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TESTING FOR COINTEGRATION WITH THRESHOLD ADJUSTMENT IN THE PRESENCE OF STRUCTURAL BREAKS

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Testing for cointegration with threshold adjustment in the presence of structural breaks

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Abstract

In this paper, we develop new threshold cointegration tests with SETAR and MTAR adjustment allowing for the presence of structural breaks in the equilibrium equation. We propose a simple procedure to simultaneously estimate the previously unknown breakpoint and test the null hypothesis of no cointegration. Thereby, we extend the well-known residual-based cointegration test with regime shift introduced by [Gregory and Hansen](#page-38-0) [\(1996a\)](#page-38-0) to include forms of nonlinear adjustment. We derive the asymptotic distribution of the test statistics and demonstrate the finite-sample performance of the tests in a series of Monte Carlo experiments. We find a substantial decrease of power of the conventional threshold cointegration tests caused by a shift in the slope coefficient of the equilibrium equation. The proposed tests perform superior in these situations. An application to the 'rockets and feathers' hypothesis of price adjustment in the US gasoline market provides empirical support for this methodology.

Keywords: Cointegration, threshold autoregression, structural breaks, SETAR, MTAR, asymmetric price transmission **MSC Codes:** 62H15, 62M10, 62P20 **JEL Codes:** C12, C32, C34, Q41

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1 Introduction

The residual-based threshold cointegration models developed by [Enders and Siklos](#page-37-0) [\(2001\)](#page-37-0) are a useful addition to the toolbox of researchers working with multivariate time series. They are easy to apply, allow for discontinuous adjustment to a long-run equilibrium and nest linear cointegration in the sense of [Engle and Granger](#page-38-1) [\(1987\)](#page-38-1) as a special case. The dynamics of the adjustment process are described by a two-regime threshold autoregressive (TAR) model which partitions the residual process according to a threshold value and specifies different coefficients of the leading autoregressive lag for each regime. It can therefore be considered a restricted model under the general class of TAR models described by [Tong](#page-39-0) [\(1983,](#page-39-0) [1990\)](#page-39-1). A prominent application in the economics literature is the empirical analysis of asymmetric price transmissions in which case non-stationary price series form a cointegrating relationship and may feature asymmetric adjustment to long-run equilibrium. The speed of adjustment is usually assumed to depend on the sign and magnitude of the deviations from the long-run equilibrium. While threshold cointegration models are suitable to study these cases, they do not account for possible structural change in the long-run relationship. It is well-known that conventional residual-based cointegration tests perform poorly when a cointegration relationship has structural breaks (see, for example, [Gregory et al.](#page-38-2) [\(1996\)](#page-38-2)). [Maki](#page-38-3) [\(2012\)](#page-38-3) found that the power property of threshold cointegration tests is more robust to structural breaks than, for example, the Engle-Granger cointegration tests assuming linear adjustment. Nevertheless, the power of all residual-based cointegration tests is impaired if the tests do not model the structural breaks explicitly. Consequently, it is difficult to provide evidence for the existence of a cointegration relationship. Furthermore, the estimated adjustment coefficients are biased if the cointegrating vector does not account for structural change.

An extensive body of literature exists on the problem of structural instability in time series. Based on the seminal work of [Perron](#page-39-2) [\(1989\)](#page-39-2), several unit root tests accounting for structural change have been developed (see, inter alia, [Zivot and Andrews](#page-40-0) [\(1992\)](#page-40-0), [Lumsdaine and Papell](#page-38-4) [\(1997\)](#page-38-4) and [Lee and Strazicich](#page-38-5) [\(2003\)](#page-38-5)). Structural breaks in linear cointegration models are addressed in [Gregory and Hansen](#page-38-0) [\(1996a,](#page-38-0)[b\)](#page-38-6), [Arai](#page-37-1) [and Kurozumi](#page-37-1) [\(2007\)](#page-37-1), [Carrion-i Silvestre and Sanso](#page-37-2) [\(2006\)](#page-37-2), [Westerlund and Edgerton](#page-39-3) [\(2007\)](#page-39-3) and [Hatemi-J](#page-38-7) [\(2008\)](#page-38-7). For a comprehensive survey on structural change in time series models, see [Perron](#page-39-4) [\(2006\)](#page-39-4). [Gregory and Hansen](#page-38-0) [\(1996a\)](#page-38-0), henceforth GH, propose a residual-based cointegration test with structural break. Their test does not require a pre-specified breakpoint which is rarely known in empirical applications. Instead, a single unknown breakpoint is determined from the data based on one of three structural break models. However, the GH test is only suitable for cointegration models with linear adjustment.^{[1](#page-5-0)} We contribute to the literature by extending the GH test to include two forms of non-linear adjustment. These new tests are residual-based and use either a self-exciting threshold autoregressive (SETAR) model or a momentum threshold autoregressive (MTAR) to describe the adjustment toward equilibrium. Thereby, we also provide an extension to the Enders-Siklos cointegration tests which are robust to a structural break in the cointegrating vector.

We derive the limiting distributions of the test statistics considered in this paper and provide a formal proof. The properties of the proposed test are investigated by Monte Carlo experiments for a variety of models ranging from linear adjustment with no structural break to non-linear adjustment with structural break in the intercept and slope coefficients. The results suggest that a break in the intercept does not influence the power of the threshold cointegration tests enough to justify modelling the structural break. However, a break in the slope coefficients reduces the power of the Enders-Siklos tests substantially such that our proposed tests perform clearly better than their benchmarks. In addition, we find that the unknown breakpoints are estimated accurately by the new procedure.

The methodology is applied to empirical data in the context of the 'rockets and feathers' hypothesis. We use US gasoline market data covering the Financial Crisis. We illustrate that empirical evidence for the existence of a long-run relationship between neighbouring stages of the gasoline value-chain can only be provided if we control for a structural break in the cointegrating vector. Using a cointegration model with SETAR adjustment and the possibility of structural breaks, we find evidence for asymmetric adjustment from spot gasoline to retail gasoline prices. The MTAR model yields similar results.

The paper is organized as follows. [Section 2](#page-6-0) describes the models and the cointegration testing procedure, [Section 3](#page-9-0) presents the asymptotic distributions of the test statistics. [Section 4](#page-13-0) is devoted to the Monte Carlo simulation study. [Section 5](#page-16-0) reports the results of the empirical application, and [Section 6](#page-18-0) summarizes the study.

¹The effects on the power properties of linear cointegration tests, if the equilibrium error follows a nonlinear adjustment process, are reported in [Pippenger and Goering](#page-39-5) [\(2000\)](#page-39-5).

2 Models and cointegration testing

The long-run equilibrium equation of Engle-Granger cointegration models is given by

$$
y_t = \mu + \alpha_1 x_{1t} + \alpha_2 x_{2t} + \dots + \alpha_m x_{mt} + e_t
$$

=
$$
\mu + \alpha' x_t + e_t
$$
 (1)

where $t = 1, 2, \ldots, T$ is the time series index, y_t and $x_t = (x_{it}, x_{2t}, \ldots, x_{mt})'$ are $I(1)$ variables, μ is an intercept, $\alpha' = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a vector of slope coefficients and e_t is the equilibrium error. The null hypothesis of no cointegration is rejected if the residuals obtained from least squared estimation of [\(1\)](#page-6-1) are mean-zero stationary. Since the parameters μ and α are time-invariant, a residual-based cointegration test based on [\(1\)](#page-6-1) becomes invalid if the long-run equilibrium is subject to structural change.

Following [Perron](#page-39-2) [\(1989\)](#page-39-2) and [Gregory and Hansen](#page-38-0) [\(1996a\)](#page-38-0), we consider three forms of structural change.^{[2](#page-6-2)} First, in the *C* model, a break in the intercept μ is considered. This model captures events that cause a parallel shift of the equilibrium equation. Second, the *C/T* model adds an additional trend term to the equilibrium equation. Third, in the *C/S* model, a simultaneous break in the constant and slope parameters is specified. This model allows for the possibility of a complete regime shift at one point in time. The three models are given as follows,

(C)
$$
y_t = \mu_1 + \mu_2 \varphi_{t,\tau} + \alpha' x_t + e_{t\tau}
$$

\n(C/T) $y_t = \mu_1 + \mu_2 \varphi_{t,\tau} + \delta t + \alpha' x_t + e_{t\tau}$
\n(C/S) $y_t = \mu_1 + \mu_2 \varphi_{t,\tau} + \alpha'_1 x_t + \alpha'_2 x_t \varphi_{t,\tau} + e_{t\tau}$ (2)

where μ_1, μ_2 are constants, $\alpha_1 = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1m})'$ and $\alpha_2 = (\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2m})'$ are slope coefficients. The dummy variable $\varphi_{t,\tau}$ is defined as

$$
\varphi_{t,\tau} = \begin{cases} 1 & \text{if } t \geq [T\tau] \\ 0 & \text{if } t < [T\tau] \end{cases},\tag{3}
$$

where $\tau \in (0,1)$ denotes the relative timing of the breakpoint (break fraction), and $[\cdot]$ denotes integer part. The timing of the breakpoint is rarely known in empirical applica-

²We restrict our analysis to these three models. However, our methodology can easily be adapted for other structural break models, as for example given in [Gregory and Hansen](#page-38-6) [\(1996b\)](#page-38-6) and [Hatemi-J](#page-38-7) [\(2008\)](#page-38-7).

tions so that the GH test is constructed without the need of pre-specified breakpoints. More specifically, a grid search over all possible breakpoint is employed, i.e. the structural change model is repeatedly estimated for each possible break fraction $\tau \in \mathcal{T}$. The set $\mathcal T$ can be any compact subset of $(0,1)$ which excludes endpoint results. GH suggest a lateral trimming of 15 percent $(\mathcal{T} = (0.15, 0.85))$ and, for computational reasons, consider only integer steps. Estimating one of the structural break models in [\(3\)](#page-6-3) by least squares for each breakpoint yields a sequence of residuals. The GH test applies the ADF test to each sequence and evaluates the null hypothesis of no cointegration based on the smallest values of the *t* ratios across all $\tau \in \mathcal{T}$. The infimum statistic is chosen since it puts the most weight on the alternative hypothesis. If the null hypothesis is rejected, the break fraction *τ*ˆ corresponding to the infimum statistic is considered to be the most likely breakpoint.

In order to account for asymmetric adjustment, the two-regime SETAR model is now used to describe the adjustment toward equilibrium. The SETAR model for the breakpoint-specific equilibrium error process $e_{t\tau}$ is given by

$$
\Delta e_{t\tau} = \rho_1 e_{t-1\tau} \mathbb{1}\{e_{t-1\tau} \ge \lambda\} + \rho_2 e_{t-1\tau} \mathbb{1}\{e_{t-1\tau} < \lambda\} + \sum_{j=1}^K \gamma_j \Delta e_{t-j\tau} + \epsilon_{t\tau K},\qquad(4)
$$

where $1\{\cdot\}$ denotes the Heaviside indicator function, the parameter λ is a possibly nonzero threshold value and $\epsilon_{t\tau K}$ is a stationary mean zero error term. The coefficient ρ_1 measures the mean-reversion toward the attractor after a shock greater than or equal to λ whereas ρ_2 measures the mean-reversion toward the cointegrating vector after a shock less than λ . The indicator function in this case is set according to the level of $e_{t-1\tau}$.

In an alternative specification, suggested by [Enders and Granger](#page-37-3) [\(1998\)](#page-37-3) and [Caner](#page-37-4) [and Hansen](#page-37-4) [\(2001\)](#page-37-4), the indicator function is set depending on $\Delta e_{t-1\tau}$. The two-regime MTAR model is given by

$$
\Delta e_{t\tau} = \rho_1 e_{t-1\tau} \mathbb{1} \{ \Delta e_{t-1\tau} \ge \lambda \} + \rho_2 e_{t-1\tau} \mathbb{1} \{ \Delta e_{t-1\tau} < \lambda \} + \sum_{j=1}^K \gamma_j \Delta e_{t-j\tau} + \epsilon_{t\tau K}.
$$

In this specification, ρ_1 measures the mean-reversion toward the attractor if a shock has momentum greater than or equal to λ whereas ρ_2 measures the mean-reversion toward the cointegrating vector if a shock has momentum less than λ .

