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A new linguistic aggregation operator and its application to multiple attribute decision making



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ABSTRACT

In this paper, a new linguistic aggregation operator in linguistic environment is established and the desirable properties: monotonic, focus effect, idempotent, commutative and bounded are studied. Then, a new restricted ordering relation on the n -dimensional linguistic scales is proposed which satisfies strict pareto-dominance and is restricted by a weighting vector. A practical multiple attribute decision making methodology for an uncertain linguistic environment is proposed based on the proposed operator. An example is given to illustrate the rationality and validity of the new approach to decision making application.

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1. Introduction

Due to the complexities of objectivity and the vagueness of human thinking, it is often more appropriate to use qualitative information rather than quantitative information when dealing with practical evaluation and decision making problems (see for instance [1–8], etc.).

In general, experts deal with many problems using qualitative information usually expressed through linguistic terms [9]. For instance, when evaluating the speed of a car, linguistic terms such as “slow”, “medium”, “fast”, or “very fast” can be used; similarly, when evaluating the “comfort” or “design” of a car, linguistic terms such as “bad”, “tolerable”, “average”, or “good” are usually used. In essence, practical problem solving using qualitative information involves a procedure in which the computing is conducted using words. These linguistic approaches generally represent the qualitative aspects of a decision and, therefore, require the specification of linguistic values using linguistic variables, which are words or sentences in a natural or artificial language [10,11].

In recent years, decision-making problems have become a popular research field [12–18]. Such problems require the collection of decision-making information about the attribute values and attribute weights from a group of decision makers. One important concept in decision-making problems is the compilation of decision-making information and the sorting and fusing of a representative output. The process of merging all collected information

into a concrete representative value is known as information aggregation [19]. Therefore, aggregation operators are essential tools when seeking to deal with information fusion in decision-making problems and have, themselves, become an extensively studied field. From the literature review, there are generally two main types of methods for modeling and aggregating qualitative linguistic information.

(1) The computational model. Several computational models have been proposed to manipulate linguistic information and the main ones as follows: (1) The approximate model based on extension principle [20–23], which converts the linguistic information into fuzzy numbers by means of the corresponding membership functions. However, this model needs to assume membership function, and so it is often difficult to choose an appropriate membership function in an actual application [24,25]. (2) The symbolic method based on ordinal scales [26–29], which uses the order of the linguistic term set and makes computations on the subscripts of linguistic terms. However, this model requires a linguistic approximation of the finally computed result in the initial term sets, which can lead to inaccuracy and result in a loss of information [20,30]. (3) The 2-tuple linguistic model [5,30–32], in which the linguistic information is represented by means of 2-tuple (l, α) , in which l is a linguistic term and α is a numerical value supporting the information of the symbolic translation. However, as Herrera and Martínez themselves have pointed out in Ref. [33], this method is only suitable for linguistic variables with equidistant labels. (4) The model that computes with words directly [34–37]. This method is straightforward and very convenient for the calculations. However, while aggregating the information, the aggregated results may not match any of the initial term sets; for example, $0.5 \otimes l_3 = l_{1.5}$. Because virtual term sets only appear in

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the operation and in the comparisons, what is the meaning of the $l_{1.5}$ in the initial term set L ? Therefore, the results using this method have caused controversy in some cases. Given a linguistic term set, aggregate results which are also in the initial linguistic terms may be more appropriate and more easily comprehended.

(II) Aggregation of Linguistic Information. Several types of aggregation operators have been proposed as follows [38–40]: conjunctive operators (which lie under the minimum, such as t -norms, copulas and quasi-copulas), disjunctive operators (which lie over the maximum, such as t -conorms, dual quasi-copulas and dual copulas), averaging or median operators (which lie between the minimum and maximum, such as means, geometric means on the unit interval, and OWA's), and mixed operators (those not included in any of the previous classes, such as uninorms and nullnorms). Clearly, the basic aggregation operations of the min and max have been shown to cause some non-compensatory difficulties and have also been shown to violate pareto-dominance. T -norms generalize the intersection *and* operator while the t -conorms generalize the intersection *or* operator. However, the boundary conditions create counterintuitive ties in the process of aggregation based on t -norms and t -conorms. To recover from the pareto-dominance violation limitation for the min and max, Dubois has proposed the aggregation operations Discrimin(Discrimax) and Leximin (Leximax) through a refinement of the min(max)-ordering [40]. However, these operators have also been found to have some drawbacks as they are non-compensatory and conflict with intuition. For instance, $(1, 2, 3, 3, 3, 3) >_{\text{Discrimin}} (1, 1, 6, 6, 6, 6)$: the many 6's are unable to compensate for a 1 and beat as many 3's.

Based upon the above analyses, the aim of this paper is to develop a new linguistic aggregation operator to deal with qualitative linguistic information. The remainder of this paper is organized as follows. Section 2 introduces the basic definitions and operations used in the remaining parts of the paper. In Section 3, we develop the distance measures for a linguistic term set based on linguistic scale functions. In Section 4, we propose a new uncertain linguistic aggregation operator based on linguistic evaluation scales and study the desirable properties of the proposed aggregation operator. In Section 5, a new restricted ordering relation on n -dimension linguistic scales is proposed. In Section 6, an approach to multiple attribute decision making with linguistic information is given based on the proposed aggregation operator. Finally, an illustrative example is given to show the rationality and validity of the proposed method in Section 7 and conclusions are discussed in Section 8.

2. Preliminaries

In this section, we briefly review some basic definitions and operations of linguistic terms (linguistic variables). These definitions and operations are used in the remaining parts of the paper.

2.1. The linguistic term sets and their extension

Let $L = \{l_i | i = 0, 1, \dots, g\}$ ($g = 2h, h \in N$) be a totally ordered discrete and finite term set, in which l_i represents a possible value for a linguistic variable, N is the set of natural numbers. It is necessary for the linguistic term set L to satisfy the following additional characteristics [41–43]:

- (1) Order relation: $l_i \leq l_j$, if $i \leq j$;
- (2) Negation operator: $neg(l_i) = l_j$, where $i = g - j$;
- (3) Maximization operator: $\max\{l_i, l_j\} = l_i$, if $i \geq j$;
- (4) Minimization operator: $\min\{l_i, l_j\} = l_i$, if $i \leq j$.

