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The Role of Budget Constraints in Sequential Elimination Tournaments*

Malin Arve† and Olga Chiappinelli‡

December 11, 2018

Abstract

Motivated by the EU concept of Pre-Commercial Procurement and the massive presence of SMEs in the European economy, we study how budget constraints affect R&D effort in sequential elimination tournaments. We show that introducing budget constraints leads to a non-monotonicity in unconstrained contestants' effort. Furthermore, we show that if the budget asymmetry is not too large, unconstrained contestants exert higher effort than when faced with unconstrained contestants only.

Keywords: Pre-Commercial Procurement; Contests; Budget constraints.

JEL Classification: D72, H57.

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1 Introduction

In this paper we study how asymmetric budget constraints affect firm behavior in sequential elimination tournaments. The research is motivated by a new concept called Pre-Commercial Procurement (PCP) that has been introduced in the European Union to stimulate innovation through public procurement.[1]

PCP concerns the R&D phase of an innovation and is a new approach compared to standard auction-like practices of public procurement. An important feature of PCP is that it is organised as a competitive stepwise process. In PCP a number of firms enter the competition and start developing alternative solutions. At the end of each phase, intermediate evaluations are conducted, which form the basis for the sequential elimination of participating firms. This makes it suitable to model the procurement process as a sequential contest game or elimination contest.

Importantly, we take into account asymmetries across firms. Paradoxically, EU public procurement is mostly undertaken by big firms[2], whereas the large majority of all businesses in the EU are small and medium-sized enterprises (SMEs)[3]. Moreover, SMEs are in many cases thought to contribute importantly to innovation activities and growth[4]. This raises a very natural question of what might happen in PCP when big firms that are typically present in public contracting compete against small firms that are desirable to involve in such innovative environments. An important feature of SMEs is their limited access to finance and, thus, their budget constraints.[5] In PCP, stimulating participation among these firms will inevitably lead them to compete against bigger and less constrained firms.[6]

The analysis in this paper is motivated by our desire to understand incentives for financially asymmetric firms to exert R&D effort in PCP-like contests. We explicitly take into account this characteristic of SMEs and show that a two-stage contest where all

[1] See European Commission (2007), *Pre-commercial Procurement: Driving Innovation to Ensure Sustainable High Quality Public Services in Europe, http://ec.europa.eu/invest-in-research/pdf/download_en/com_2007_799.pdf.*

[2] Between 2009 and 2011, only 29% of the value of EU public procurement went to small and medium-sized enterprises (European Commission, 2014).

[3] 99,8% of all EU registered firms are SMEs and they produce more than a half of European GDP (European Commission, 2014).

[4] SMEs being a driver of innovation and growth goes, at least, back to Schumpeter (1910, 1943) and is one of the important arguments behind the US Small Business Act and the Small Business Act of Europe. See Rothwell (1989) for an introduction to the topic.

[5] See Abraham and Schmukler (2017) and references therein.

[6] See Arve (2014) for an analysis of the interaction between financially constrained and unconstrained firms in traditional public procurement.

contestants compete against each other at each stage and where only a limited number of contestants are accepted to the next stage gives increasing incentives to exert effort over time.

Our model also suggests that the asymmetry between budget-constrained and unconstrained contestants fundamentally changes competition within the contest. Introducing budget constraints mechanically limits the ability to invest in effort and R&D. But, more importantly, it leads to a non-monotonicity in the unconstrained firms' first-stage effort. For very tight budgets, the weakness of the budget-constrained contestants lowers the incentives of the unconstrained contestants to exert effort relative to an environment with no budget-constraints. However, there always exists an interval of values for the budget constraints, which we can think of as *moderate and minor budget constraints*, for which the unconstrained firms exert higher effort than when faced with unconstrained contestants only.

In our model we abstract away from some firms (for instance SMEs) being better at innovation or R&D and simply assume that everyone has the same ex ante ability to win the innovation contest. Allowing SMEs to have a higher ability to come up with original ideas would only strengthen our result of increased effort in contests that include such firms.

In fact the driver of our result is the following: When exerting effort in the first stage, the unconstrained contestants not only influence their likelihood of making it to the final stage, but also whom to face in the final stage. Because the expected payoff of an unconstrained contestant is decreasing in the budget-constrained firm's potential to exert effort in the final stage, we obtain a non-monotonicity in the unconstrained contestants effort. This effect is present regardless of whether we allow for spillover effects across stages or not. Spillovers is a natural assumption in an innovation environment, including PCP. However, the fact that the result is not driven by them broadens the scope of our result to more general contest environments.

Through our non-monotonicity result we differ from previous results in the contest theory literature that suggest that even contests stimulate effort more than uneven contests. In their seminal paper on rank-order tournaments, Lazear and Rosen (1981) argue that contests with asymmetric types are inefficient and suggest a handicap scheme to level the playing field between heterogeneous contests. Clark and Riis (2000) obtain a similar result in a bribe tournament. In a one-stage Tullock (1980) contest, Baik (1994) shows that both individual and total effort is maximised in an even contest.

PCP has not yet received a lot of attention in the economic literature. One exception

is Che et al (2015) who investigate the extent to which the pure unbundling between R&D and commercial phases of procurement, as it is called for in PCP, can be justified on contract-theoretical grounds. Our model is very different in that it assumes that PCP is the selection method and investigates how effort in this tender competition is affected by the characteristics of the contestants.

Furthermore, in the contest literature, we build on Clark and Riis (1996) and Fu and Lu (2012), who consider an all-against-all contest, and extend their analysis to also include a final stage with only one winner.[7] In fact, most papers on multi-stage contests (for instance Rosen (1986), Gradstein (1998), Gradstein and Konrad (1999), Amegashie (1999), Amegashie (2000), Baik and Lee (2000), Stein and Rapoport (2005) and Moldovanu and Sela (2006)) focus on situations in which at each stage, subgroups of contestants compete to become the unique group winner. In some settings such as sport tournaments this is of course a more appropriate assumption, but in PCP all firms compete against all other firms at each stage making an all-against-all contest more appropriate.

Budget constraints in contest environments have been studied by Che and Gale (1997), Che and Gale (1998) and Gavious et al (2002) in one stage contests and by Stein and Rapoport (2005), Amegashie (2004), Harbaugh and Klumpp (2005) and Ghosh and Stong (2018) in a two-stage contest where at the first stage there are two or more groups and there is only one winner within each group.[8] We extend this literature to also consider two-stage contests with all-against-all contests at each stage.

There is also a literature that looks at contests and all-pay auctions with head starts. The closest paper to ours in Cohen et al (2016) who study the optimal head start in two-stage elimination contests. In a related paper Klein and Schmutzler (2017) study optimal head starts (vs prizes) in two-stage rank-order tournaments. Neither of these papers consider budget constraints. Furthermore, they model spillovers as the first-stage effort directly affecting the outcome of the second stage, regardless of what that first-stage effort actually lead to. We think of spillovers as a fixed advantage coming from having generated the best concept in the first stage and model the phenomenon as in Franke et al (2018). This way of modeling spillovers is close to models of all-pay auctions

[7]Formally, they consider a contest in which effort is chosen once and for all, but the losers at each of the first K stages continue to compete. The winners of the contest are all the winners of the different stages where previous winners have been sequentially eliminated. In our model, this is how we model the contest within each stage.

[8] Klumpp and Polborn (2006) and Sela and Erez (2013) consider the role of budget constraints in sequential contests with a different structure, namely where the two same players compete against each other over several stages.

where the score of each firm includes both the effort and potential head start or other heterogeneities across contestants (see for instance, Kovenock and Roberson (2009) and Clark and Nilssen (2017)) and it is also similar to contest models with homogeneous spillovers or noise (Dasgupta and Nti, 1998; Amegashie, 2006; Myerson and Wärneryd, 2006).[9]

In our model environment, we consider that the spillovers are exogenous. As they measure the benefits from having offered the best concept in the first-stage, they cannot be considered as an instrument that the contest designer can choose. This is in contrast to a small literature that looks at the optimal choice for head starts (Kirkekgaard, 2012; Seel and Wasser, 2014). Relatedly, Konrad (2002) considers a contest where participants can spend resources at earlier stages to increase the value of the prize. We touch upon this topic in Section 5.3.

The rest of the paper is structured as follows: The model as well as benchmark results in the case of no budget constraints are presented in Section 2. Our main results in the case with both budget-constrained and unconstrained contestants are presented in Section 3. In Section 4 we extend our model to allow for spillovers across periods. Section 5 discusses extensions of our model to different pools of contestants, observability of contestants' type as well as some design issues.

2 The model

We consider a two-stage elimination contest with 3 contestants. Players are risk neutral and can exert costly effort to influence their probability of winning the contest. We use a standard Tullock contest function (Tullock, 1980) to model the probability of winning a stage contest. Effort at stage t by contestant i is denoted x_{ti}, where $t \in \{1, 2\}$ and $i \in \{1, 2, 3\}$. The winner of the contest is the winner of the final stage. This contestant obtains a prize of value v.

In the first stage, contestants exert effort levels to become one of two finalists who continue to the second stage of the contest. We follow Clark and Riis (1996) and Fu and Lu (2012) and assume that the probability of becoming a finalist equals the probability of coming first or second in the first-stage contest. We can therefore define $\tilde{q}_{ij}$ as the

[9] An alternative way of modeling spillovers is to assume that the level of first-stage effort directly influences the second-stage win probability as in Baye et al. (2012) and Baik and Lee (2000) or the cost of exerting effort in the second stage, as in Clark and Nilssen (2013) and Clark et al (2012). However, for our purpose we feel that modeling spillovers as a fixed advantage from generating the best solution in the first-stage captures important features of PCP.

probability of contestants i and j continuing to the final and q_{1i} as the probability that contestant i makes it to the final (regardless of the identity of his opponent in the final).

Given a vector of first-period effort levels (x_{11}, x_{12}, x_{13}), the probability that contestants 1 and 2 make it to the final, $\tilde{q}_{12}$, is defined as

$$\tilde{q}_{12}(x_{11}, x_{12}, x_{13}) = \underbrace{\frac{x_{11}}{(x_{11}+x_{12}+x_{13})}\frac{x_{12}}{(x_{12}+x_{13})}}_{\substack{\text{Prob. that 1 is ranked first} \\ \text{and 2 is ranked second}}} + \underbrace{\frac{x_{12}}{(x_{11}+x_{12}+x_{13})}\frac{x_{11}}{(x_{11}+x_{13})}}_{\substack{\text{Prob. that 2 is ranked first} \\ \text{and 1 is ranked second}}}. \quad (1)$$

The formulas for other contestants are symmetrically defined so that the probability that contestants i and j make it to the final, $\tilde{q}_{ij}$, is

$$\tilde{q}_{ij}(x_{11}, x_{12}, x_{13}) = \frac{x_{1i}}{\sum_{k=1}^{3} x_{1k}}\frac{x_{1j}}{\sum_{l\neq i} x_{1l}} + \frac{x_{1j}}{\sum_{k=1}^{3} x_{1k}}\frac{x_{1i}}{\sum_{l\neq j} x_{1l}}, \quad i,j,k,l \in \{1,2,3\}. \quad (2)$$

Given first-period efforts (x_{11}, x_{12}, x_{13}), contestant 1's probability of making it to the final, q_{11}, is the sum of $\tilde{q}_{12}$ and $\tilde{q}_{13}$. This can be rewritten as

$$q_{11}(x_{11}, x_{12}, x_{13}) = \underbrace{\frac{x_{11}}{x_{11}+x_{12}+x_{13}}}_{\substack{\text{Prob. of being} \\ \text{ranked number one}}} + \underbrace{\frac{x_{12}}{x_{11}+x_{12}+x_{13}}\frac{x_{11}}{x_{11}+x_{13}}}_{\substack{\text{Prob. that 2 wins} \\ \text{and 1 comes second}}} + \underbrace{\frac{x_{13}}{x_{11}+x_{12}+x_{13}}\frac{x_{11}}{x_{11}+x_{12}}}_{\substack{\text{Prob. that 3 wins} \\ \text{and 1 comes second}}}. \quad (3)$$

or more compactly as

$$q_{11}(x_{11}, x_{12}, x_{13}) = \frac{x_{11}}{\sum_{j=1}^{3} x_{1j}} + \sum_{k\neq 1}\frac{x_{1k}}{\sum_{j=1}^{3} x_{1j}}\frac{x_{11}}{\sum_{j\neq k} x_{1j}}, \quad i,j,k \in \{1,2,3\}. \quad (4)$$

Again, the formulas for the other contestants are symmetrically defined so that the probability of contestant i continuing to the second stage, q_{1i} can be written as

$$q_{1i}(x_{11}, x_{12}, x_{13}) = \frac{x_{1i}}{\sum_{j=1}^{3} x_{1j}} + \sum_{k\neq i}\frac{x_{1k}}{\sum_{j=1}^{3} x_{1j}}\frac{x_{1i}}{\sum_{j\neq k} x_{1j}}, \quad i,j,k \in \{1,2,3\}. \quad (5)$$

For notational simplicity we sometimes write (x_{11}, x_{12}, x_{13}) as (x_{1i}, x_{1-i}).