Under the null hypothesis of no cointegration, $\rho_1 = \rho_2 = 0$, the data-generating

process (DGP) of $e_{t\tau}$ is symmetric and a unit root is present in both regimes. Models [\(4\)](#page-7-0) and [\(5\)](#page-7-1) are a special case of the general class of threshold autoregressive models in that they do not allow for regime-specific deterministic terms and regime-specific dynamics beyond the leading autoregressive lag. This restriction is convenient since it circumvents the problem of having an identified threshold under the null hypothesis resulting in an asymptotic distribution of the test statistic that depends on nuisance parameters (see [Caner and Hansen](#page-37-4) [\(2001\)](#page-37-4) for a more detailed discussion in the context of MTAR processes with a unit root). Furthermore, the Engle-Granger test for symmetric adjustment $(\rho_1 = \rho_2)$ is itself a special case of [\(4\)](#page-7-0) and [\(5\)](#page-7-1). [Petruccelli and Woolford](#page-39-6) [\(1984\)](#page-39-6) show that the stationarity of the SETAR process is ensured if $\rho_1 < 0$, $\rho_2 < 0$ and $(1 + \rho_1)(1 + \rho_2)$ < 1 for any value λ . In the case of MTAR processes, [Lee and](#page-38-8) [Shin](#page-38-8) [\(2000\)](#page-38-8) prove that stationarity is ensured if $\rho_1 < 0$, $\rho_2 < 0$, $(1 + \rho_1)(1 + \rho_2) < 1$, $(1+\rho_1)(1+\rho_2)^2 < 1$ and $(1+\rho_1)^2(1+\rho_2) < 1$. Assuming stationarity, [Tong](#page-39-0) [\(1983,](#page-39-0) [1990\)](#page-39-1) demonstrated that least squares estimators of ρ_1 and ρ_2 are asymptotically normally distributed. [Enders and Siklos](#page-37-0) [\(2001\)](#page-37-0) recommend a Wald-type *F*-test to test the null hypothesis of no cointegration in their model without structural breaks. However, since the *F*-test can lead to rejection of the null hypothesis when only one coefficient is negative, the test should only be applied if both point estimates suggest a meanreversion behaviour. In other words, the one-sided alternative $\rho_1 < 0 \wedge \rho_2 \geq 0$ or ρ_2 < 0 \wedge ρ_1 ≥ 0 should not lead to rejection of the null hypothesis.

In the case of a cointegration model with potential structural break, we propose the following cointegration test: First, an appropriate structural break model is selected from [\(3\)](#page-6-3) and the cointegrating regression is estimated by least squares for each break fraction $\tau \in \mathcal{T}$. Then, the SETAR or MTAR regression is estimated and the *F*statistic, F_{τ} , is computed for each sequence of residuals. Since the null hypothesis of no cointegration is naturally rejected for large values of the *F*-statistic, the supremum statistic,

$$
F^* = \sup_{\tau \in \mathcal{T}} F_{\tau},\tag{5}
$$

is used to evaluate the null hypothesis of no cointegration against the alternative of threshold cointegration with possible structural break. The largest value found in this grid search also determines the most likely breakpoint.

3 Asymptotic distribution

In the following, we present the asymptotic distributions of the test statistics as functionals of Brownian motion. The asymptotic theory for SETAR processes with a unit root was developed in [Seo](#page-39-7) [\(2008\)](#page-39-7) and the asymptotic theory for MTAR processes with a unit root was developed in [Caner and Hansen](#page-37-4) [\(2001\)](#page-37-4). [Gregory and Hansen](#page-38-0) [\(1996a\)](#page-38-0) provide important results for cointegration test statistics which are functions of the break fraction parameter τ and serves as the building block for our residual-based tests.

For notational convenience we use ' \Rightarrow ' to signify weak convergence of the associated probability measures. Continuous stochastic processes such as the Brownian motion *B*(*s*) on [0,1] are simply written as *B* if no confusion will be caused. We also write integrals with respect to the Lebesgue measure such as \int_0^1 $\boldsymbol{0}$ $B(s)ds$ simply as \int_0^1 $\boldsymbol{0}$ *B*.

Let $\{z_t\}_0^{\infty}$ be an $(m+1)$ -vector integrated process whose data generating process is

$$
z_t = z_{t-1} + \xi_t, \qquad t = 1, 2, \dots \tag{6}
$$

where it is assumed that $T^{-1/2}z_0 \stackrel{p}{\to} 0$ so that z_0 can be treated as either fixed or random and the results do not depend on the initial condition. The $(m + 1)$ -vector random sequence $\{\xi_t\}_{1}^{\infty}$ is defined on the probability space (X, \mathcal{F}, P) and is assumed to be strictly stationary and ergodic with zero mean and finite variance. $\{\xi_t\}_{1}^{\infty}$ satisfies the following regularity conditions:

Assumption 1. ξ_t *is a stationary ARMA process with* $\xi_t = \sum_{i=1}^{\infty}$ $\sum_{j=0} C_j \nu_{t-j}, \ C_0 = I_n,$ $\sum_{i=1}^{\infty}$ $\sum_{j=0}$ *j* $||C_j|| < \infty$ and $\nu_t \sim \text{iid}(0, \Sigma)$, where Σ *is a positive definite variance matrix and ν*_t have absolutely continuous distribution^{[3](#page-9-1)}. Further, $E|\nu_t|^r < \infty$ for some $r \geq 4$.

The partial sum process constructed from $\{\xi_t\}$ satisfies the functional central limit theorem (FCLT) for Reyni-mixing processes, described in [Hall and Heyde](#page-38-9) [\(1980\)](#page-38-9). For $s \in [0, 1]$ and as $T \to \infty$, it holds that

$$
X_T(s) = T^{-1/2} \sum_{t=1}^{[Ts]} \xi_t \Rightarrow B(s),\tag{7}
$$

³A stationary ARMA process is not necessarily strong-mixing. But if the innovations have absolutely continuous distribution, the strong-mixing condition is ensured (see, for example [Andrews](#page-37-5) [\(1984\)](#page-37-5) and [Mokkadem](#page-39-8) [\(1988\)](#page-39-8)).

where $B(s)$ is $(m+1)$ -vector Brownian motion with covariance matrix

$$
\Omega = \lim_{T \to \infty} T^{-1} E\left(\left(\sum_{t=1}^{T} \xi_t \right) \left(\sum_{t=1}^{T} \xi'_t \right) \right). \tag{8}
$$

We partition $z_t = (y_t, x_t')'$ into the scalar variate y_t and the *m*-vector x_t with conformable partitions of Ω and *B*:

$$
B = \begin{bmatrix} B_y \\ B_x \end{bmatrix} \qquad \Omega = \begin{bmatrix} \omega_{11} & \omega'_{21} \\ \omega_{21} & \Omega_{22} \end{bmatrix} . \tag{9}
$$

We assume $\Omega_{22} > 0$ and decompose Ω as $\Omega = L'L$, where *L* is given by

$$
L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & L_{22} \end{bmatrix},
$$
\n(10)

with $l_{11} = (\omega_{11} - \omega_{21}^{\prime} \Omega_{22}^{-1} \omega_{21})^{1/2}$, $l_{21} = \Omega_{22}^{-1/2} \omega_{21}$, and $L_{22} = \Omega_{22}^{1/2}$. Further, we define $W(s)$ to be $(m+1)$ -vector standard Brownian motion and from Lemma 2.2 of [Phillips](#page-39-9) [and Ouliaris](#page-39-9) [\(1990\)](#page-39-9) it follows that $B = L'W$.

Residual-based cointegration tests seek to test the null hypothesis of no cointegration using unit root tests applied to the residuals of the cointegrating regression. Hence, we estimate the cointegrating regression according to one of the structural break models [\(3\)](#page-6-3) using least squares and apply the SETAR model (4) to the residuals $\hat{e}_{t\tau}$ given that the threshold parameter λ is known, i.e. a fixed value. We make the following assumption about λ to ensure a sufficiently large number of observations in each regime. The cointegration residual series $\hat{e}_{t\tau}$ follows a stochastic trend under the null hypothesis and has no stable distribution. Hence, the exact threshold value is negligible asymptotically. Still, we have to specify a threshold value for which the SETAR model can be estimated using finite samples.

Assumption 2.a. *A fixed value for* λ *is specified which satisfies the condition* 0.15 \leq $P(\hat{e}_{t\tau} \leq \lambda) \leq 0.85$ *for all* $\tau \in \mathcal{T}$ *.*

Alternatively, we use the MTAR specification in [\(5\)](#page-7-1) and change the assumptions about λ slightly. Since $\Delta \hat{e}_{t\tau}$ has a stationary distribution under the null hypothesis and the alternative, we can directly specify the threshold with respect to the probability distribution of its asymptotic counterpart.

Assumption 2.b. *A fixed value for* λ *is specified which satisfies the condition* 0.15 \leq

 $P(\Delta \hat{e}_{t\tau} \leq \lambda) \leq 0.85$ *for all* $\tau \in \mathcal{T}$ *. Alternatively, the probability,* $u \in [0.15, 0.85]$ *, of the asymptotic counterpart to* $\Delta \hat{e}_{t\tau}$ *being greater than a threshold* λ *is specified directly.*

We assume the lag order *K* in the auxiliary regression to be large enough to capture the correlation structure of the cointegration residuals. Similar to [Said and Dickey](#page-39-10) [\(1984\)](#page-39-10), we approximate the infinite order process $\epsilon_{t\tau}$ by a TAR model with finite lag order $\epsilon_{t\tau K}$. Since $\epsilon_{t\tau}$ might have a nonzero MA component, it is necessary to increase *K* with the sample size $(K \to \infty \text{ as } T \to \infty)$. In practice, we can use order selection rules such as AIC, BIC or a general-to-specific pretesting procedure to determine the lag truncation parameter. We follow [Chang and Park](#page-37-6) [\(2002\)](#page-37-6) and state:

Assumption 3. *K increases with T in such a way that* $K = o(T^{1/2})$ *.*

Since the indicators $1\{\cdot \geq \lambda\}$ and $1\{\cdot \leq \lambda\}$ are orthogonal, we can write the test statistic as

$$
F_{\tau} = \frac{t_1^2 + t_2^2}{2},\tag{11}
$$

where t_1 and t_2 are the *t* ratios for $\hat{\rho}_1$ and $\hat{\rho}_2$ from regression [\(4\)](#page-7-0) or [\(5\)](#page-7-1). F_τ is computed for each possible break fraction $\tau \in \mathcal{T}$ and the sup *F*-statistic is computed to evaluate the null hypothesis of no cointegration against the alternative of threshold cointegration with possible structural break.

The following theorem presents the asymptotic distributions of the sup *F* test statistic for model specifications *C*, *C/T* and *C/S* and SETAR adjustment:

Theorem 1. *If* $\{z_t\}_0^\infty$ *is generated by* [\(6\)](#page-9-2)*, Assumptions* [\(1\)](#page-9-3)*,* [\(2.a\)](#page-9-3)*,* [\(3\)](#page-11-0) *hold and* τ *belongs to a compact subset of* $(0, 1)$ *, then as* $T \to \infty$

$$
F_{SETAR}^{*} \Rightarrow \frac{1}{2} \sup_{\tau \in \mathcal{T}} \left\{ \frac{\left(\int_{0}^{1} \mathbb{1}\{Q_{\kappa\tau} \ge 0\} Q_{\kappa\tau} dQ_{\kappa\tau}\right)^{2}}{\kappa_{\tau}' D_{\tau} \kappa_{\tau} \int_{0}^{1} \mathbb{1}\{Q_{\kappa\tau} \ge 0\} Q_{\kappa\tau}^{2}} + \frac{\left(\int_{0}^{1} \mathbb{1}\{Q_{\kappa\tau} < 0\} Q_{\kappa\tau} dQ_{\kappa\tau}\right)^{2}}{\kappa_{\tau}' D_{\tau} \kappa_{\tau} \int_{0}^{1} \mathbb{1}\{Q_{\kappa\tau} < 0\} Q_{\kappa\tau}^{2}} \right\}
$$

where

$$
Q_{\kappa\tau} = W_y - \left(\int_0^1 W_{x\tau} W'_{x\tau}\right)^{-1} \left(\int_0^1 W_y W'_{x\tau}\right) W_{x\tau}
$$

$$
\kappa_{\tau} = \left(1, -\left(\int_0^1 W_{x\tau} W'_{x\tau}\right)^{-1} \left(\int_0^1 W_y W'_{x\tau}\right)\right)
$$

Under the alternative of cointegration with two-regime SETAR adjustment, $F_{SETAR}^* \rightarrow$ ∞ *as T* → ∞ *. Q*_{*κτ*} *depends on the model:*

a) If the residuals are obtained from least squares estimation of model C, then

$$
W_{x\tau} = (W'_x, 1, \varphi_\tau)'
$$

$$
D_\tau = \begin{bmatrix} I_{m+1} & 0 \\ 0 & 0 \end{bmatrix}.
$$

b) If the residuals are obtained from least squares estimation of model C/T, then

$$
W_{x\tau} = (W'_x, 1, s, \varphi_\tau)'
$$

$$
D_\tau = \begin{bmatrix} I_{m+1} & 0 \\ 0 & 0 \end{bmatrix}.
$$

c) If the residuals are obtained from least squares estimation of model C/S, then

$$
W_{x\tau} = (W'_x, 1, W'_x \varphi_\tau, \varphi_\tau)'
$$

$$
D_\tau = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & (1 - \tau)I_m & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (1 - \tau)I_m & 0 & (1 - \tau)I_m & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

.