To avoid a loss of information, the discrete linguistic term set L can be extended to a continuous set $\tilde{L} = \{l_x | 0 \leq x \leq g\}$, where $l_x \in L$ implies l_x belongs to the original linguistic term set and $l_x \notin L$ implies l_x belongs to a virtual linguistic term set.

Definition 1. For any $l_x, l_y \in \tilde{L}$, a binary relation \leq is defined by $l_x \leq l_y \Leftrightarrow x \leq y$ and $l_x = l_y \Leftrightarrow x = y$.

Property 1. For any $l_x, l_y, l_z \in \tilde{L}$, \leq satisfies the following four properties:

- (1) (Comparability) Either $l_x < l_y$ or $l_y < l_x$ or $l_x = l_y$ holds.
- (2) (Reflexivity) $l_x \leq l_x$, for any $x \in [0, g]$.
- (3) (Antisymmetry) $l_x \leq l_y$ and $l_y \leq l_x$ imply $l_x = l_y$.
- (4) (Transitivity) $l_x \leq l_y$ and $l_y \leq l_z$ imply $l_x \leq l_z$.

Proof. (1) For any $x, y \in [0, g]$, we have either $x < y$ or $y < x$ or $x = y$.

If $x < y$, then $l_x < l_y$.

If $y < x$, then $l_y < l_x$.

If $x = y$, then $l_x = l_y$.

(2) For any $x \in [0, g]$, since $x \leq x$, then $l_x \leq l_x$.

(3) $l_x \leq l_y \Rightarrow x \leq y$, and $l_y \leq l_x \Rightarrow y \leq x$, so we have $x = y$ and $l_x = l_y$.

(4) $l_x \leq l_y \Rightarrow x \leq y$ and $l_y \leq l_z \Rightarrow y \leq z$, so we have $x \leq z$ and $l_x \leq l_z$.

Therefore, (\tilde{L}, \leq) is a totally ordered set.

2.2. Linguistic scale functions and their extension

In linguistic term sets, to use the data more efficiently and to flexibly express the semantics, it is necessary to explore and define some new linguistic scale functions to transform the linguistic information into a numeric value.

Definition 2 (See [25]). Let $L = \{l_i | i = 0, 1, \dots, g\}$ be a linguistic term set and $\theta_i \in R^+$ ($R^+ = \{r | r \geq 0, r \in R\}$) be a numeric value, respectively. The linguistic scale function v that conducts the mapping from l_i to θ_i ($i = 0, 1, 2, \dots, g$) is defined as follows:

$$v : l_i \rightarrow \theta_i \quad (i = 0, 1, 2, \dots, g)$$

where $0 \leq \theta_0 < \theta_1 < \dots < \theta_g$.

Clearly, function v is a strictly monotonically increasing function with regard to subscript i . The value θ_i , in fact, denotes the semantics of the linguistic terms [25]. Without a loss of generality, the value θ_i can be constrained in the unit interval $[0, 1]$. To avoid a loss of information and to facilitate the calculation, the above linguistic scale function v can be expanded to the extended continuous linguistic term set \tilde{L} as follows:

Definition 3. Let \tilde{L} be the extended continuous linguistic term set and $l_x \in \tilde{L}$, then the numeric value θ_x can be determined using the following function:

$$u(l_x) = \theta_x, \quad x \in [0, g].$$

Here $u : \tilde{L} \rightarrow [0, 1]$ is monotonic with the following properties:

- (1) $u(l_0) = 0$;
- (2) $u(l_g) = 1$;
- (3) $u(l_y) \geq u(l_x)$ if $y \geq x$.

Similarly, u is a strictly monotonically increasing and continuous function. The inverse function of u exists and is denoted as u^{-1} .

In [25], Wang and Wu developed two kinds of composite assessment scale expressions based on the exponential scale. To reflect the changed rate in the numerical value of the term set, three kinds of scale functions can be considered depending on the function convexity.

(1) Neutral linguistic scale function:

$$u_1(l_x) = \frac{x}{g}, \quad (x \in [0, g]).$$

In this case, the evaluation scale of the linguistic information given above is divided by the average. The absolute deviation between the adjacent linguistic subscripts remains unchanged.

(2) Optimistic linguistic scale function:

$$u_2(l_x) = \left(\frac{x}{g}\right)^t, \quad (x \in [0, g]).$$

Here $1 < t$. In this case, the absolute deviation between the adjacent linguistic subscripts increases.

(3) Pessimistic linguistic scale function:

$$u_3(l_x) = \left(\frac{x}{g}\right)^s, \quad (x \in [0, g]).$$

Here $0 < s < 1$. In this case, the absolute deviation between the adjacent linguistic subscripts will decrease.

In the above three scale functions, as the linguistic subscripts i ($i = 0, 1, 2, \dots, g$) increase, the absolute deviation between the adjacent linguistic subscripts may increase, decrease or remain unchanged. As is well known, decision-makers need to assume certain risks in practical decision making processes, but different decision-makers have different attitudes toward risk. This situation is handled by the utility function from expected utility theory, which, in some cases, can be considered when selecting linguistic scale functions.

In practical decision making, if the decision-maker is a risk taker, then an optimistic linguistic scale function is adopted. Conversely, if the decision-maker is risk averse, then a pessimistic linguistic scale function is adopted. If the decision-maker has a medium risk perspective, then a neutral linguistic scale function is adopted.

3. Distance measures for linguistic term set

Distance measures have been used widely to measure the deviation degrees between different arguments. Based on the aforementioned linguistic scale functions, distance measures for the linguistic term set can be defined as follows.

Definition 4. Let $L = \{l_i | i = 0, 1, \dots, g\}$ be a linguistic term set and $d : L \times L \rightarrow [0, 1]$ be a binary function. For any $l_i, l_j \in L$, d is defined as follows:

$$d(l_i, l_j) = |u(l_i) - u(l_j)|. \tag{1}$$

Here $u : \tilde{L} \rightarrow [0, 1]$ is an extension of the linguistic scale functions such that $u(l_0) = 0, u(l_g) = 1$.