In the second, and final, stage, the two finalists compete to win the prize v. In PCP, the tender for the R&D contract is separated from the tender for the commercialization of the innovative product. Thus, the contest is for the value of the R&D contract, not the firm specific value of the innovation itself.

Given the two finalists, i and j, and their effort levels, (x_{2i}, x_{2j}), the probability of i winning is equal to

$$q_{2i}(x_{2i}, x_{2j}) = \frac{x_{2i}}{x_{2i} + x_{2j}}. \tag{6}$$

In the baseline model we ignore spillovers across periods. This is merely for expositional reasons since winning the first round in an innovation contest probably means that that contestant has a better idea or design and this should increase his chance of winning the entire contest. We account for this more realistic assumption in Section 4 and show that our results from the baseline model straightforwardly carry over to the setting with spillovers.

The solution concept is subgame perfect Nash equilibrium. Szidarovszky and Okuguchi (1997) ensures that we have existence of a unique equilibrium under these assumptions.

2.1 Benchmark without budget constraints

Without any budget concerns, in the final stage, the two finalists choose effort levels that solve

$$\max_{x_{2i}} v\frac{x_{2i}}{x_{2i} + x_{2j}} - x_{2i}. \tag{7}$$

The equilibrium effort level for each finalist is $x_2^{no} = \frac{v}{4}$. This is in fact the well-known n-player Tullock contest formula for a prize equal to v.[10] This yields a second-period expected payoff of $\Pi^{no} = \frac{v}{4}$.

Compared to a one-stage contest (where equilibrium effort would be equal to $\frac{2}{9}v$), the two-stage mechanism gives the finalists greater incentives to exert effort. This is a direct consequence of the well-known result in Tullock contests which says that the higher the number of contestants the lower the individual effort.

Given this equilibrium behaviour, each contestant i chooses first-period effort that solves

$$\max_{x_{1i}} q_{1i}(x_{1i}, x_{1-i})\Pi^{no} - x_{1i}. \tag{8}$$

Straightforward computations of first-order conditions yield an equilibrium level of first-stage effort $x_1^{no} = \frac{5}{18}\frac{v}{4}$. Since there are two ways of "winning" the first stage (coming first or second in the first stage), the standard n-player Tullock contest formula mentioned previously does not apply. Clark and Riis (1996) provides a generalisation of this formula to the many-prize setting.[11] This formula applies in our setting where the prizes for

[10] The general n-player formula is $x = \frac{n-1}{n^2}v$.

[11] The general n-contestant-k-prize formula in Clark and Riis (1996) is $x =$

coming first or second in the first-stage is Π^{no}, the expected payoff from the second stage. The individual first-stage effort is higher than the effort had there only been one first-period prize of value Π^{no}, but it is lower than the total effort from two contests each with a single prize of Π^{no}. This is a direct consequence of there being two ways of "winning" the first stage in our setting.

3 Two-stage contest with both budget-constrained and unconstrained contestants

To illustrate how asymmetric budget constraints influence contestants' incentives to choose effort x_{ti}, we limit the analysis to the case where one contestant's budget-constraint is indeed a relevant concern.[12] The case of two budget-constrained contestants is provided as an extension in Section 5.1 but does not qualitatively alter the results. To focus on the interesting case of a binding budget constraint, we assume in the following that the budget is less than the sum of total equilibrium effort when there are no budget constraint.

Assumption 1. *Contestants 1 and 2 face no budgetary restrictions whereas contestant 3's total budget, w_3, satisfies*

$$w_3 < w_3^{max} \equiv \left[1 + \frac{5}{18}\right] \frac{v}{4}. \tag{9}$$

This assumption implies that contestant 3 cannot choose first-period effort equal to $x_2^{no} = \frac{v}{4}$ and second-period effort equal to $x_1^{no} = \frac{5}{18}\frac{v}{4}$.

For completeness, we provide the solution to the alternative benchmark, in which all contestants are budget-constrained in Appendix B. However, this is less relevant for the analysis of settings in which SMEs are introduced into PCP contests where they will face larger firms.

Second stage: The equilibrium outcome of the final stage now depends on the identity of the finalists and on their budgets. If the two finalists are contestants 1 and 2, then the second-period effort is the same as in the benchmark, i.e., the symmetric second-stage effort is $x_{2,ij} = \frac{v}{4}$. However, if contestant 3 is one of the finalists, he can only exert effort

$\frac{1}{n}\left[\frac{k(n-1)}{n} - \sum_{j=1}^{k-1} \frac{k-j}{n-j}\right] V(k)$, where $V(k)$ is a contestant's valuation of winning one of the k prizes.

[12] As much as possible we use the notation x_{ti} for effort of contestant i in period t. Whenever it is necessary to clarify who the other contestant (in the second period) is, we extend the notation to $x_{2,ij}$.

up to the level $\min\{w_3 - x_{13}, \frac{v}{4}\}$. The following observation will be useful to obtain our results.

Lemma 1. *Contestant 3 will always be (weakly) budget constrained in the second stage, i.e.,*

$$w_3 - x_{13} \leq \frac{v}{4}. \tag{10}$$

Lemma 1 establishes that the budget constraint binds in the second period. This is a straightforward proof by contradiction. If this does not hold, then contestant 3 should spend more in the first-stage contest as this would increase his expected payoff and could still choose the first-best level of effort in the second stage. Using this observation, we can apply results from Che and Gale (1997) that shows that, in this budget-constrained case, it is indeed optimal for contestant 3 to spend his entire remaining budget in the final stage.

Lemma 2. *(Che and Gale (1997), Section 3.2) When contestant 3 is budget constrained in the final stage, it is optimal for him to spend his entire budget:*

$$x_{23} = w_3 - x_{13}. \tag{11}$$

Contestant 3 would like to spend $\frac{v}{4}$ in the final stage, but does not have enough budget. He therefore chooses to spend as much as he can to get as close as possible to this optimal, but unattainable, level of effort. Importantly, this constraint on x_{23} has an effect on the other finalist's choice of effort; their optimal response is given by the first-order condition associated with (7) but where $x_{23} = w_3 - x_{13}$. Thus, the optimal level of effort for an unconstrained contestant facing a budget-constrained contestant is $x_{2,i3} = \sqrt{v(w_3 - x_{13})} - (w_3 - x_{13})$. It is straightforward to check that $x_{2,i3} < x_{2,ij} = \frac{v}{4}$. The unconstrained contestant optimally bids less against the weaker contestant than against an equal competitor. Furthermore, to increase his probability of winning, an unconstrained contestant spends more than his constrained counterpart ($x_{2,i3} > x_{23}$) and the larger the budget of the constrained contestant, the more effort is exerted in the second-stage, by both contestants.

Contestant 3's expected payoff when entering the final stage is

$$\Pi_3(w_3 - x_{13}) = \sqrt{v(w_3 - x_{13})} - (w_3 - x_{13}). \tag{12}$$

When facing contestant 3 in the final stage, unconstrained contestant i's expected second-

period payoff is:[13]

$$\Pi_{i3}(w_3 - x_{13}) = \left(\sqrt{v(w_3 - x_{13})} - (w_3 - x_{13})\right)\left(\sqrt{\frac{v}{w_3 - x_{13}}} - 1\right), \tag{13}$$

Otherwise, it is $\Pi^{no} = \frac{v}{4}$.

The three levels of expected payoff in the final stage can be ordered as follows.

Lemma 3. *For all* $x_{13} \in (0, w_3)$,

$$\Pi_3(w_3 - x_{13}) \leq \frac{v}{4} \leq \Pi_{i3}(w_3 - x_{13}). \tag{14}$$

Contestant 3 is limited in his expenditures and can never earn as much as he could without this constraint. This gives us the first inequality. For an unconstrained contestant, facing a budget-constrained contestant weakens competition and, consequently, leads to higher expected payoffs than when facing an equal competitor.

First stage: Contestant 3 chooses first-period effort x_{13} that solves

$$\max_{x_{13} \leq w_3} q_{13}(x_{11}, x_{12}, x_{13})\Pi_3(w_3 - x_{13}) - x_{13}. \tag{15}$$

His best-response to contestants 1 and 2's symmetric[14] strategy x_{1i} is characterized in the following lemma.

Lemma 4. *Contestant 3's optimisation problem is concave and his best response* x_{13} *to the other contestants' (symmetric) strategy* x_{1i} *solves*

$$\left[\frac{2x_{1i}^2(3x_{1i} + 2x_{13})}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2}\right]\Pi_3(w_3 - x_{13}) = 1 + \frac{x_{13}}{2x_{1i} + x_{13}}\left(1 + \frac{2x_{1i}}{x_{1i} + x_{13}}\right)\left[\sqrt{\frac{v}{4\,(w_3 - x_{13})}} - 1\right]. \tag{16}$$

Contestant 3's best response equalises the marginal benefit from the first-stage effort to the marginal cost of exerting such effort. The marginal benefit is the (marginal) increase in expected payoff from a higher probability of winning. There are two costs associated with effort in the first stage. It involves the actual cost of effort, but it also affects the expected payoff in the final stage. A marginal increase in first-stage effort reduces

[13] The subscript indicates that i is facing contestant 3.

[14] By anticipation, we use the result from Lemma 5, that shows that contestants 1 and 2 choose symmetric strategies.

the final-stage effort because of the budget constraint. Less effort in the final stage is associated with a lower expected payoff.

For contestant $i \in \{1, 2\}$, the expected final-stage payoff depends on the identity of the other finalist. With our notation $\tilde{q}_{ij}$ for the probability that i and j make it to the final, the first stage problem becomes:

$$\max_{x_{1i}} \tilde{q}_{ij}(x_{1i}, x_{1-i})\frac{v}{4} + \tilde{q}_{i3}(x_{1i}, x_{1-i})\Pi_{i3}(w_3 - x_{13}) - x_{1i}. \tag{17}$$

Lemma 5. *Contestant i's optimisation problem is concave and his best response solves*

$$\frac{x_{1i}x_{13}(3x_{1i} + 2x_{13})}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2}\frac{v}{4} + \frac{x_{13}(4x_{1i} + x_{13})}{4x_{1i}(2x_{1i} + x_{13})^2}\Pi_{i3}(w_3 - x_{13}) = 1. \tag{18}$$

For each unconstrained contestant, the associated first-order condition also balances the marginal cost and benefit from effort. In this case, notice that the expected second-stage payoff depends on the identity of the second finalist. As pointed out in Lemma 3, facing a constrained finalist yields a higher expected second-period payoff. This observation allows us to show that first-stage equilibrium effort is higher for the unconstrained contestants than for contestant 3.

Proposition 1. *The unique equilibrium levels of effort, (x^*_{1i}, x^*_{13}), solve (16) and (18) and are such that*

$$x^*_{1i} > x^*_{13}. \tag{19}$$

The constrained contestant always exerts less effort than his unconstrained counterparts. This is because he has a limited budget and must stay within it. The two unconstrained contestants can exert more effort in order to improve their chances of getting to the final. This is illustrated in Figure 1.

Figure 1 also suggests a less straightforward result: first-stage effort by unconstrained contestants is not monotone in the strength of the weak contestant (as measured by w_3). The remainder of this section formally proves this non-monotonicity and shows that changes to the budget constraint, w_3, has two opposing effects on the first-stage effort by unconstrained contestants.

