral allocation variables appear to be identical. The striking feature lies in the negative ROC at the beginning of the transition [Figure 3 (f) and (h)]. This points to the fact that the respective allocation variables initially

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<sup>17</sup> There are two negative eigenvalues with  $\lambda_1$  denoting the smaller, in absolute terms, of the stable eigenvalues, i.e.,  $\lambda_2 < \lambda_1 < 0$ .

diverge from their long-run values [compare to Figure 2 (c) and (d)]. Although per capita output converges monotonically ( $\psi_y > 0$  for all  $t$ ), the resource-reallocation mechanism initially contributes negatively to the overall ROC. The economic intuition behind this pattern has been described above.

Figure 3 (a), (c), (e) and (g) displays the time paths of the weights of the ROC, which appear in Decomposition 1. It is obvious that the weights differ quite substantially in magnitude. More precisely, the weights associated with  $\psi_k$  and  $\psi_a$  are about ten times the weights associated with  $\psi_\theta$  and  $\psi_\phi$ . To understand the source of this difference compare the weight of  $\psi_k$  ( $\sigma_K b_k$ ) to the weight of  $\psi_\phi$  ( $\sigma_K d_k$ ). Since both contain the same parameter  $\sigma_K$ , the difference must stem from divergence in  $b_k$  and  $d_k$ . Recall that these show the (proportional) distance of the respective “input factor” (including allocation variables) relative to the overall (proportional) distance from the steady state. It follows that the difference in the weights ( $\sigma_K b_k$  and  $\sigma_K d_k$ ) is due to the fact that, at every point in time, the (proportional) distance of capital ( $k$ ) is higher than the (proportional) distance of the capital allocation variable ( $\phi$ ) from its steady state. In summary, although the ROC are of similar magnitude (ignoring the initial stage of the transition process), the accumulation-decumulation mechanism appears much more important than the resource-reallocation mechanism.

The finding that the resource-reallocation mechanism is of minor importance compared to the accumulation-decumulation mechanism requires two comments. First, the analysis conducted above is restricted to movements along the stable manifold only. The decomposition of the ROC does not take the contribution of the initial jump of the allocation variables into account. As a result, the importance of the resource-reallocation mechanism is underestimated. Second, the model under study does not capture costs of capital adjustment (Hayashi, 1982) nor does it include resource-reallocation costs. It is clear that the relative importance of these two types of costs is likely to influence the relative importance of the accumulation-decumulation vis-à-vis the resource reallocation mechanism.

Both the accumulation-decumulation and the resource-reallocation mechanism contribute directly to the convergence of output and should, therefore, be considered as equally important in this respect. From an economic perspective, however, the resource-reallocation mechanism is of instrumental character with respect to the accumulation

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<sup>18</sup> In case of capital being unproductive in R&D, the long-run level of technology is independent of the shock under consideration.

or decumulation of resources. Precisely, the intersectoral reallocation of resources is one of many possibilities to achieve the desired economy-wide combination of input stocks. In order to accomplish, say, an accumulation of physical capital there are different mechanisms available. Each of these mechanisms has an intratemporal as well as an intertemporal component and is associated with an intertemporal substitution of consumption. For the model under study, there are three such mechanisms: (1) the direct consumption-saving mechanism (output can be either consumed or invested in physical capital); (2) the intersectoral allocation of labour to final-output and R&D and (3) the intersectoral allocation of physical capital to final-output and R&D.

### FIGURE 3

#### Decomposition 2

Let us now turn to Decomposition 2 given by  $\psi_{x_i} \cong \sum_{j=1}^n a_{i,j} \lambda_j$ . The ROC of each “input factor” accordingly is a linear combination of the stable eigenvalues. The weights ( $a_{i,j}$ ) depend critically on the initial deviation of the input factors from their respective steady state values. Therefore, the shock under consideration determines the relative size of the weights.

In order to illustrate the effects of different initial conditions most clearly consider Figure 4, which shows the two-dimensional stable manifold projected into the  $(k, a)$ -plane. The variables have been transformed into their proportional distances from the stationary point, i.e.,  $a_p := \frac{a - \tilde{a}}{\tilde{a}}$  and  $k_p := \frac{k - \tilde{k}}{\tilde{k}}$ . As a consequence, the steady state is shifted to the origin. The dashed lines moving from south-west to north-east represent the  $\dot{k}_p = 0$ -locus and  $\dot{a}_p = 0$ -locus, respectively. A number of trajectories starting on a circle around the stationary solution with radius 0.5 are displayed. It can be recognised that a trajectory starting at point D, for example, converges more slowly compared to one starting at point E. Notice that the arrows indicate the state of the economy after the first 5 year intervals. This pattern results from the two-dimensional stable manifold with (stable) eigenvalues which differ quite substantially in magnitude. For the baseline set of parameters the stable eigenvalues are  $\lambda_1 = -0.017$  and  $\lambda_2 = -0.15$ . The second eigenvalue is nearly 10 times larger than the first. As a result, the economy converges comparably slow along the direction determined by the

eigenvector associated with  $\lambda_1$ . In contrast, convergence is comparably fast along the direction given by the eigenvector associated with  $\lambda_2$ . Of course, this is a rather mechanical explanation of the phenomenon under study. An explanation in economic terms would require to understand how the difference in the stable eigenvalues results from the technology and preference parameters of the underlying model. This task is left for future research.