A formal proof of Theorem [1](#page-11-1) is provided in the Appendix. Accordingly, the asymptotic distribution of the sup *F* test statistic for cointegration models with MTAR adjustment is given in Theorem [2.](#page-12-0)

Theorem 2. *If* $\{z_t\}_0^\infty$ *is generated by* [\(6\)](#page-9-2)*, Assumptions* [\(1\)](#page-9-3)*,* [\(2.b\)](#page-10-0) *and* [\(3\)](#page-11-0) *hold and* τ *belongs to a compact subset of* $(0,1)$ *, then as* $T \to \infty$

$$
F_{MTAR}^* \Rightarrow \frac{1}{2} \sup_{\tau \in \mathcal{T}} \left\{ \frac{\left(\int\limits_0^1 Q_{\kappa \tau}(s) dW(s, u)\right)^2}{u \int\limits_0^1 Q_{\kappa \tau}^2(s) ds} + \frac{\left(\int\limits_0^1 Q_{\kappa \tau}(s) \left(dW(s, 1) - dW(s, u)\right)\right)^2}{(1 - u) \int\limits_0^1 Q_{\kappa \tau}^2(s) ds} \right\}
$$

where

$$
Q_{\kappa\tau} = W_y - \left(\int_0^1 W_{x\tau} W'_{x\tau}\right)^{-1} \left(\int_0^1 W_y W'_{x\tau}\right) W_{x\tau}
$$

Under the alternative of cointegration with two-regime MTAR adjustment, $F^*_{MTAR} \rightarrow \infty$ $as T \rightarrow \infty$ *.* $Q_{\kappa\tau}$ depends on the model:

a) If the residuals are obtained from least squares estimation of model C, then

$$
W_{x\tau}=(W'_x,1,\varphi_\tau)'
$$

b) If the residuals are obtained from least squares estimation of model C/T, then

$$
W_{x\tau} = (W'_x, 1, s, \varphi_\tau)'
$$

c) If the residuals are obtained from least squares estimation of model C/S, then

$$
W_{x\tau} = (W'_x, 1, W'_x \varphi_\tau, \varphi_\tau)'
$$

A formal proof of Theorem [2](#page-12-0) is provided in the Appendix.[4](#page-13-1)

4 Simulation results

Critical values and finite sample properties of the sup *F* tests are examined by Monte Carlo experiments. In the absence of a structural break, we use a DGP according to [Engle and Granger](#page-38-1) [\(1987\)](#page-38-1) and [Banerjee et al.](#page-37-7) [\(1986\)](#page-37-7) which is given for one regressor $(m = 1)$ in the form of

$$
y_t = \mu + \alpha x_{1,t} + e_t \quad \Delta e_t = \rho e_{t-1} + \vartheta_t \quad \vartheta_t \sim N(0, 1) \n y_t = x_{1,t} + \eta_t \quad \eta_t = \eta_{t-1} + \omega_t \quad \omega_t \sim N(0, 1),
$$
\n(12)

⁴[Enders and Siklos](#page-37-0) [\(2001\)](#page-37-0) do not provide an asymptotic theory for their tests. The theorems given here are easily adapted to provide the asymptotic distributions for models without structural breaks using $W_{x\tau} = (W'_x, 1)'$. Hence, the asymptotic distribution of their *F*-statistic using fixed threshold values and a SETAR model is given as a special case of Theorem [1](#page-11-1) of this paper and as a special case $(\lambda_1 = \lambda_2)$ $(\lambda_1 = \lambda_2)$ $(\lambda_1 = \lambda_2)$ of Theorem 2 in [Maki and Kitasaka](#page-38-10) [\(2015\)](#page-38-10). Theorem 2 is new in this context. It shows that the cointegration test using MTAR adjustment in [Enders and Siklos](#page-37-0) [\(2001\)](#page-37-0) depends on the nuisance parameter *u*. However, critical values obtained for different *u* are very similar for the model without structural breaks.

where the parameters of the equilibrium equation are $\mu = 1$ and $\alpha = 2$. First, the null hypothesis of no cointegration is simulated with $\rho = 0$. This enables us to obtain quantiles of the sup *F* distribution for different sample sizes. The BIC is used to determine the lag truncation parameter *K*. Critical values are computed for 10*,* 000 draws for each sample size. The results are reported in [Table 1,](#page-29-0) [Table 2](#page-30-0) and [Table 3.](#page-31-0)

The power of the sup F test under structural change is evaluated with a DGP designed in line with [Gregory and Hansen](#page-38-0) [\(1996a\)](#page-38-0). A slight modification was, however, necessary to allow for asymmetric adjustment to the long-run equilibrium. The following DGP is employed for a bivariate cointegrated system,

$$
y_{t} = \mu_{t} + \alpha_{t}x_{1,t} + e_{t} \quad \Delta e_{t} = \begin{cases} \rho_{1}e_{t-1} + \vartheta_{t} & \text{if } e_{t-1} \geq 0 \\ \rho_{2}e_{t-1} + \vartheta_{t} & \text{if } e_{t-1} < 0 \end{cases} \qquad \vartheta_{t} \sim N(0, 1)
$$
\n
$$
y_{t} = x_{1,t} + \eta_{t} \qquad \eta_{t} = \eta_{t-1} + \omega_{t} \qquad \omega_{t} \sim N(0, 1)
$$
\n
$$
\left[\mu_{t} = \mu_{1}, \quad \alpha_{t} = \alpha_{1}, \quad t \leq [T\tau] \right],
$$
\n
$$
\mu_{t} = \mu_{2}, \quad \alpha_{t} = \alpha_{2}, \quad t > [T\tau] \qquad (13)
$$

in which symmetric adjustment is nested as $\rho_1 = \rho_2$. In the case of MTAR adjustment, the speed of adjustment depends on whether the previous periods change was greater than the median of Δe_t . Thus, we investigate the power for $u = 0.5$. A change in the intercept is modelled by means of an increase from $\mu_1 = 1$ to $\mu_2 = 4$ at the breakpoint, whereas a change in the slope is modelled as an increase from $\alpha_1 = 2$ to $\alpha_2 =$ 4. The simulation set-up used for cointegrated systems with symmetric adjustment directly follows [Gregory and Hansen](#page-38-0) [\(1996a\)](#page-38-0) so that the results for the sup *F* test can be compared with the results for the GH test.

[Table 4](#page-32-0) reports displays the rejection rates under cointegration with symmetric adjustment and break in either the intercept or slope. The power of the tests is investigated by generating 2,500 draws for every specification. We find that the sup *F* tests have generally higher rejection rates than either the Engle-Granger test using the ADF test statistic or threshold cointegration tests without breakpoint estimation. The simulation reveals that the sup *F* test with SETAR adjustment has comparable power properties to the GH test. The MTAR specification of the sup *F* test has slightly lower power against the alternative than the GH test. The Enders-Siklos test with SETAR adjustment seems to be rather robust to a break in the intercept but suffers from a drastic reduction in power if a break in the slope is considered. The sup *F* tests appear to have sufficient power at sample sizes above $T = 100$ and moderate adjustment rate $\rho = -0.5$. As expected, the model *C* outperforms model C/T and C/S if a break in the intercept is considered, while *C/S* performs best if the slope changes at one point in the sample.

The simulation results under symmetric adjustment can also be used to analyze the estimation accuracy of the pre-specified breakpoint in the DGP. The timing of the break is varied and takes place either at the beginning, the middle or near the end of the series. The results are summarized in [Table 5](#page-33-0) and reveal that breakpoint estimates are in large parts very accurate. In general, it seems that breaks at the beginning of the sample are most difficult to detect and the sup *F* tests often indicate a later breakpoint. Breaks in the intercept and the slope are estimated with equal accuracy as long as the correct structural break model is applied. The SETAR model seems to produce slightly more accurate breakpoint estimates than the MTAR model.

The upper panel of [Table 6](#page-34-0) displays the rejection rates under structural stability and asymmetric adjustment. For each combination of autoregressive coefficients, we generate series with sample size $T = 100$. If the series are generated under asymmetric adjustment with a stable cointegrating vector, we find that the sup *F* tests operate with less power than the threshold cointegration tests by [Enders and Siklos](#page-37-0) [\(2001\)](#page-37-0). Falsely incorporating breaks in form of additional dummy variables in the equilibrium equation thus reduces the power against the null hypothesis. Accordingly, the most parsimonious model *C* performs best among the three structural break models.

Finally, the behaviour of the sup *F* test is evaluated under parameter instability and asymmetric adjustment. For that matter, we draw from the DGP in [\(13\)](#page-14-0). We consider SETAR adjustment in [Table 6](#page-34-0) and MTAR adjustment in [Table 7,](#page-35-0) respectively. In the second panel, we model a break in the intercept. The sup *F* tests have poor power properties and are outperformed by the Enders-Siklos test in each parameter combination. The loss in power of the original threshold cointegration test due to a break in the intercept does not justify the additional parameter estimation and grid search of the *C* model. The *C/T* and *C/S* models involve an additional parameter and, as expected, have lower rejection rates. With a break in the slope (third panel of [Table 6](#page-34-0) and [Table 7\)](#page-35-0), we find the picture to be quite different. All structural break models have more power against the null hypothesis than the Enders-Siklos test. As expected, the power of the correctly specified *C/S* model exceeds all other structural change models for each parameter combination. Further, we evaluate the power of the GH test and find that the power is lower than the sup *F* tests' power if the adjustment is asymmetric.^{[5](#page-16-1)}

In the last panel, we display the results for a simultaneous break in the intercept and the slope. Again, the *C/S* model performs best among the structural break models and far exceeds the benchmark Enders-Siklos test. In general, we find a break in the slope to have a more substantial impact on the power function than a break in the intercept. In practice, we have to assume that structural change involves all parameters of the equilibrium equation. Since the sup *F* tests based on the *C/S* model perform best in those situations, it has to be considered the preferred model for cointegration relationships with asymmetric adjustment which are subject to parameter instability at an unknown point in time.

5 Empirical application

In this section, we apply the sup F test methodology to study the 'rockets and feathers' hypothesis^{[6](#page-16-2)} in the US gasoline market. The 'rockets and feathers' hypothesis describes the adjustment behaviour of prices faced with input price shocks. More precisely, the hypothesis states that prices adjust faster to input price increases than to input price decreases. In the terms of [Bacon](#page-37-8) [\(1991\)](#page-37-8)'s seminal paper, the price goes up like a rocket, but falls down like a feather. While early studies on the matter [\(Bacon](#page-37-8) [\(1991\)](#page-37-8), [Manning](#page-38-11) [\(1991\)](#page-38-11), [Borenstein et al.](#page-37-9) [\(1997\)](#page-37-9)) focused on the short-run asymmetry in the pricing process, the focus quickly shifted to the economically meaningful long-run asymmetry estimated by asymmetric error correction models [\(Bachmeier and Griffin](#page-37-10) [\(2002\)](#page-37-10)).

For the empirical illustration, we examine the fuel prices transmission at two points of the production chain. First, we analyze the speed of adjustment for deviations from the long-run relationship between crude oil prices and gasoline spot prices (*first stage*). Second, we analyze the pass-through from gasoline spot prices to retail prices (*second stage*). Finally, the direct link between crude oil prices and retail prices is analyzed (*single stage*). Naturally, we expect the speed of adjustment at the first and second stage to be faster than at the single stage transmission. Long-run asymmetry in the sense of the 'rockets and feathers' hypothesis is found if negative deviations from the long-run equilibrium are adjusted faster than positive deviations, i.e. $\rho_1 = \rho^- < \rho^+$ ρ_2 . Alternatively, we use the MTAR model to investigate whether a shock having

⁵The results are not reported in this paper but can be obtained from the author upon request.

⁶The name originates from the [Bacon](#page-37-8) [\(1991\)](#page-37-8) paper entitled: 'Rockets and feathers: the asymmetric speed of adjustment of UK retail gasoline prices to cost changes'

Figure 1: WTI crude oil prices, spot gasoline prices and retail gasoline prices from January 2006 to December 2013

momentum greater than or equal to its median is adjusted faster than a shock with less momentum.