Proposition 2. *The effect of an increase in the constrained contestant's budget w_3 on*

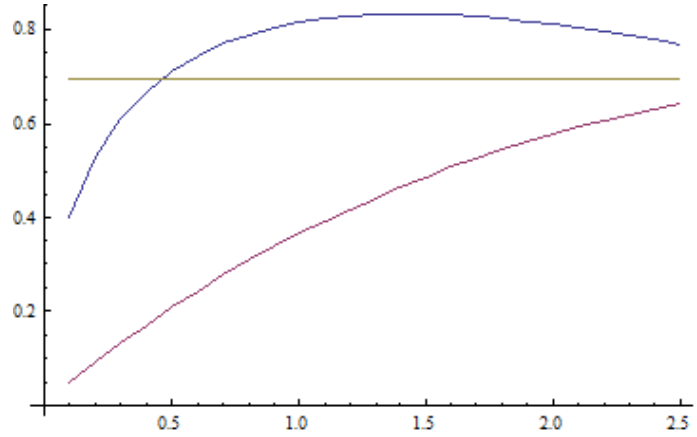

Figure 1: First-stage effort as a function of w_3 (x_{1i} in blue, x_{13} in red and x^{sym} in yellow) for $v = 10$.

the unconstrained contestants' effort can be decomposed a follows:

$$\frac{dx_{1i}^*}{dw_3} = \underbrace{\frac{\partial x_{1i}}{\partial x_{13}}\frac{\partial x_{13}}{\partial w_3}}_{\textit{Competition Effect}} + \underbrace{\frac{\partial x_{1i}}{\partial w_3}}_{\textit{Payoff Effect}} \tag{20}$$

- *A competition effect:*

$$\frac{\partial x_{1i}}{\partial x_{13}} \geq 0, \textit{ and } \frac{\partial x_{13}}{\partial w_3} \geq 0. \tag{21}$$

  An increase in w_3 increases the best response x_{13}, which in equilibrium increases x_{1i}.

- *A payoff effect:*

$$\frac{\partial x_{1i}}{\partial w_3} \leq 0. \tag{22}$$

  An increase in w_3 decreases the best response x_{1i}.

The intuition for Proposition 2 is as follows. First, when w_3 increases, as a response to a more *competitive* contestant 3, the unconstrained contestants increase their first-stage effort in order to ensure a high enough probability of making it to the final stage. This is the competition effect. In fact, when w_3 is close to zero, the unconstrained bidders need to exert very little effort to reach the final since competition from the constrained contestant is virtually non-existing. As the budget w_3 increases, more effort needs to be exerted as a response to competition. This can be seen in Figure 1 where first-stage effort is increasing for small enough w_3.

However, the first-stage effort also depends on whom they face in the final. This is because expected payoff in the final stage depends on the identity of the finalists. For the unconstrained contestant, the expected payoff when facing a budget-constrained contestant is decreasing in w_3. This means that as the budget w_3 increases the expected gains in the final stage decreases, thus making it less attractive to exert effort. This is the payoff effect. As suggested in Figure 1, for high enough values of w_3 the payoff effect is sufficiently negative so that first-stage effort x_{1i} is decreasing in w_3.

The following two corollaries confirm the result in Figure 1 and show that both effects are always relevant and may dominate for some values of w_3.

Corollary 1. *The competition effect dominates for small enough values of* w_3*:*

$$\lim_{w_3 \to 0} \frac{dx_{1i}^*}{dw_3} \geq 0. \tag{23}$$

When contestant 3 has a very small budget, the expected payoff from facing this contestant in a final is very high. The decrease in expected payoff from a small increase in contestant 3's budget is negligible. An unconstrained contestant will increase his effort as w_3 increases so as to optimise his chance of making it to the final stage of the contest.

Corollary 2. *The payoff effect dominates for high enough values of* w_3*:*

$$\lim_{w_3 \to w_3^{max}} \frac{dx_{1i}^*}{dw_3} \leq 0. \tag{24}$$

For high enough values of w_3, the expected payoff from facing a budget-constrained contestant in a final is not very different from the expected payoff from facing an equal competitor. It becomes less attractive to exert effort as the gains from facing a *disadvantaged* contestant decreases. Therefore x_{1i} decreases in w_3 for high enough values of w_3.

These two corollaries taken together establish the non-monotonicity of x_{1i} in w_3.

Corollary 3. x_{1i}^* *is non-monotone in* w_3*.*

This corollary suggests that when first-period effort is important for the contest designer, an interior value of the budget constraint may be preferable to no budget constraints as it gives unconstrained contestants an extra incentive via the larger gains they can expect when facing a weaker finalist. This, along with other design issues, are discussed in Section 5.3.

4 Spillovers between stages

In the previous section we provided results in the case where effort at one stage of the contest does not influence the outcome of other stages of the contest. However, in an innovation contest like PCP this might seem very restrictive and, perhaps, it is more reasonable to assume the winner of the initial stage has an advantage in terms of the idea or design that this contestant has produced and his chances of winning the entire contest has therefore increased. In this section we account for this more realistic assumption and show that our results from the baseline model straightforwardly carry over to the setting with spillovers.

In this section we follow Franke et al (2018) and assume that winning the first stage increases the winner's probability of winning the second stage[15]:

$$q_{2i}^{s}(x_{2i},x_{2j}) = \begin{cases} q_{2i}^{h}(x_{2i},x_{2j}) = \frac{x_{2i}+h}{x_{2i}+x_{2j}+h} & \text{if } i \text{ is the first-stage winner,} \\ q_{2i}^{nh}(x_{2i},x_{2j}) = \frac{x_{2i}}{x_{2i}+x_{2j}+h} & \text{if not.} \end{cases} \tag{25}$$

This definition of winning probabilities implies that for given (x_{2i}, x_{2j}), winning the first stage increases the winner's probability of winning the second stage relative to the benchmark case with no spillovers ($q_{2i}^{h}(x_{2i},x_{2j}) > q_{2i}(x_{2i},x_{2j})$), while losing the first stage, reduces it ($q_{2i}^{nh}(x_{2i},x_{2j}) < q_{2i}(x_{2i},x_{2j})$).[16] For the second-stage effort to still have some influence on the outcome of the contest we impose the following restriction on h:[17]

Assumption 2. $0 < h < \frac{v}{4}$.

In the benchmark case without budget-constraints it is straightforward to show that effort levels in the final stage are:

$$x_{2i}^{no} = \begin{cases} \frac{v}{4} - h & \text{if } i \text{ is the first-stage winner,} \\ \frac{v}{4} & \text{if not.} \end{cases} \tag{26}$$

In the second stage, both contestants equalise marginal benefit from exerting effort to marginal cost. However, due to the first-stage winner enjoying a head start, this contestant

[15]This way of modeling a head start is also very close to how a head start is modeled in all-pay auctions. For instance Kovenock and Roberson (2009) and Clark and Nilssen (2017) assume that the win probability in an all-pay auction depends on the comparison between effort levels and head start h in the following way: $x_{1i} + h \lesseqgtr x_{2j}$.

[16]The former is true as $\frac{\partial q_{2i}^{h}(x_{2i},x_{2j})}{\partial h} = \frac{x_{2j}}{(x_{2i}+x_{2j}+h)^2} > 0$. The latter is straightforward.

[17]With this assumption we rule out the uninteresting case where x_2 is optimally set to zero.

does not need to exert as much effort. These effort levels are associated with the following expected second-period payoffs

$$\Pi_i^{no} = \begin{cases} \Pi^h = \frac{v}{4} + h & \text{if } i \text{ is the first-stage winner,} \\ \Pi^{nh} = \frac{v}{4} & \text{if not.} \end{cases} \tag{27}$$

The lower optimal effort level for the contestant with the head start implies that he enjoys a higher second-period payoff, both relative to the contestant with no head start and relative to the benchmark with no spillovers (while the equilibrium probability of winning is the same in the three cases and equal to 1/2).

In the first stage without any budget constraints, we can also show that the possibility of obtaining a head start in the final gives contestants incentives to exert higher effort in the initial stage. Namely,

$$x_1^{no} = \frac{5}{18}\frac{v}{4} + \frac{2}{9}h. \tag{28}$$

As winning the first contest gives the contestant an advantage in the second-stage competition which results in an extra payoff, the contestants are willing to exert an extra effort in the first stage, relative to the benchmark case (where $x_1^{no} = \frac{5}{18}\frac{v}{4}$).

When contestant 3 faces a budget constraint, the same reasoning as in the previous section applies, but we have to take into account which contestant got a head start. Analogously to the case with no spillovers, we adopt the following assumption which ensures that h is small enough for the budget constraint in the final stage to always be an issue:

Assumption 3.

$$w_3 < \left[1 + \frac{5}{18}\right]\frac{v}{4} - \frac{7}{9}h. \tag{29}$$

The following lemma summarizes final-stage strategies x_{2j}.

Lemma 6. *Second-stage strategies depend on the identity of the finalist and who has a head start. These strategies are summarized as follows:*

- *For the constrained contestant,* $x_{23}^h = x_{23}^{nh} = w_3 - x_{13}$
- *For an unconstrained contestant* $i \in \{1, 2\}$*,*

– *facing the other unconstrained contestant:*

$$x_{2,ij} = \begin{cases} x_{2,ij}^{h} = \frac{v}{4} - h & \textit{if } i \textit{ is the first-stage winner,} \\ x_{2,ij}^{nh} = \frac{v}{4} & \textit{if not.} \end{cases} \tag{30}$$

– *facing the constrained contestant:*

$$x_{2,i3} = \begin{cases} x_{2,i3}^{h} = \sqrt{v(w_3 - x_{13})} - (w_3 - x_{13}) - h & \textit{if } i \textit{ is the first-stage winner,} \\ x_{2,i3}^{nh} = \sqrt{v(w_3 - x_{13} + h)} - (w_3 - x_{13}) - h & \textit{if not.} \end{cases} \tag{31}$$

Contestant 3 is budget constrained and can therefore not choose his optimal level of effort. He therefore chooses the highest admissible effort level given the budget constraint (which is the same regardless of whether he has a head start or not). As in the case without budget constraints, an unconstrained contestant with the head start optimally exerts a lower level of effort. It can be shown that, for a given x_{13}, when a contestant has the head start the optimal level of effort is lower relative to the benchmark case, while if he does not have the head start his optimal effort is higher: $x_{2,i3}^{h} < x_{2,i3} < x_{2,i3}^{nh}$.[18] Also, analogously to the benchmark with no spillovers, it holds that $x_{2,i3}^{h} < x_{2,ij}^{h}$ and $x_{2,i3}^{nh} < x_{2,ij}^{nh}$.

These second-stage strategies give rise to the following expected second-stage payoffs:

- For the constrained contestant:

$$\Pi_3 = \begin{cases} \Pi_3^{h} = \sqrt{v(w_3 - x_{13} + h)} - (w_3 - x_{13}) & \text{if 3 is the first-stage winner,} \\ \Pi_3^{nh} = \sqrt{v(w_3 - x_{13})} - (w_3 - x_{13}) & \text{if not.} \end{cases} \tag{32}$$

- For an unconstrained contestant $i \in \{1, 2\}$,

  – facing the other unconstrained contestant:

$$\Pi_{ij} = \begin{cases} \Pi_{ij}^{h} = \frac{v}{4} + h & \text{if } i \text{ is the first-stage winner,} \\ \Pi_{ij}^{nh} = \frac{v}{4} & \text{if not.} \end{cases} \tag{33}$$

[18] The first inequality is straightforward and the latter is true as $\frac{\partial x_{2,i3}^{nh}}{\partial h} = \sqrt{\frac{v}{4(w_3 - x_{13} + h)}} - 1 > 0$.

– facing the constrained contestant:

$$\Pi_{i3} = \begin{cases} \Pi_{i3}^{h} = \Pi_{3}^{nh}\left(\sqrt{\frac{v}{w_3 - x_{13}}} - 1\right) + h & \text{if } i \text{ is the first-stage winner,} \\ \Pi_{i3}^{nh} = \left(\Pi_{3}^{h} - h\right)\left(\sqrt{\frac{v}{w_3 - x_{13} + h}} - 1\right) & \text{if not.} \end{cases} \tag{34}$$

It is straightforward to check that a contestant's expected payoff against a given opponent is always higher when the contestant in question has a head start. This result is obvious since a head start in an innovation contest is always beneficial. Despite exerting the same effort regardless whether he has the head start or not, the constrained contestant has a higher chance of winning in case of a head start, which results in a higher expected payoff relative to the benchmark with no spillovers: $\Pi_3^h > \Pi_3^{nh} = \Pi_3(w_3 - x_{13})$. A similar result holds for an unconstrained meeting another unconstrained contestant $\Pi_{ij}^h > \Pi_{ij}^{nh} = \Pi_{ij}$, while when he meets a constrained contestant $\Pi_{i3}^h > \Pi_{i3}(w_3 - x_{13}) > \Pi_{i3}^{nh}$ (where $\Pi_{i3}(w_3 - x_{13})$ is defined by (13)).[19]

The following lemma extends the comparison of expected payoffs along the same lines as Lemma 3.