#### FIGURE 4

Figure 4 illustrates another basic property of the transition process. It shows that along a two-dimensional stable manifold a wide range of different adjustment patterns is possible. Specifically, the trajectories starting at point A and D show monotonic adjustments. In contrast, the trajectories beginning at point B and E are characterised by non-monotonicity. Finally, the trajectories originating from point C and F display non-monotonicity together with overshooting.<sup>19</sup>

### 5. Summary and conclusion

The question whether real-world growth processes represent primarily transitional dynamics or balanced growth dynamics is one of the basic and still open research topics in growth theory and empirics (e.g., Ben-David and Papell, 1995; Jones, 2002). A lot of economic research has focused on the properties of the balanced growth path. On the other hand, our understanding of the transition path is clearly less developed. As a prerequisite for answering the question raised above it is, therefore, necessary to better understand the transition process. In order to accomplish this task, the paper at hand follows Solow (1957) who introduces the powerful tool of growth accounting. This procedure enables the identification of the driving forces behind growth of output.<sup>20</sup> By applying this basic idea to the issue of convergence, the determinants of the rate of convergence and the driving forces behind the convergence process are identified.

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<sup>19</sup> It must be stressed that the trajectories displayed in Figure 4 result from the solution to the linearised problem. The solution of the non-linear problem can be expected to be characterised by an even more pronounced degree of non-monotonicity and overshooting.

<sup>20</sup> Of course, growth accounting is not a substitute for a theory of economic growth. Nonetheless, it represents an important first step and leads to a better understanding of the sources of growth.

The main insights from the convergence-accounting formula may be summarised as follows. The instantaneous rate of convergence of final output depends on four components: (1) The elasticities of the input factors in the production of final output; (2) the distance of the economy's position from its steady state; (3) the direction of the economy's deviation from its steady state and, of course, (4) the stable eigenvalues of the dynamic system under consideration. The first component captures the obvious importance of the underlying final output technology. The second and third component indicate the meaning of the economy's (initial) position relative to its steady state and thus point to the importance of the specific shock under consideration. The last component describes the significance of the stable manifold of the underlying dynamic system.

Furthermore, if one allows for an explicit multi-sectoral framework, it becomes evident that two basic convergence mechanisms can be distinguished: the accumulation-decumulation mechanism and the resource-reallocation mechanism. Both mechanisms contribute directly to the convergence of output and should, therefore, be considered as equally important with respect to the convergence issue. From an economic perspective, however, the resource-reallocation mechanism is of instrumental character with respect to the accumulation or decumulation of resources.

The decomposition of the rate of convergence reveals that the specific shock under study may be of major importance for the speed at which an economy converges to its steady state. This aspect is especially important for multi-sectoral dynamic models with a multi-dimensional stable manifold. In this case, the rate of convergence may depend critically on the direction of the initial deviation from the steady state. This point has been largely ignored within the literature on convergence since most theoretical investigations focus on the asymptotic rate of convergence only. Moreover, this aspect may be important for understanding the international distribution of income. The widely applied  $\beta$ -convergence approach regresses the growth rate of per capita income on the initial level of per capita income controlling for differences in the balanced growth path due to parameter heterogeneity. It does not, however, account for the effect of different initial conditions of the state variables (e.g., physical capital, human capital and technological knowledge) on the speed of adjustment.

In order to demonstrate the usefulness of the proposed decomposition and to better understand the mechanisms behind the convergence process, the convergence-accounting formula has been employed to investigate the transition process of a general R&D-based

endogenous growth model. The analysis yields the following insights: (1) The accumulation-decumulation mechanism appears more important than the resource-reallocation mechanism. This result must, however, be qualified in an important direction. The decomposition does not take the initial jump of the intersectoral allocation variables into account since the investigation is restricted to the stable manifold. As a result, the importance of the resource-reallocation mechanism is underestimated. (2) Despite the fact that we observe permanent convergence in output, some variables may contribute negatively to the overall rate of convergence (at least for some period of time). For the example considered, this observation is due to non-monotonic adjustments of the intersectoral allocation variables. (3) Convergence is generally slow if the economy starts out in the direction given by the eigenvector associated with the smaller, in absolute terms, of the stable eigenvalues. This finding supports the hypothesis according to which the shock under consideration may be of major importance for the speed of convergence. (4) It has been demonstrated that along a two-dimensional stable manifold a wide range of different adjustment patterns is possible. This includes monotonic as well as non-monotonic transitions with and without overshooting.

The paper points to a number of interesting issues for future research. The first is theoretical in nature, while the second and third concern empirical research. First, it has been shown that the direction of the initial deviation might be crucial for the speed at which the economy converges to its steady state. This implication arises when the stable manifold is multi-dimensional and the eigenvalues differ substantially in magnitude. Since this constellation bears important implications for the nexus between the speed of convergence and the initial conditions, it would be clearly instructive to identify the sources of this pattern in terms of the coefficients of the underlying dynamic system. Second, as has been stated above, the standard cross-sectional convergence regression framework can be extended to capture the systematic influence of initial conditions of the state variables on the estimated rate of convergence. Finally, the convergence-accounting formula derived in this paper can be employed to analyse the rate of convergence empirically. This research may give rise to a profound understanding of the driving forces behind the convergence processes in the real world.