Our sample reaches from January 2006 to December 2013 to include the collapse of commodity prices in 2009 and their subsequent recovery. We observe prices at a monthly frequency yielding a total of 96 observations. The West Texas Intermediate prices (p_t^c) , regular gasoline spot prices (p_t^s) and regular gasoline retail prices (p_t^g) *t*) are all obtained from the U.S. Energy Information Administration (EIA). [Figure 1](#page-17-0) depicts the trajectory of the prices and shows volatile behaviour of prices for petroleum products during the Financial Crisis. Although the times series are affected by global events, it does not immediately follow that the long-run relationship between them changes. However, from our simulation study, we know that an existing instability of the cointegrating vector can severely decrease the power of a threshold cointegration test.

First, we estimate a threshold cointegration model according to [Enders and Siklos](#page-37-0) [\(2001\)](#page-37-0). We specify the long-run equilibrium equations

$$
(I) \quad p_t^s = \mu + \alpha p_t^c + e_t
$$

\n
$$
(II) \quad p_t^g = \mu + \alpha p_t^s + e_t
$$

\n
$$
(S) \quad p_t^g = \mu + \alpha p_t^c + e_t
$$

\n(14)

where the (I), (II), (S) denote *first stage*, *second stage* and *single stage*, respectively. The

coefficients of the cointegrating vector are estimated using least squares and a threshold model is applied to the residuals. The results for the SETAR model are reported in panel (a) of [Table 8](#page-36-0) and reveal significant asymmetry in the adjustment process only in the *second stage*. The results for the MTAR model with $u = 0.5$ are reported in panel (b) of [Table 8.](#page-36-0) Here, we also find significant asymmetries in the transmission from spot gasoline to retail gasoline prices. Surprisingly, we do not find sufficient evidence for a long-run relationship between crude oil prices and gasoline spot prices. In contrast, retail gasoline prices and crude oil prices seem to maintain a long-run equilibrium which is a less likely result from an economic perspective than the existence of a crude/spot relationship.

Second, we estimate the long-run equilibrium equations again using the *C/S* specification since this specification of the sup F tests performed best in the simulation study if the slope coefficient changed at one point in time and is best-suited for modelling unspecific regime shift events. The results are reported in panel (b) and panel (d) of [Table 8.](#page-36-0) The null hypothesis of no cointegration can now be rejected at all stages along the gasoline value-chain. The breakpoint is located either at the peak crude oil prices during the Financial Crisis or after the prices had recovered in 2011. Closer inspection of the time series reveals that the spread between crude oil prices and spot gasoline prices increased substantially around 2011. We do not find statistical evidence for asymmetric adjustment processes in the *first stage*. The asymmetry results for the *second stage* and *single stage* remain unchanged.

6 Summary

This paper proposed an extension to the GH test to include SETAR and MTAR adjustment. Thereby, we constructed threshold cointegration tests which endogenously determine the location of a structural break in the cointegrating vector and test the null of no cointegration. We derived the limiting distribution for the structural break models *C*, *C/T* and *C/S* and tabulated their critical values which were obtained by Monte Carlo simulations. Analysis of the finite sample properties under the alternative of linear and threshold cointegration revealed that the tests exhibit considerable power gains over the conventional Enders-Siklos tests if a break in the slope coefficient is present. We applied the sup *F* tests to US gasoline market data and found evidence for a long-run relationship between prices along the value-chain after we accounted for

structural breaks. The results for the SETAR and MTAR models provided evidence for asymmetric price transmission from spot gasoline to retail gasoline.

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Appendix

Proof of Theorem 1. The asymptotic distribution is derived by adapting the results of [Gregory and Hansen](#page-38-12) [\(1992\)](#page-38-12) to match the *F*-statistic process involving a threshold indicator function using results in [Maki and Kitasaka](#page-38-10) [\(2015\)](#page-38-10). However, [Maki and](#page-38-10) [Kitasaka](#page-38-10) [\(2015\)](#page-38-10) use a different definition of the threshold parameter space in their SETAR model. The threshold parameter in our model is fixed, i.e. belongs to a trivial compact subset of $\mathbb R$ whereas the parameter space in [Maki and Kitasaka](#page-38-10) [\(2015\)](#page-38-10) is data dependent (see the discussion on threshold parameter space in Section 2.2 of their paper). Indicator functions with threshold parameters defined on compact sets are treated in [Seo](#page-39-7) [\(2008\)](#page-39-7). The proof only refers to model *C/S* while the results for the remaining models can be deduced from the results obtained for this model. Hence, we consider the cointegrating regression,

$$
y_t = \hat{\alpha}_1' x_t + \hat{\mu}_1 + \hat{\alpha}_2' x_t \varphi_{t,\tau} + \hat{\mu}_2 \varphi_{t,\tau} + \hat{e}_{t\tau}, \tag{15}
$$

where $\hat{e}_{t\tau}$ is an integrated process under the null hypothesis of no cointegration and $z_t = (y_t, x_t')'$ is generated according to [\(6\)](#page-9-2).

Define the $(2m + 3)$ -vector $X_{t\tau} = (y_t, x_t', 1, x_t\varphi_{t,\tau'}, \varphi_{t,\tau})'$ and partition $X_{t\tau} =$ $(X_{1t\tau}, X_{2t\tau})'$ where $X_{1t\tau} = y_t$ and $X_{2t\tau}$ contains all regressors of [\(15\)](#page-20-0). De- $\delta_T = diag(T^{-1/2}I_{m+1}, 1, T^{-1/2}I_m, 1), \quad \varphi_{\tau}(s) = 1\{s > \tau\}$ and $X_{\tau}(s) = 1$ $(B(s)', 1, B_x(s)\varphi_{\tau}(s)', \varphi_{\tau}(s))'$. Partition $\delta_T = (\delta_{1T}, \delta_{2T})$ in conformity to $X_{t\tau}$.

Next, we partition the $(m + 1)$ -vector standard Brownian Motion *W* as *W* = $(W_y, W'_x)'$ where

$$
W_y = l_{11}^{-1} \left(B_y - \omega_{21}^{\prime} \Omega_{22}^{-1} B_x \right)
$$

\n
$$
W_x = \Omega_{22}^{-1/2} B_x.
$$
\n(16)

Furthermore, we define

$$
W_{x\tau} = (W_x', 1, W_x \varphi_\tau', \varphi_\tau)'
$$
\n⁽¹⁷⁾

and $W_{\tau} = (W_y, W_{x\tau})'$.

First, we consider the least squares estimator of the parameters of the cointegrating regression. It is shown in [Gregory and Hansen](#page-38-12) [\(1992\)](#page-38-12) using the FCLT for vector processes in [Phillips and Durlauf](#page-39-11) [\(1986\)](#page-39-11) and the continuous mapping theorem (CMT, see [Billingsley](#page-37-11) [\(1999\)](#page-37-11), Theorem 2.7) that

$$
T^{-1}\delta_T \sum_{t=1}^T X_{t\tau} X_{t\tau} \delta_T \Rightarrow \int_0^1 X_\tau X_{\tau}' \tag{18}
$$

where the weak convergence is with respect to the uniform metric over $\tau \in \mathcal{T}$. In the remainder of the proof, we refer to weak convergence results involving the break fraction parameter τ as holding uniformly over τ (see also [Arai and Kurozumi](#page-37-1) [\(2007\)](#page-37-1) for a similar application).

We define the vector $\hat{\theta}_{\tau} = (\hat{\alpha}'_1, \hat{\mu}_1, \hat{\alpha}'_2, \hat{\mu}_2)$ as the least squares estimator of [\(15\)](#page-20-0) for each τ . It follows from [\(18\)](#page-21-0) and the CMT that

$$
T^{-1/2}\delta_{2T}^{-1}\hat{\theta}_{\tau} = \left(T^{-1}\delta_{2T}\sum_{t=1}^{T} X_{2t\tau} X_{2t\tau}' \delta_{2T}\right)^{-1} \left(T^{-1}\delta_{2T}\sum_{t=1}^{T} X_{2t\tau} X_{1t\tau} \delta_{1T}\right)
$$

$$
\Rightarrow \left(\int_{0}^{1} X_{2\tau} X_{2\tau}'\right)^{-1} \left(\int_{0}^{1} X_{2\tau} X_{1\tau}\right). \tag{19}
$$

When we set $\hat{\eta}_{\tau} = T^{-1/2} \delta_T^{-1} (1, -\hat{\theta}'_{\tau})' = (1, -\delta_{2T}^{-1} \hat{\theta}'_{\tau})'$, it follows that

$$
\hat{\eta}_{\tau} \Rightarrow \left(1, -\left(\int_{0}^{1} X_{1\tau} X_{2\tau}'\right) \left(\int_{0}^{1} X_{2\tau} X_{2\tau}'\right)^{-1}\right)' = \eta_{\tau}.
$$
 (20)

Next, we state some useful convergence results for the residuals of the cointegrating regression. We define the residual series $\hat{e}_{t\tau} = y_t - \hat{\alpha}'_1 x_t - \hat{\mu}_1 - \hat{\alpha}'_2 x_t \varphi_{t,\tau} - \hat{\mu}_2 \varphi_{t,\tau}$ which is dependent on τ . Note that $\hat{e}_{t\tau}$ can be expressed as

$$
\hat{e}_{t\tau} = T^{1/2} \hat{\eta}_{\tau}' \delta_T X_{t\tau}.\tag{21}
$$

Using Lemma 2.2 of [Phillips and Ouliaris](#page-39-9) [\(1990\)](#page-39-9) yields

$$
T^{-1/2}\hat{e}_{t\tau} \Rightarrow \eta_{\tau}' X_{\tau} = l_{11} \kappa_{\tau}' W_{\tau} = l_{11} Q_{\kappa \tau},\tag{22}
$$

where

$$
\kappa_{\tau} = \left(1, -\left(\int_{0}^{1} W_{y} W'_{x\tau}\right) \left(\int_{0}^{1} W_{x\tau} W'_{x\tau}\right)^{-1}\right)'
$$
\n
$$
L\eta_{\tau} = l_{11} \kappa_{\tau}
$$
\n
$$
Q_{\kappa\tau} = W_{y} - \left(\int_{0}^{1} W_{y} W'_{x\tau}\right) \left(\int_{0}^{1} W_{x\tau} W'_{x\tau}\right)^{-1} W_{x\tau}.
$$
\n(23)

The first-differenced residuals are expressed as $\Delta \hat{e}_{t\tau} = T^{1/2} \hat{\eta}'_{\tau} \delta_T \Delta X_{t\tau}$, where

$$
\Delta X_{t\tau} = \Delta(y_t, x_t', 1, x_t \varphi_{t,\tau'}, \varphi_{t,\tau})'
$$

\n
$$
= (\xi_{1t}, \xi_{2t'}, 0, x_{t-1} \Delta \varphi_{t,\tau'} + \Delta x_t \varphi_{t,\tau'}, \Delta \varphi_{t,\tau})'
$$

\n
$$
= (\xi_{1t}, \xi_{2t'}, 0, x_{t-1} \Delta \varphi_{t,\tau} + \xi_{2t} \varphi_{t,\tau'}, \Delta \varphi_{t,\tau})'
$$
\n(24)

and

$$
\Delta \varphi_{t,\tau} = \begin{cases} 1 & \text{if } t = [T\tau] \\ 0 & \text{if } t \neq [T\tau] \end{cases} . \tag{25}
$$

The asymptotic counterpart to $\Delta \varphi_{t,\tau}$ is the differential $d\varphi_{\tau}$, a Dirac function concentrating the unit mass at the point $t = \tau$ so that

$$
\int_{a}^{b} f d\varphi_{\tau} = \lim_{z \uparrow \tau} f(z), \qquad a < \tau < b,
$$

for all functions with left-limits. Then, we can define the differential dX_τ by

$$
dX_{\tau}(s) = (dB(s)', 0, B_x(s)'d\varphi_{\tau}(s) + dB_x(s)\varphi_{\tau}(s), d\varphi_{\tau}(s))'. \tag{26}
$$

Under Assumption (1) , ξ_t is a stationary VARMA process and consequently, the scalar process $T^{1/2} \hat{\eta}_\tau^{\prime} \delta_T \Delta X_{t\tau} \Rightarrow T^{1/2} \eta_\tau^{\prime} \delta_T \Delta X_{t\tau}$ is also a stationary ARMA process with an intervention outlier at $t = [T\tau]$. Moreover, under Assumption [\(3\)](#page-11-0) the lag truncation parameter $K \to \infty$ for $T \to \infty$. This means that the error of approximating $\epsilon_{t\tau}$ by a finite AR process becomes small as *K* grows large. Following [Phillips and Ouliaris](#page-39-9) [\(1990\)](#page-39-9) we write the infinite order AR representation of the SETAR error term process as $\epsilon_{t\tau} = \sum^{\infty}$ $\sum_{j=0}^{\infty} D_j (T^{1/2} \delta_T \Delta X_{t-j\tau})' \eta_{\tau} = D(L) (T^{1/2} \delta_T \Delta X_{t\tau})' \eta_{\tau}$. The lag structure is chosen in a way that $\epsilon_{t\tau}$ is an orthogonal $(0, \sigma^2(\eta, \tau))$ sequence with long-run variance $\sigma^2(\eta, \tau)$ =

 $D(1)^2 \eta'_\tau \Omega_\tau \eta_\tau$. From Lemma 2.1 of [Phillips and Ouliaris](#page-39-9) [\(1990\)](#page-39-9), it follows that

$$
T^{-1/2} \sum_{t=1}^{[Ts]} \epsilon_{t\tau K} = D(L)\eta_{\tau}' \left(T^{-1/2} \sum_{t=1}^{[Ts]} T^{1/2} \delta_T \Delta X_{t\tau} \right) + o_p(1) \Rightarrow D(1)\eta_{\tau}' X_{\tau}(s), \qquad (27)
$$

where $D(1) = \sum_{n=1}^{\infty}$ $\sum_{j=0}$ D_j .

