

Aaberge, Rolf; Bjerve, Steinar; Doksum, Kjell

Working Paper

Modeling Concentration and Dispersion in Multiple Regression

Discussion Papers, No. 412

Provided in Cooperation with:

Research Department, Statistics Norway, Oslo

Suggested Citation: Aaberge, Rolf; Bjerve, Steinar; Doksum, Kjell (2005) : Modeling Concentration and Dispersion in Multiple Regression, Discussion Papers, No. 412, Statistics Norway, Research Department, Oslo

This Version is available at:

<https://hdl.handle.net/10419/192394>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

*Rolf Aaberge, Steinar Bjerve and Kjell
Doksum*

Modeling Concentration and Dispersion in Multiple Regression

Abstract:

We consider concepts and models that are useful for measuring how strongly the distribution of a positive response Y is concentrated near a value $y_0 > 0$ with a focus on how concentration varies as a function of covariates. We combine ideas from statistics, economics and reliability theory. Lorenz introduced a device for measuring inequality in the distribution of incomes that indicate how much the incomes below the u th quantile fall short of the egalitarian situation where everyone has the same income. Gini introduced an index that is the average over u of the difference between the Lorenz curve and its values in the egalitarian case. More generally, we can think of the Lorenz and Gini concepts as measures of concentration that applies to other response variables in addition to incomes, e.g. wealth, sales, dividends, taxes, test scores, precipitation, and crop yield. In this paper we propose modified versions of the Lorenz and Gini measures of concentration that we relate to statistical concepts of dispersion. Moreover, we consider the situation where the measures of concentration/dispersion are functions of covariates. We consider the estimation of these functions for parametric models and a semiparametric model involving regression coefficients and an unknown baseline distribution. In this semiparametric model, which combines ideas from Pareto, Lehmann and Cox, we find partial likelihood estimates of the regression coefficients and the baseline distribution that can be used to construct estimates of the various measures of concentration/dispersion.

Keywords: Spread, concentration, Lorenz curve, Gini index, Lehmann model, Cox regression, Pareto model.

JEL classification: C14, D31, D63

Acknowledgement: We would like to thank Anne Skoglund for typing and editing the paper.

Address: Rolf Aaberge, Statistics Norway, Research Department. E-mail: rolf.aaberge@ssb.no

Steinar Bjerve, Department of Mathematics, University of Oslo.
E-mail: steinar@math.uio.no

Kjell Doksum, Department of Statistics, University of Wisconsin.
E-mail: doksum@stat.wisc.edu

Discussion Papers

comprise research papers intended for international journals or books. A preprint of a Discussion Paper may be longer and more elaborate than a standard journal article, as it may include intermediate calculations and background material etc.

Abstracts with downloadable Discussion Papers
in PDF are available on the Internet:

<http://www.ssb.no>

<http://ideas.repec.org/s/ssb/dispap.html>

For printed Discussion Papers contact:

Statistics Norway
Sales- and subscription service
NO-2225 Kongsvinger

Telephone: +47 62 88 55 00

Telefax: +47 62 88 55 95

E-mail: Salg-abonnement@ssb.no

1. Introduction

Regression models typically postulate how the location parameter of a response variable Y changes with covariates X_1, \dots, X_d . In the case of heteroscedastic models, the spread of Y is also modelled as a function of X_1, \dots, X_d . In this paper we model the “concentration” of the distribution of Y as a function of the covariates. By concentration we mean spread relative to location. Besides the coefficient of variation, two famous concentration measures are the Lorenz curve and Gini coefficient.

In this section we first present the Lorenz curve and the closely related Bonferroni curve, which can be considered as devices for measuring inequality or concentration. We also propose two alternative curves of concentration and the corresponding summary measures of concentration that relate to statistical concepts of dispersion. We then present orderings that order distributions according to their degree of concentration. We propose and analyze regression versions in the rest of the paper.

1.1. Defining concentration

The Lorenz curve (LC) $L(u)$ is defined (Lorenz (1905)) to be the proportion of the total amount of income that is owned by the “poorest” $100 \times u$ percent of the population. More precisely, let the random income $Y > 0$ have the distribution function $F(y)$, let $F^{-1}(u) = \inf \{y : F(y) \geq u\}$ denote the left inverse, and assume that $0 < \mu < \infty$, where

$$\mu = \mu_F = E(Y) = \int_0^{\infty} F^{-1}(u) du .$$

Then the LC (see e.g. Gastwirth (1971)) is defined by

$$L(u) = L_F(u) = \mu^{-1} \int_0^u F^{-1}(s) ds, \quad 0 \leq u \leq 1 .$$

When F is continuous we can write

$$L(u) = \mu^{-1} E \left\{ Y I \left[Y \leq F^{-1}(u) \right] \right\}$$

where $I[A]$ denotes the indicator of the event A .

When the population consists of incomes of people, the LC measures deviation from the *egalitarian* case $L(u) = u$ corresponding to where everyone has the same income $a > 0$ and the distribution of Y is point-mass at a . The other extreme occurs when one person has all the income. The

income of a person drawn at random is then zero with probability 1^- , which corresponds to $L(u) = 0, 0 \leq u < 1$. The intermediate case where Y is uniform on $[0, b]$, $b > 0$, corresponds to $L(u) = u^2$.

In general $L(u)$ will be non-decreasing, convex, below the line $L(u) = u, 0 \leq u \leq 1$, and the greater the “distance” from u , the greater are the inequality in the population. If the population consists of companies providing a certain service or product, the LC measures to what extent a few companies dominate the market with the extreme case corresponding to monopoly. More generally, we can think of the LC as a measure of concentration of a nonnegative random variable Y .