## 6. Appendix: A general R&D-based endogenous growth model

### 6.1. Firms

#### Final output sector

The final output sector is assumed competitive and produces a homogeneous good that can be used universally for consumption or investment purposes. The original production function may be expressed as  $Y = \bar{\bar{F}}[\theta L, \phi(i)x(i), A]$ , where  $Y$  denotes final output,  $L$  the stock of labour and  $x(i)$ , with  $i$  real valued and  $i \in [0, A]$ , is the quantity of the differentiated intermediate good (capital goods) of type  $i$ .  $A$  indicates the number of differentiated intermediate goods available at every point in time. A characteristic feature of this class of models is that this number is an endogenous variable; the law of motion of  $A$  is described below. The allocation variables  $\theta$  and  $\phi(i)$  [ $0 \leq \theta, \phi(i) \leq 1$ ] represent the shares of labour and intermediate goods allocated to final output production respectively. The final output technology satisfies  $\bar{\bar{F}}_{\theta L}(\cdot) > 0$ ,  $\bar{\bar{F}}_{\phi(i)x(i)}(\cdot) > 0$ ,  $\bar{\bar{F}}_A(\cdot) > 0$ , where  $\bar{\bar{F}}_{\theta L}(\cdot) := \frac{\partial \bar{\bar{F}}(\cdot)}{\partial \theta L}$ . The differentiated intermediate goods enter the production function symmetrically. Hence, the original production function,  $\bar{\bar{F}}(\cdot)$ , may be expressed as  $Y = \bar{\bar{F}}(\theta L, \phi x, A)$ . The production function is assumed to satisfy three further restrictions: (1) Constant returns to scale in the private inputs ( $L$  and  $x$ ); (2) the elasticity of substitution among the intermediate goods is finite; (3) an increase in the number of intermediate inputs causes total factor productivity to rise (Smith-Ethier effect; Ethier, 1982).

Aggregate capital can be expressed as  $K := qAx$ , where  $q$  represents a constant technology parameter. By substituting  $x = K/(qA)$ , the production function,  $Y = \bar{\bar{F}}(\theta L, \phi x, A)$ , can be transformed to read  $Y = F(\theta L, \phi K, A)$ . The dynamics of the aggregate capital stock is given by  $\dot{K} = F(\theta L, \phi K, A) - \delta_K K - C$ , where  $\delta_K \geq 0$  denotes the constant rate of capital depreciation and  $C$  total consumption.

#### Intermediate goods sector

There is an infinite number of firms measured on the interval  $[0, A]$  manufacturing differentiated intermediate goods. Each producer must at first invest in blueprints as the technical prerequisite of production. The owner of a blueprint is the only producer of the respective intermediate good (effective patent protection). The representative intermediate

goods producer can convert  $q > 0$  units of final output into one unit of the differentiated intermediate good. Operating profits may be expressed as  $\pi(x) = [p(x) - qr]x$ . The gross interest rate is denoted by  $r$ , i.e.,  $r = r_n + \delta_K$  with  $r_n$  representing the net interest rate.

The typical intermediate goods producer faces demand from final output producers and from R&D firms. The elasticities of substitution among the intermediate goods are constant for both the final output as well as the R&D sector. Since there is a large number of firms in both sectors, the elasticities of substitution equal the respective price elasticities of demand denoted by  $\varepsilon_1$  (final output) and  $\varepsilon_2$  (R&D). To simplify matters we assume  $\varepsilon_1 = \varepsilon_2 = \varepsilon$ . With constant marginal costs ( $qr$ ) and a price elasticity given by  $\varepsilon$ , the solution to the underlying monopoly pricing problem implies a supply price of  $p_s = \frac{\varepsilon}{\varepsilon - 1}qr$  ( $1 < \varepsilon < \infty$ ).

The typical final-goods producer is willing to pay the marginal products for his inputs. The inverse demand functions for intermediate goods originating from the final-output sector are given by  $p_D(i) = \bar{F}_{\phi(i)x(i)}[\theta L, \phi(i)x(i), A]$  for all  $i$  with  $p_D(i)$  denoting the demand price. Since all  $x(i)$  enter the production function symmetrically, we can write  $p_D = \bar{F}_{\phi x}[\theta L, \phi x, A]$ . Moreover, we can substitute  $x = K/(qA)$  into  $\bar{F}_{\phi x}(\theta L, \phi x, A)$  to get  $G(\theta L, \phi K, A)$ , the marginal product of one specific variety of the intermediate good in the production of final output in terms of  $K$ .

From  $\pi(x) = [p(x) - qr]x$ ,  $p_D = G(\theta L, \phi K, A)$ ,  $p_s = \frac{\varepsilon}{\varepsilon - 1}qr$ ,  $p(x) = p_D = p_s$  and  $x = K/(qA)$  operating profits can be written as  $\pi = \frac{G(\theta L, \phi K, A)K}{\varepsilon q A}$ . From equilibrium in the intermediate goods market ( $p_D = p_s$ ), we have  $G(\theta L, \phi K, A) = \frac{\varepsilon}{\varepsilon - 1}qr$  and hence the interest rate may be expressed as  $r = \frac{\varepsilon - 1}{\varepsilon} \frac{G(\theta L, \phi K, A)}{q}$ .

### R&D sector

There is a large number of R&D firms searching for new designs. The R&D technology is of the following shape  $\dot{A} = \bar{J} \left\{ A, (1 - \theta)L, \overline{(1 - \theta)L}, [1 - \phi(i)]x(i), \overline{[1 - \phi(i)]x(i)} \right\}$ . Several points should be observed at this stage. First, this production function generalises the usual R&D

technology in that intermediate goods  $[x(i)]$  are considered to be productive in R&D as well.