Example 1. Assume that $L = \{l_0, l_1, l_2, l_3, l_4\}, u_1(l_x) = \frac{x}{4}, u_2(l_x) = (\frac{x}{4})^2, u_3(l_x) = (\frac{x}{4})^{\frac{1}{2}}$, then

$$d_{u_1}(l_0, l_1) = \left|0 - \frac{1}{4}\right| = \frac{1}{4}.$$

Similarly, we have $d_{u_1}(l_1, l_2) = d_{u_1}(l_2, l_3) = d_{u_1}(l_3, l_4) = \frac{1}{4}$.
Similarly, we have

$$d_{u_2}(l_0, l_1) = \frac{1}{16}, \quad d_{u_2}(l_1, l_2) = \frac{3}{16},$$

$$d_{u_2}(l_2, l_3) = \frac{5}{16}, \quad d_{u_2}(l_3, l_4) = \frac{7}{16}$$

and

$$d_{u_3}(l_0, l_1) = \frac{1}{2}, \quad d_{u_3}(l_1, l_2) = \frac{\sqrt{2} - 1}{2},$$

$$d_{u_3}(l_2, l_3) = \frac{\sqrt{3} - \sqrt{2}}{2}, \quad d_{u_3}(l_3, l_4) = \frac{2 - \sqrt{3}}{2}.$$

Property 2. For any $l_i, l_j, l_k \in L$, d satisfies the following four conditions:

- (1) $0 \leq d(l_i, l_j) \leq 1$;
- (2) $d(l_i, l_j) = 0$ if and only if $l_i = l_j$;
- (3) (Commutativity) $d(l_i, l_j) = d(l_j, l_i)$;
- (4) (Triangle inequality) $d(l_i, l_k) \leq d(l_i, l_j) + d(l_j, l_k)$.

Proof. The proofs are obvious and thus omitted.

4. New aggregation operator in linguistic environment

In a multiple attribute decision making problem with linguistic information, there are generally a finite set of alternatives and a collection of attributes. It collects the information about attribute values and attribute weights, needs weighted aggregation of the attribute values across all attributes for each alternative to get an overall attribute value.

Similarly, in a group decision making problem with linguistic information, for each alternative, the decision makers generally need to provide their evaluations by means of linguistic variables. Then, all the individual evaluation information is fused to become a group opinion, which can sufficiently reflect the opinion of every member of the group. As a result, the group evaluation of the alternatives should be as close as possible to all the decision makers' individual opinion. This means that the computation of the distance between all individual evaluations and the aggregated group evaluation on a given alternative is an integral part of the solution to a linguistic decision making problem.

For convenience, we consider a linguistic decision making problem with n ($n \geq 1$) decision makers. For a given alternative, each decision maker gives a preference value l_{x_i} ($i = 1, 2, \dots, n$), $l_{x_i} \in L, L = \{l_i | i = 0, 1, \dots, g\}$. $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of decision makers, where $w_i \geq 0$ ($i = 0, 1, \dots, n$), $\sum_{i=1}^n w_i = 1$. To obtain the group opinion, we construct the following objective function

$$f(u(l_x)) = \sum_{i=1}^n w_i (d(l_{x_i}, l_x))^2$$

$$= \sum_{i=1}^n w_i (u(l_{x_i}) - u(l_x))^2$$

$$= u^2(l_x) - 2 \left(\sum_{i=1}^n w_i u(l_{x_i}) \right) u(l_x) + \sum_{i=1}^n w_i u^2(l_{x_i}), \tag{2}$$

where $x \in [0, g]$, that is, $l_x \in \tilde{L}$ and $u : \tilde{L} \rightarrow [0, 1]$ is an extension of the linguistic scale functions.

Obviously, the symmetrical axis of $f(u(l_x))$ is

$$u(l_{\bar{x}}) = \sum_{i=1}^n w_i u(l_{x_i}). \tag{3}$$

Since u is a strictly monotonic increasing and continuous function which satisfies: $u(l_0) = 0, u(l_g) = 1$. Thus,

$$l_{\bar{x}} = u^{-1}(u(l_{\bar{x}})) = u^{-1} \left(\sum_{i=1}^n w_i u(l_{x_i}) \right). \tag{4}$$

Consider the mathematical programming model as follows:

$$(MP1) \min_{l_x \in L} F(l_x) = \sum_{i=1}^n w_i (d(l_{x_i}, l_x))^2 = \sum_{i=1}^n w_i (u(l_{x_i}) - u(l_x))^2. \tag{5}$$

Theorem 1. Let l^* be the optimal solution for (MP1).

- (i) If $l_{\bar{x}} \in L$, then $l^* = l_{\bar{x}} = u^{-1}(\sum_{i=1}^n w_i u(l_{x_i}))$.
- (ii) If $l_{\bar{x}} \notin L$, then there exists a non-negative integer j ($0 \leq j \leq g - 1$), such that $l_j < l_{\bar{x}} < l_{j+1}$. There are three cases as follows:
 - If $u(l_{j+1}) - u(l_{\bar{x}}) < u(l_{\bar{x}}) - u(l_j)$, then $l^* = l_{j+1}$.
 - If $u(l_{j+1}) - u(l_{\bar{x}}) > u(l_{\bar{x}}) - u(l_j)$, then $l^* = l_j$.
 - If $u(l_{j+1}) - u(l_{\bar{x}}) = u(l_{\bar{x}}) - u(l_j)$, then $l^* = l_j$ or $l^* = l_{j+1}$.

Proof. Based on the operational properties of the quadratic model, it is easy to prove, so the proof steps are omitted here.

Now, we present the linguistic aggregation operator generated by the quadratic model.

Definition 5. Let $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \in L^n$. An uncertain linguistic aggregation operator (ULA_F) of dimension n is a mapping $ULA_F : L^n \rightarrow L$ according to the following formula:

$$ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) = \min\{l_j | F(l_j) = \min_{l_x \in L} F(l_x)\}. \tag{6}$$

The ULA_F operator is monotonic, focus effect, idempotent, commutative and bounded. These desirable properties can be provided as in the following property.