Now, we consider the auxiliary regression. We apply the SETAR model to the residuals according to [\(4\)](#page-7-0) and compute the test statistics F_τ . Note that the estimated adjustment coefficients might be correlated with the estimated coefficients of the additional lagged differences. Therefore, we write the least squares estimator of $\rho = (\rho_1, \rho_2)'$ in the breakpoint specific notation under the null hypothesis $\rho_1 = \rho_2 = 0$ as $\hat{\rho} = (U_{\tau}'Q_KU_{\tau})^{-1}U_{\tau}'Q_K\epsilon_{\tau}$, where

$$
U_{\tau} = \begin{bmatrix} \hat{e}_{0\tau} \mathbb{1} \{ \hat{e}_{0\tau} \ge \lambda \} & \hat{e}_{0\tau} \mathbb{1} \{ \hat{e}_{0\tau} < \lambda \} \\ \hat{e}_{1\tau} \mathbb{1} \{ \hat{e}_{1\tau} \ge \lambda \} & \hat{e}_{1\tau} \mathbb{1} \{ \hat{e}_{1\tau} < \lambda \} \\ \vdots & \vdots \\ \hat{e}_{T-1\tau} \mathbb{1} \{ \hat{e}_{T-1\tau} \ge \lambda \} & \hat{e}_{T-1\tau} \mathbb{1} \{ \hat{e}_{T-1\tau} < \lambda \} \end{bmatrix},
$$
(28)

 $\epsilon_{\tau} = (\epsilon_{1\tau}, \epsilon_{2\tau}, \dots, \epsilon_{T\tau})'$ and $Q_K = I - M_K (M'_K M_K)^{-1} M'_K$ is the projection matrix onto the space orthogonal to the regressors $M_K = (\Delta \hat{e}_{t-1\tau}, \dots, \Delta \hat{e}_{t-K\tau}).$

Partition the matrix U_{τ} as $U_{\tau} = (U_{1\tau}, U_{2\tau})$, then the *t* ratio of $\hat{\rho}_1$ can be expressed as

$$
t_1 = \frac{\hat{\rho}_1}{se(\hat{\rho}_1)} = \frac{\hat{\rho}_1}{(\hat{\sigma}^2 (U_{1\tau}' Q_K U_{1\tau})^{-1})^{1/2}} = \frac{U_{1\tau}' Q_K \epsilon_\tau}{\hat{\sigma} (U_{1\tau}' Q_K U_{1\tau})^{1/2}}
$$
(29)

and similarly the *t* ratio of $\hat{\rho}_2$ can be expressed as

$$
t_2 = \frac{U'_{2\tau} Q_K \epsilon_\tau}{\hat{\sigma} (U'_{2\tau} Q_K U_{2\tau})^{1/2}}.
$$
\n(30)

In the remainder of the proof, we focus on t_1 . Scaling the t ratio appropriately yields the numerator

$$
T^{-1}U'_{1\tau}Q_K\epsilon_{\tau} = T^{-1}U'_{1\tau}\epsilon_{\tau} - T^{-1/2} \cdot T^{-1}U'_{1\tau}M_K(T^{-1}M'_K M_K)^{-1}T^{-1/2}M'_K\epsilon_{\tau}
$$

=
$$
T^{-1}U'_{1\tau}\epsilon_{\tau} + o_p(1) = N_T(\lambda, \tau) + o_p(1)
$$
(31)

and the term

$$
T^{-2}U'_{1\tau}Q_KU_{1\tau} = T^{-2}U'_{1\tau}U_{1\tau} - T^{-1} \cdot T^{-1}U'_{1\tau}M_K(T^{-1}M'_K M_K)^{-1}T^{-1}M'_K U_{1\tau}
$$

=
$$
T^{-2}U'_{1\tau}U_{1\tau} + o_p(1) = D_T(\lambda, \tau) + o_p(1).
$$
 (32)

Finally, we need convergence results for $N_T(\lambda, \tau)$, $D_T(\lambda, \tau)$ and $\hat{\sigma}^2$. Since $x \mapsto$ $x1\{x \geq \lambda\}$ is a regular function, it follows from [\(22\)](#page-21-1) and Theorem 3.1 of [Park and](#page-39-12) [Phillips](#page-39-12) [\(2001\)](#page-39-12) that

$$
T^{-1/2}\hat{e}_{t-1\tau}1\{\hat{e}_{t-1\tau} \ge \lambda\} = \hat{\eta}_{\tau}'\delta_T X_{t-1\tau}1\{T^{1/2}\hat{\eta}_{\tau}'\delta_T X_{t-1\tau} \ge \lambda\}
$$

\n
$$
= \hat{\eta}_{\tau}'\delta_T X_{t-1\tau}1\{\hat{\eta}_{\tau}'\delta_T X_{t-1\tau} \ge T^{-1/2}\lambda\}
$$

\n
$$
\Rightarrow \eta_{\tau}' X_{\tau}1\{\eta_{\tau}' X_{\tau} \ge 0\} = l_{11}Q_{\kappa\tau}1\{Q_{\kappa\tau} \ge 0\}.
$$
 (33)

Thus, Theorem 2.2 of [Kurtz and Protter](#page-38-13) [\(1991\)](#page-38-13) combined with results [\(27\)](#page-23-0) and [\(33\)](#page-24-0) yields

$$
N_T(\lambda, \tau) = T^{-1} \sum_{t=1}^T \mathbb{1} \{\hat{e}_{t-1\tau} \ge \lambda\} \hat{e}_{t-1\tau} \epsilon_{t\tau}
$$

\n
$$
= \hat{\eta}_{\tau}' \delta_T \sum_{t=1}^T \mathbb{1} \{\delta_T \hat{\eta}_{\tau}' X_{t-1\tau} \ge T^{-1/2} \lambda\} X_{t-1\tau} D(L) (\Delta X_{t\tau})' \delta_T \eta_{\tau}
$$

\n
$$
\Rightarrow D(1) \eta_{\tau}' \int_{0}^{1} \mathbb{1} \{\eta_{\tau}' X_{\tau} \ge 0\} X_{\tau} dX_{\tau}' \eta_{\tau}
$$

\n
$$
= D(1) l_{11}^2 \int_{0}^{1} \mathbb{1} \{Q_{\kappa\tau} \ge 0\} Q_{\kappa\tau} dQ_{\kappa\tau},
$$
\n(34)

while (27) , (33) and the CMT yield

$$
D_T(\lambda, \tau) = T^{-2} \sum_{t=1}^T \mathbb{1} \{\hat{e}_{t-1\tau} \ge \lambda\} \hat{e}_{t-1\tau}^2
$$

\n
$$
= \hat{\eta}_\tau' \delta_T T^{-1} \sum_{t=1}^T \mathbb{1} \{\delta_T \hat{\eta}_\tau' X_{t-1\tau} \ge T^{-1/2} \lambda\} X_{t-1\tau} X_{t-1\tau}' \delta_T \hat{\eta}_\tau
$$

\n
$$
\Rightarrow \eta_\tau' \int_0^1 \mathbb{1} \{\eta_\tau' X_\tau \ge 0\} X_\tau X_\tau' \eta_\tau
$$

\n
$$
= l_{11}^2 \int_0^1 \mathbb{1} \{Q_{\kappa\tau} \ge 0\} Q_{\kappa\tau}^2.
$$
\n(35)

For the variance estimate, $\hat{\sigma}^2$, we note that $\hat{\rho}_1 = O_p(T^{-1})$ and $\hat{\rho}_2 = O_p(T^{-1})$, but $(\hat{\gamma}_j - \gamma_j) = O_p(T^{-1/2})$. Using Lemma 2.2 of [Phillips and Ouliaris](#page-39-9) [\(1990\)](#page-39-9) yields

$$
\hat{\sigma}^2 = T^{-1} \sum_{t=1}^T \left(\Delta \hat{e}_{t\tau} - \hat{\rho}_1 \hat{e}_{t-1\tau} \mathbb{1} \{ \hat{e}_{t-1\tau} \ge \lambda \} - \hat{\rho}_2 \hat{e}_{t-1\tau} \mathbb{1} \{ \hat{e}_{t-1\tau} < \lambda \} - \sum_{j=1}^K \hat{\gamma}_j \Delta \hat{e}_{t-j\tau} \right)^2
$$

=
$$
T^{-1} \sum_{t=1}^T \epsilon_{t\tau}^2 + o_p(1) \Rightarrow D(1)^2 \eta_\tau' \Omega_\tau \eta_\tau = D(1)^2 l_{11}^2 \kappa_\tau' D_\tau \kappa_\tau,
$$
 (36)

where the long-run covariance matrix is given by

$$
\Omega_{\tau} = \begin{bmatrix}\n\omega_{11} & \omega'_{21} & 0 & (1 - \tau)\omega'_{21} & 0 \\
\omega_{21} & \Omega_{22} & 0 & (1 - \tau)\Omega_{22} & 0 \\
0 & 0 & 0 & 0 & 0 \\
(1 - \tau)\omega_{21} & (1 - \tau)\Omega_{22} & 0 & (1 - \tau)\Omega_{22} & 0 \\
0 & 0 & 0 & 0 & 0\n\end{bmatrix}
$$
\n(37)

and

$$
D_{\tau} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & (1-\tau)I_m & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\tau)I_m & 0 & (1-\tau)I_m & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .
$$
 (38)

Similar results can be obtained for t_2 so that the results (34) , (35) , (36) combine with the CMT to proof the theorem under the null hypothesis.

Under the alternative, the system is cointegrated so that we have $\hat{\eta}_{\tau} \stackrel{p}{\to} \eta_{\tau}$ and

$$
\hat{\eta}_{\tau} = \eta_{\tau} + O_p(T^{-1}) \tag{39}
$$

from [Phillips and Durlauf](#page-39-11) [\(1986\)](#page-39-11), Theorem 4.1. Thus, for the residual series it holds that

$$
\hat{e}_{t\tau} = \hat{\eta}'_{\tau} z_t = \eta'_{\tau} z_t + O_p(T^{-1/2}) = q_{t\eta\tau} + O_p(T^{-1/2}). \tag{40}
$$

By assumption a stationary SETAR representation of $q_{t\eta\tau}$ exists and is given by

$$
q_{t\eta\tau} = a_{11}q_{t-1\eta\tau} \mathbb{1}\{q_{t-1\eta\tau} \ge \lambda\} + a_{12}q_{t-1\eta\tau} \mathbb{1}\{q_{t-1\eta\tau} < \lambda\} + \sum_{j=2}^{\infty} a_j q_{t-j\eta\tau} + \epsilon^*_{t\eta\tau},\qquad(41)
$$

where $\epsilon_{t\eta\tau}^{*}$ is an orthogonal $(0, \sigma_{\epsilon_{\eta\tau}^{*}})$ sequence. This can alternatively be written as

$$
\Delta q_{t\eta\tau} = \psi_{11} q_{t-1\eta\tau} \mathbb{1}\{q_{t-1\eta\tau} \ge \lambda\} + \psi_{12} q_{t-1\eta\tau} \mathbb{1}\{q_{t-1\eta\tau} < \lambda\} + \sum_{j=2}^{\infty} \psi_j \Delta q_{t-j\eta\tau} + \epsilon_{t\eta\tau}^*.
$$
 (42)

If we consider the *t* ratio of $\hat{\rho}_1$ and use the expression

$$
t_1 = \frac{1}{\hat{\sigma}} \left(\hat{\rho}_1 \left(U'_{1\tau} Q_K U_{1\tau} \right)^{1/2} \right), \tag{43}
$$

we find that $\hat{\rho}_1 \stackrel{p}{\to} \psi_{11} \neq 0$ and $\hat{\sigma}^2 \stackrel{p}{\to} \sigma_{\epsilon_{\eta\tau}}^2$. Further, we observe

$$
U_{1\tau}' Q_K U_{1\tau} = U_{1\tau}' U_{1\tau} - U_{1\tau}' M_K (M'_K M_K)^{-1} M'_K U_{1\tau} = O_p(T)
$$
\n(44)

which yields $t_1 = O_p(T^{1/2})$ and similarly $t_2 = O_p(T^{1/2})$. Hence, we immediately see that $F_{SETAR}^* \to \infty$ as $T \to \infty$.