A closely related curve is the Bonferroni curve (BC) $B(u)$ which is defined (Aaberge (1982) and Giorgi and Mondani (1995)) as

$$B(u) = B_F(u) = u^{-1}L(u), \quad 0 \leq u \leq 1.$$

When F is continuous the BC is the LC except truncation is replaced by conditioning

$$B(u) = \mu^{-1}E[Y | Y \leq F^{-1}(u)].$$

The BC possesses several attractive properties. First, it provides a convenient alternative interpretation of the information content of the Lorenz curve. For a fixed u , $B(u)$ is the ratio between the mean income of the poorest $100u$ per cent of the population and the overall mean. Thus, the BC may also yield essential information on poverty provided that we know the poverty rate. Second, the BC of a uniform $(0, a)$ distribution proves to be the diagonal line joining the points $(0, 0)$ and $(1, 1)$ and thus represents a useful reference line, in addition to the two well-known standard reference lines. The egalitarian reference line, coincides with the horizontal line joining the points $(0, 1)$ and $(1, 1)$. At the other extreme, when one person holds all income, the BC coincides with the horizontal axis except for $u = 1$. The uniform case yields $B(u) = u$, which is exactly in the middle between the egalitarian and extreme non-egalitarian cases.

In the next subsection we will consider concepts of dispersion from the statistics literature. It turns out that those concepts lead to measures that are modifications of $L(\cdot)$ and $B(\cdot)$ and motivates the introduction of the following measures of concentration

$$C(u) = C_F(u) = \int_0^u \left[\frac{F^{-1}(s)}{F^{-1}(u)} \right] ds = \mu_F \frac{L_F(u)}{F^{-1}(u)}, \quad 0 < u < 1$$

and

$$D(u) = D_F(u) = \frac{1}{u} \int_0^u \left[\frac{F^{-1}(s)}{F^{-1}(u)} \right] ds = \mu_F \frac{B_F(u)}{F^{-1}(u)}, \quad 0 < u < 1$$

Accordingly, $C(u)$ and $D(u)$ emerge by replacing the overall mean μ in the dominators of $L(u)$ and $B(u)$ by the u^{th} quantile $y_u = F^{-1}(u)$ and $C(u)$ (resp. $D(u)$) is equal to the ratio between the income share (resp. mean income) of those with lower income than the u^{th} quantile and the u -quantile income. Thus, $C(u)$ and $D(u)$ measure how strongly the income below the u^{th} quantile is concentrated near y_u . They satisfy $C(u) \leq u, D(u) \leq 1, 0 < u < 1$, and $C(u)$ equals u and 0 while $D(u)$ equals 1 and 0 in the egalitarian and extreme non-egalitarian cases, respectively, and they equal $u/2$ and $1/2$ in the uniform case.

To summarize the information content of $C(\cdot)$ and $D(\cdot)$ we introduce the following dispersion indices

$$C = 2 \int_0^1 [u - C(u)] du$$

$$D = \int_0^1 [1 - D(u)] du.$$

The dispersion indices C and D measure the distances from $C(u)$ and $D(u)$ to their values in the most concentrated cases, that is, the egalitarian case. Note that C and D can be considered as modified versions of the Gini and Bonferroni coefficients (see Aaberge (2000) for a normative justification of the Bonferroni coefficient as a measure of income inequality). As the Gini and Bonferroni coefficients they take values between 0 and 1 and are increasing with increasing inequality. If all units have the same income then $C = D = 0$, and in the extreme non-egalitarian case where one unit has all the income and the others zero, $G = B = C = D = 1$. When F is uniform on $[0, b]$, $B = C = D = 1/2$. $L(u)$, $B(u)$, $C(u)$, $D(u)$, G , B , C and D are scale invariant, that is, they remain the same if Y is replaced by aY , $a > 0$.

G , B , C and D resemble “spread” divided by “location” scaled to go from zero to one as the distribution moves from the egalitarian to the extreme non-egalitarian case. These properties resemble that of the coefficient of variation, $CV = \sigma/\mu$, or its scaled version

$$CV^* = \sqrt{3}CV [1 + \sqrt{3}CV]^{-1}$$

which goes from zero to one as we move from the egalitarian to the extreme non-egalitarian case and equals $1/2$ in the uniform case.

1.2. Ordering concentration

When we are interested in how covariates influence concentration we may ask whether larger values of a covariate leads to more or less inequality. For instance, is there less inequality among the higher educated? To answer such questions we consider orderings that order distributions according to how concentrated they are. In statistics and reliability engineering, orderings are plentiful, e.g. Lehmann (1955), van Zwet (1964), Barlow and Proschan (1965), Birnbaum, Esary and Marshall (1966), Doksum (1969), Yanagimoto and Sibuya (1976), Bickel and Lehmann (1979), Rojo and He (1991), Rojo (1992) and Shaked and Shanthikumar (1994). In statistics, orderings are often discussed in terms of spread or dispersion. Thus, for non-negative random variables, using van Zwet (1964) we could define Y to have a distribution which is more spread than that of Y_0 if Y can be written as $Y = h(Y_0)$ for some non-negative, nondecreasing convex function h . It turns out to be more general and more convenient to replace “convex” with “starshaped” (convex functions are starshaped and concave functions are anti-starshaped):

Weakening the convexity condition

$$g(\lambda x_0 + (1-\lambda)x_1) \leq \lambda g(x_0) + (1-\lambda)g(x_1), \quad 0 \leq \lambda \leq 1,$$

we call a function g defined on the interval $I \subset [0, \infty)$ *starshaped* on I if $g(\lambda x) \leq \lambda g(x)$ whenever $x \in I$, $\lambda x \in I$ and $0 \leq \lambda \leq 1$. Thus if $I = (0, \infty)$, then the graph of g initially lies on or below any straight line through the origin, and then lies on or above it. If $g(\lambda x) \geq \lambda g(x)$, g is anti-starshaped.

On the class \mathcal{F} of continuous distributions F with $F(0) = 0$, the (Doksum (1969)) following ordering (partial) is defined: $F <_* H$ (F is *starshaped* with respect to H) if $H^{-1}F$ is starshaped on

$\{x: 0 < F(x) < 1\}$, where $H^{-1}(u) = \inf\{x: H(x) \geq u\}$. Thus if $F <_* H$ and X has distribution F , then

$Z = H^{-1}[F(X)]$ has distribution H and is a starshaped transformation of X ; hence we say that the

distribution of Z is more dispersed than the distribution of X . This interpretation is valid when

$\{x: 0 < F(x) < 1\} = (0, \infty)$, because when $F <_* H$, there exists a nondecreasing function $g(x)$ such

that Z has the same distribution as $g(X)X$. To see this take $g(x) = H^{-1}(F(x))$ and see the proof of

Proposition 1.1.