Lemma 7. *For all $x_{13} \in (0, w_3)$, expected payoffs can be ordered as follows:*

$$\Pi_3^h(w_3 - x_{13}) < \Pi_{ij}^h < \Pi_{i3}^h(w_3 - x_{13}), \tag{35}$$

and

$$\Pi_3^{nh}(w_3 - x_{13}) < \Pi_{ij}^{nh} \leq \Pi_{i3}^{nh}(w_3 - x_{13}). \tag{36}$$

In terms of expected payoff, a contestant with a head start has the highest payoff if he is unconstrained and faces a constrained contestant. But a favored contestant facing an equal contestant also has a higher expected payoff than a favored constrained contestant. The same payoff ranking holds for the unfavored contestant.

These rankings of expected payoffs, along with the best responses from the first stage give us the equivalent to Proposition 1 in the presence of spillovers.

Proposition 3. *The unique equilibrium levels of first-stage effort in the presence of spillovers, $(x_{1i}^{s*}, x_{13}^{s*})$, are such that*

$$x_{1i}^{s*} > x_{13}^{s*}. \tag{37}$$

[19]The latter inequality is true as $\frac{\partial \Pi_{i3}^{nh}}{\partial h} = -\left(\sqrt{\frac{v}{w_3 - x_{13} + h}} - 1\right) < 0$ and $\Pi_{i3}^{nh}|_{h=0} = \Pi_{i3}(w_3 - x_{13})$.

Furthermore, in the presence of spillovers we can still decompose the effect of changes in the constrained contestant's budget as in the model without spillover and the signs of the effects remain unchanged.

Proposition 4. *In the presence of spillovers in the form of a head start to the winner of the first stage, the effect of an increase in the constrained contestant's budget w_3 on the unconstrained competitors' effort can still be decomposed as follows:*

$$\frac{dx_{1i}^{s*}}{dw_3} = \underbrace{\frac{\partial x_{1i}}{\partial x_{13}}\frac{\partial x_{13}}{\partial w_3}}_{Competition\ Effect} + \underbrace{\frac{\partial x_{1i}}{\partial w_3}}_{Payoff\ Effect} \tag{38}$$

- *A competition effect:*

$$\frac{\partial x_{1i}}{\partial x_{13}} \geq 0, \text{ and } \frac{\partial x_{13}}{\partial w_3} \geq 0. \tag{39}$$

  An increase in w_3 increases the best response x_{13}, which in equilibrium increases x_{1i}.

- *A payoff effect:*

$$\frac{\partial x_{1i}}{\partial w_3} \leq 0. \tag{40}$$

  An increase in w_3 decreases the best response x_{1i},

An increase in the budget of the constrained contestant on the one hand raises the optimal effort of the unconstrained contestant as a response to the increase in competitiveness of the latter (competition effect). On the other hand it lowers the optimal effort since the expected payoff of competing against the constrained in the second stage (which depends negatively on w_3) is lower (payoff effect).

Also, we can show that the following result holds:

Proposition 5. *The optimal first-stage effort of both constrained and unconstrained contestants is higher than in the case with no spillovers*

$$x_{1i}^{s*} > x_{1i}^{*}, \quad x_{13}^{s*} > x_{13}^{*}. \tag{41}$$

5 Discussion

5.1 Two budget-constrained contestants

To show that the results in the main model do not depend on just having one budget-constrained firm, we now focus on the case where two players are (symmetrically) budget constrained, such that $w_2 = w_3 = w$. Again, to focus on the interesting case of binding budget constraints we assume that w satisfies Assumption 1.

Second stage: The equilibrium outcome of the final stage again depends on the identity of the finalists and on their budgets. If the two finalists are 2 and 3, their second period effort would be their remaining budget, as in the benchmark with symmetric budgets i.e., $x_{2i} = w - x_{1i}$, $i = 2, 3$ (See Appendix B). On the other hand, if the unconstrained player 1 is one of the finalists, he will optimally exert effort as in Section 3:

$$x_{21} = \sqrt{v(w - x_{1i})} - (w - x_{1i}). \tag{42}$$

Second-period expected payoffs are therefore as follows:

- if the unconstrained contestant 1 enters the final stage and faces contestant i with remaining budget $w - x_{1i}$, his expected payoff is (the same as in (13))

$$\Pi_1(w - x_{1i}) = \left(\sqrt{v(w - x_{1i})} - (w - x_{1i})\right)\left(\sqrt{\frac{v}{w - x_{1i}}} - 1\right), \tag{43}$$

  where $i = 2, 3$.

  Notice that because contestant 1 is not budget constrained, his second-period payoff does not directly depend on his first-period choice of effort.

- if constrained contestant $i = 2, 3$ enters the final stage, his expected payoff is

$$\Pi_i(w - x_{1i}) = \begin{cases} \sqrt{v(w - x_{1i})} - (w - x_{1i}) & \text{if facing contestant 1,} \\ \frac{v}{2} - (w - x_{1i}) & \text{otherwise.} \end{cases} \tag{44}$$

For notational reasons we define

$$\Pi_{i1}(w - x_{1i}) = \sqrt{v(w - x_{1i})} - (w - x_{1i}). \tag{45}$$

$$\Pi_{ij}(w - x_{1i}) = \frac{v}{2} - (w - x_{1i}). \tag{46}$$

The ranking of expected payoff levels in the final stage is stated in the following lemma

Lemma 8. *For all* $x_{1i} \in (0, w)$,

$$\Pi_{i1} \leq \Pi_{ij} \leq \Pi_1(w - x_{1i}). \tag{47}$$

Intuitively, a constrained contestant expects a larger second-period payoff from competing against another constrained player than against an unconstrained one. This gives the first inequality. An unconstrained contestant gets a higher expected payoff than an unconstrained competitor regardless of competition.

First stage: Turning to the first stage of the contest, we focus on the case where the two constrained contestants play symmetric equilibrium strategies.[20]

Contestant 1 chooses first-period effort x_{11} to maximize[21]:

$$q_{11}(x_{11}, x_{1i}, x_{1i})\Pi_1(w - x_{1i}) - x_{11}. \tag{48}$$

The following lemma characterizes his best response against contestants 2 and 3' symmetric strategy x_{1i}:

Lemma 9. *Contestant 1's best response,* x_{11}*, solves*

$$\Pi_1(w - x_{1i}) \left[\frac{2x_{1i}(x_{11} + x_{1i})^2 - 2x_{1i}x_{11}(x_{11} + x_{1i}) + 2x_{1i}^2(x_{11} + 2x_{1i})}{(x_{11} + 2x_{1i})^2(x_{11} + x_{1i})^2} \right] = 1. \tag{49}$$

Again, the unconstrained contestant's optimal effort equalises the marginal cost of effort and the marginal benefit of effort in terms of increased probability of making it to the final stage.

Simultaneously, for $i, j = \{2, 3\}$, $j \neq i$, constrained contestant i chooses effort x_{1i} to solve:

$$\begin{array}{ll} \max\limits_{x_{1i}} & \tilde{q}_{ij}(x_{1i}, x_{1-i})\Pi_{ij}(w - x_{1i}) + \tilde{q}_{i1}(x_{1i}, x_{1-i})\Pi_{i1}(w - x_{1i}) - x_{1i} \\ \text{subject to} & x_{1i} \leq w. \end{array} \tag{50}$$

[20] This can easily be shown to be equilibrium behaviour.

[21] In the statement of expected payoff of contestant 1 we have replaced x_{12} and x_{13} by x_{1i} to simplify notations. Since this simplification is not on contestant 1 strategy x_{1i} it is without mathematical consequence, but greatly simplifies the notation.

The symmetric best response of constrained contestants 2 and 3 to unconstrained contestant 1 is characterized in the following lemma.

Lemma 10. *Contestants 2 and 3's best response, x_{1i}, solves*

$$\frac{2x_{1i}^2}{(x_{11}+2x_{1i})(x_{11}+x_{1i})}+\Pi_{ij}(w-x_{1i})\left[\frac{x_{1i}(x_{11}+x_{1i})^2-x_{1i}^2(x_{11}+x_{1i})+x_{11}x_{1i}(x_{11}+2x_{1i})}{(x_{11}+2x_{1i})^2(x_{11}+x_{1i})^2}\right]$$
$$+\Pi_{i1}(w-x_{1i})\left[\frac{2x_{1i}x_{11}+(x_{11}+2x_{1i})x_{11}}{(x_{11}+2x_{1i})^2 4x_{1i}}\right]=1+\left(\sqrt{\frac{v}{4(w-x_{1i})}}-1\right)\left[\frac{2x_{11}x_{1i}+x_{11}(x_{11}+x_{1i})}{2(x_{11}+2x_{1i})(x_{11}+x_{1i})}\right] \tag{51}$$

The interpretation is analogous to the one of Lemma 4, namely the best response of the constrained contestants equalises the marginal benefit of an increase of effort, in terms of increased probability of winning (left-hand side) and its marginal cost, in terms of actual cost (first term on the right-hand side) plus decrease in the expected payoff of the second stage due to spending a lower effort in the second stage (second term on the right-hand side).

The additional effect in Lemma 10 complicates calculations of the equilibrium, but as can be seen from the illustration in the following figure, equilibrium effort in the first stage of the contest is similar to the case of one budget-constrained contestant and two unconstrained contestants.

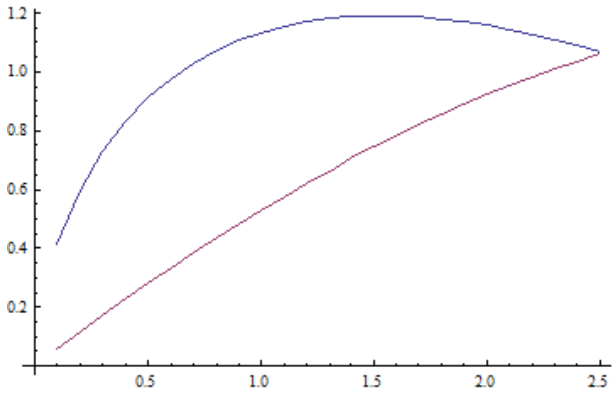

Figure 2: First-stage effort as a function of w (x_{11} in blue and x_{1i} in red) for $v=10$.

As before, a contestant without a budget constraint can clearly exert more first-stage effort and increase, his chances of making it to the final stage. With only one unconstrained contestant, there is no incentive to choose your fellow-finalist as both other

contestants are symmetric and budget-constrained. However, as the budget of these contestants increase the unconstrained contestant still decreases his first-stage effort. This is now solely to give the budget-constrained contestants incentives to exert more effort in the first stage and thus become less fierce in the second stage.

In a more general framework with a larger number of contestants, we would have all the effects described in this section as well as different levels of second-period payoff depending on the type of the finalists as in Lemma 5. However, this extension is beyond the scope of this paper and does not add something fundamentally new to our main insights.

5.2 Non-observability of finalist identity

In our baseline model, the identities of the finalists are common knowledge. That implies that each contestant in the final knows the type (budget-constrained or not) of his competitor and can fine-tune his second-period strategy to this information. That is why the second-stage effort by an unconstrained bidder is higher when facing an equal contestant compared to when he faces a budget-constrained finalist. This is in line with the Principle of Transparency in European public procurement. From a theoretical perspective, it is nevertheless interesting to study how incentives to exert effort change when the identity of the finalists is not common knowledge.

For a budget-constrained finalist, nothing changes as he knows that he faces one of the two unconstrained contestants and with a limited budget it will still be optimal to spend it all. The following proposition compares the optimal strategy of an unconstrained contestant who does not know whether he faces another unconstrained, and thus equal, contestant or a weak, budget-constrained contestant.

Proposition 6. *When the identity of the finalists remains secret, an unconstrained finalist chooses effort x_2^u between the levels of effort he would have chosen had he known the identity of his competitor:*

$$x_{2,i3} \leq x_2^u \leq x_2^{no}.$$

Since an unconstrained finalist takes into account the probability of the type of the other finalist, his best-response lies between the strategy he would choose when facing an equal finalist for sure and the one chosen when facing a budget-constrained finalist for sure. The exact level of this second-stage effort, depends on the beliefs about the identity of the competitor. This in turn depends on effort in the first-stage as they determine the likelihood of each contestant making it to the final stage.

Turning to expected second-stage payoffs, we get a similar ranking as in the previous cases. The constrained contestant cannot expect to gain as much as had he been unconstrained and the constrained contestant makes more than had he faced an equally constrained contestant.

Lemma 11. *Denote the expected second-stage payoff of a budget-constrained contestant* $\tilde{\Pi}_3$ *and the expected second-stage payoff of an unconstrained contestant* $\tilde{\Pi}_i$*. These expected payoffs can be ranked in the following way:*

$$\tilde{\Pi}_3 \leq \frac{v}{4} \leq \tilde{\Pi}_i.$$

In the first stage of the contest, the budget-constrained contestant still solves the same maximization problem as in the baseline model where finalists' types are observable, except that expected payoff from the final is lower since the other finalist exerts more effort.