Property 3. Let $(l_{x_1}, l_{x_2}, \dots, l_{x_n})$ and $(l_{y_1}, l_{y_2}, \dots, l_{y_n})$ be two linguistic argument collections, then the ULA_F operator satisfies:

- (i) (Monotonic) If $l_{x_i} \leq l_{y_i}$ for all i ($0 \leq i \leq n$), then $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \leq ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n})$.
- (ii) (Focus effect (See [44])) For the weighting vector w , if w_k is sufficiently close to 1, and $l_{x_k} < l_{y_k}$, then $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) < ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n})$.
- (iii) (Idempotent) If $l_{x_i} = l$ for all i ($0 \leq i \leq n$) and $l \in L$, then $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) = l$.
- (iv) (Commutative) If $w = (w_1, w_2, \dots, w_n)^T = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) = ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n})$, where $(l_{y_1}, l_{y_2}, \dots, l_{y_n})$ is any a permutation of $(l_{x_1}, l_{x_2}, \dots, l_{x_n})$.
- (v) (Bounded) For any the weighting vector w , the following inequality holds: $\min\{l_{x_1}, l_{x_2}, \dots, l_{x_n}\} \leq ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \leq \max\{l_{x_1}, l_{x_2}, \dots, l_{x_n}\}$.

Proof. (i) Let us give the proof of monotonic firstly. Since $l_{x_i} \leq l_{y_i}$ for all i ($0 \leq i \leq n$), then

$$u(l_{\bar{x}}) = \sum_{i=1}^n w_i u(l_{x_i}) < u(l_{\bar{y}}) = \sum_{i=1}^n w_i u(l_{y_i}).$$

If $l_{\bar{x}} \in L$, then $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) = l_{\bar{x}} < l_{\bar{y}}$, so, $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \leq ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n})$.

If $l_{\bar{x}} \notin L$, then a non-negative integer j ($0 \leq j \leq g - 1$) exists, such that $l_j < l_{\bar{x}} < l_{j+1}$. We distinguish the two cases,

(1) If $u(l_{\bar{x}}) - u(l_j) \geq (l_{j+1}) - u(l_{\bar{x}})$, then $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) = l_{j+1}$.

Obviously, $u(l_{\bar{y}}) - u(l_j) > u(l_{j+1}) - u(l_{\bar{y}})$ and $l_j < l_{\bar{y}}$. In this case, if $l_{j+1} < l_{\bar{y}}$, then

$$ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \leq ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n}).$$

Else if $l_j < l_{\bar{y}} \leq l_{j+1}$, then $ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n}) = l_{j+1}$ and

$$ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) = ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n}).$$

(2) If $u(l_{\bar{x}}) - u(l_j) < (l_{j+1}) - u(l_{\bar{x}})$, then $ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) = l_j$.

Since $l_j < u^{-1}(\sum_{i=1}^n w_i u(l_{x_i})) \leq u^{-1}(\sum_{i=1}^n w_i u(l_{y_i}))$, it is clear that $l_j \leq ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n})$, therefore

$$ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \leq ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n}).$$

This concludes the proof of monotonic.

(ii) Note that the proofs of focus effect, idempotent, commutative and bounded are straightforward and thus omitted.

Example 2. Let us consider a simple example to illustrate the focus effect of the ULA_F operator. Let $(l_{x_1}, l_{x_2}, l_{x_3}, l_{x_4}, l_{x_5}) = (l_5, l_5, l_5, l_1, l_5)$, $(l_{y_1}, l_{y_2}, l_{y_3}, l_{y_4}, l_{y_5}) = (l_1, l_2, l_4, l_3, l_1)$. Take $w = (0.004, 0.002, 0.002, 0.99, 0.002)^T$ and $u(l_x) = \frac{x}{6}$. Obviously, $l_{x_4} = l_1 < l_{y_4} = l_3$ and w_4 is sufficiently close to 1. We have $u(l_{\bar{x}}) \approx 0.173$ and $u(l_{\bar{y}}) = 0.498$. So, we can obtain $ULA_F(l_{x_1}, l_{x_2}, l_{x_3}, l_{x_4}, l_{x_5}) = ULA_F(l_5, l_5, l_5, l_1, l_5) = l_1$. Similarly, we have $ULA_F(l_{y_1}, l_{y_2}, l_{y_3}, l_{y_4}, l_{y_5}) = ULA_F(l_1, l_2, l_4, l_3, l_1) = l_3$. So

$$ULA_F(l_5, l_5, l_5, l_1, l_5) < ULA_F(l_1, l_2, l_4, l_3, l_1).$$

From a practical point of view, the focus effect means that if one alternative satisfies the most important criterion where the ratings of the two alternatives differ then it should be preferred [44].

Property 4. Let $(l_{x_1}, l_{x_2}, \dots, l_{x_n})$, $(l_{y_1}, l_{y_2}, \dots, l_{y_n}) \in L^n$.

If $\sum_{i=1}^n w_i u(l_{x_i}) \leq \sum_{i=1}^n w_i u(l_{y_i})$, then

$$ULA_F(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \leq ULA_F(l_{y_1}, l_{y_2}, \dots, l_{y_n}).$$

Proof. Since Property 4 can be easily proven, the proof is omitted.

Let $\vec{u} = (l_{u_1}, l_{u_2}, \dots, l_{u_n})$, $\vec{v} = (l_{v_1}, l_{v_2}, \dots, l_{v_n}) \in L^n$. Strict pareto-dominance is defined as $\vec{u} <_p \vec{v}$ if and only if $\forall i = 1, \dots, n$, $l_{u_i} \leq l_{v_i}$ and $\exists j$, $l_{u_j} < l_{v_j}$ (see [44]).

Example 3. Let $(l_{x_1}, l_{x_2}, l_{x_3}, l_{x_4}, l_{x_5}) = (l_1, l_2, l_3, l_2, l_1)$, $(l_{y_1}, l_{y_2}, l_{y_3}, l_{y_4}, l_{y_5}) = (l_1, l_2, l_3, l_3, l_1)$. Obviously, $l_{x_i} \leq l_{y_i}$ ($i = 1, 2, 3, 5$) and $l_{x_4} < l_{y_4}$.

Take $w = (0.2, 0.2, 0.2, 0.2, 0.2)^T$ and $u(l_x) = \frac{1}{6}x$, we have $\sum_{i=1}^5 w_i u(l_{x_i}) = 0.3$, $\sum_{i=1}^5 w_i u(l_{y_i}) = \frac{1}{3}$. So $ULA_F(l_1, l_2, l_3, l_2, l_1) = ULA_F(l_1, l_2, l_3, l_3, l_1) = l_2$.

Obviously, $(l_1, l_2, l_3, l_2, l_1) <_p (l_1, l_2, l_3, l_3, l_1)$. Hence, the above example shows that the result of the aggregation operator ULA_F does not satisfy strict pareto-dominance and conflicts with the intuition.