When $F <_* H$ we also call the distribution of X more *concentrated* than that of Z . That is, if X and Z are random variables that represent incomes under two different conditions, the condition generating X corresponds to less inequality.

We next show that the preceding definition of concentration leads to the corresponding ordering of the concentration curves $C_F(\cdot)$ and $D_F(\cdot)$ as well as of the dispersion indices C and D .

Proposition 1.1. *Suppose $F, H \in \mathcal{F}$ and $F <_* H$, then $C_F(u) \geq C_H(u)$ and $D_F(u) \geq D_H(u), 0 < u < 1$. Moreover, $C_F \geq C_H$ and $D_F \geq D_H$.*

Proof. Note that the condition $g(\lambda x) \leq \lambda g(x)$ is equivalent to $\left[\frac{g(\lambda x)}{\lambda x} \right] \leq \frac{g(x)}{x}$, that is $g(x)/x$ is non-decreasing. It follows that by setting $u = F(x)$, $v = F(x')$, $x < x'$, we obtain

$$H^{-1}(u)/F^{-1}(u) \leq H^{-1}(v)/F^{-1}(v) \text{ for } 0 < u < v < 1.$$

That is

$$H^{-1}(u)/H^{-1}(v) \leq F^{-1}(u)/F^{-1}(v) \text{ for } 0 < u < v < 1.$$

If we integrate this inequality over $u \in (0, v)$, we obtain $C_F(v) \geq C_H(v), 0 < v < 1$. The other inequalities follow from this. □

2. Regression

Next consider the case where the distribution of Y depends on covariates such as education, work experience, status of parents, sex, etc. Let $\mathbf{X} = (X_1, \dots, X_d)^T$ denote the covariates, let $F(y|\mathbf{x})$ denote the conditional distribution of Y given $\mathbf{X} = \mathbf{x}$ and define $F^{-1}(u|\mathbf{x}) = \inf \{y : F(y|\mathbf{x}) \geq u\}$.

We define the *conditional C- and D- curves* as

$$C(u|\mathbf{x}) = \int_0^u \left[\frac{F^{-1}(s|\mathbf{x})}{F^{-1}(u|\mathbf{x})} \right] ds, \quad 0 < u < 1$$

and

$$D(u|\mathbf{x}) = \frac{C(u|\mathbf{x})}{u}, \quad 0 < u < 1.$$

We define the corresponding conditional dispersion indices as

$$C(\mathbf{x}) = 2 \int_0^1 (u - C(u|\mathbf{x})) du$$

and

$$D(\mathbf{x}) = \int_0^1 (1 - D(u|\mathbf{x})) du.$$

3. Parametric Regression Models

3.1. Transformation regression models

Let Y_0 denote a baseline variable which corresponds to the case where the covariate vector \mathbf{x} has no effect on the distribution of income. We assume that $F(y|\mathbf{x})$ depends on \mathbf{x} through some real valued function $\Delta(\mathbf{x}) = g(\mathbf{x}, \boldsymbol{\beta})$ which is known up to a vector $\boldsymbol{\beta}$ of unknown parameters. Let $Y \sim Z$ denote “ Y is distributed as Z ”. As we have seen in Section 1.2, if large values of $\Delta(\mathbf{x})$ corresponds to a more egalitarian distribution of income, then it is reasonable to model this as

$$Y \sim h(Y_0),$$

for some increasing concave function h depending on $\Delta(\mathbf{x})$ because an increasing concave transformation brings values closer together relative to their mean. On the other hand, an increasing convex h would correspond to income being less concentrated.

Set $\mathbf{x} = (1, x_1, \dots, x_d)^T$ and $\boldsymbol{\beta} = (\beta_0, \dots, \beta_d)^T$, then a convenient parametric form of h is

$$(3.1) \quad Y \sim Y_0^\Delta.$$

Here $0 < \Delta < 1$ corresponds to covariates that lead to a more egalitarian distribution of income while $\Delta > 1$ is the opposite case. Note that

$$(3.2) \quad \log Y \sim \Delta \log Y_0.$$

Thus (3.1) is a scale model in $Z = \log Y$ and Δ is a scale parameter for log income.

Example 3.1. Suppose $Y_0 \sim F_0$ where F_0 is the Pareto standardized distribution

$$F_0(y) = 1 - \left(\frac{1}{y}\right)^a, \quad a > 1, y \geq 1.$$

Note that here wage has been standardized by dividing by the minimum wage, that is, one is the smallest possible value of Y . Then $Y = Y_0^\Delta$ has the Pareto distribution

$$F(y|\mathbf{x}) = F_0\left(y^{\frac{1}{\Delta}}\right) = 1 - \left(\frac{1}{y}\right)^{\alpha(\mathbf{x})}, \quad y \geq 1,$$

where $\alpha(\mathbf{x}) = \Delta(\mathbf{x})/a$. Provided $\alpha(\mathbf{x}) > 1$ and F_0 is the baseline distribution of Y , the corresponding conditional regression *C-curve* and *C-coefficient* is easily found to be given by

$$C(u|\mathbf{x}) = \frac{\alpha(\mathbf{x})}{1-\alpha(\mathbf{x})} \left[(1-u)^{\frac{1}{\alpha(\mathbf{x})}} - (1-u) \right], \quad 0 < u < 1,$$

and

$$C(\mathbf{x}) = \frac{1}{\alpha(\mathbf{x}) + 1}.$$

By choosing the parametrization $\alpha(\mathbf{x}) = (\exp(-\beta^T \mathbf{x}) - 1)$ we have $C(\mathbf{x}) = \exp(\mathbf{x}^T \boldsymbol{\beta})$, where $\boldsymbol{\beta}$ may be estimated by maximum likelihood.