However, the first-stage choice of effort for the unconstrained bidders gets more complicated. The expected second-stage payoff now depends on the beliefs about the type of the other finalist. The first-order condition does not simply equalise the marginal change in probability of winning multiplied by the expected second-stage payoff to marginal cost. It also takes into account how a change in first-stage effort changes the beliefs about the other finalist and thus second-stage expected payoff.

5.3 Design issues

While an in-depth analysis of design issues is beyond the scope of this paper, in this section we provide some illustrations that the design of PCP presented in our model (i.e., two stages, final prize only and asymmetric contestants) may perform well for some objectives of the principal.

We start by showing that the presence of a budget-constrained contestant may induce higher total effort in the first stage of the contest relative to the symmetric case with no constraints. As it is possible to see in Figure 3, provided the budget asymmetry is not too large (i.e., w_3 is high enough) total first-stage effort is larger in the asymmetric contest. This is the case because for w_3 high enough, the two unconstrained contestants' additional effort (relative to the symmetric case) more than compensates for contestant 3's reduced effort (see Figure 4 for the difference between individual effort in our main model and the benchmark). The illustration suggests that if the principal is concerned

with the total first-stage effort, then small budget constraints is in fact not a concern, but rather beneficial.

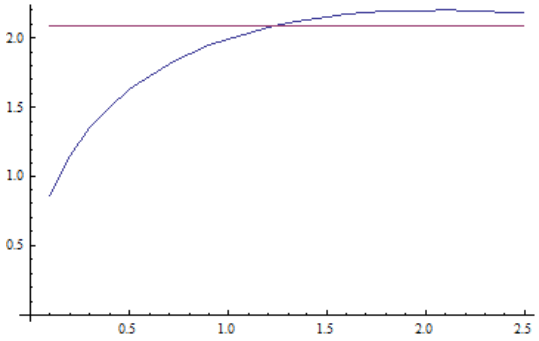

Figure 3: Total effort in the first stage as a function of w_3: asymmetric case ($2x_{1i}+x_{13}$, in blue) vs symmetric case ($3x_1^{no}$, in red) for $v = 10$.

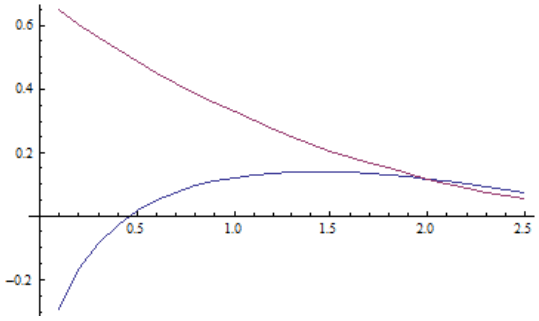

Figure 4: Contestant i' s additional effort ($x_{1i} - x_1^{no}$, in blue) compensating for 3' s reduced effort ($x_1^{no} - x_{13}$, in red) in the first stage, as a function of w_3, for $v = 10$.

Second, we show that the expected level of total effort for the unconstrained player is higher relative to the symmetric case. As shown in Figure 5 the unconstrained player exerts an overall larger expected effort in the asymmetric case than in the symmetric case. This is mainly because expected second-stage effort is higher in the asymmetric case relative to the symmetric case. Although the actual level of effort in the asymmetric case is lower or equal to the one in the symmetric case, the likelihood of making it to the second stage is much higher. For w_3 high enough, the first-stage effort is also higher. However this level of effort is much smaller than the second-stage level[22] and the lower level of first-stage effort for low w_3 does not outweigh the positive effect of the increased expected second-stage effort. This result suggests that a principal who expects the unconstrained contestants to win and therefore cares about their expected effort throughout the contest should not be worried about introducing weaker contestants and thus making the contest asymmetric.

The principal could of course have other objectives, including minimizing costly duplication of effort. Our main point is to show that introducing asymmetries into an all-against-all elimination contest has a non-monotone effect on effort and provide some casual evidence that this might also be beneficial to the principal. A deeper discussion of the design issues and what the principal's objective is, or should be, is beyond the scope of this paper.

[22]Recall that in the symmetric case $x_2^{no} = 2.5$ and $x_1^{no} = \frac{25}{36} < 1$.

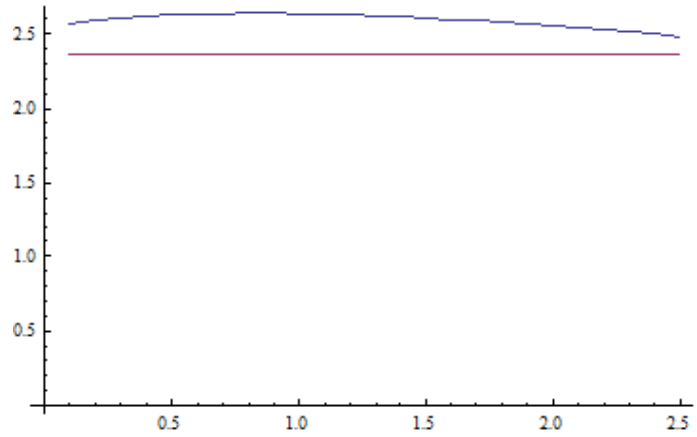

Figure 5: Sum of efforts for the unconstrained player in the asymmetric case ($x_{1i} + E[x_{2i}]$, in blue) vs symmetric case ($x_1^{no} + E[x_2^{no}]$, in red), as a function of w_3, for $v = 10$.

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Appendix A

Proof of Lemma 1. Assume that $w_3 - x_{13} > \frac{v}{4}$. This implies that contestant 3's budget constraint is not binding and he can choose is optimal level of effort $x_2^{no} = \frac{v}{4}$ in the final stage and still have some budget left. Denote this budget $\epsilon \equiv w_3 - x_{13} - \frac{v}{4}$.

By Assumption 1 the total budget does not allow the contestant to choose his unconstrained, optimal level of effort in both periods. That means that $x_{13} < x_1^{no}$. However, by spending the budget ϵ in the first stage, contestant 3 can still choose the optimal level of effort $x_2^{no} = \frac{v}{4}$ in the second stage. But doing so increases his first-period expected utility since $q_{13}(x_{13}, x_{1-3})\Pi^{no} - x_{13}$ reaches its maximum at x_1^{no}. Thus, we necessarily have $w_3 - x_{13} \leq \frac{v}{4}$. □

Proof of Lemma 3. The assumption $w_3 \leq \frac{v}{4}$ implies that $w_3 - x_{13}$ is also less than $\frac{v}{4}$.

To prove that $\frac{v}{4} \leq \Pi_{i3}(w_3 - x_{13})$, it suffices to prove that $\frac{d\Pi_{i3}(w_3 - x_{13})}{d(w_3 - x_{13})} < 0$ and that $\Pi_{i3}(\frac{v}{4}) \geq \frac{v}{4}$. Since $w_3 - x_{13} \leq \frac{v}{4}$, $\Pi_{i3}(\frac{v}{4})$ gives a lower bound for expected payoff with a budget-constraint w_3.

Differentiating (13) with respect to $w_3 - x_{13}$ and simplifying yields $\frac{d\Pi_{i3}(w_3 - x_{13})}{d(w_3 - x_{13})} = -\left(\sqrt{\frac{v}{w_3 - x_{13}}} - 1\right) < 0$. Furthermore, $\Pi_{i3}\left(\frac{v}{4}\right) = \frac{v}{4}$ and we can conclude that $\frac{v}{4} \leq \Pi_{i3}(w_3 - x_{13})$.

The proof of $\Pi_3(w_3 - x_{13}) \leq \frac{v}{4}$ follows in the same way. In fact when $w_3 - x_{13} \leq \frac{v}{4}$, $\frac{d\Pi_3(w_3 - x_{13})}{d(w_3 - x_{13})} \geq 0$. It can easily be checked that $\Pi_3\left(\frac{v}{4}\right) = \frac{v}{4}$ and we can conclude that $\Pi_3(w_3 - x_3) \leq \frac{v}{4}$. □

Proof of Lemma 4. The first-order condition associated with contestant 3's maximization problem is

$$-1 + \left[\frac{2x_{1i}}{(2x_{1i} + x_{13})^2} - \frac{2x_{1i}x_{13}}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})} + \frac{2x_{1i}^2}{(2x_{1i} + x_{13})(x_{1i} + x_{13})^2}\right]\Pi_3(w_3 - x_{13}) - \frac{x_{13}}{2x_{1i} + x_{13}}\left(1 + \frac{2x_{1i}}{x_{1i} + x_{13}}\right)\left[\sqrt{\frac{v}{4\,(w_3 - x_{13})}} - 1\right] = 0. \tag{A.1}$$

This can easily be rearranged to give (16).

Denoting this derivative as $LHS_3(x_3)$ gives

$$LHS_3(x_{13}) = -1 + \left[\frac{2x_{1i}^2(3x_{1i} + 2x_{13})}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2}\right]\Pi_3(w_3 - x_{13}) - \frac{x_{13}}{2x_{1i} + x_{13}}\left(1 + \frac{2x_{1i}}{x_{1i} + x_{13}}\right)\left[\sqrt{\frac{v}{4\,(w_3 - x_{13})}} - 1\right]. \tag{A.2}$$

To ensure concavity of the maximization problem, we need $\frac{\partial LHS_3}{\partial x_{13}} \leq 0$:

$$\begin{aligned}\frac{\partial LHS_3}{\partial x_{13}} =& -\left(\sqrt{\frac{v}{4(w_3-x_{13})}}-1\right)\left(\frac{2x_{1i}^2(3x_{1i}+2x_{13})}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})^2}\right) \\ &- \frac{2x_{1i}^2(3x_{1i}+2x_{13})}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})^2}\left(\sqrt{\frac{v}{4(w_3-x_{13})}}-1\right) - \frac{x_{13}\left(1+\frac{2x_{1i}}{x_{1i}+x_{13}}\right)}{2(w_3-x_{13})(2x_{1i}+x_{13})}\sqrt{\frac{v}{4(w_3-x_{13})}} \\ &+ \frac{\Pi_3(w_3-x_{13})}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\left[-28x_{1i}^4-36x_{1i}^3x_{13}-12x_{1i}^2x_{13}^2\right] \leq 0.\end{aligned} \tag{A.3}$$

□

Proof of Lemma 5. The first-order condition associated with contestant i's optimization problem (where after derivation we have used $x_{11} = x_{12} = x_{1i}$) is

$$\begin{aligned}&-1+\left[\frac{x_{13}x_{1i}}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}+\frac{x_{13}x_{1i}}{(2x_{1i}+x_{13})(x_{1i}+x_{13})^2}\right]\frac{v}{4} \\ &+\left[\frac{(x_{1i}+x_{13})x_{13}}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}-\frac{x_{13}x_{1i}}{(2x_{1i}+x_{13})^2 2x_{1i}}+\frac{x_{13}x_{1i}}{(2x_{1i}+x_{13})(2x_{1i})^2}\right]\Pi_{i3}=0.\end{aligned} \tag{A.4}$$

This can be rearranged as (18).
It is straightforward to check that the second-order condition (obtained before using $x_{11} = x_{12}$) holds. □

Proof of Proposition 1. The concavity of the contestants' first-stage optimisation problem as shown in the proofs of Lemma 4 and 5 ensures the existence and uniqueness of the equilibrium following arguments in Szidarovszky and Okuguchi (1997).

Define the left-hand side of (A.4) as $LHS(x_{1i})$:

$$LHS(x_{1i}) = \frac{x_{1i}x_{13}(3x_{1i}+2x_{13})}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})^2}\frac{v}{4}+\frac{x_{13}(4x_{1i}+x_{13})}{4x_{1i}(2x_{1i}+x_{13})^2}\Pi_{i3}(w_3-x_{13})-1. \tag{A.5}$$

Differentiating this yields

$$\frac{\partial LHS(x_{1i})}{\partial x_{1i}} = \frac{x_{13}(-12x_{1i}^3-12x_{1i}^2x_{13}+2x_{13}^3)}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\frac{v}{4}+\frac{x_{13}(-16x_{1i}^2-6x_{1i}x_{13}-x_{13}^2)}{4x_{1i}^2(2x_{1i}+x_{13})^3}\Pi_{i3}(w_3-x_{13}). \tag{A.6}$$

Since from Lemma 3 $\Pi_{i3}(w_3 - x_{13}) \geq \frac{v}{4}$, we get:

$$\begin{aligned}\frac{\partial LHS(x_{1i})}{\partial x_{1i}} \leq& \frac{-12x_{13}(x_{1i}^3+x_{1i}^2x_{13})}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\frac{v}{4} \\ &- \frac{x_{13}\Pi_{i3}(w_3-x_{13})(16x_{1i}^5+54x_{1i}^4x_{13}+67x_{1i}^3x_{13}^2+29x_{1i}^2x_{13}^3+9x_{1i}x_{13}^4+x_{13}^5)}{4x_{1i}^2(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3},\end{aligned} \tag{A.7}$$

and we can conclude that $\frac{\partial LHS(x_{1i})}{\partial x_{1i}} < 0$ (since $x_{1i} > 0$ and $x_{13} > 0$).