Second, it is assumed that  $\bar{\bar{J}}_A(\cdot) > 0$  which captures two distinct effects:  $A$  indicates the net effect of (intertemporal) knowledge spill-overs and “fishing out” effects (Jones and Williams, 2000). Moreover, in case of capital being productive in R&D,  $A$  additionally reflects the specialisation effect due to the use of differentiated producer goods (Smith-Ethier effect).

Third,  $\bar{\bar{J}}_{(1-\theta)L}(\cdot) > 0$  and  $\bar{\bar{J}}_{[1-\phi(i)]x(i)}(\cdot) > 0$  denote the private marginal products of labour and differentiated capital goods respectively. There are constant returns to scale at the level of the individual firm. Fourth, following Jones (1995) and Jones and Williams (2000) we allow for negative externalities associated with the economy-wide averages of the private resources employed in R&D which are denoted as  $\overline{(1-\theta)L}$  and  $\overline{[1-\phi(i)]x(i)}$ . These are indicated by

$\bar{\bar{J}}_{\overline{(1-\theta)L}}(\cdot) \leq 0$  and  $\bar{\bar{J}}_{\overline{[1-\phi(i)]x(i)}}(\cdot) \leq 0$  and capture (intratemporal) duplication externalities (accidental or intentional). Since the  $x(i)$  enter the production function symmetrically, we

may write  $\dot{A} = \bar{\bar{J}}\left[A, (1-\theta)L, \overline{(1-\theta)L}, (1-\phi)x, \overline{(1-\phi)x}\right]$ . Moreover, using  $x = K/(qA)$  leads

to  $\dot{A} = \bar{\bar{J}}\left[A, (1-\theta)L, \overline{(1-\theta)L}, (1-\phi)K, \overline{(1-\phi)K}\right]$ . Since in equilibrium  $(1-\theta)L = \overline{(1-\theta)L}$

and  $(1-\phi)K = \overline{(1-\phi)K}$  we may express the preceding function as  $\dot{A} = J\left[A, (1-\theta)L, (1-\phi)K\right]$ .

The price of one design is given by  $v(t) = \int_t^\infty \pi(\tau) e^{-R(t)} d\tau$  with  $R(t) := \int_t^\tau r_n(u) du$ .

Here we observe a further market distortion since only private returns are taken into account.

Differentiating the preceding integral equation with respect to time gives  $\dot{v} = r_n v - \pi$ .

Inserting the expressions for  $\pi$  and  $r$  derived above, one obtains the differential equation in

$$v \text{ as } \dot{v} = \left[ \frac{\varepsilon - 1}{\varepsilon} \frac{G(\theta L, \phi K, A)}{q} - \delta_K \right] v - \frac{G(\theta L, \phi K, A) K}{\varepsilon q A}.$$

Let us now turn to the factor allocation conditions. Profit-maximising firms reward the factors of production according to their (private) marginal product. Moreover, in equilibrium wages are equalised across the two sectors so that  $w = F_{\theta L}(\cdot) = v J_{(1-\theta)L}(\cdot)$ . This intersectoral

labour allocation condition may be expressed as  $\theta = \theta(A, K, L, v)$ . As for the differentiated

capital goods, we have  $p_D = \bar{\bar{F}}_{\phi x}(\cdot) = v \bar{\bar{J}}_{(1-\phi)x}(\cdot)$ ; notice that  $\bar{\bar{J}}_{(1-\phi)x}(\cdot)$  requires to differentiate  $\bar{\bar{J}}(\cdot)$  with respect to  $[1-\phi(i)]x(i)$  and then drop the index  $i$ . Substituting  $x = K/(qA)$  into

the preceding equation gives the allocation condition for intermediate goods in terms of (aggregate) capital as  $\phi = \phi(A, K, L, v)$ .

## 6.2. Households

The representative household inelastically supplies one unit of labour during every period of time and maximises her intertemporal utility. The instantaneous utility function is of the constant-intertemporal-elasticity-of-substitution type (CIES). The dynamic optimisation problem reads as follows:

$$\max_{\{C/L\}} \int_0^{\infty} \frac{(C/L)^{1-\gamma} - 1}{1-\gamma} e^{-\rho t} dt$$

s.t.  $\dot{K} = r_n K + wL + A\pi - v\dot{A} - C; K(0) > 0,$  (17)

where  $\rho > 0$ ,  $\gamma > 0$  and  $w$  denote the constant time preference rate, a constant preference parameter and the wage rate, respectively. From the first-order conditions we get the Keynes-Ramsey rule describing the optimal consumption profile.<sup>21</sup>

$$\dot{C} = \frac{C}{\gamma} [r_n - \rho - (1-\gamma)n]$$
 (18)

The preceding discussion of the general R&D-based endogenous growth model can be summarised by the dynamic system (31) to (36) shown in the main text.

## 6.3. The dynamic system

The preceding discussion can be summarised by the subsequent differential-algebraic system which governs the dynamics of the market solution. The system shown below is valid for a broad class of R&D-based endogenous growth models of the increasing-variety type.