Example 3.2. Another interesting case is obtained by setting F_0 equal to the log normal distribution $\Phi([\log(y) - \mu_0]/\sigma_0)$, $y > 0$. In this case we also get an explicit form of the *conditional concentration curve*:

Proposition 3.1. *In the model (3.2) with F_0 log normal*

$$(3.3) \quad C(u|\mathbf{x}) = \sigma_0^{-1} \Phi\left(\Phi^{-1}(u) - [\sigma_0/\Delta(\mathbf{x})]\right) \exp\left\{\frac{1}{2}[\sigma_0^2/\Delta^2(\mathbf{x})] - \sigma_0 \Phi^{-1}(u)/\Delta(\mathbf{x})\right\}.$$

Proof. Because μ_0 is a scale parameter for Y , it will cancel in the concentration curve. Thus we can set

$\mu_0 = 0$. In the proof we write σ for σ_0 . Here $F^{-1}(u|\mathbf{x}) = [F_0^{-1}(u)]^{\frac{1}{\Delta}}$, where $\Delta = \Delta(\mathbf{x})$, thus

$$\begin{aligned} \int_0^u F^{-1}(s|\mathbf{x}) ds &= \int_0^u [F_0^{-1}(s)]^{\frac{1}{\Delta}} ds = \int_0^{F_0^{-1}(u)} y^{\frac{1}{\Delta}} dF_0(y) \\ &= \int_{-\infty}^{\log F_0^{-1}(u)} e^{\frac{z}{\Delta}} dF_0(e^z) = \sigma^{-1} \int_{-\infty}^{\sigma \Phi^{-1}(u)} e^{\frac{z}{\Delta}} \varphi\left(\frac{z}{\sigma}\right) dz \\ &= \sigma^{-1} \int_{-\infty}^{\Phi^{-1}(u)} e^{\frac{\sigma v}{\Delta}} \varphi(v) dv = \sigma^{-1} \Phi\left(\Phi^{-1}(u) - [\sigma/\Delta]\right) e^{\frac{1}{2}\left(\frac{\sigma}{\Delta}\right)^2}, \end{aligned}$$

where the last equality follows from

$$e^{\frac{\sigma v}{\Delta}} e^{-\frac{1}{2}v^2} = e^{-\frac{1}{2}\left(v - \frac{\sigma}{\Delta}\right)^2} \cdot e^{\frac{1}{2}\left(\frac{\sigma}{\Delta}\right)^2}.$$

The result follows because

$$F^{-1}(u|\mathbf{x}) = \exp\left\{\sigma_0 \Phi^{-1}(u) / \Delta(\mathbf{x})\right\}.$$

Suppose we choose the parametrization $\Delta(\mathbf{x}) = \exp(\mathbf{x}^T \boldsymbol{\beta})$. To estimate $\boldsymbol{\beta}$ for this lognormal model we set $Z_i = \log Y_i$. Then Z_i has a $N(\mu_0 \Delta(\mathbf{x}_i), \sigma_0^2 \Delta^2(\mathbf{x}_i))$ distribution, where $\mathbf{x}_i = (1, x_{i1}, \dots, x_{id})^T$. Here only $d + 2$ of the $d + 3$ parameters are identifiable because in

$$\mu_0 \Delta(\mathbf{x}) = \mu_0 e^{\beta_0} \exp\left\{\sum_{j=1}^d \beta_j x_j\right\},$$

μ_0 and β_0 are not both identifiable. Thus we absorb μ_0 into e^{β_0} and replace $\mu_0 \Delta(\mathbf{x}_i)$ by $\Delta(\mathbf{x}_i)$.

When Y_1, \dots, Y_n are independent, this gives the log likelihood function (leaving out the constant term)

$$l(\boldsymbol{\beta}, \sigma^2) = -n \log \sigma_0 - \sum_{i=1}^n \mathbf{x}_i^T \boldsymbol{\beta} - \frac{1}{2} \sigma_0^{-2} \sum_{i=1}^n (-2 \mathbf{x}_i^T \boldsymbol{\beta}) \left\{ Z_i - \exp(\mathbf{x}_i^T \boldsymbol{\beta}) \right\}^2.$$

See Anscombe (1961), Bickel (1978), and Carroll and Ruppert (1982, 1988) for estimation based on such likelihoods. Bickel suggests modifications that result in more robust estimates.

3.2. Models based on the income improvement rate. The Weibull model

Poverty in undeveloped regions of the world is in part measured by the incomes earned by the people in these regions, and the success of aid and programs to decrease poverty is also measured by income. It would be helpful to have a measure of the odds of income improvement of a person whose income is Y . Suppose this person goes looking for a new job without acquiring any new skills and without there being new types of job opportunities being developed in the region. Let Y' denote the new income, where Y and Y' have the same distribution and are independent. Then in the discrete case we define the income improvement rate as the odds of improving on the wage $Y = y$, that is,

$$(3.4) \quad \frac{P(Y' > Y | Y = y)}{P(Y = y)} = \frac{1 - F(y)}{f(y)}.$$

Note that we assume $P(Y' < Y) = 0$, that is, the person would refuse a lesser paying job. We extend (3.4) in the natural way to the continuous case and write the IIR as

$$r(y) \equiv \frac{1 - F(y)}{f(y)}$$

for $y \in \{y : f(y) > 0\}$.

For the Pareto distribution $F(y) = 1 - y^{-a}$, $a > 1$, $y \geq 1$, the IIR is $r_p(y) \equiv a^{-1}y$, $y \geq 1$.

Thus the odds on improving ones income is proportional to the current income. As seen in Example 3.1, the Pareto power regression model where $\log Y = \Delta(\mathbf{x}) \log Y_0$ with Y_0 Pareto has

$$r_p(y|\mathbf{x}) = a^{-1} \Delta(\mathbf{x}) y, y \geq 1.$$

For the exponential distribution $F(y) = 1 - \exp\{-\lambda y\}$, we have a constant IIR

$r_E(y) \equiv \lambda^{-1}$. However, most empirical wage distributions have heavier right tails than the exponential distribution. The Weibull distribution $F(y) = 1 - \exp\{-\lambda y^a\}$, $a > 0$, $y \geq 0$, is a more flexible choice. In

this case $r_W(y) = \lambda^{-1} a^{-1} y^{1-a}$, and for the Weibull power regression model where $\log Y = \Delta(\mathbf{x}) \log Y_0$ with Y_0 Weibull, we have

$$r_W(y|\mathbf{x}) = \lambda^{-1} a^{-1} \Delta(\mathbf{x}) y^{1-(1/\Delta(\mathbf{x}))}, y > 0.$$

In this case the IIR is increasing or decreasing in y according as $(\Delta(\mathbf{x})/a)$ is greater than or smaller than 1. Note that $r_W(y+1|\mathbf{x})$ approximates $r_p(y|\mathbf{x})$ for $a/\Delta(\mathbf{x})$ close to 0.