5. Ordering relation on L^n

From Example 3 and the aforementioned analysis, the uncertain linguistic aggregation operator ULA_F violates strict pareto-dominance, so the aggregated result conflicts with the intuition in

some cases. To overcome this drawback, a new restricted ordering relation is proposed as follows:

Definition 6. Let $L = \{l_0, l_1, \dots, l_g\}$ be a linguistic term set. For any $(l_{x_1}, l_{x_2}, \dots, l_{x_n}), (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \in L^n$, a binary relation \lesssim_w on L^n is defined by

$$(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \Leftrightarrow \sum_{i=1}^n w_i u(l_{x_i}) \leq \sum_{i=1}^n w_i u(l_{y_i})$$

and

$$(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \sim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \Leftrightarrow \sum_{i=1}^n w_i u(l_{x_i}) = \sum_{i=1}^n w_i u(l_{y_i}).$$

Here $u : \tilde{L} \rightarrow [0, 1]$ is an extension of linguistic scale functions such that $u(l_0) = 0, u(l_g) = 1$. $w = (w_1, w_2, \dots, w_n)^T$ is the weighting vector, such that $0 \leq w_i \leq 1$ ($i = 1, 2, \dots, n$) and $\sum_{i=1}^n w_i = 1$.

Property 5. For any $(l_{x_1}, l_{x_2}, \dots, l_{x_n}), (l_{y_1}, l_{y_2}, \dots, l_{y_n}), (l_{z_1}, l_{z_2}, \dots, l_{z_n}) \in L^n$, the binary relation \lesssim_w satisfies the following five properties:

- (1) (Comparability) Either $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) <_w (l_{y_1}, l_{y_2}, \dots, l_{y_n})$ or $(l_{y_1}, l_{y_2}, \dots, l_{y_n}) <_w (l_{x_1}, l_{x_2}, \dots, l_{x_n})$ or $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \sim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n})$ holds.
- (2) (Reflexivity) $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{x_1}, l_{x_2}, \dots, l_{x_n})$.
- (3) (Antisymmetry) If $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n}), (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \lesssim_w (l_{x_1}, l_{x_2}, \dots, l_{x_n})$, then $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \sim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n})$.
- (4) (Transitivity) If $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n}), (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \lesssim_w (l_{z_1}, l_{z_2}, \dots, l_{z_n})$, then $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{z_1}, l_{z_2}, \dots, l_{z_n})$.
- (5) (Strict pareto-dominance) If $l_{x_i} \leq l_{y_i}$ ($i = 1, \dots, n$) and j exists such that $l_{x_j} < l_{y_j}$, then $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) <_w (l_{y_1}, l_{y_2}, \dots, l_{y_n})$.

Proof. Let $(l_{x_1}, l_{x_2}, \dots, l_{x_n}), (l_{y_1}, l_{y_2}, \dots, l_{y_n}), (l_{z_1}, l_{z_2}, \dots, l_{z_n})$ be any collection of L .

(1) Since $(l_{x_1}, l_{x_2}, \dots, l_{x_n}), (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \in L^n$, we have

$$\text{either } \sum_{i=1}^n w_i u(l_{x_i}) < \sum_{i=1}^n w_i u(l_{y_i})$$

$$\text{or } \sum_{i=1}^n w_i u(l_{y_i}) < \sum_{i=1}^n w_i u(l_{x_i})$$

$$\text{or } \sum_{i=1}^n w_i u(l_{x_i}) = \sum_{i=1}^n w_i u(l_{y_i}).$$

If $\sum_{i=1}^n w_i u(l_{x_i}) < \sum_{i=1}^n w_i u(l_{y_i})$, then $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) <_w (l_{y_1}, l_{y_2}, \dots, l_{y_n})$.

If $\sum_{i=1}^n w_i u(l_{y_i}) < \sum_{i=1}^n w_i u(l_{x_i})$, then $(l_{y_1}, l_{y_2}, \dots, l_{y_n}) <_w (l_{x_1}, l_{x_2}, \dots, l_{x_n})$.

If $\sum_{i=1}^n w_i u(l_{y_i}) = \sum_{i=1}^n w_i u(l_{x_i})$, then $(l_{y_1}, l_{y_2}, \dots, l_{y_n}) \sim_w (l_{x_1}, l_{x_2}, \dots, l_{x_n})$.

(2) Since $\sum_{i=1}^n w_i u(l_{x_i}) \leq \sum_{i=1}^n w_i u(l_{x_i})$, we have

$$(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{x_1}, l_{x_2}, \dots, l_{x_n}).$$

(3) From Definition 6, we have

$$(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \Rightarrow \sum_{i=1}^n w_i u(l_{x_i}) \leq \sum_{i=1}^n w_i u(l_{y_i})$$

and

$$(l_{y_1}, l_{y_2}, \dots, l_{y_n}) \lesssim_w (l_{x_1}, l_{x_2}, \dots, l_{x_n}) \Rightarrow \sum_{i=1}^n w_i u(l_{y_i}) \leq \sum_{i=1}^n w_i u(l_{x_i}).$$

So we have $\sum_{i=1}^n w_i u(l_{y_i}) = \sum_{i=1}^n w_i u(l_{x_i})$, then

$$(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \sim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n}).$$

(4) From Definition 6, we have

$$(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{y_1}, l_{y_2}, \dots, l_{y_n}) \Rightarrow \sum_{i=1}^n w_i u(l_{x_i}) \leq \sum_{i=1}^n w_i u(l_{y_i})$$

and

$$(l_{y_1}, l_{y_2}, \dots, l_{y_n}) \lesssim_w (l_{z_1}, l_{z_2}, \dots, l_{z_n}) \Rightarrow \sum_{i=1}^n w_i u(l_{y_i}) \leq \sum_{i=1}^n w_i u(l_{z_i}).$$

We have $\sum_{i=1}^n w_i u(l_{x_i}) \leq \sum_{i=1}^n w_i u(l_{z_i})$, then

$$(l_{x_1}, l_{x_2}, \dots, l_{x_n}) \lesssim_w (l_{z_1}, l_{z_2}, \dots, l_{z_n}).$$

(5) Since u is a strictly monotonic increasing and continuous function, $l_{x_i} \leq l_{y_i}$ ($i = 1, \dots, n$) and j exists such that $l_{x_j} < l_{y_j}$, we have $u(l_{x_i}) \leq u(l_{y_i})$ and $u(l_{x_j}) < u(l_{y_j})$.