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<sup>21</sup> It is assumed that the sufficient conditions are equally satisfied and that the transversality condition holds, i.e.,  $-\rho + \lim_{t \rightarrow \infty} \hat{\lambda} + \lim_{t \rightarrow \infty} \hat{K} < 0$ , where  $\lambda$  denotes the current-value shadow price of capital.

$$\dot{K} = F[A, \theta L, \phi K] - \delta_K K - C \quad (19)$$

$$\dot{A} = J[A, (1-\theta)L, (1-\phi)K] \quad (20)$$

$$\dot{C} = \frac{C}{\gamma} \left[ \frac{\varepsilon-1}{\varepsilon} \frac{G(\theta L, \phi K, A)}{q} - \delta_K - \rho - (1-\gamma)n \right] \quad (21)$$

$$\dot{v} = \left[ \frac{\varepsilon-1}{\varepsilon} \frac{G(\theta L, \phi K, A)}{q} - \delta_K \right] v - \frac{G(\theta L, \phi K, A)K}{\varepsilon q A} \quad (22)$$

$$\theta = \theta(A, K, L, v) \quad (23)$$

$$\phi = \phi(A, K, L, v), \quad (24)$$

where the size of population ( $L$ ) is assumed to grow at exponential rate, i.e.,  $\dot{L} = nL$ .

#### 6.4. The balanced growth path

A BGP is defined by constant growth rates of the endogenous variables. This implies that the allocation variables ( $\theta$  and  $\phi$ ) must be constant along the BGP. In accordance with the stylised facts, we assume  $\hat{Y} = \hat{K}$  along the BGP (Romer, 1989).<sup>22</sup> From  $\hat{K} = Y/K - \delta_K - C/K$  it then follows that balanced growth requires  $\hat{K} = \hat{C}$ . The balanced growth rates of  $K$  and  $A$  can be derived from  $\frac{d}{dt} \frac{F(\cdot)}{K} = 0$  and  $\frac{d}{dt} \frac{J(\cdot)}{A} = 0$ . Carrying out the preceding instructions yields.

$$(1-\sigma_K)\hat{K} - \sigma_A\hat{A} = \sigma_L\hat{L} \quad (25)$$

$$(1-\eta_A)\hat{A} - \eta_K\hat{K} = \eta_L\hat{L} \quad (26)$$

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<sup>22</sup> As usual, a variable with a hat denotes the growth rate of the respective variable.

The elasticities of production  $\sigma_z$  and  $\eta_z$  are defined by  $\sigma_z := \frac{F_z(\cdot)z}{F(\cdot)}$  and  $\eta_z := \frac{J_z(\cdot)z}{J(\cdot)}$  for  $z = A, L, K$ . Provided that  $\hat{L} = n > 0$ , equations (25) and (26) uniquely determine  $\hat{K}$  and  $\hat{A}$ :

$$\hat{K} = \beta_K n \quad \text{with} \quad \beta_K := \frac{\sigma_L(1-\eta_A) + \eta_L \sigma_A}{(1-\eta_A)(1-\sigma_K) - \eta_K \sigma_A} \quad (27)$$

$$\hat{A} = \beta_A n \quad \text{with} \quad \beta_A := \frac{\eta_L(1-\sigma_K) + \eta_K \sigma_L}{(1-\eta_A)(1-\sigma_K) - \eta_K \sigma_A} \quad (28)$$

Notice that balanced growth is characterised by non-scale growth, i.e., the size of the economy does not enter the growth rates.<sup>23</sup>

## 6.5. Dynamic system in scale adjusted variables

The last step consists in an adjustment of scale in order to get a stationary dynamic system. With the balanced-growth rates shown in (27) and (28), the appropriate scale adjustments are given by  $y := Y/L^{\beta_K}$ ,  $k := K/L^{\beta_K}$ ,  $c := C/L^{\beta_K}$ ,  $a := A/L^{\beta_A}$ ,  $j := J/L^{\beta_A}$  and  $v_a := v/L^{\beta_K - \beta_A}$ .<sup>24</sup> For the class of models considered, the marginal product of one specific intermediate good in terms of aggregate capital may be expressed as  $G(\theta L, \phi K, A) = \frac{\sigma_K Y q}{\phi K}$ . In addition, taking into account that  $(\varepsilon - 1)/\varepsilon = \sigma_K$  the dynamic system in scale-adjusted variables may be expressed as shown in (9) to (14) in the main text.

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<sup>23</sup> Moreover, the balanced growth rates of the market and the social solution coincide provided that  $\hat{L} = n > 0$ .

<sup>24</sup> The scale-adjusted price  $v_a := v/L^{\beta_K - \beta_A}$  results from the following consideration. From (22) together with  $G(\cdot) = \frac{\sigma_K Y q}{\phi K}$ , the growth rate of  $v$  may be expressed as  $\hat{v} = \frac{\varepsilon - 1}{\varepsilon} \frac{\sigma_K Y q}{\phi K} - \delta_K - \frac{\sigma_K Y}{\varepsilon A \phi K v}$ . Along the BGP, the first term on the RHS is constant. Hence,  $v$  must grow at a rate equal to  $\hat{v} = \hat{Y} - \hat{A} = (\beta_K - \beta_A)n$  along the BGP.