Since $LHS(x_{1i})$ is strictly decreasing and $LHS(x^*_{1i}) = 0$, it suffices to prove that $LHS(x^*_{13}) > 0$.

From Lemma 3, we have both $\Pi_3(w_3 - x_{13}) < \frac{v}{4}$ and $\Pi_3(w_3 - x_{13}) < \Pi_{i3}(w_3 - x_{13})$. This implies:

$$LHS(x_{1i}) \geq -1 + \left[\frac{x_{1i}x_{13}(3x_{1i} + 2x_{13})}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2} + \frac{x_{1i}x_{13}(4x_{1i} + x_{13})}{4x_{1i}^2(2x_{1i} + x_{13})^2} \right] \Pi_3(w_3 - x_{13}). \tag{A.8}$$

Using the definition of LHS_3 we can write:

$$LHS(x_{1i}) > LHS_3(x_{13}) + \frac{\Pi_3(w_3 - x_{13})}{4x_{1i}^2(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2} \left[17x_{1i}^3x_{13}^2 + 6x_{1i}^2x_{13}^3 + x_{1i}x_{13}^4 - 24x_{1i}^5 \right]. \tag{A.9}$$

Assume that $x_{13} > x_{1i}$. This implies that the last bracket is positive and we get a contradiction (because $LHS(x_{1i}) = 0 = LHS_3(x_{13})$). □

Proof of Proposition 2. The proof of Proposition 2 relies on a series of derivatives of LHS and LHS_3:

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$$\frac{\partial LHS}{\partial w_3} = -\frac{x_{13}(4x_{1i} + x_{13})}{4x_{1i}(2x_{1i} + x_{13})^2} \left(\sqrt{\frac{v}{w_3 - x_{13}}} - 1 \right) < 0. \tag{A.10}$$

· Straightforward differentiation and simplification yield:

$$\begin{aligned} \frac{\partial LHS}{\partial x_{13}} =& \frac{v}{4(2x_{1i} + x_{13})^3(x_{1i} + x_{13})^3} \left[6x_{1i}^4 - x_{1i}^3x_{13} - 9x_{1i}^2x_{13}^2 - 4x_{1i}x_{13}^3 \right] \\ &+ \frac{2x_{1i}\Pi_{i3}(w_3 - x_{13})}{(2x_{1i} + x_{13})^3} + \frac{x_{13}(4x_{1i} + x_{13})}{4x_{1i}(2x_{1i} + x_{13})^2} \left(\sqrt{\frac{v}{w_3 - x_{13}}} - 1 \right). \end{aligned} \tag{A.11}$$

The last term is positive and since from Lemma 3 we have $\frac{v}{4} \leq \Pi_{i3}(w_3 - x_{13})$:

$$\frac{\partial LHS}{\partial x_{13}} \geq \frac{v}{4(2x_{1i} + x_{13})^3(x_{1i} + x_{13})^3} \left[8x_{1i}^4 + 5x_{1i}^3x_{13} - 3x_{1i}^2x_{13}^2 - 2x_{1i}x_{13}^3 \right]. \tag{A.12}$$

Since $x_{1i} \geq x_{13} > 0$, we can conclude that $\frac{\partial LHS}{\partial x_{13}} > 0$.

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$$\begin{aligned} \frac{\partial LHS_3}{\partial w_3} =& \left(\frac{2x_{1i}^2(3x_{1i} + 2x_{13})}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2} \right) \left(\sqrt{\frac{v}{4(w_3 - x_{13})}} - 1 \right) \\ &+ \frac{x_{13}}{2x_{1i} + x_{13}} \left(1 + \frac{2x_{1i}}{x_{1i} + x_{13}}\right) \frac{1}{2(w_3 - x_{13})} \sqrt{\frac{v}{4(w_3 - x_{13})}} > 0. \end{aligned} \tag{A.13}$$

Recall that from previous proofs, we also have $\frac{\partial LHS}{\partial x_{1i}} < 0$ and $\frac{\partial LHS_3}{\partial x_{13}} \leq 0$. The payoff effect is obtained by differentiating (18) with respect to x_{1i} and w_3. This yields:

$$\frac{\partial x_{1i}}{\partial w_3} = -\frac{\frac{\partial LHS}{\partial w_3}}{\frac{\partial LHS}{\partial x_{1i}}} \leq 0. \tag{A.14}$$

The competition effect is obtained in two steps. First from (16) we get

$$\frac{\partial x_{13}}{\partial w_3} = -\frac{\frac{\partial LHS_3}{\partial w_3}}{\frac{\partial LHS_3}{\partial x_{13}}} \geq 0. \tag{A.15}$$

Finally, from (18) we get

$$\frac{\partial x_{1i}}{\partial x_{13}} = -\frac{\frac{\partial LHS}{\partial x_{13}}}{\frac{\partial LHS}{\partial x_{1i}}} \geq 0. \tag{A.16}$$

□

Proof of Corollary 1. From the proof of Proposition 2 we know that the payoff effect is measured by

$$\frac{\partial x_{1i}}{\partial w_3} = -\frac{\frac{\partial LHS}{\partial w_3}}{\frac{\partial LHS}{\partial x_{1i}}}. \tag{A.17}$$

Its value, compared to the competition effect depends on $\frac{\partial LHS}{\partial w_3}$ (since both effects are *normalized*, i.e., divided by $\frac{\partial LHS}{\partial x_{1i}} > 0$.).

Notice that for any equilibrium values (x_{13}, x_{23}), we can write $x_{13} = \alpha w_3$ and $x_{23} = (1-\alpha)w_3$ for some $\alpha \in (0,1)$. We can use this to write (A.10) as

$$\frac{\partial LHS}{\partial w_3} = -\frac{\sqrt{\frac{\alpha}{1-\alpha}w_3}(4x_{1i} + \alpha w_3)(\sqrt{v} - \sqrt{(1-\alpha)w_3})}{4x_{1i}(2x_{1i} + \alpha w_3)^2} \tag{A.18}$$

Even if the value of α varies with w_3, it always remains bounded within $(0,1)$. It is thus straightforward to conclude that

$$\lim_{w_3 \to 0} \frac{\partial LHS}{\partial w_3} = 0. \tag{A.19}$$

For small enough w_3, the payoff effect becomes negligible and, using Proposition 2 we can conclude that $\lim_{w_3 \to 0} \frac{dx_{1i}}{dw_3} \geq 0$. □

Proof of Corollary 2. Recall from the proof of Lemma 3 that for $w_3 < w_3^{max}$, we have $\Pi_{13}(x_{23}) < \frac{v}{4}$. Thus, we can obtain the following inequality from the first-order condition with respect to

x_{1i} defined in (A.5)

$$LHS(x_{1i}) > \left[\frac{x_{1i}x_{13}(3x_{1i}+2x_{13})}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})^2} + \frac{x_{1i}x_{13}(4x_{1i}+x_{13})}{4x_{1i}^2(2x_{1i}+x_{13})^2}\right]\frac{v}{4} - 1. \tag{A.20}$$

In the limit when w_3 is close to w_3^{max}, it is easily checked that x_{1i} goes to x_1^{no} and the right-hand side of the above inequality goes to $\frac{5}{18x_1^{no}}\frac{v}{4} - 1$; This is the first-order condition for the symmetric solution x_1^{no}. Since both problems are concave and the first-order conditions are continuous functions in all their arguments, we can conclude that there always exists an interval $(\hat{w}_3, w_3^{max})$ where $x_{1i} > x_1^{no}$. Since $lim_{\{w_3 \to w_3^{max}\}} x_{1i} = x_1^{no}$, this means that x_{1i} needs to be strictly decreasing on this interval. With continuity of x_{1i}, we can finally conclude that $\frac{dx_{1i}}{dw_3} < 0$ on $(\hat{w}_3, w_3^{max})$ and thus the payoff effect dominates for large enough budgets. □

Proof of Corollary 3. This is a direct consequence of Corollaries 1 and 2. □

Proof of Lemma 6. When the budget-constrained contestant 3 makes it to the final, he is by assumption limited by his budget and following Che and Gale (1997) it is optimal for him to spend his entire budget. As in Lemmas 1 and 2 we get $x_{23} = w_3 - x_{13}$ regardless of whether this contestant has a head start or not.

A finalist i with a head start facing the budget-constrained contestant chooses $x_{2,i3}^h$ to maximize the following:

$$\max_{x_{2,i3}^h} v\frac{x_{2,i3}^h + h}{x_{2,i3}^h + x_{23} + h}v - x_{2,i3}^h. \tag{A.21}$$

The first-order condition for this yields

$$\frac{vx_{23}}{(x_{2,i3}^h + x_{23} + h)^2} = 1. \tag{A.22}$$

Rearranging terms yields

$$x_{2,i3}^h = \sqrt{vx_{23}} - x_{23} - h. \tag{A.23}$$

A finalist i who does not have a head start (but is facing contestant 3 with a head start) chooses $x_{2,i3}^{nh}$ to maximize the following:

$$\max_{x_{2,i3}^{nh}} v\frac{x_{2,i3}^{nh}}{x_{2,i3}^{nh} + x_{23} + h} - x_{2,i3}^{nh}. \tag{A.24}$$

The first-order condition for this yields

$$\frac{v(x_{23}+h)}{(x_{2,i3}^{nh} + x_{23} + h)^2} = 1. \tag{A.25}$$

Rearranging terms yields

$$x_{2,i3}^{nh} = \sqrt{v(x_{23}+h)} - x_{23} - h. \tag{A.26}$$

When the budget-constrained contestant is not in the final, we are back to the benchmark without budget constraints and it is straightforward to see that the finalist with a head start chooses $\frac{v}{4} - h$ and the finalist without a head start chooses $\frac{v}{4}$. □

Proof of Lemma 7. To establish the expected-payoff ranking for the contestant with a head start, we first notice that Π_3^h is increasing and Π_{i3}^h is decreasing in the second-stage effort of contestant 3 (recall that $x_{23} = w_3 - x_{13}$):

$$\frac{\partial \Pi_3^h}{\partial(w_3 - x_{13})} = \left(\sqrt{\frac{v}{2(w_3 - x_{13} + h)}} - 1\right) > 0. \tag{A.27}$$

$$\frac{\partial \Pi_{i3}^h}{\partial(w_3 - x_{13})} = -\left(\sqrt{\frac{v}{w_3 - x_{13}}} - 1\right) < 0. \tag{A.28}$$

In the limit when the budget-constraint becomes irrelevant and contestant 3 can play his unconstrained optimal strategy, we get that Π_3^h and Π_{i3}^h are equal to Π_{ij}^h:

$$\lim_{x_{23} \to \frac{v}{4}} \Pi_3^h(x_{23}) = \Pi_{ij}^h. \tag{A.29}$$

$$\lim_{x_{23} \to \frac{v}{4}} \Pi_{i3}^h(x_{23}) = \Pi_{ij}^h. \tag{A.30}$$

However since this limit is excluded we obtain the first ranking in Lemma 7.

Similarly, in the case of the non-favored contestant, we show that Π_3^{nh} is increasing and Π_{i3}^{nh} is decreasing in the second-stage effort of contestant 3:

$$\frac{\partial \Pi_3^{nh}}{\partial(w_3 - x_{13})} = \left(\sqrt{\frac{v}{4(w_3 - x_{13})}} - 1\right) > 0. \tag{A.31}$$

$$\frac{\partial \Pi_{i3}^{nh}}{\partial(w_3 - x_{13})} = -\left(\sqrt{\frac{v}{4(w_3 - x_{13} + h)}} - 1\right) < 0. \tag{A.32}$$

In the limit when the budget-constraint becomes irrelevant and contestant 3 can play his unconstrained optimal strategy, we get that Π_3^{nh} and Π_{i3}^{nh} are equal to Π_{ij}^h. This allows us to obtain the second ranking in Lemma 7. □

Proof of Proposition 3. Before proving that $x_{1i}^{s*} > x_{13}^{s*}$ (where we, for simplicity, drop the superscript in the proof), we state the first-stage optimisation problems and the first-order conditions that characterize these equilibrium values. Notice also that the concavity of the optimisation

problems ensures the existence and uniqueness of the equilibrium (Szidarovszky and Okuguchi, 1997).