Proof of Theorem 2. The proof is structured similarly to the proof of Theorem [1.](#page-11-1) Using the results for the cointegrating regression, we write the AR representation of the MTAR error term process as $\epsilon_{t\tau} = \sum^{\infty}$ $\sum_{j=0}^{\infty} a_j (T^{-1/2} \delta_T \Delta X_{t-j\tau})' \eta_{\tau} = a(L) (T^{-1/2} \delta_T \Delta X_{t-j\tau})' \eta_{\tau}$ and $\epsilon_{t\tau}$ is an orthogonal $(0, \sigma^2(\eta, \tau))$ sequence with $\sigma^2(\eta, \tau) = a(1)^2 \eta_\tau' \Omega_\tau \eta_\tau$. From Lemma 2.1 of [Phillips and Ouliaris](#page-39-9) [\(1990\)](#page-39-9), it follows that

$$
T^{-1/2} \sum_{t=1}^{[Ts]} \epsilon_{t\tau K} = a(L)\eta_{\tau}' \left(T^{-1/2} \sum_{t=1}^{[Ts]} T^{1/2} \delta_T \Delta X_{t\tau} \right) + o_p(1) \Rightarrow a(1)\eta_{\tau}' X_{\tau}, \qquad (45)
$$

where $a(1) = \sum_{n=1}^{\infty}$ $\sum_{j=0} a_j$.

Now, we apply the MTAR model to the residuals according to [\(5\)](#page-7-1) and compute the test statistics F_{τ} . The *t* ratio of $\hat{\rho}_1$ is written as

$$
t_1 = \frac{U_{1\tau}' Q_K \epsilon_\tau}{\hat{\sigma} (U_{1\tau}' Q_K U_{1\tau})^{1/2}}
$$
(46)

and the *t* ratio of $\hat{\rho}_2$ is written as

$$
t_2 = \frac{U'_{2\tau} Q_K \epsilon_\tau}{\hat{\sigma} (U'_{2\tau} Q_K U_{2\tau})^{1/2}},\tag{47}
$$

where

$$
U_{\tau} = (U_{1\tau}, U_{2\tau}) = \begin{bmatrix} \hat{e}_{0\tau} 1\{\Delta \hat{e}_{0\tau} \ge \lambda\} & \hat{e}_{0\tau} 1\{\Delta \hat{e}_{0\tau} < \lambda\} \\ \hat{e}_{1\tau} 1\{\Delta \hat{e}_{1\tau} \ge \lambda\} & \hat{e}_{1\tau} 1\{\Delta \hat{e}_{1\tau} < \lambda\} \\ \vdots & \vdots \\ \hat{e}_{T-1\tau} 1\{\Delta \hat{e}_{T-1\tau} \ge \lambda\} & \hat{e}_{T-1\tau} 1\{\Delta \hat{e}_{T-1\tau} < \lambda\} \end{bmatrix} \tag{48}
$$

Finally, we need convergence results for $N_T(\lambda, \tau)$, $D_T(\lambda, \tau)$ and $\hat{\sigma}^2$. The main difference between the asymptotic distribution for the SETAR and the MTAR models lies in the fact that the indicator variable $\Delta \hat{e}_{t\tau}$ has a stationary distribution under the null hypothesis and the alternative. Further, the MTAR decomposition of $\hat{e}_{t-1\tau}$ is not regular and Theorem 3.1 of [Park and Phillips](#page-39-12) [\(2001\)](#page-39-12) cannot be used. However, from Theorem 1 in [Caner and Hansen](#page-37-4) [\(2001\)](#page-37-4) it follows that

$$
T^{-1/2} \sum_{t=1}^{[Ts]} \mathbb{1} \{ \Delta \hat{e}_{t-1\tau} \ge \lambda \} \epsilon_{t\tau} = T^{-1/2} \sum_{t=1}^{[Ts]} \mathbb{1} \{ G \left(\Delta \hat{e}_{t-1\tau} \right) \ge G \left(\lambda \right) \} \epsilon_{t\tau}
$$

$$
= T^{-1/2} \sum_{t=1}^{[Ts]} \mathbb{1} \{ U_t \ge G \left(\lambda \right) \} \epsilon_{t\tau}
$$

$$
\Rightarrow Q_{\kappa\tau}(s, u) = \sigma(\eta, \tau) W(s, u)
$$
(49)

$$
= a(1)l_{11}(\kappa'_{\tau} D_{\tau} \kappa_{\tau})^{1/2} W(s, u),
$$

where $G(\cdot)$ is the marginal distribution of $\Delta \hat{e}_{t-1\tau}$ so that $G(\Delta \hat{e}_{t-1\tau}) = U_t \sim U[0,1]$ and $G(\lambda) = u$. The standard two-parameter Brownian motion $W(s, u)$ is defined on $(s, u) \in [0, 1]^2$. Using Theorem 2.2 of [Kurtz and Protter](#page-38-13) [\(1991\)](#page-38-13) and [\(49\)](#page-27-0) yields

$$
N_T(\lambda, \tau) = T^{-1} \sum_{t=1}^T \mathbb{1} \{ \Delta \hat{e}_{t-1\tau} \ge \lambda \} \hat{e}_{t-1\tau} \epsilon_{t\tau}
$$

\n
$$
= \hat{\eta}_{\tau}' \delta_T \sum_{t=1}^T \mathbb{1} \{ G \left(\Delta \hat{e}_{t-1\tau} \right) \ge G \left(\lambda \right) \} X_{t-1\tau} \epsilon_{t\tau}
$$

\n
$$
\Rightarrow a(1) l_{11} (\kappa_{\tau}' D_{\tau} \kappa_{\tau})^{1/2} \eta_{\tau}' \int_{0}^{1} X_{\tau}(s) dW(s, u)
$$

\n
$$
= a(1) l_{11}^2 (\kappa_{\tau}' D_{\tau} \kappa_{\tau})^{1/2} \int_{0}^{1} Q_{\kappa \tau}(s) dW(s, u)
$$

\n(50)

and Theorem 3 of [Caner and Hansen](#page-37-4) [\(2001\)](#page-37-4) yields

$$
D_T(\lambda, \tau) = T^{-2} \sum_{t=1}^T \mathbb{1} \{ \Delta \hat{e}_{t-1\tau} \ge \lambda \} \hat{e}_{t-1\tau}^2
$$

\n
$$
= \hat{\eta}_\tau' \delta_T T^{-1} \sum_{t=1}^T \mathbb{1} \{ G \left(\Delta \hat{e}_{t-1\tau} \right) \ge G \left(\lambda \right) \} X_{t-1\tau} X_{t-1\tau'} \delta_T \hat{\eta}_\tau
$$

\n
$$
\Rightarrow u \eta_\tau' \int_0^1 X_\tau(s) X_\tau'(s) ds \eta_\tau
$$

\n
$$
= u l_{11}^2 \int_0^1 Q_{\kappa \tau}^2(s) ds.
$$

\n(51)

For the variance estimate, $\hat{\sigma}^2$, Lemma 2.2 of [Phillips and Ouliaris](#page-39-9) [\(1990\)](#page-39-9) yields

$$
\hat{\sigma}^2 = T^{-1} \sum_{t=1}^T \epsilon_{t\tau}^2 + o_p(1)
$$

\n
$$
\Rightarrow a(1)^2 l_{11}^2 \kappa_{\tau}^{\prime} D_{\tau} \kappa_{\tau}.
$$
\n(52)

The results (50) , (51) , (52) combine with the CMT to proof

$$
t_1 \Rightarrow \frac{\int\limits_0^1 Q_{\kappa\tau}(s)dW(s,u)}{\left(u\int\limits_0^1 Q^2_{\kappa\tau}(s)ds\right)^{1/2}}.
$$
\n(53)

Analogously, we can show that

$$
t_2 \Rightarrow \frac{\int_{0}^{1} Q_{\kappa\tau}(s) (dW(s, 1) - dW(s, u))}{\left((1 - u) \int_{0}^{1} Q_{\kappa\tau}^{2}(s) ds \right)^{1/2}}
$$
(54)

holds. Finally, we observe that taking the supremum over all $\tau \in \mathcal{T}$ is a continuous transformation so that we can use the CMT to proof the theorem under the null hypothesis. The proof of the theorem under the alternative is a straightforward adaptation of the results given in the proof of Theorem [1.](#page-11-1)

			$\cal C$			C/T			C/S	
	$\mathbf T$	90%	95%	99%	90%	95%	99%	90%	95%	99%
$m=1$										
	50	16.01	18.48	24.22	18.80	21.38	27.10	17.52	20.16	25.83
	100	12.73	14.66	19.24	15.58	17.75	22.49	14.44	16.68	21.40
	250	10.80	12.29	15.70	12.99	14.59	18.16	12.52	14.36	17.82
	500	10.13	11.42	14.30	12.11	13.46	16.37	11.76	13.24	16.39
	∞	9.48	10.70	13.45	11.53	12.86	15.74	11.20	12.71	15.69
$m=2\,$										
	50	17.63	20.14	26.26	19.84	22.42	28.47	20.49	23.47	29.57
	100	16.19	18.21	23.21	18.69	20.90	25.94	19.24	21.56	26.54
	250	13.33	15.02	18.93	15.50	17.39	21.69	16.47	18.39	23.03
	500	12.22	13.68	17.08	14.06	15.63	19.07	15.22	16.85	20.18
	∞	12.18	13.60	16.88	14.22	15.82	19.33	15.30	16.86	20.45
$m=3\,$										
	50	19.80	22.49	28.40	21.71	24.56	30.57	23.94	27.05	34.08
	100	18.20	20.51	25.37	20.40	22.81	28.00	22.87	25.43	30.89
	250	15.37	17.16	21.21	17.31	19.24	23.42	19.81	22.00	26.48
	500	14.15	15.71	19.11	15.88	17.57	21.14	18.44	20.30	24.11
	∞	14.12	15.65	19.03	16.00	17.66	21.23	18.60	20.44	24.09
$m=4$										
	50	21.19	23.92	29.90	23.22	26.11	32.96	27.33	30.40	37.89
	100	20.13	22.56	27.61	22.42	24.80	29.47	25.98	28.49	34.16
	250	17.36	19.27	23.87	19.21	21.23	26.12	23.26	25.81	30.78
	500	15.77	17.41	20.70	17.41	19.13	22.73	21.46	23.44	27.80
	∞	16.04	17.69	21.28	17.81	19.51	23.12	21.75	23.83	27.95

Table 1: Approximate critical values of F_{SETAR}^*

Note: *C*, *C/T* and *C/S* denote the sup *F* tests using the structural break models in [\(3\)](#page-6-3). *m* refers to the number of columns of the regressor matrix *xt*. The lag truncation parameter is determined using the BIC and maximum lag length *K*max = 8. Critical values for different order selection rules are not reported but can be obtained from the author upon request.