## 6.6. Restrictions on the set of parameters

There are a number of (technical) restrictions which must be taken into account. First, necessary and sufficient conditions for positive per capita growth are  $(1-\eta_A)(1-\sigma_K)-\eta_K\sigma_A > 0$  and  $\sigma_K < 1$ . Since both sectors are characterised by Cobb-Douglas technologies, the preceding conditions are also sufficient for balanced growth (Eicher and Turnovsky, 1999a, pp. 402). Second, convergence of the utility integral demands for  $-\rho+(1-\gamma)(n\beta_K-n) < 0$ . Third, the transversality condition to hold  $\tilde{r} < n\beta_K$  has to be assumed, where  $\tilde{r}$  is the real rate of return to physical capital along the BGP determined by  $\tilde{r} = \frac{\sigma_K^2 \tilde{y}}{\phi \tilde{k}} - \delta_K$ . Fourth, the assumption of perfect competition calls for the returns to scale in the private inputs to equal unity in both the final output as well as the R&D sector. Accordingly, we set  $\sigma_L + \sigma_K = 1$  and  $\eta_L^p + \eta_K^p = 1$ . Fifth, the derivation of the market equilibrium assumes that intermediate goods producers have no incentive to differentiate their supply price vis-à-vis their two groups of customers (final output and R&D producers). This simplifying assumption requires  $\sigma_K = \eta_K^p$ . Moreover, the narrow economic justification of  $\sigma_A > 0$  demands for  $\sigma_A = 1 - \sigma_K$  (the Smith-Ethier effect). Following Eicher and Turnovsky (2001) this additional restriction has been relaxed.

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Table 1: Notation for the general R&D-based endogenous growth model.

$Y$ : output of final-output sector ( $y$ : scale-adjusted output)	$\alpha_F$ : exogenous final-output technology parameter
$J$ : output of R&D sector ( $j$ : scale-adjusted output)	$\alpha_J$ : exogenous R&D technology parameter
$L$ : population (supply of labour)	$v$ : price of one idea ( $v_a$ : scale-adjusted price)
$A$ : number of ideas ( $a$ : scale-adjusted number of ideas)	$\varepsilon$ : price elasticity of demand (intermediate goods)
$K$ : aggregate capital stock ( $k$ : scale-adjusted capital)	$\delta_K$ : depreciation rate of capital
$C$ : consumption ( $c$ : scale-adjusted consumption)	$\gamma$ : elasticity of marginal utility w.r.t. consumption
$\theta$ : share of labour allocated to final output ( $0 \leq \theta \leq 1$ )	$\rho$ : time preference rate
$\phi$ : share of capital allocated to final output ( $0 \leq \phi \leq 1$ )	$n$ : growth rate of population
$\sigma_z$ : elasticity of factor $Z$ in final output production	$\pi$ : profit of typical intermediate good producer
$\eta_z$ : elasticity of factor $Z$ in R&D	$r$ : gross interest rate

Table 2: Baseline set of parameters.

FO technology (IO technology)*	$\alpha_F = 1$ ; $\sigma_L = 0.6$ ; $\sigma_K = 0.4$ ; $\sigma_A = 0.2$ ; $\delta_K = 0.05$ ; $q = 1$
R&D technology	$\alpha_J = 1$ ; $\eta_A = 0.5$ $\eta_L^p = 0.6$ ; $\eta_L^e = -0.3\eta_L^p$ $\eta_K^p = 0.4$ ; $\eta_K^e = -0.2\eta_K^p$
Preferences and population growth	$\rho = 0.04$ ; $\gamma = 1$ ; $n = 0.015$
Shock considered	$\alpha_F : 0.5 \rightarrow 1$ (permanent shock)

\*FO: final output; IO: intermediate output

Table 3: Growth rates, key economic ratios and rate of convergence.

$\hat{Y}$	$\sigma_A \hat{A}^*$	$\widetilde{Y/K}$	$\widetilde{C/Y}$	$\tilde{\theta}$	$\tilde{\phi}$	$\tilde{\psi}_{Y/L}$
0.025	0.006	0.64	0.88	0.90	0.90	0.0084

\*Note:  $\sigma_A \hat{A}$  equals the growth rate of TFP.

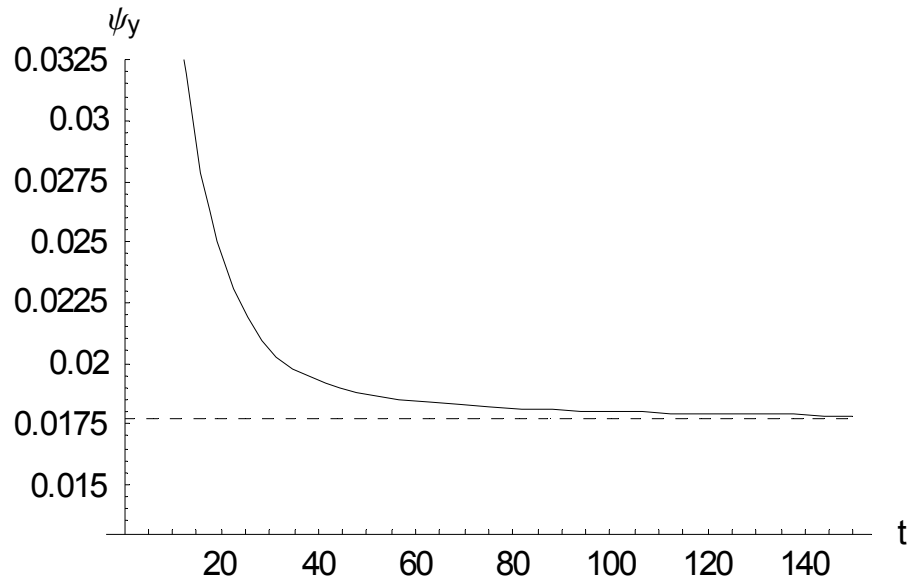


Figure 1: The instantaneous and overall rate of convergence.