If we choose the parametrization $\Delta(\mathbf{x}) = \exp\{\mathbf{x}^T \boldsymbol{\beta}\}$, then the parameters of the Weibull model can be estimated by maximum likelihood software which also provides standard errors.

4. Lehmann-Cox type models. Partial likelihood

4.1. The distribution transformation model

Let $Y_0 \sim F_0$ be a baseline income distribution and let $Y \sim F(y|\mathbf{x})$ denote the distribution of income for given covariate vector \mathbf{x} . One way to express that $F(y|\mathbf{x})$ is less concentrated than $F_0(y)$ is to use the model

$$F(y|\mathbf{x}) = h(F_0(y))$$

for a convex transformation h depending on \mathbf{x} . This interpretation is valid when

$\{y : 0 < F_0(y) < 1\} = (0, \infty)$, because the density of $F(y|\mathbf{x})$ is $h'(F_0(y))f_0(y)$ where $h'(F_0(y))$ is increasing. Note the similarity with Section 1.1 where multiplying X with an increasing function defined less concentration. Similarly, g concave corresponds to more egalitarian income. A model of the form $F_2(y) = g(F_1(y))$ was considered for the two-sample case by Lehmann (1953) who noted that $F_2(y) = F_1^\Delta(y)$, $\Delta > 0$, was a convenient choice of h . Similarly, for regression experiments, we consider a regression version of this model which we define as

$$(4.1) \quad F(y|\mathbf{x}) = F_0^\Delta(y),$$

where $\Delta = \Delta(\mathbf{x}) = g(\mathbf{x}, \boldsymbol{\beta})$ with a real valued parametric function and where $\Delta > 1$ or $\Delta < 1$ corresponds to more or less egalitarian respectively. Since

$$\log F(y|\mathbf{x}) = \Delta \log F_0(y)$$

this model assumes that the log of the income distributions of Y and Y_0 are proportional with Δ being the proportionality constant.

If we set $Z_i = 1 - F_0(Y_i)$, then Z has the distribution

$$H(u) = 1 - (1 - u)^\Delta, \quad 0 < u < 1.$$

Since the rank R_i of Y_i equals $n + 1 - S_i$, where S_i is the rank of $1 - F_0(Y_i)$, we can use rank methods, or partial likelihood methods, to estimate β without knowing F_0 . In fact, because the Cox partial likelihood is a rank likelihood we can apply the likelihood in the next subsection to estimate the parameters in the current model provided we reverse the ordering of the Y 's.

4.2. The income function transformation model

In this section we show how the Pareto parametric regression model for income can be extended to a semiparametric model where the shape of the income distribution is completely general. Let the incomes Y_1, \dots, Y_n be independent and let $F(y|\Delta_i)$ be the distribution of Y_i , where

$$\Delta_i = \exp\{\mathbf{x}_i^T \beta\}.$$

One convenient model is a regression version of the Pareto model which we define as

$$F(y|\mathbf{x}_i) = 1 - \left(\frac{c}{y}\right)^{\Delta_i}, \quad y \geq c; \Delta_i > 0,$$

where c , the minimum salary in the population, is known. This model satisfies

$$(4.2) \quad 1 - F(y|\mathbf{x}_i) = [1 - F_0(y)]^{\Delta_i},$$

where $F_0(t) = 1 - \frac{c}{t}$, $y \geq c$. When F_0 is an arbitrary continuous distribution on $[0, \infty)$, the model (4.2)

for the two sample case was called the Lehmann alternative by Savage (1956, 1980) because if V satisfies model (4.1), then $Y = -V$ satisfies model (4.2). Cox (1972) introduced proportional hazard models for regression experiments in survival analysis which also satisfy (4.2) and introduced partial likelihood methods that can be used to analyse such models even in the presence of time dependent covariates (in our case, wage dependent covariates).

Cox introduced the model equivalent to (4.2) as a generalization of the exponential model where $F_0(y) = 1 - \exp\{-y\}$ and $F(y|\mathbf{x}_i) = F_0(\Delta_i y)$. That is, (4.2) is in the Cox case a generalization of a scale model with scale parameter Δ_i . However, in our case we regard (4.2) as a shape model which generalizes the Pareto model, and Δ_i represents the degree of concentration of the variable Y for a given covariate vector \mathbf{x}_i .

If we call the probability $\bar{F}(y) = P(Y > y) = 1 - F(y)$ of income greater than y the *income function*, then (4.2) is a model with proportional log income functions. Note that $\Delta_i < 1$ corresponds to $F(y|\mathbf{x})$ more concentrated than $F_0(y)$ while $\Delta_i > 1$ corresponds to F_0 less concentrated.

The Cox (1972) partial likelihood to estimate $\boldsymbol{\beta}$ for (4.2) is (see also Kalbfleisch and Prentice (2002), page 102),

$$L(\boldsymbol{\beta}) = \prod_{i=1}^n \left\{ \frac{\exp(\boldsymbol{\beta}^T \mathbf{x}_{(i)})}{\sum_{k \in R(Y_{(i)})} \exp(\boldsymbol{\beta}^T \mathbf{x}_{(k)})} \right\},$$

where $Y_{(i)}$ is the i -th order statistic, $\mathbf{x}_{(i)}$ is the covariate vector for the subject with response $Y_{(i)}$, and $R(Y_{(i)}) = \{k : Y_{(k)} \geq Y_{(i)}\}$. Here $\hat{\boldsymbol{\beta}} = \arg \max L(\boldsymbol{\beta})$ can be found in many statistical packages. These packages also give the standard errors of the $\hat{\boldsymbol{\beta}}$. Note that $L(\boldsymbol{\beta})$ does not involve F_0 .