		\mathcal{C}				C/T			C/S	
	\boldsymbol{u}	90%	95%	99%	90%	95%	99%	90%	95%	99%
						$m=1$				
$T=50$										
	0.15	17.49	20.18	26.82	18.72	$21.25\,$	26.65	18.12	20.62	26.60
	0.25	16.89	19.41	25.74	18.56	20.86	26.52	17.98	20.43	26.32
	0.50	16.56	19.03	24.57	18.51	20.94	26.52	17.88	20.34	26.00
$T = 100$										
	0.15	17.95	21.03	28.71	18.16	20.77	26.43	18.04	21.17	28.37
	0.25	15.56	18.26	$24.05\,$	17.03	19.32	24.00	16.55	18.97	24.92
	0.50	14.58	16.86	21.30	16.47	18.92	23.86	15.92	18.28	23.42
$T=250\,$										
	0.15	19.21	23.19	32.09	18.82	21.85	28.79	19.45	23.15	31.94
	0.25	15.08	17.60	24.11	16.11	18.20	23.65	16.13	18.65	25.27
	0.50	$12.85\,$	14.72	18.87	14.75	16.50	20.70	14.33	16.21	20.98
$T = 500$										
	0.15	20.83	24.76	35.04	20.90	24.37	32.96	21.56	25.51	35.91
	0.25 0.50	$15.35\,$	17.67	24.79	16.60	18.98	24.46	16.55	18.95	25.93
$T = \infty$		12.49	14.04	17.37	14.34	15.89	19.51	13.96	15.66	19.37
	0.15	21.59	25.94	36.86	21.52	25.25	34.64	22.52	26.71	37.94
	0.25	15.30	17.48	25.12	16.37	18.67	24.70	16.56	19.01	26.38
	0.50	11.81	13.13	16.39	13.65	14.82	18.08	13.35	14.79	18.27
						$m=2$				
$T=50$										
	0.15	18.18	20.74	26.68	19.72	22.13	28.33	20.23	22.86	29.18
	0.25	17.95	20.36	26.17	19.64	21.99	27.80	20.37	23.05	29.61
	0.50	17.77	20.24	26.40	19.91	22.40	28.33	20.44	23.38	29.22
$T = 100$										
	0.15	18.74	21.57	27.97	19.57	22.19	28.05	20.01	22.43	28.48
	$0.25\,$	17.21	19.08	25.53	18.84	21.23	26.08	19.62	21.93	27.02
	0.50	16.84	19.15	24.07	18.84	21.09	25.82	19.62	21.13	27.15
$T = 250$										
	0.15	19.85	23.35	32.90	19.81	22.51	28.71	20.89	23.90	31.19
	0.25	16.43	18.77	24.80	17.53	19.73	24.53	18.64	21.00	26.47
	0.50	14.70	16.64	20.91	$16.48\,$	18.35	22.44	17.46	19.56	24.40
$T=500$										
	0.15	21.24	25.01	35.89	21.49	24.69	31.92	22.52	26.01	35.26
	0.25	16.54	18.96	26.23	17.70	19.77	24.76	18.62	21.19	27.14
	0.50	14.26	15.90	19.66	15.86	17.51	21.06	16.97	18.75	23.10
$T = \infty$										
	0.15	21.88	26.00	38.63	21.79	25.05	32.61	23.17	27.05	37.19
	0.25	16.23	18.99	26.06	17.12	19.07	24.19	18.20	20.87	27.27
	0.50	13.20	14.64	18.18	14.69	16.14	19.47	15.88	17.47	21.79

Table 2: Approximate critical values of F_{MTAR}^*

Note: *C*, *C/T* and *C/S* denote the structural break models in [\(3\)](#page-6-3). *m* refers to the number of columns of the regressor matrix *xt*. The lag truncation parameter is determined using the BIC and maximum lag length *K*max = 8. Critical values for different order selection rules are not reported but can be obtained from the author upon request. Critical values for $u = \{0.75, 0.85\}$ are not reported to conserve space. Since the distribution is symmetric in *u*, the values can easily be inferred from the table.

			$\,C$			C/T			C/S	
	\boldsymbol{u}	90%	95%	99%	90%	95%	99%	90%	95%	99%
						$m=3$				
$T=50$										
	$0.15\,$	19.72	22.32	27.81	21.40	24.06	30.13	23.59	26.77	33.19
	0.25	19.73	22.47	27.92	21.55	24.10	29.93	23.72	26.87	33.77
	0.50	19.86	22.37	28.41	21.60	24.49	30.78	24.13	27.36	34.55
$T = 100$										
	0.15	19.74	22.37	28.26	21.05	23.53	28.71	22.89	25.54	30.77
	0.25	18.90	$21.17\,$	26.30	20.50	22.95	27.94	22.77	25.36	30.68
	0.50	18.66	21.03	25.79	20.51	22.89	28.32	22.90	25.71	31.06
$T = 250$										
	0.15	20.12	23.50	31.76	20.66	23.36	28.98	22.49	25.25	31.63
	0.25	17.66	20.05	26.40	18.94	21.00	25.55	21.07	23.50	29.01
	0.50	16.52	18.40	22.78	18.17	20.05	24.18	20.52	22.73	27.76
$T=500$										
	0.15	21.79	25.58	34.78	22.22	25.74	33.58	23.45	26.54	34.99
	0.25	17.92	20.37	26.21	19.01	21.17	27.07	20.88	23.27	28.54
	0.50	15.94	17.76	21.83	17.45	19.21	23.49	19.77	21.75	26.03
$T = \infty$										
	0.15	22.14	26.37	36.87	22.20	25.94	34.35	23.42	26.67	36.10
	0.25	$17.39\,$	19.98	26.48	18.33	20.30	26.20	20.08	22.46	28.03
	0.50	14.91	16.48	20.49	16.28	17.83	21.55	18.60	20.29	24.59
						$m=4$				
$T=50$										
	0.15	21.12	23.88	29.68	22.90	25.88	31.96	26.96	30.13	36.84
	0.25	21.08	23.84	29.93	23.04	25.89	31.99	27.21	30.20	37.04
	0.50	21.33	24.07	30.30	23.29	26.13	32.80	27.65	30.68	37.73
$T = 100$										
	0.15	21.05	23.51	29.54	22.43	24.71	30.01	25.57	28.29	33.88
	0.25	20.52	22.93	27.47	22.31	24.65	29.63	25.77	28.31	34.02
	0.50	20.49	22.93	27.85	22.43	24.91	30.01	26.15	28.71	34.65
$T = 250$										
	0.15	21.03	23.88	31.30	21.75	24.39	$29.32\,$	24.56	27.12	32.56
	0.25	19.15	$21.26\,$	26.57	20.50	22.62	27.36	23.84	26.21	31.76
	0.50	18.24	20.24	24.79	19.83	21.87	26.58	23.61	26.01	31.68
$T = 500$										
	$0.15\,$	$22.38\,$	25.85	35.15	22.59	25.57	33.44	25.12	28.03	35.57
	0.25	18.89	21.28	27.64	19.98	22.11	27.03	23.28	25.52	30.62
	0.50	17.43	19.27	23.32	18.82	20.71	24.71	22.37	24.53	29.27
$T = \infty$										
	0.15	22.50	26.28	36.47	22.37	25.67	33.72	24.82	27.74	35.55
	0.25	18.21	20.51	27.60	19.06	21.15	26.11	22.39	24.59	29.73
	0.50	16.26	17.90	21.89	17.45	19.14	23.09	21.10	23.22	27.89

Table 3: Approximate critical values of F_{MTAR}^* , continued

	$\mu_1 = 1, \mu_2 = 4, \alpha_1 = 2, \alpha_2 = 2$							$\mu_1 = 1, \mu_2 = 1, \alpha_1 = 2, \alpha_2 = 4$						
	$T = 50$				$T = 100$			$T = 50$		$T = 100$				
τ	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75		
SETAR														
\mathcal{C}	0.372	0.377	0.380	0.970	0.970	0.972	0.211	0.217	0.421	0.686	0.627	0.880		
C/T	0.224	0.234	0.223	0.875	0.878	0.874	0.107	0.113	0.157	0.485	0.529	0.744		
C/S	0.278	0.277	0.290	0.922	0.921	0.934	0.343	0.293	0.334	0.980	0.972	0.966		
MTAR														
\mathcal{C}	0.297	0.299	0.319	0.893	0.879	0.892	0.167	0.178	0.355	0.516	0.447	0.759		
C/T	0.231	0.221	0.218	0.790	0.800	0.797	0.122	0.127	0.172	0.398	0.443	0.653		
C/S	0.247	0.238	0.259	0.851	0.819	0.842	0.279	0.265	0.281	0.911	0.892	0.879		
EG(c)	0.139	0.096	0.096	0.391	0.274	0.277	0.089	0.060	0.086	0.126	0.100	0.145		
$EG(c + t)$	0.124	0.125	0.116	0.397	0.481	0.434	0.076	0.058	0.096	0.109	0.122	0.187		
GH(C)	0.364	0.369	0.374	0.970	0.970	0.973	0.176	0.170	0.375	0.606	0.545	0.870		
GH(C/T)	0.240	0.248	0.239	0.879	0.879	0.878	0.101	0.108	0.150	0.411	0.470	0.709		
GH(C/S)	0.271	0.271	0.283	0.922	0.921	0.934	0.296	0.257	0.293	0.968	0.963	0.962		
Φ_{SETAR}	0.216	0.184	0.186	0.738	0.600	0.727	0.111	0.095	0.148	0.196	0.189	0.303		
Φ_{MTAR}	0.194	0.193	0.183	0.699	0.565	0.619	0.092	0.082	0.132	0.182	0.169	0.245		

Table 4: Size-adjusted power of the sup F test under structural change and symmetric adjustment

Note: *C*, *C*/*T* and *C*/*S* denote the structural break models in [\(3\)](#page-6-4). EG (c) and EG (c + t) refer to the Engle-Granger test with intercept and intercept plus trend, respectively. GH denotes the Gregory-Hansen test. Φ_{SETAR} and Φ_{MTAR} denote the Enders-Siklos cointegration test with SETAR and MTAR adjustment, respectively.
The table is based on 2.500 replieding of the DCB described The table is based on 2,500 replications of the DGP described in [\(13\)](#page-14-1). The autoregressive coefficients are $\rho_1 = \rho_2 = -0.5$, i.e. the adjustment is constant and symmetric.

SETAR														
		$\mu_1 = 1, \mu_2 = 4, \alpha_1 = 2, \alpha_2 = 2$						$\mu_1 = 1, \mu_2 = 1, \alpha_1 = 2, \alpha_2 = 4$						
	$T = 50$			$T = 100$			$T = 50$			$T = 100$				
τ	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75		
\mathcal{C}	0.32(0.15) 0.28(0.04)	0.53(0.11) 0.52(0.04)	0.70(0.15) 0.74(0.04)	0.28(0.10) 0.26(0.02)	0.51(0.08) 0.51(0.02)	0.74(0.11) 0.76(0.02)	0.34(0.18) 0.28(0.05)	0.55(0.13) 0.54(0.04)	0.72(0.13) 0.76(0.04)	0.28(0.12) 0.26(0.02)	0.54(0.11) 0.52(0.02)	0.75(0.10) 0.77(0.02)		
C/T	0.38(0.19) 0.28(0.26)	0.50(0.16) 0.50(0.08)	0.66(0.22) 0.74(0.34)	0.33(0.16) 0.27(0.03)	0.51(0.11) 0.51(0.02)	0.69(0.16) 0.75(0.03)	0.39(0.19) 0.28(0.26)	0.53(0.15) 0.52(0.10)	0.65(0.20) 0.74(0.22)	0.31(0.15) 0.27(0.02)	0.53(0.11) 0.52(0.02)	0.73(0.13) 0.75(0.02)		
C/S	0.35(0.16) 0.28(0.18)	0.53(0.12) 0.54(0.04)	0.68(0.16) 0.76(0.12)	0.30(0.11) 0.25(0.02)	0.51(0.07) 0.51(0.02)	0.72(0.12) 0.76(0.03)	0.33(0.14) 0.26(0.14)	0.54(0.09) 0.54(0.04)	0.71(0.13) 0.78(0.08)	0.27(0.07) 0.25(0.02)	0.51(0.05) 0.51(0.02)	0.75(0.07) 0.77(0.01)		
MTAR														
		$\mu_1 = 1, \mu_2 = 4, \alpha_1 = 2, \alpha_2 = 2$						$\mu_1 = 1, \mu_2 = 1, \alpha_1 = 2, \alpha_2 = 4$						
	$T = 50$			$T = 100$			$T = 50$			$T = 100$				
τ	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75		
\mathcal{C}	0.35(0.18)	0.52(0.14)	0.68(0.18)	0.29(0.11)	0.51(0.08)	0.74(0.10)	0.38(0.22)	0.55(0.21)	0.67(0.21)	0.30(0.17)	0.54(0.17)	0.72(0.17)		
	0.28(0.14)	0.52(0.04)	0.74(0.10)	0.26(0.02)	0.51(0.02)	0.75(0.02)	0.28(0.30)	0.54(0.20)	0.74(0.04)	0.27(0.02)	0.52(0.09)	0.77(0.02)		
C/T	0.40(0.20)	0.50(0.16)	0.60(0.22)	0.34(0.17)	0.51(0.12)	0.68(0.17)	0.42(0.21)	0.53(0.18)	0.64(0.21)	0.35(0.18)	0.54(0.15)	0.72(0.15)		
C/S	0.28(0.34) 0.38(0.18) 0.30(0.26)	0.50(0.08) 0.53(0.14) 0.52(0.06)	0.72(0.38) 0.67(0.17) 0.74(0.18)	0.27(0.04) 0.31(0.12) 0.26(0.04)	0.51(0.02) 0.51(0.08) 0.52(0.02)	0.75(0.05) 0.72(0.11) 0.76(0.03)	0.32(0.32) 0.36(0.17) 0.30(0.22)	0.54(0.14) 0.52(0.16) 0.54(0.06)	0.74(0.26) 0.66(0.22) 0.76(0.16)	0.27(0.13) 0.27(0.09) 0.25(0.02)	0.52(0.07) 0.50(0.10) 0.52(0.02)	0.76(0.02) 0.72(0.16) 0.77(0.02)		

Table 5: Estimates of the breakpoint under symmetric adjustment

Note: *C*, *C*/*T* and *C*/*S* denote the structural break models in [\(3\)](#page-6-4). The left panel and right panel report the estimates of the break fraction following a shift in the intercept and ^a shift in the slope, respectively. Upper rows contain the mean breakpoint estimate and the empirical standard deviation. Lower row contain the medianbreakpoint and the interquartile range. The autoregressive coefficients are $\rho_1 = \rho_2 = -0.5$, i.e. the adjustment is constant and symmetric.