So $\sum_{i=1}^n w_i u(l_{x_i}) < \sum_{i=1}^n w_i u(l_{y_i})$, then $(l_{x_1}, l_{x_2}, \dots, l_{x_n}) <_w (l_{y_1}, l_{y_2}, \dots, l_{y_n})$.

This proves that the ordering relation \lesssim_w satisfies strict pareto-dominance. This completes the proof. Hence, (L^n, \lesssim_w) is a totally ordered set.

Example 4. Let $(l_{x_1}, l_{x_2}, l_{x_3}, l_{x_4}, l_{x_5}) = (l_5, l_2, l_2, l_1, l_1), (l_{y_1}, l_{y_2}, l_{y_3}, l_{y_4}, l_{y_5}) = (l_1, l_2, l_3, l_2, l_5)$. Take $w_1 = (0.6, 0.1, 0.1, 0.1, 0.1)^T$ and $u(l_x) = \frac{1}{6}x$, then we have $\sum_{i=1}^5 w_{1i} u(l_{y_i}) = 0.3 < \sum_{i=1}^5 w_{1i} u(l_{x_i}) = 0.6$.

So $(l_1, l_2, l_3, l_2, l_5) <_{w_1} (l_5, l_2, l_2, l_1, l_1)$.

Take $w_2 = (0.1, 0.1, 0.1, 0.1, 0.6)^T$ and keep u unchanged. We have

$$\sum_{i=1}^5 w_{2i} u(l_{x_i}) \approx 0.267 < \sum_{i=1}^5 w_{2i} u(l_{y_i}) \approx 0.633.$$

So $(l_5, l_2, l_2, l_1, l_1) <_{w_2} (l_1, l_2, l_3, l_2, l_5)$.

This example shows that the ordering relation \lesssim_w is affected by the weighting vector w .

Example 5. $(l_{x_1}, l_{x_2}, l_{x_3}, l_{x_4}, l_{x_5}), (l_{y_1}, l_{y_2}, l_{y_3}, l_{y_4}, l_{y_5})$ remain the same as those in Example 3, and then $l_{x_i} \leq l_{y_i}$ ($i = 1, 2, 3, 5$), $l_{x_4} < l_{y_4}$.

For any $w = (w_1, w_2, w_3, w_4, w_5)^T$, such that $0 \leq w_i \leq 1, \sum_{i=1}^5 w_i = 1$, take $u(l_x) = \frac{1}{6}x$, obviously, $\sum_{i=1}^5 w_i u(l_{x_i}) < \sum_{i=1}^5 w_i u(l_{y_i})$. So

$$(l_1, l_2, l_3, l_2, l_1) <_w (l_1, l_2, l_3, l_3, l_1).$$

Therefore, the ordering relation \lesssim_w satisfies strict pareto-dominance.

6. Application in multiple attribute decision making

Through an analysis of the applicable condition for the ULA_F operator, we can see that it can be applied to similar situations where information is assessed using linguistic variables. For example, the ULA_F operator can be applied to decision making in areas such as statistics, engineering, economics, decision theory and soft computing. Below, we focus on an application of the ULA_F operator to a decision-making problem. Specifically, we consider a multi-attribute decision making problem, in which the attribute weights are completely unknown.

Let $A = \{A_1, A_2, \dots, A_m\}$ be the discrete set of alternatives, and $C = \{C_1, C_2, \dots, C_n\}$ be the set of attributes, and $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of attributes, which satisfies $0 \leq w_j \leq 1$ and $\sum_{j=1}^n w_j = 1$.

For each alternative $A_i \in A$, the expert gives a preference value a_{ij} with respect to attribute $C_j \in C$. Thus, a_{ij} represents the assessment information given by the expert about the j th attribute of alternative A_i , where a_{ij} is a linguistic variable. That is $a_{ij} \in L$, so all the preference values of the alternatives make up the decision-making matrix $A = (a_{ij})_{m \times n}$.

The maximizing deviation method was proposed by Wang [45] to deal with multiple attribute decision making problems with numerical information. If the performance values of each alternative have little difference under an attribute, this indicates that such an attribute plays a small role in the priority procedure. Conversely, if some attribute makes the performance values among all the alternatives have obvious differences, such an attribute plays an important role in choosing the best alternative. So to the view of sorting the alternatives, if one attribute has similar attribute values across alternatives, it should be assigned a small weight; otherwise, the attribute which makes larger deviations should be evaluated a bigger weight, in spite of the degree of its own importance. Especially, if all available alternatives score about equally with respect to a given attribute, then such an attribute will be judged unimportant by most decision makers. In other words, such an attribute should be assigned a very small weight. Wang [45] suggested that zero should be assigned to the attribute of this kind.

The deviation method is selected here to compute the differences of the performance values of each alternative. Under linguistic environment, for the attribute $C_j \in C$, the deviation of alternative A_i to all the other alternatives can be defined as follows:

$$H_{ij} = \sum_{k=1}^m d(a_{ij}, a_{kj}), \quad i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}. \quad (7)$$

Let

$$H_j = \sum_{i=1}^m H_{ij}, \quad i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}. \quad (8)$$

H_j represents the deviation of all alternatives to other alternatives for the attribute C_j .

Based on the aforementioned analysis, the weight vector w is used to maximize all deviation values for all the attributes that should be selected. So, we have

$$w_j = \frac{H_j}{\sum_{j=1}^n H_j}. \quad (9)$$

Note that H_j represents the deviation from all alternatives to other alternatives for the attribute C_j . The larger the H_j , the more important the attribute C_j . Eq. (9) is obtained directly using each H_j divided by the sum of H_j , which is considered as an explanation to the maximizing method. The theoretical foundation of this

method is that the attribute providing more information should be evaluated a bigger weight [46].

To some extent, w_j represents the weight of attribute C_j , therefore, the weight vector of attributes is $w = (w_1, w_2, \dots, w_n)^T$. Using (7)–(9), a simple and exact formula for determining the attribute weights has also been obtained.