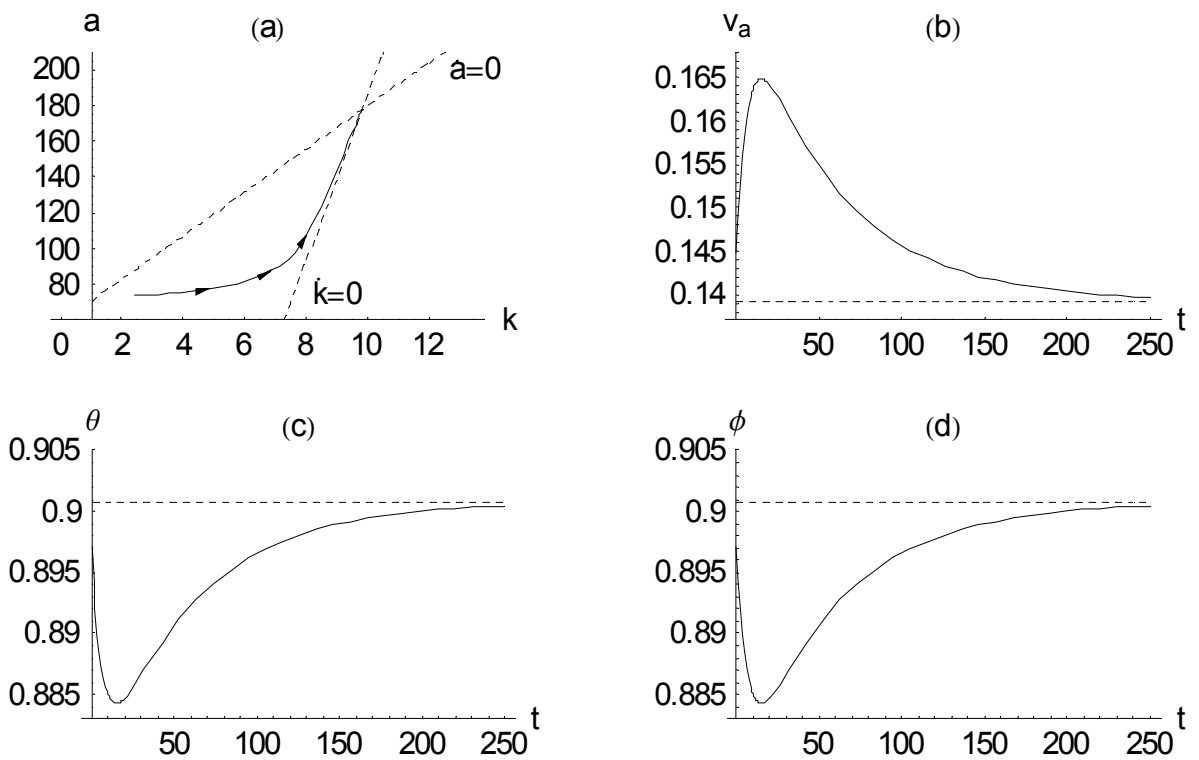


Figure 2: Illustration of the transition process.