	$\mu_1 = 1, \mu_2 = 1, \alpha_1 = 2, \alpha_2 = 2$												
			\overline{C}			C/T			C/S			Φ_{SETAR}	
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
-0.025	-0.05	0.109	0.060	0.015	0.098	0.052	0.010	0.112	0.055	0.010	0.123	0.070	0.019
	-0.15	0.130	0.072	0.022	0.121	0.065	0.012	0.125	0.066	0.014	0.160	0.086	0.023
	-0.25	0.149	0.084	0.025	0.129	0.069	0.014	0.137	0.075	0.019	0.185	0.113	0.031
-0.05	-0.10 -0.25	0.133 0.178	0.076 0.104	0.021 0.034	0.128 0.163	0.068 0.089	0.012 0.019	0.127 0.168	0.067 0.093	0.014 0.024	0.174 0.276	0.093 0.170	0.024 0.052
-0.10	-0.15 -0.25	0.195 0.258	0.114 0.156	0.035 0.051	0.176 0.228	0.103 0.131	0.022 0.029	0.176 0.236	0.103 0.142	0.025 0.036	0.338 0.477	0.205 0.313	0.060 0.114
Size:		0.135	0.073	0.023	0.143	0.085	0.022	0.126	0.080	0.021	0.120	0.060	0.012
	$\mu_1=1,\,\mu_2=4,\,\alpha_1=2,\,\alpha_2=2$												
			$\cal C$			C/T			C/S			Φ_{SETAR}	
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
-0.025	-0.05	0.104	0.052	0.013	0.108	0.051	0.009	0.106	0.054	0.013	0.119	0.062	0.014
	-0.15	0.125	0.062	0.014	0.120	0.061	0.012	0.117	0.061	0.014	0.143	0.067	0.018
	-0.25	0.140	0.072	0.016	0.132	0.068	0.017	0.128	0.068	0.016	0.153	0.084	0.022
-0.05	-0.10	0.129	0.062	0.013	0.122	0.065	0.012	0.118	0.063	0.014	0.147	0.080	0.019
	-0.25	0.174	0.094	0.023	0.159	0.086	0.022	0.155	0.082	0.020	0.182	0.108	0.032
-0.10	-0.15	0.187	0.109	0.023	0.166	0.090	0.018	0.166	0.084	0.023	0.210	0.118	0.037
	-0.25	0.244	0.145	0.041	0.211	0.120	0.029	0.209	0.113	0.032	0.258	0.158	0.053
	$\mu_1=1,\,\mu_2=1,\,\alpha_1=2,\,\alpha_2=4$												
			\mathcal{C}			C/T			C/S			Φ_{SETAR}	
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
-0.025	-0.05	0.249	0.168	0.098	0.210	0.140	0.064	0.339	0.266	0.146	0.114	0.057	0.012
	-0.15	0.272	0.196	0.111	0.225	0.159	0.070	0.386	0.295	0.177	0.117	0.059	0.014
	-0.25	0.282	0.203	0.116	0.237	0.167	0.077	0.410	0.315	0.188	0.119	0.064	0.016
-0.05	-0.10	0.277	0.202	0.114	0.226	0.166	0.072	0.399	0.316	0.182	0.114	0.059	0.013
	-0.25	0.313	0.227	0.137	0.257	0.187	0.085	0.474	0.369	0.225	0.126	0.067	0.017
-0.10	-0.15 -0.25	0.328 0.370	0.252 0.281	0.140 0.161	0.265 0.297	0.184 0.212	0.090 0.105	0.505 0.566	0.400 0.459	0.246 0.279	0.136 0.193	0.072 0.110	0.018 0.037
	$\mu_1 = 1, \mu_2 = 4, \alpha_1 = 2, \alpha_2 = 4$		\boldsymbol{C}			C/T			C/S			Φ_{SETAR}	
									5%				
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%		1%	10%	5%	1%
-0.025	-0.05 -0.15	0.247 0.275	0.176 0.192	0.096 0.108	0.206 0.225	0.141 0.160	0.061 0.069	0.339 0.383	0.262 0.297	0.148 0.169	0.111 0.112	0.057 0.058	0.011 0.013
	-0.25	0.283	0.200	0.110	0.235	0.165	0.076	0.408	0.313	0.186	0.118	0.062	0.015
-0.05	-0.10	0.280	0.203	0.117	0.232	0.162	0.072	0.398	0.307	0.176	0.116	0.058	0.011
	-0.25	0.316	0.231	0.131	0.261	0.180	0.082	0.465	0.362	0.220	0.130	0.068	0.017
-0.10	-0.15	0.335	0.249	0.138	0.261	0.189	0.091	0.505	0.398	0.240	0.134	0.069	0.017
	-0.25	$\,0.365\,$	0.282	0.162	0.295	0.208	0.107	0.565	0.450	0.282	0.143	0.081	0.022

Table 6: Size-adjusted power of the sup *F* test (SETAR) under structural change and asymmetric adjustment

Note: *C*, *C/T* and *C/S* denote the structural break models in [\(3\)](#page-6-3). Φ*SET AR* denotes the Enders-Siklos cointegration test with SETAR adjustment. The table is based on 2,500 replications of the DGP described in [\(13\)](#page-14-0) with sample size $T = 100$. The breakpoint occurs mid-sample, i.e. $\tau = 0.5$. The test with the highest rejection rates is highlighted in boldface.

	$\mu_1=1,\,\mu_2=1,\,\alpha_1=2,\,\alpha_2=2$												
			\overline{C}			C/T			C/S			Φ_{MTAR}	
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
-0.025	-0.05	0.090	0.043	0.007	0.096	0.045	0.008	0.094	0.050	0.008	0.112	0.054	0.013
	-0.15	0.098	0.050	0.008	0.117	0.064	0.014	0.107	0.054	0.009	0.206	0.109	0.030
	-0.25	0.135	0.073	0.014	0.152	0.082	0.016	0.142	0.072	0.013	0.378	0.224	0.066
-0.05	-0.10 $-0.25\,$	0.090 0.152	0.048 0.077	0.009 0.015	0.115 0.161	0.059 0.089	0.014 0.017	0.099 0.153	0.056 0.081	0.012 0.015	0.168 0.430	0.091 0.269	0.024 0.083
$-0.10\,$	-0.15 -0.25	0.115 0.189	0.064 0.100	0.012 0.019	0.139 0.198	0.074 0.106	0.017 0.020	0.129 0.192	0.067 0.110	0.013 0.016	0.323 0.557	0.192 0.365	0.055 0.119
Size:		0.085	0.042	0.007	$\,0.074\,$	0.039	0.006	0.093	0.041	0.009	0.109	0.060	$\,0.014\,$
	$\mu_1 = 1, \mu_2 = 4, \alpha_1 = 2, \alpha_2 = 2$ C												
						${\cal C}/T$			C/S			Φ_{MTAR}	
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
-0.025	-0.05	0.090	0.044	0.009	0.110	0.057	0.010	0.099	0.051	0.012	0.108	0.051	0.013
	-0.15	0.104	0.053	0.013	0.117	0.064	0.010	0.106	0.050	0.012	0.155	0.083	0.014
	-0.25	0.151	0.070	0.018	0.151	0.086	0.014	0.146	0.074	0.014	0.233	0.128	0.033
-0.05	-0.10 -0.25	0.151 0.166	0.070 0.078	0.018 0.017	0.151 0.159	0.086 0.091	0.014 0.016	0.146 0.160	0.074 0.078	0.014 0.017	0.233 0.255	0.128 0.142	0.033 0.039
-0.10	-0.15 -0.25	0.134 0.205	0.067 0.105	0.015 0.023	0.136 0.196	0.073 0.103	0.015 0.025	0.143 0.194	0.066 0.104	0.013 0.021	0.212 0.305	0.115 0.178	0.026 0.047
	$\mu_1 = 1, \mu_2 = 1, \alpha_1 = 2, \alpha_2 = 4$												
			\overline{C}			C/T			C/S			Φ_{MTAR}	
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
-0.025	-0.05	0.193	0.130	0.071	0.191	0.133	0.062	0.310	0.228	0.114	0.080	0.040	0.009
	-0.15	0.224	0.156	0.086	0.221	0.149	0.070	0.369	0.274	0.141	0.100	0.049	0.007
	-0.25	0.261	0.190	0.104	0.245	0.173	0.082	0.431	0.331	0.178	0.120	0.062	0.012
-0.05	-0.10	0.217	0.155	0.084	0.214	0.148	0.072	0.355	0.261	0.134	0.096	0.050	0.008
	-0.25	0.276	0.200	0.112	0.258	0.184	0.090	0.455	0.359	0.199	0.126	0.067	0.014
-0.10	-0.15	0.261	0.192	0.106	0.245	0.176	0.088	0.424	0.328	0.178	0.113	0.060	0.012
	-0.25	0.300	0.221	0.125	0.287	0.204	0.100	0.504	0.397	0.230	0.133	0.076	0.018
	$\mu_1 = 1, \mu_2 = 4, \alpha_1 = 2, \alpha_2 = 4$												
			\boldsymbol{C}			${\cal C}/T$			${\cal C}/S$			Φ_{MTAR}	
ρ_1	ρ_2	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
-0.025	-0.05		0.196 0.132 0.073		0.192	0.135	0.060		0.303 0.232 0.113		0.085	0.040	0.009
	-0.15 -0.25	0.219 0.256	0.156 0.193	0.082 0.107	0.223 0.250	0.154 0.173	0.071 0.084	0.367 0.435	0.281 0.330	0.140 0.181	0.096 0.122	0.050 0.063	0.009 0.013
-0.05	-0.10		0.156	0.082			0.070	0.348	0.262	0.136	0.093	0.046	
	-0.25	0.213 0.273	0.204	0.115	0.219 0.257	0.145 0.182	0.090	0.458	0.350	0.200	0.126	0.069	0.010 0.016
-0.10	-0.15	0.265	0.191	0.106	0.240	0.175	0.084	0.422	0.326	0.180	0.115	0.062	0.012
	-0.25	0.301	0.220	0.132	0.282	0.202	0.100	0.503	0.397	0.228	0.134	0.079	0.018

Table 7: Size-adjusted power of the sup *F* test (MTAR) under structural change and asymmetric adjustment

Note: *C*, *C/T* and *C/S* denote the structural break models in [\(3\)](#page-6-3). Φ*MT AR* denotes the threshold cointegration test with MTAR adjustment. The table is based on 2,500 replications of the DGP described in (13) with sample size $T = 100$. The breakpoint occurs mid-sample, i.e. *τ* = 0*.*5. The test with the highest rejection rates is highlighted in boldface.

Table 8: Long-run adjustment along the gasoline value-chain

Note: μ (α) denotes the intercept (slope coefficient) of the long-run equilibrium equation without structural break. $\mu_1(\alpha_1)$ and $\mu_2(\alpha_2)$ denote the intercept (slope coefficient) of the long-run equilibrium equation before the break and after the break, respectively. δ is the linear trend coefficient. Φ_{SETAR} and Φ_{MTAR} denote the *F*-statistic based on the null hypothesis $H_0: \rho^+ = \rho^- = 0$, respectively. We conduct bootstrap *F*-tests with 600 replications to test the null hypothesis $\rho^+ = \rho^-$. $*** p < 0.01, ** p < 0.05, * p < 0.1$

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