At the first stage, the unconstrained contestants $i \in \{1,2\}$ face the following optimisation problem:

$$\max_{x_{1i}} \frac{x_{1i}}{x_{1i}+x_{1j}+x_{13}}\frac{x_{1j}}{x_{1j}+x_{13}}\Pi_{ij}^{h} + \frac{x_{1i}}{x_{1i}+x_{1j}+x_{13}}\frac{x_{13}}{x_{1j}+x_{13}}\Pi_{i3}^{h} \\ +\frac{x_{1j}}{x_{1i}+x_{1j}+x_{13}}\frac{x_{1i}}{x_{1i}+x_{13}}\Pi_{ij}^{nh} + \frac{x_{13}}{x_{1i}+x_{1j}+x_{13}}\frac{x_{1i}}{x_{1i}+x_{1j}}\Pi_{i3}^{nh} - x_{1i} \tag{A.33}$$

The associated first-order condition (when at equilibrium $x_{1i} = x_{1j}$) is

$$\frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{1i}}{x_{1i}+x_{13}}\Pi_{ij}^{h} + \frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{13}}{x_{1i}+x_{13}}\Pi_{i3}^{h} \\ +\left[\frac{x_{1i}x_{13}}{(2x_{1i}+x_{13})(x_{1i}+x_{13})^2} - \frac{x_{1i}^2}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}\right]\Pi_{ij}^{nh} \\ +\left[\frac{x_{1i}x_{13}}{4x_{1i}^2(2x_{1i}+x_{13})} - \frac{x_{1i}x_{13}}{2x_{1i}(2x_{1i}+x_{13})^2}\right]\Pi_{i3}^{nh} - 1 = 0 \tag{A.34}$$

All of the four coefficients in front of the expected second-stage payoffs can be shown to be positive.

Denote the left-hand side of (A.34) as $FOC(x_{1i})$. To ensure concavity of the maximization problem we need $\frac{\partial FOC(x_{1i})}{\partial x_{1i}} \leq 0$.

Differentiating $FOC(x_{1i})$ yields:

$$\frac{\partial FOC(x_{1i})}{\partial x_{1i}} = \frac{x_{13}^2 - 2x_{1i}^2 - x_{1i}x_{13}}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})}\Pi_{ij}^{h} - \frac{4x_{13}(x_{1i}+x_{13})}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})}\Pi_{i3}^{h} \\ -\frac{x_{13}^4 + 2x_{1i}^4 - x_{1i}x_{13}^3 - 9x_{1i}^2x_{13}^2 - 7x_{1i}^3x_{13}}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\Pi_{ij}^{nh} - \frac{x_{13}^2(6x_{1i}+x_{13})}{4x_{1i}^2(2x_{1i}+x_{13})^3}\Pi_{i3}^{nh}. \tag{A.35}$$

Since from Lemma 7 we have $\Pi_{i3}^{h} > \Pi_{ij}^{h}$ and $\Pi_{i3}^{nh} \geq \Pi_{ij}^{nh}$, plus $\Pi_{ij}^{h} > \Pi_{ij}^{nh}$ holds, we get:

$$\frac{\partial FOC(x_{1i})}{\partial x_{1i}} < \frac{-4x_{1i}x_{13} - 3x_{13}^2}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})}\Pi_{i3}^{h} - \frac{5x_{1i}^3x_{13} + 4x_{1i}^2x_{13}^2 + x_{1i}x_{13}^3}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\Pi_{ij}^{h} \\ -\frac{x_{1i}x_{13}^3 + 9x_{1i}^2x_{13}^2 + 7x_{1i}^3x_{13}}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\Pi_{ij}^{nh} - \frac{17x_{1i}^2x_{13}^4 + 6x_{1i}^4x_{13}^2 + 19x_{1i}^3x_{13}^3 + 9x_{1i}x_{13}^5 + x_{13}^6}{4x_{1i}^2(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\Pi_{i3}^{nh} < 0. \tag{A.36}$$

At the first stage of the contest, the budget-constrained contestant 3 has the following

maximization problem:

$$\max_{x_{13}} \frac{x_{13}}{x_{1i}+x_{1j}+x_{13}}\Pi_3^h + \frac{x_{13}}{x_{1i}+x_{1j}+x_{13}}\left[\frac{x_{11}}{x_{12}+x_{13}}+\frac{x_{12}}{x_{11}+x_{13}}\right]\Pi_3^{nh} - x_{13}. \tag{A.37}$$

The associated first-order condition (when at equilibrium $x_{1i}=x_{1j}$) is

$$\begin{aligned}\frac{2x_{1i}}{(2x_{1i}+x_{13})^2}\Pi_3^h + \left[\frac{2x_{1i}}{(2x_{1i}+x_{13})^2}\frac{2x_{1i}}{x_{1i}+x_{13}} - \frac{2x_{13}}{(2x_{1i}+x_{13})}\frac{x_{1i}}{(x_{1i}+x_{13})^2}\right]\Pi_3^{nh} \\ -1 - \frac{x_{13}}{(2x_{1i}+x_{13})}\left(\sqrt{\frac{v}{4(w_3-x_{13}+h)}}-1\right) - \frac{2x_{13}}{(2x_{1i}+x_{13})}\frac{x_{1i}}{x_{1i}+x_{13}}\left(\sqrt{\frac{v}{4(w_3-x_{13})}}-1\right)=0,\end{aligned} \tag{A.38}$$

where all the three last terms are negative.

Denote the left-hand side of (A.38) as $FOC_3(x_{13})$. Differentiation and simplification yield:

$$\begin{aligned}\frac{\partial FOC_3(x_{13})}{\partial x_{13}} = &-\frac{4x_{1i}}{(2x_{1i}+x_{13})^3}\Pi_3^h + \frac{-24x_{1i}^4-24x_{1i}^3x_{13}+4x_{1i}x_{13}^4}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\Pi_3^{nh} \\ &-\frac{x_{13}}{(2x_{1i}+x_{13})}\frac{\sqrt{v}}{2(w_3-x_{13}+h)^{3/2}} - \frac{2x_{1i}x_{13}}{(2x_{1i}+x_{13})(x_{1i}+x_{13})}\frac{\sqrt{v}}{2(w_3-x_{13})^{3/2}}\end{aligned}$$

Since $\Pi_3^h > \Pi_3^{nh}$ and the two last terms are negative, the following holds:

$$\begin{aligned}\frac{\partial FOC_3(x_{13})}{\partial x_{13}} < &\frac{-4x_{1i}^4-12x_{1i}^3x_{13}-12x_{1i}^2x_{13}^2}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\Pi_3^h \\ &-\frac{24x_{1i}^4x_{13}+24x_{1i}^4}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}\Pi_3^{nh} < 0.\end{aligned} \tag{A.39}$$

So we can conclude that also the problem of the constrained contestant is concave.

From Lemma 7 we obtain the following inequality:

$$\begin{aligned}FOC(x_{1i}) > &\left[\frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{1i}}{x_{1i}+x_{13}} + \frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{13}}{x_{1i}+x_{13}}\right]\Pi_3^h \\ &+\left[\frac{x_{1i}x_{13}}{(2x_{1i}+x_{13})(x_{1i}+x_{13})^2} - \frac{x_{11}^2}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}\right. \\ &\left.+\frac{x_{1i}x_{13}}{4x_{1i}^2(2x_{1i}+x_{13})} - \frac{x_{1i}x_{13}}{2x_{1i}(2x_{1i}+x_{13})^2}\right]\Pi_3^{nh} - 1.\end{aligned} \tag{A.40}$$

Adding $FOC_3(x_{13})$ and subtracting its value gives the following inequality

$$FOC(x_{1i}) > \left[\frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{1i}}{x_{1i}+x_{13}} + \frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{13}}{x_{1i}+x_{13}}\right]\Pi_3^h \tag{A.41}$$
$$+\left[\frac{x_{1i}x_{13}}{(2x_{1i}+x_{13})(x_{1i}+x_{13})^2} - \frac{x_{11}^2}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}\right.$$
$$\left. + \frac{x_{1i}x_{13}}{4x_{1i}^2(2x_{1i}+x_{13})} - \frac{x_{1i}x_{13}}{2x_{1i}(2x_{1i}+x_{13})^2}\right]\Pi_3^{nh}$$
$$+FOC_3(x_{13}) - \frac{2x_{1i}}{(2x_{1i}+x_{13})^2}\Pi_3^h - \left[\frac{2x_{1i}}{(2x_{1i}+x_{13})^2}\frac{2x_{1i}}{x_{1i}+x_{13}} - \frac{2x_{13}}{(2x_{1i}+x_{13})}\frac{x_{1i}}{(x_{1i}+x_{13})^2}\right]\Pi_3^{nh}$$
$$+\frac{x_{13}}{(2x_{1i}+x_{13})}\left(\sqrt{\frac{v}{4(w_3-x_{13}+h)}}-1\right) + \frac{2x_{13}}{(2x_{1i}+x_{13})}\frac{x_{1i}}{x_{1i}+x_{13}}\left(\sqrt{\frac{v}{4(w_3-x_{13})}}-1\right).$$

The last two terms are clearly positive and rearranging the rest yields

$$FOC(x_{1i}) > FOC_3(x_{13}) + \left[\frac{(x_{1i}+x_{13})[x_{13}-x_{1i}]}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}\right]\Pi_3^h \tag{A.42}$$
$$+\left[\frac{-20x_{1i}^5+4x_{1i}^4x_{13}+13x_{1i}^3x_{13}^2+2x_{1i}^2x_{13}^3+x_{1i}x_{13}^4}{4x_{1i}^2(2x_{1i}+x_{13})^2(x_{1i}+x_{13})^2}\right]\Pi_3^{nh}$$

Assume that $x_{1i} < x_{13}$. This implies that the coefficient in front of Π_3^h is strictly positive Under this assumption, the coefficient in front of Π_3^{nh} is also strictly positive because

$$-20x_{1i}^5 < 4x_{1i}^4x_{13}+13x_{1i}^3x_{13}^2+2x_{1i}^2x_{13}^3+x_{1i}x_{13}^4. \tag{A.43}$$

This implies that $FOC(x_{1i}) > FOC_3(x_{13})$. However, for the optimal values of x_{1i} and x_{13} these are both zero and we obtain our contradiction ($0 > 0$). Thus $x_{1i}^{s*} \geq x_{13}^{s*}$.

□

Proof of Proposition 4 . The proof of this proposition mirrors that of the proof of Proposition 2. We therefore only need to prove the sign of the following derivatives: $\frac{\partial FOC}{\partial w_3} < 0$, $\frac{\partial FOC_3}{\partial w_3} > 0$ and $\frac{\partial FOC}{\partial x_{13}} > 0$ (from the previous proof we have $\frac{\partial FOC}{\partial x_{1i}} < 0$ and $\frac{\partial FOC_3}{\partial x_{13}} < 0$). The rest follows from the arguments in the proof of Proposition 2.

$$\frac{\partial FOC}{\partial w_3} = \frac{x_{13}(x_{1i}+x_{13})}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}\frac{\partial \Pi_{i3}^h}{\partial w_3} \tag{A.44}$$
$$+\frac{x_{1i}x_{13}^2}{4x_{1i}^2(2x_{1i}+x_{13})^2}\frac{\partial \Pi_{i3}^{nh}}{\partial w_3}.$$

It is straightforward to check that the derivative of both of these expected payoffs is strictly

negative and therefore we can conclude that $\frac{\partial FOC}{\partial w_3} < 0$.

$$\frac{\partial FOC_3}{\partial w_3} = \frac{2x_{13}}{(2x_{1i}+x_{13})^2}\frac{\partial \Pi_3^h}{\partial w_3} + \frac{2x_{1i}(2x_{1i}^2 - x_{13}^2)}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})^2}\frac{\partial \Pi_3^{nh}}{\partial w_3} \tag{A.45}$$
$$+\frac{x_{13}}{(2x_{1i}+x_{13})}\frac{\sqrt{v}}{4(w_3 - x_{13} + h)^{3/2}} + \frac{2x_{1i}x_{13}}{(2x_{1i}+x_{13})(x_{1i}+x_{13})}\frac{\sqrt{v}}{4(w_3 - x_{13} + h)^{3/2}}.$$

It is straightforward to check that the derivative of both of these expected payoffs is strictly positive. In addition the two last terms are strictly positive and we can conclude that $\frac{\partial FOC_3}{\partial w_3} > 0$.