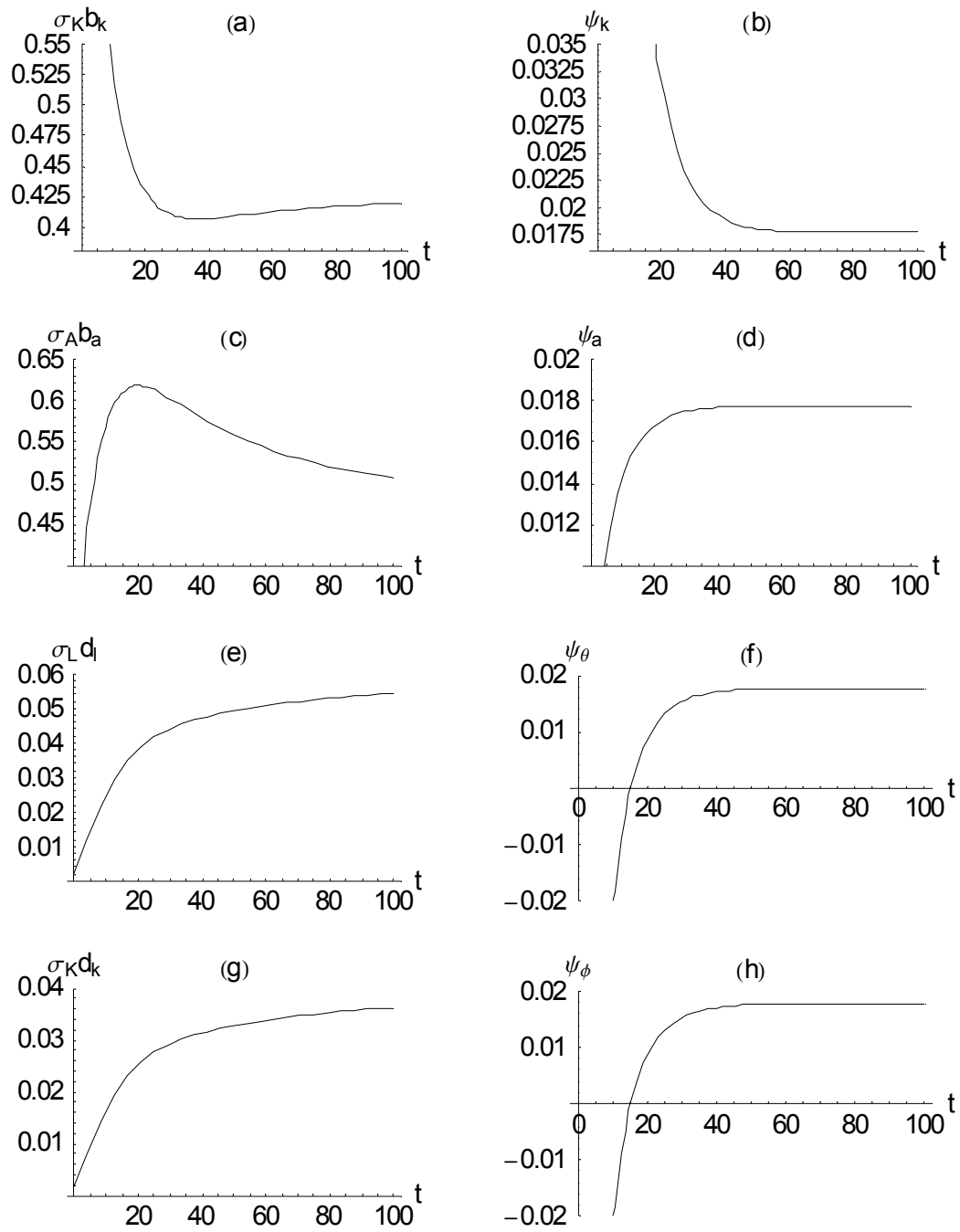


Figure 3: Illustration of Decomposition 1:  $\psi_y = \sum_{i=1}^n \sigma_i b_i \psi_{x_i} + \sum_{i=1}^n \sigma_i d_i \psi_{\theta_i}$ .

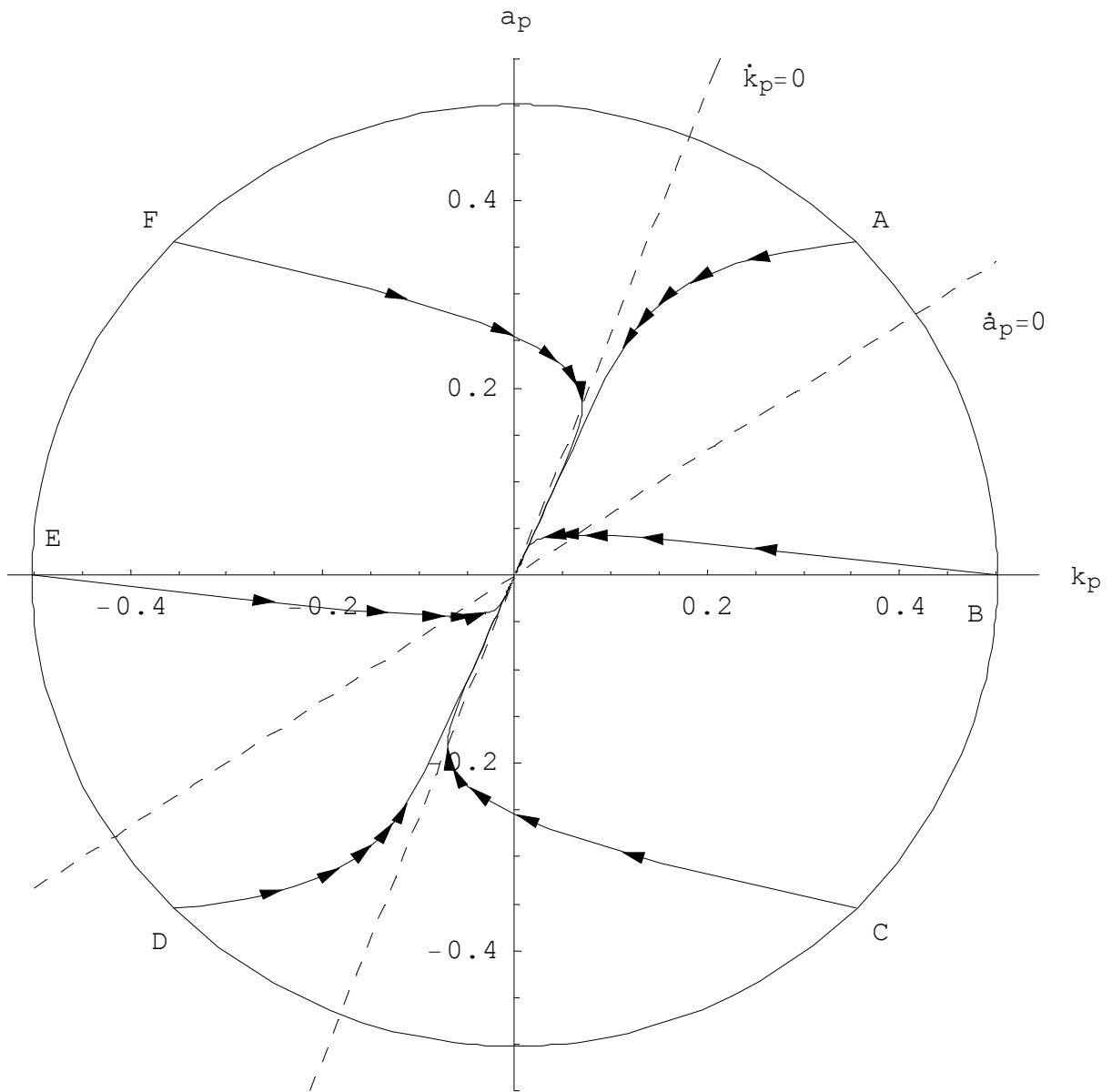


Figure 4: Convergence along a two-dimensional stable manifold (Decomposition 2 :  $\psi_{x_i} \cong \sum_{j=1}^n a_{i,j} \lambda_j$ ).