Many estimates are available for F_0 in model (4.2), again in packages. If we maximize the likelihood keeping $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$ fixed, we find (e.g., Kalbfleisch and Prentice (2002), page 116)

$$\hat{F}_0(Y_{(i)}) = 1 - \prod_{j=1}^i \hat{\alpha}_j \text{ where}$$

$$\hat{\alpha}_j = \left(1 - \frac{\exp(\hat{\boldsymbol{\beta}}^T \mathbf{x}_{(j)})}{\sum_{k \in R(Y_{(j)})} \exp(\hat{\boldsymbol{\beta}}^T \mathbf{x}_{(k)})} \right).$$

We can now give empirical expressions for the conditional C -curve and the coefficient C . Using (4.2), we find

$$(4.3) \quad F^{-1}(u|\mathbf{x}_i) = F_0^{-1} \left(1 - (1-u)^{\frac{1}{\Delta_i}} \right),$$

$$(4.4) \quad \mu(u|\mathbf{x}_i) = \int_0^u F^{-1}(t|\mathbf{x}_i) dt = \int_0^u F_0^{-1} \left(1 - (1-v)^{\frac{1}{\Delta_i}} \right) dv.$$

We set $t = F_0^{-1} \left(1 - (1-v)^{\frac{1}{\Delta_i}} \right)$ and obtain

$$\mu(u|\mathbf{x}_i) = \Delta_i \int_0^{\delta_i(u)} t [1 - F_0(t)]^{\Delta_i - 1} dF_0(t)$$

where $\delta_i(u) = F_0^{-1}\left(1 - (1-u)^{\frac{1}{\Delta_i}}\right)$. Note that when all $\beta_j = 0, j \geq 1$, then $\Delta_i = 1$ and $C(u|\mathbf{x})$ and $D(u|\mathbf{x})$ reduce to the *C- and D- curves* without covariates. To estimate the *C- and D- curves*, we let

$$b_i = \hat{F}_0(Y_{(i)}) - \hat{F}_0(Y_{(i-1)}) = \prod_{j=1}^{i-1} \hat{\alpha}_j = [1 - \hat{\alpha}_i] \prod_{j=1}^{i-1} \hat{\alpha}_j$$

be the jumps of $\hat{F}_0(\cdot)$; then

$$\hat{\mu}(u|\mathbf{x}_i) = \hat{\Delta}_i \sum_j b_j Y_{(j)} [1 - \hat{F}_0(Y_{(j)})]^{\hat{\Delta}_i - 1},$$

where the sum is over j with $\hat{F}_0(Y_{(j)}) \leq 1 - (1-u)^{\frac{1}{\hat{\Delta}_i}}$. Finally, $\hat{C}(u|\mathbf{x}) = \hat{\mu}(u|\mathbf{x}) / \hat{F}^{-1}(u|\mathbf{x})$ and $\hat{D}(u|\mathbf{x}) = \hat{C}(u|\mathbf{x}) / u$ where $\hat{F}^{-1}(u|\mathbf{x})$ is the estimate of the conditional quantile function obtained from (4.3) by replacing Δ_i with $\hat{\Delta}_i$.

Remark. We can obtain nonparametric estimates of $C(u|\mathbf{x})$ and $D(u|\mathbf{x})$ by using nonparametric estimates of $F^{-1}(u|\mathbf{x})$ in (4.3) and (4.4). These could then be compared with the estimates based on the semiparametric model (4.2). See Chaudhuri (1991) and Dabrowska (1992) for nonparametrically estimated $F^{-1}(u|\mathbf{x})$.

References

- Aaberge, R. (2000): Characterizations of Lorenz Curves and Income Distributions. *Social Choice and Welfare* **17**, 639-653.
- Aaberge, R. (1982), On the Problem of Measuring Inequality. (In Norwegian). Rapport 82/9, Statistics Norway.
- Anscombe, F.J. (1961), Examination of residuals. *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **1** 1-36. Univ. California Press, Berkeley.
- Barlow, R.E. and Proschan, F. (1965), *Mathematical Theory of Reliability*. Wiley, New York.
- Bickel, P.J. (1978), Using residuals robustly I: Tests for heteroscedasticity, nonlinearity, *Ann. Statist.* **6** 266-291.
- Bickel, P.J. and Lehmann, E.L. (1979), Descriptive measures for nonparametric models IV, Spread. In J. Juneckova (ed.): *Contributions to Statistics, Hajek Memorial Volume*. Reidel, London, 33-40.
- Birnbaum, S.W., Esary, J.D. and Marshall, A.W. (1966), A stochastic characterization of wear-out for components and systems, *Ann. Math. Statist.* **37** 816-826.
- Carroll, R.J. and Ruppert, D. (1982), Robust estimation in heteroscedastic linear models, *Ann. Statist.* **10** 429-441.
- Carroll, R.J. and Ruppert, D. (1988), *Transformation and Weighting in Regression*, Chapman and Hall, New York.
- Chaudhuri, P. (1991), Nonparametric estimates of regression quantiles and their local Bahadur representation, *Ann. Statist.* **19** 760-777.
- Cox, D.R. (1972), Regression models and life tables (with discussion), *J. R. Stat. Soc. B* **34** 187-220.
- Dabrowska, D. (1992), Nonparametric quantile regression with censored data, *Sankhya Ser. A* **54** 252-259.
- Doksum, K.A. (1969), Starshaped transformations and the power of rank tests. *Ann. Math. Statist.* **40**, 1167-1176.
- Gastwirth, J.L. (1971), A general definition of the Lorenz curve, *Econometrica* **39** 1037-1039.
- Giorgi, G.M. and Mondani, R. (1995), Sampling distribution of the Bonferroni inequality index from exponential population, *Sankhya* **57**, 10-18.
- Kalbfleisch, J.D. and Prentice, R.L. (2002), *The Statistical Analysis of Failure Time Data*, 2nd edition, New York: Wiley.
- Lehmann, E.L. (1953), The power of rank tests, *Ann. Math. Statist.* **24** 23-43.
- Lehmann, E.L. (1955), Ordered families of distributions, *Ann. Math. Statist.* **37** 1137-1153.

- Lorenz, M.C. (1905), Methods of measuring the concentration of wealth, *J. Amer. Statist.* **9** 209-219.
- Rojo, J. and He, G.Z. (1991), New Properties and Characterizations of the Dispersive Orderings. *Statistics and Probability Letters* **11**, 365-372.
- Rojo, J. (1992), A pure-tail ordering based on the ratio of the quantile functions. *Ann. Statist.* **20**, 570-579.
- Savage, I.R. (1956), Contributions to the theory of rank order statistics - the two-sample case, *Ann. Math. Statist.* **27** 590-615.
- Savage, I.R. (1980), *Lehmann alternatives*, Colloquia Mathematica Societatis János Bolyai, Nonparametric Statistical Inference, Proceedings, Budapest, Hungary.
- Shaked, M. and Shanthikumar, J.G. (1994), *Stochastic Orders and Their Applications*. Academic Press, San Diego.
- van Zwet, W.R. (1964), *Convex Transformations of Random Variables*, Math. Centre, Amsterdam.
- Yanagimoto, T. and Sibuya, M. (1976), Isotonic tests for spread and tail. *Annals of Statist. Math.* **28** 329-342.