In the following, a new approach based on the uncertain linguistic aggregation operator ULA_F is proposed to solving the multiple attribute decision making problems, in which the information about attribute weights is completely unknown, and the attribute values take the form of linguistic variables. The new approach involves the following steps:

- Step 1: Let $A = (a_{ij})_{m \times n}$ be a decision-making matrix, where $a_{ij} \in L$ of each alternative $A_i \in A$ of each attribute $C_j \in C$, and $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of attributes, where $w_j \in [0, 1], j = 1, 2, \dots, n$.
- Step 2: Suppose the weights of the attributes are completely unknown, then Eq. (9) is used to calculate the weight vector of attributes $w = (w_1, w_2, \dots, w_n)^T$.
- Step 3: The approximate function expression of $u(l_i)$ can be determined based on decision-maker's attitude toward risk. Utilize the ULA_F operator to consolidate the decision-making information to determine the collective evaluation value of each alternative.
- Step 4: Rank all the alternatives $A_i (i = 1, 2, \dots, m)$ (if there is no difference in the collective evaluation value between two alternatives A_i and A_j , according to the rules in Definition 6, we can rank the alternatives A_i and A_j). Select the best alternative(s).
- Step 5: End.

7. Illustrative example

In this section, an illustrative example is provided to show the feasibility and validity of the proposed aggregation operator. We consider an investment selection decision making problem.

Example 6. Suppose an engineering investment company wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money:

- A_1 A car company;
- A_2 A food company;
- A_3 A computer company;
- A_4 An arms company.

The engineering investment company must make a decision according to the following four attributes:

- C_1 Risk analysis;
- C_2 Growth analysis;
- C_3 Social-political impact analysis;
- C_4 Environmental impact analysis;
- C_5 Technological analysis.

The four alternatives $A_i (i = 1, 2, 3, 4)$ are evaluated using the linguistic term set

$$L = \{l_0 = \text{None}, l_1 = \text{Very low}, l_2 = \text{Low}, l_3 = \text{Medium}, l_4 = \text{High}, l_5 = \text{Very high}, l_6 = \text{Perfect}\}.$$

The performance evaluations for each alternative $A_i (i = 1, 2, 3, 4)$ are listed in Table 1.

Table 1
Linguistic decision-making matrix R.

a_{ij}	c_1	c_2	c_3	c_4	c_5
A_1	l_1	l_2	l_5	l_3	l_3
A_2	l_2	l_3	l_1	l_4	l_1
A_3	l_4	l_4	l_3	l_5	l_4
A_4	l_5	l_2	l_4	l_4	l_2

7.1. Illustration of the proposed approach

The approach proposed in Section 6 is employed to determine the best alternative(s), as described in the following steps. **Case**

1: Taking the linguistic scale function $u(l_x) = x/6$ ($x \in [0, 6]$), the evaluation scale of the linguistic information given above is divided by the average. The absolute deviation between the adjacent linguistic subscripts remains unchanged, so then, the linguistic label L is equidistant.

Step 1: Calculate the weight vector of the attributes.

Applying Eqs. (7)–(8), we get the deviation value H_j of all alternatives to other alternatives for the attribute C_j :

$$H_1 = \frac{14}{3}, \quad H_2 = \frac{7}{3}, \quad H_3 = \frac{13}{3},$$

$$H_4 = 2, \quad H_5 = \frac{10}{3}.$$

Use Eq. (9) to calculate the weight vector of the attributes

$$w = (0.28, 0.14, 0.26, 0.12, 0.2)^T.$$

Step 2: Use Eq. (3) and the weight vector of the attributes w to calculate

$$u(l_x^{A_1}) = w_1u(l_1) + w_2u(l_2) + w_3u(l_5) + w_4u(l_3) + w_5u(l_3)$$

$$= 0.28 \times \frac{1}{6} + 0.14 \times \frac{2}{6} + 0.26 \times \frac{5}{6} + 0.12 \times \frac{3}{6} + 0.2 \times \frac{3}{6}$$

$$= 0.47,$$

$$u(l_x^{A_2}) = w_1u(l_2) + w_2u(l_3) + w_3u(l_1) + w_4u(l_4) + w_5u(l_1)$$

$$= 0.28 \times \frac{2}{6} + 0.14 \times \frac{3}{6} + 0.26 \times \frac{1}{6} + 0.12 \times \frac{4}{6} + 0.2 \times \frac{1}{6}$$

$$= 0.32,$$

$$u(l_x^{A_3}) = w_1u(l_4) + w_2u(l_4) + w_3u(l_3) + w_4u(l_5) + w_5u(l_4)$$

$$= 0.28 \times \frac{4}{6} + 0.14 \times \frac{4}{6} + 0.26 \times \frac{3}{6} + 0.12 \times \frac{5}{6} + 0.2 \times \frac{4}{6}$$

$$\approx 0.643,$$

$$u(l_x^{A_4}) = w_1u(l_5) + w_2u(l_2) + w_3u(l_4) + w_4u(l_4) + w_5u(l_2)$$

$$= 0.28 \times \frac{5}{6} + 0.14 \times \frac{2}{6} + 0.26 \times \frac{4}{6} + 0.12 \times \frac{4}{6} + 0.2 \times \frac{2}{6}$$

$$= 0.6.$$

Step 3: Utilize the ULA_F operator to aggregate the linguistic argument collections to determine the collective evaluation

value of each alternative A_i ($i = 1, 2, 3, 4$) as follows:

$$A_1 : ULA_F(l_1, l_2, l_5, l_3, l_3) = l_3,$$

$$A_2 : ULA_F(l_2, l_3, l_1, l_4, l_1) = l_2,$$

$$A_3 : ULA_F(l_4, l_4, l_3, l_5, l_4) = l_4,$$

$$A_4 : ULA_F(l_5, l_2, l_4, l_4, l_2) = l_4.$$

Step 4: Although A_3 and A_4 have the same collective evaluation value, according to Definition 6 and $u(l_x^{A_2}) < u(l_x^{A_1}) < u(l_x^{A_4}) < u(l_x^{A_3})$, then

$$A_2 \prec_w A_1 \prec_w A_4 \prec_w A_3.$$

Thus the best alternative is A_3 .

Case 2: Taking the linguistic scale function $u(l_x) = (x/6)^2$ ($x \in [0, 6]$), the absolute deviation between the adjacent linguistic subscripts increases, so the linguistic label L is non-equidistant. In such a case, as Herrera and Martínez themselves have pointed out in Ref. [33], the 2-tuple linguistic representation model is unable to deal with the non-equidistant labels. However, the proposed method in this paper is able to overcome this limitation, as described in the following steps.