Finally, we also need to prove that $\frac{\partial FOC}{\partial x_{13}} > 0$.

$$\frac{\partial FOC}{\partial x_{13}} = -\frac{\partial FOC}{\partial w_3} + X_1\Pi_{ij}^h + X_2\Pi_{i3}^h + X_3\Pi_{ij}^{nh} + X_4\Pi_{i3}^{nh}, \tag{A.46}$$

where

$$X_1 = \frac{-2x_{1i}}{(2x_{1i}+x_{13})^3},$$
$$X_2 = \frac{-2x_{1i}-x_{13}}{(2x_{1i}+x_{13})^3},$$
$$X_3 = \frac{8x_{1i}^4 + 5x_{1i}^3x_{13} - 3x_{1i}^2x_{13}^2 - 2x_{1i}x_{13}^3}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3},$$
$$X_4 = \frac{x_{13}}{(2x_{1i}+x_{13})^3}.$$

Since $\frac{\partial FOC}{\partial w_3} < 0$, the first term is clearly positive.

From previous results we also have that $\Pi_{ij}^h = \Pi_{ij}^{nh} + h < \Pi_{i3}^h$ and $\Pi_{i3}^{nh} > \Pi_{ij}^{nh}$. Therefore,

$$\frac{\partial FOC}{\partial x_{13}} > \Pi_{ij}^{nh}(X_1 + X_2 + X_4) + X_3\Pi_{ij}^{nh} + hX_1. \tag{A.47}$$

One can easily check that $X_1 + X_2 + X_4 = 0$. Furthermore, by Assumption 2, $\Pi_{ij}^{nh} > h$ so that

$$\frac{\partial FOC}{\partial x_{13}} > (X_1 + X_3)\Pi_{ij}^{nh}. \tag{A.48}$$

To prove that $\frac{\partial FOC}{\partial x_{13}} > 0$, it suffices to prove that $X_1 + X_3 > 0$.

$$X_1 + X_3 = \frac{8x_{1i}^4 + 4x_{1i}^3x_{13} - 6x_{1i}^2x_{13}^2 - 5x_{1i}x_{13}^3 - x_{13}^4}{(2x_{1i}+x_{13})^3(x_{1i}+x_{13})^3}. \tag{A.49}$$

Since $x_{13} \leq x_{1i}$, this is positive. □

Proof of Proposition 5. Notice that if the head start h is zero, then the first-order condition for the unconstrained player in the benchmark (equation (18)) and the first-order condition in the model with spillovers (equation (A.34)) are the same.

If we can prove that h increases the marginal benefit (marginal cost is constant), then because of concavity of the problem we can conclude that the best-response function $x_{1i}(x_{13})$ shifts up as h increases (in particular the best-response with h is higher than without).

Recall that the first-order condition associated with the unconstrained contestant's problem in the spillover model is

$$\begin{aligned} FOC^h = & \frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{1i}}{x_{1i}+x_{13}}\Pi_{ij}^h + \frac{x_{1i}+x_{13}}{(2x_{1i}+x_{13})^2}\frac{x_{13}}{x_{1i}+x_{13}}\Pi_{i3}^h \\ & + \left[\frac{x_{1i}x_{13}}{(2x_{1i}+x_{13})(x_{1i}+x_{13})^2} - \frac{x_{11}^2}{(2x_{1i}+x_{13})^2(x_{1i}+x_{13})}\right]\Pi_{ij}^{nh} \\ & + \left[\frac{x_{1i}x_{13}}{4x_{1i}^2(2x_{1i}+x_{13})} - \frac{x_{1i}x_{13}}{2x_{1i}(2x_{1i}+x_{13})^2}\right]\Pi_{i3}^{nh} - 1. \end{aligned} \tag{A.50}$$

Differentiating this function with respect to h gives us

$$\begin{aligned} FOC^h = & \frac{x_{1i}}{(2x_{1i}+x_{13})^2} + \frac{x_{13}}{(2x_{1i}+x_{13})^2} \\ & + \frac{x_{13}\left(x_{13}-2x_{1i}\right)}{4x_{1i}(2x_{1i}+x_{13})^2}\frac{\partial \Pi_{i3}^{nh}}{\partial h}. \end{aligned} \tag{A.51}$$

From Proposition 5, we know that in the relevant domain $x_{1i} > x_{13}$. Thus if proving $\frac{\partial \Pi_{i3}^{nh}}{\partial h} < 0$, allows us to complete the proof and conclude that $x_{1i}(x_{13}) = x_{1i}^h(x_{13})|_{h=0} \leq x_{1i}^h(x_{13})$ (where the last inequality is strict for $h > 0$).

Proof of $\frac{\partial \Pi_{i3}^{nh}}{\partial h} < 0$:

$$\frac{\partial \Pi_{i3}^{nh}}{\partial h} = \left(\frac{\partial \Pi_3^h}{\partial h} - 1\right)\left(\sqrt{\frac{v}{w_3 - x_{13} + h}} - 1\right) - \left(\Pi_3^h - h\right)\sqrt{\frac{v}{4(w_3 - x_{13} + h)}}\frac{1}{w_3 - x_{13} + h}. \tag{A.52}$$

Straightforward computations give

$$\frac{\partial \Pi_3^h}{\partial h} = \sqrt{\frac{v}{4(w_3 - x_{13} + h)}}, \tag{A.53}$$

and

$$\Pi_3^h - h = \sqrt{v(w_3 - x_{13} + h)} - (w_3 - x_{13} + h) = \left(\sqrt{\frac{v}{(w_3 - x_{13} + h)}} - 1\right)(w_3 - x_{13} + h)\,. \tag{A.54}$$

Inserting these results into $\frac{\partial \Pi_{i3}^{nh}}{\partial h}$ yields

$$\begin{aligned}
\frac{\partial \Pi_{i3}^{nh}}{\partial h} &= \left(\sqrt{\frac{v}{4(w_3 - x_{13} + h)}} - 1\right)\left(\sqrt{\frac{v}{w_3 - x_{13} + h}} - 1\right) && \text{(A.55)}\\
&\quad - \left(\sqrt{\frac{v}{(w_3 - x_{13} + h)}} - 1\right)\sqrt{\frac{v}{4(w_3 - x_{13} + h)}}\frac{w_3 - x_{13} + h}{w_3 - x_{13} + h} \\
&= \left(\sqrt{\frac{v}{(w_3 - x_{13} + h)}} - 1\right)\left[\sqrt{\frac{v}{4(w_3 - x_{13} + h)}} - 1 - \sqrt{\frac{v}{4(w_3 - x_{13} + h)}}\right] && \text{(A.56)}\\
&= -\left(\sqrt{\frac{v}{(w_3 - x_{13} + h)}} - 1\right) < 0. && \text{(A.57)}
\end{aligned}$$

Similarly, for the constrained player if the head start h is zero, then the first-order condition in the benchmark (equation (16)) and the first-order condition in the model with spillovers (equation (A.38)) are the same. If we can show that h increases the marginal benefit and decreases the marginal cost, then because of concavity we can conclude that the best-response $x_{13}(x_{1i})$ is higher with h than without. Recall that the marginal benefit in the spillover case (writing in more compact way first line of equation (A.38)) is

$$MB^h = \frac{2x_{1i}}{(2x_{1i} + x_{13})^2}\Pi_3^h + \left[\frac{4x_{1i}^3 - 2x_{1i}x_{13}^2}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2}\right]\Pi_3^{nh} \tag{A.58}$$

Since $\Pi_3^h > \Pi_3^{nh} = \Pi_3(w_3 - x_{13})$ the following holds

$$MB^h > \left[\frac{2x_{1i}}{(2x_{1i} + x_{13})^2} + \frac{4x_{1i}^3 - 2x_{1i}x_{13}^2}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2}\right]\Pi_3(w_3 - x_{13}) = \frac{2x_{1i}(3x_{1i} + 2x_{13})}{(2x_{1i} + x_{13})^2(x_{1i} + x_{13})^2}\Pi_3(w_3 - x_{13}) \tag{A.59}$$

Where it is straightforward to see that the last term is equal to the marginal benefit in the benchmark case (LHS equation (16)). On the other hand, the marginal cost in the spillover case is

$$MC^h = 1 + \frac{x_{13}}{(2x_{1i} + x_{13})}\left(\sqrt{\frac{v}{4(w_3 - x_{13} + h)}} - 1\right) + \frac{2x_{13}}{(2x_{1i} + x_{13})}\frac{x_{1i}}{x_{1i} + x_{13}}\left(\sqrt{\frac{v}{4(w_3 - x_{13})}} - 1\right) \tag{A.60}$$

Since $\frac{\partial\sqrt{\frac{v}{4(w_3-x_{13}+h)}}}{\partial h} = -\sqrt{\frac{v}{4(w_3-x_{13}+h)}}\frac{1}{w_3-x_{13}+h} < 0$, the following inequality is true:

$$MC^h < 1 + \frac{x_{13}}{(2x_{1i}+x_{13})}\left(\sqrt{\frac{v}{4(w_3-x_{13})}}-1\right) + \frac{2x_{13}}{(2x_{1i}+x_{13})}\frac{x_{1i}}{x_{1i}+x_{13}}\left(\sqrt{\frac{v}{4(w_3-x_{13})}}-1\right) \tag{A.61}$$

where the RHS is equal to the marginal cost in the benchmark case (RHS equation (16)). □

Appendix B

Suppose that each contestant $i \in \{1,\ldots,n\}$ has wealth $w_i = w$, which is publicly known. To focus on the interesting case of a binding budget constraint, we assume in the following that w is less than the sum of total equilibrium effort when there are no budget constraints.

Assumption 4.

$$w < \left[\frac{(2n-1)(n-2)}{n^2(n-1)} + 1\right]\frac{v}{4}. \tag{B.1}$$

In the final stage the two finalists solve the following maximization problem:

$$\begin{aligned} \max_{x_{2i}} \quad & v\frac{x_{2i}}{x_{2i}+x_{2j}} - x_{2i} \\ \text{subject to} \quad & x_{2i} \leq w - x_{2i}. \end{aligned} \tag{B.2}$$

Following Che and Gale (1997), it can easily be shown that equilibrium effort in this situation is:

$$x_{2i} = \begin{cases} \frac{v}{4} & \text{if } v < 4(w-x_{1i}), \\ w - x_{1i} & \text{if } v \geq 4(w-x_{1i}). \end{cases} \tag{B.3}$$

In the first case the contestants are de facto unconstrained, so the solution is the same as in the previous benchmark. This case occurs when the value of the contract is not high enough for the player to be willing to spend the entire budget left after the first stage. The reverse is true in the second case. As in the main analysis, we can show that the second case is the only relevant case.

Lemma 1. *The remaining budget in the second period is always less than* $\frac{v}{4}$*:*

$$w \leq \frac{v}{4}.$$

The proof follows the same line of argument as the proof of Lemma 1 and is therefore omitted.

In this case when second-period budgets are binding, a contestant's expected payoff in the second stage is therefore:

$$\Pi^2_{sym}(w - x_{1i}) = \frac{v}{2} - (w - x_{1i}). \tag{B.4}$$

Given the equilibrium behavior in the final stage, each contestant i chooses a first-period effort level to solve

$$\begin{aligned} &\max_{x_i^1} && q_i^1(x_{1i}, x_{1-i})\Pi^2_{sym}(w - x_{1i}) - x_{1i} \\ &\text{subject to} && x_{1i} \leq w. \end{aligned} \tag{B.5}$$

The symmetric equilibrium yields:

$$x_1^{sym} = \min\left\{w; \frac{2n-1}{n^2-3n+1}\left[\frac{v}{2} - w\right]\right\}. \tag{B.6}$$

For low numbers of contestants it is optimal to spend the entire budget in the first stage. In a symmetric equilibrium, the contestants all face the same probability of reaching the final stage. However, since first-stage effort limits second-stage effort, the expected payoff in the final stage is increasing in first-stage effort. It is thus optimal for the contestants to spend all the resources in the first stage and thereby maximizing the expected payoff from the second stage. This burning-out result is also derived in Amegashie (2004) who shows that burning out occurs when contestants are constrained, incentives are extreme high-powered and the playing field is even.[23] In our model, the reward is higher (and incentives higher powered) the lower the number of contestants as this increases the expected payoff from the contest.

[23]See Amegashie et al (2007) for a burning-out result in the case of an all-pay auction.