Recent publications in the series Discussion Papers

- 320 T. J. Klette and A. Raknerud (2002): How and why do Firms differ?
- 321 J. Aasness and E. Røed Larsen (2002): Distributional and Environmental Effects of Taxes on Transportation
- 322 E. Røed Larsen (2002): The Political Economy of Global Warming: From Data to Decisions
- 323 E. Røed Larsen (2002): Searching for Basic Consumption Patterns: Is the Engel Elasticity of Housing Unity?
- 324 E. Røed Larsen (2002): Estimating Latent Total Consumption in a Household.
- 325 E. Røed Larsen (2002): Consumption Inequality in Norway in the 80s and 90s.
- 326 H.C. Bjørnland and H. Hungnes (2002): Fundamental determinants of the long run real exchange rate: The case of Norway.
- 327 M. Søberg (2002): A laboratory stress-test of bid, double and offer auctions.
- 328 M. Søberg (2002): Voting rules and endogenous trading institutions: An experimental study.
- 329 M. Søberg (2002): The Duhem-Quine thesis and experimental economics: A reinterpretation.
- 330 A. Raknerud (2002): Identification, Estimation and Testing in Panel Data Models with Attrition: The Role of the Missing at Random Assumption
- 331 M.W. Arneberg, J.K. Dagsvik and Z. Jia (2002): Labor Market Modeling Recognizing Latent Job Attributes and Opportunity Constraints. An Empirical Analysis of Labor Market Behavior of Eritrean Women
- 332 M. Greaker (2002): Eco-labels, Production Related Externalities and Trade
- 333 J. T. Lind (2002): Small continuous surveys and the Kalman filter
- 334 B. Halvorsen and T. Willumsen (2002): Willingness to Pay for Dental Fear Treatment. Is Supplying Fear Treatment Social Beneficial?
- 335 T. O. Thoresen (2002): Reduced Tax Progressivity in Norway in the Nineties. The Effect from Tax Changes
- 336 M. Søberg (2002): Price formation in monopolistic markets with endogenous diffusion of trading information: An experimental approach
- 337 A. Bruvold og B.M. Larsen (2002): Greenhouse gas emissions in Norway. Do carbon taxes work?
- 338 B. Halvorsen and R. Nesbakken (2002): A conflict of interests in electricity taxation? A micro econometric analysis of household behaviour
- 339 R. Aaberge and A. Langørgen (2003): Measuring the Benefits from Public Services: The Effects of Local Government Spending on the Distribution of Income in Norway
- 340 H. C. Bjørnland and H. Hungnes (2003): The importance of interest rates for forecasting the exchange rate
- 341 A. Bruvold, T.Fæhn and Birger Strøm (2003): Quantifying Central Hypotheses on Environmental Kuznets Curves for a Rich Economy: A Computable General Equilibrium Study
- 342 E. Biørn, T. Skjerpen and K.R. Wangen (2003): Parametric Aggregation of Random Coefficient Cobb-Douglas Production Functions: Evidence from Manufacturing Industries
- 343 B. Bye, B. Strøm and T. Åvitsland (2003): Welfare effects of VAT reforms: A general equilibrium analysis
- 344 J.K. Dagsvik and S. Strøm (2003): Analyzing Labor Supply Behavior with Latent Job Opportunity Sets and Institutional Choice Constraints
- 345 A. Raknerud, T. Skjerpen and A. Rygh Swensen (2003): A linear demand system within a Seemingly Unrelated Time Series Equation framework
- 346 B.M. Larsen and R.Nesbakken (2003): How to quantify household electricity end-use consumption
- 347 B. Halvorsen, B. M. Larsen and R. Nesbakken (2003): Possibility for hedging from price increases in residential energy demand
- 348 S. Johansen and A. R. Swensen (2003): More on Testing Exact Rational Expectations in Cointegrated Vector Autoregressive Models: Restricted Drift Terms
- 349 B. Holtmark (2003): The Kyoto Protocol without USA and Australia - with the Russian Federation as a strategic permit seller
- 350 J. Larsson (2003): Testing the Multiproduct Hypothesis on Norwegian Aluminium Industry Plants
- 351 T. Bye (2003): On the Price and Volume Effects from Green Certificates in the Energy Market
- 352 E. Holmøy (2003): Aggregate Industry Behaviour in a Monopolistic Competition Model with Heterogeneous Firms
- 353 A. O. Ervik, E.Holmøy and T. Hægeland (2003): A Theory-Based Measure of the Output of the Education Sector
- 354 E. Halvorsen (2003): A Cohort Analysis of Household Saving in Norway
- 355 I. Aslaksen and T. Synnestvedt (2003): Corporate environmental protection under uncertainty
- 356 S. Glomsrød and W. Taoyuan (2003): Coal cleaning: A viable strategy for reduced carbon emissions and improved environment in China?
- 357 A. Bruvold T. Bye, J. Larsson og K. Telle (2003): Technological changes in the pulp and paper industry and the role of uniform versus selective environmental policy.
- 358 J.K. Dagsvik, S. Strøm and Z. Jia (2003): A Stochastic Model for the Utility of Income.
- 359 M. Rege and K. Telle (2003): Indirect Social Sanctions from Monetarily Unaffected Strangers in a Public Good Game.
- 360 R. Aaberge (2003): Mean-Spread-Preserving Transformation.
- 361 E. Halvorsen (2003): Financial Deregulation and Household Saving. The Norwegian Experience Revisited
- 362 E. Røed Larsen (2003): Are Rich Countries Immune to the Resource Curse? Evidence from Norway's Management of Its Oil Riches
- 363 E. Røed Larsen and Dag Einar Sommervoll (2003): Rising Inequality of Housing? Evidence from Segmented Housing Price Indices
- 364 R. Bjørnstad and T. Skjerpen (2003): Technology, Trade and Inequality
- 365 A. Raknerud, D. Rønningen and T. Skjerpen (2003): A method for improved capital measurement by combining accounts and firm investment data