Step 1: The weight vector of the attributes remains the same as in case 1, $w = (0.28, 0.14, 0.26, 0.12, 0.2)^T$.

Step 2: Use Eq. (3) and the weight vector of the attributes w to calculate

$$u(l_x^{A_1}) = w_1u(l_1) + w_2u(l_2) + w_3u(l_5) + w_4u(l_3) + w_5u(l_3)$$

$$= 0.28 \times \frac{1}{36} + 0.14 \times \frac{4}{36} + 0.26 \times \frac{25}{36} + 0.12 \times \frac{9}{36} + 0.2 \times \frac{9}{36}$$

$$\approx 0.284,$$

$$u(l_x^{A_2}) = w_1u(l_2) + w_2u(l_3) + w_3u(l_1) + w_4u(l_4) + w_5u(l_1)$$

$$= 0.28 \times \frac{4}{36} + 0.14 \times \frac{9}{36} + 0.26 \times \frac{1}{36} + 0.12 \times \frac{16}{36} + 0.2 \times \frac{1}{36}$$

$$\approx 0.132,$$

$$u(l_x^{A_3}) = w_1u(l_4) + w_2u(l_4) + w_3u(l_3) + w_4u(l_5) + w_5u(l_4)$$

$$= 0.28 \times \frac{16}{36} + 0.14 \times \frac{16}{36} + 0.26 \times \frac{9}{36} + 0.12 \times \frac{25}{36} + 0.2 \times \frac{16}{36}$$

$$\approx 0.424,$$

$$u(l_x^{A_4}) = w_1u(l_5) + w_2u(l_2) + w_3u(l_4) + w_4u(l_4) + w_5u(l_2)$$

$$= 0.28 \times \frac{25}{36} + 0.14 \times \frac{4}{36} + 0.26 \times \frac{16}{36} + 0.12 \times \frac{16}{36} + 0.2 \times \frac{4}{36}$$

$$\approx 0.401.$$

Step 3: Utilize the ULA_F operator to aggregate the linguistic argument collections to determine the collective evaluation value of each alternative A_i ($i = 1, 2, 3, 4$) as follows:

$$A_1 : ULA_F(l_1, l_2, l_5, l_3, l_3) = l_3,$$

$$A_2 : ULA_F(l_2, l_3, l_1, l_4, l_1) = l_2,$$

$$A_3 : ULA_F(l_4, l_4, l_3, l_5, l_4) = l_4,$$

$$A_4 : ULA_F(l_5, l_2, l_4, l_4, l_2) = l_4.$$

Step 4: Although A_3 and A_4 have the same collective evaluation value, according to Definition 6 and $u(l_x^{A_2}) < u(l_x^{A_1}) < u(l_x^{A_4}) < u(l_x^{A_3})$, then

$$A_2 <_w A_1 <_w A_4 <_w A_3.$$

Thus the best alternative is A_3 .

Obviously, the final results in the above two cases are the same.

7.2. Comparison analysis and discussion

To further test the validity of the multiple attribute decision making method proposed in this paper, a comparison analysis is conducted using the method proposed by Wu and Chen [46] and the analysis is based on the same illustrative example described above.

Using the multiple attribute decision making method based on the LWAA operator [46], the overall assessment of the alternative A_1 can be determined as follows:

$$\begin{aligned} LWAA(l_1, l_2, l_5, l_3, l_3) &= 0.28 \times l_1 + 0.14 \times l_2 + 0.26 \times l_5 + 0.12 \times l_3 + 0.2 \times l_3 \\ &= l_{0.28 \times 1} + l_{0.14 \times 2} + l_{0.26 \times 5} + l_{0.12 \times 3} + l_{0.2 \times 3} \\ &= l_{0.28 \times 1 + 0.14 \times 2 + 0.26 \times 5 + 0.12 \times 3 + 0.2 \times 3} = l_{2.82}. \end{aligned}$$

Analogously, the overall assessment of the other alternatives can be obtained:

$$LWAA(l_2, l_3, l_1, l_4, l_1) = l_{1.92},$$

$$LWAA(l_4, l_4, l_3, l_5, l_4) = l_{3.86},$$

$$LWAA(l_5, l_2, l_4, l_4, l_2) = l_{3.6}.$$

Then, the rank ordering among the alternative is $A_2 < A_1 < A_4 < A_3$.

Obviously, the two methods have the same ranking results. Therefore, the method proposed in this paper is rational and valid.

Although the selection order is the same, the results of the LWAA may not match any of the initial term sets. For example, $LWAA(l_1, l_2, l_5, l_3, l_3) = l_{2.82} \notin L$. In such a case, as Wang et al. [47] have pointed out in their introduction, there is an awareness that $l_{2.82}$ does not have any syntax or semantics assigned, because such a virtual linguistic term makes sense only in comparison and in operation. Moreover, in the LWAA calculation process, the product between the numerical value and the linguistic variable is usually employed to calculate the alternative collective evaluation value. For example, $0.28 \times l_1$, under the meaning of linguistic label, means “ $0.28 \times$ Very low”. However, what does “ $0.28 \times$ Very low” mean in the actual decision problem? Therefore, this method has its own weaknesses.

The main difference that the ULA_F from other operators is that it converts the linguistic variable to the unit interval $[0, 1]$ using the linguistic scale function and consequently effectively avoids the problems caused by the product between the numerical values and linguistic variables.

Furthermore, the main advantage of using the ULA_F operator is that the aggregated results belong to the initial linguistic terms, which is more appropriate and more easily comprehended. As Herrera and Martínez pointed out in Ref. [33], the 2-tuple linguistic representation model is only suitable for linguistic variables with equidistant labels. The proposed method in this paper is able to overcome this limitation as the ULA_F operator can be used to aggregate linguistic variables with non-equidistant labels. Consequently, the proposed method is effective and advantageous in dealing with multiple attribute decision making problems.

8. Conclusions

In this paper, we provided a tentative method to establish an uncertain linguistic aggregation ULA_F operator in linguistic environment and studied some desirable properties of the proposed aggregation operator. Further, a new restricted ordering relation on n -dimensional linguistic scales was proposed. The new restricted ordering relation satisfies strict pareto-dominance and is restricted by the weighting vector. Based on the proposed aggregation operator and the new ordering relation, we obtained a new multiple attribute decision making method.

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