- 366 B.J. Holtsmark and K.H. Alfsen (2004): PPP-correction of the IPCC emission scenarios - does it matter?
- 367 R. Aaberge, U. Colombino, E. Holmøy, B. Strøm and T. Wennemo (2004): Population ageing and fiscal sustainability: An integrated micro-macro analysis of required tax changes
- 368 E. Røed Larsen (2004): Does the CPI Mirror Costs of Living? Engel's Law Suggests Not in Norway
- 369 T. Skjerpen (2004): The dynamic factor model revisited: the identification problem remains
- 370 J.K. Dagsvik and A.L. Mathiassen (2004): Agricultural Production with Uncertain Water Supply
- 371 M. Greaker (2004): Industrial Competitiveness and Diffusion of New Pollution Abatement Technology – a new look at the Porter-hypothesis
- 372 G. Børnes Ringlund, K.E. Rosendahl and T. Skjerpen (2004): Does oilrig activity react to oil price changes? An empirical investigation
- 373 G. Liu (2004) Estimating Energy Demand Elasticities for OECD Countries. A Dynamic Panel Data Approach
- 374 K. Telle and J. Larsson (2004): Do environmental regulations hamper productivity growth? How accounting for improvements of firms' environmental performance can change the conclusion
- 375 K.R. Wangen (2004): Some Fundamental Problems in Becker, Grossman and Murphy's Implementation of Rational Addiction Theory
- 376 B.J. Holtsmark and K.H. Alfsen (2004): Implementation of the Kyoto Protocol without Russian participation
- 377 E. Røed Larsen (2004): Escaping the Resource Curse and the Dutch Disease? When and Why Norway Caught up with and Forged ahead of Its Neighbors
- 378 L. Andreassen (2004): Mortality, fertility and old age care in a two-sex growth model
- 379 E. Lund Sagen and F. R. Aune (2004): The Future European Natural Gas Market - are lower gas prices attainable?
- 380 A. Langørgen and D. Rønningen (2004): Local government preferences, individual needs, and the allocation of social assistance
- 381 K. Telle (2004): Effects of inspections on plants' regulatory and environmental performance - evidence from Norwegian manufacturing industries
- 382 T. A. Galloway (2004): To What Extent Is a Transition into Employment Associated with an Exit from Poverty
- 383 J. F. Bjørnstad and E. Ytterstad (2004): Two-Stage Sampling from a Prediction Point of View
- 384 A. Bruvold and T. Fæhn (2004): Transboundary environmental policy effects: Markets and emission leakages
- 385 P.V. Hansen and L. Lindholt (2004): The market power of OPEC 1973-2001
- 386 N. Keilman and D. Q. Pham (2004): Empirical errors and predicted errors in fertility, mortality and migration forecasts in the European Economic Area
- 387 G. H. Bjertnæs and T. Fæhn (2004): Energy Taxation in a Small, Open Economy: Efficiency Gains under Political Restraints
- 388 J.K. Dagsvik and S. Strøm (2004): Sectoral Labor Supply, Choice Restrictions and Functional Form
- 389 B. Halvorsen (2004): Effects of norms, warm-glow and time use on household recycling
- 390 I. Aslaksen and T. Synnøve (2004): Are the Dixit-Pindyck and the Arrow-Fisher-Henry-Hanemann Option Values Equivalent?
- 391 G. H. Bjønnes, D. Rime and H. O.Aa. Solheim (2004): Liquidity provision in the overnight foreign exchange market
- 392 T. Åvitsland and J. Aasness (2004): Combining CGE and microsimulation models: Effects on equality of VAT reforms
- 393 M. Greaker and Eirik. Sagen (2004): Explaining experience curves for LNG liquefaction costs: Competition matter more than learning
- 394 K. Telle, I. Aslaksen and T. Synnøve (2004): "It pays to be green" - a premature conclusion?
- 395 T. Harding, H. O. Aa. Solheim and A. Benedictow (2004). House ownership and taxes
- 396 E. Holmøy and B. Strøm (2004): The Social Cost of Government Spending in an Economy with Large Tax Distortions: A CGE Decomposition for Norway
- 397 T. Hægeland, O. Raaum and K.G. Salvanes (2004): Pupil achievement, school resources and family background
- 398 I. Aslaksen, B. Natvig and I. Nordal (2004): Environmental risk and the precautionary principle: "Late lessons from early warnings" applied to genetically modified plants
- 399 J. Møen (2004): When subsidized R&D-firms fail, do they still stimulate growth? Tracing knowledge by following employees across firms
- 400 B. Halvorsen and Runa Nesbakken (2004): Accounting for differences in choice opportunities in analyses of energy expenditure data
- 401 T.J. Klette and A. Raknerud (2004): Heterogeneity, productivity and selection: An empirical study of Norwegian manufacturing firms
- 402 R. Aaberge (2005): Asymptotic Distribution Theory of Empirical Rank-dependent Measures of Inequality
- 403 F.R. Aune, S. Kverndokk, L. Lindholt and K.E. Rosendahl (2005): Profitability of different instruments in international climate policies
- 404 Z. Jia (2005): Labor Supply of Retiring Couples and Heterogeneity in Household Decision-Making Structure
- 405 Z. Jia (2005): Retirement Behavior of Working Couples in Norway. A Dynamic Programming Approach
- 406 Z. Jia (2005): Spousal Influence on Early Retirement Behavior
- 407 P. Frenger (2005): The elasticity of substitution of superlative price indices
- 408 M. Mogstad, A. Langørgen and R. Aaberge (2005): Region-specific versus Country-specific Poverty Lines in Analysis of Poverty
- 409 J.K. Dagsvik (2005) Choice under Uncertainty and Bounded Rationality
- 410 T. Fæhn, A.G. Gómez-Plana and S. Kverndokk (2005): Can a carbon permit system reduce Spanish unemployment?
- 411 J. Larsson and K. Telle (2005): Consequences of the IPPC-directive's BAT requirements for abatement costs and emissions
- 412 R. Aaberge, S. Bjerve and K. Doksum (2005): Modeling Concentration and Dispersion in Multiple Regression