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**Article**

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## FINANCIAL ECONOMICS | RESEARCH ARTICLE

# Behavioral heterogeneity and financial markets: Locked/crossed quotes under informationally efficient pricing

Youzong Xu<sup>1\*</sup> and Bo Li<sup>2</sup>

**Abstract:** Market professionals and regulators have been concerned by locked/crossed quotes (negative quoted spreads) for years. To ease the concerns that locked/crossed quotes may cause confusion or instability in financial markets and to promote market efficiency, the Securities and Exchange Commission implemented rules that ban locked/crossed quotes. We consider a competitive financial market that does not contain the factors considered by market professionals and regulators consider as the origins of locked/crossed quotes. We find that if there are at least three types of traders in this financial market, locked/crossed quotes can arise naturally under efficient pricing. These locked/crossed quotes reflect information transmitted in financial markets, facilitating price discovery. Hence, regulations banning locked/crossed quotes are inappropriate, as these regulations may harm the efficiency of price discovery.

**Subjects:** Microeconomics; Finance; Investment & Securities

**Keywords:** adverse selection; locked/crossed quotes; behavioral heterogeneity

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This paper contributes to the literature on microstructure of financial markets. For market participants and regulators, the research in this paper provides useful suggestions on how to improve market efficiency through proper regulations. For researchers, though this paper mainly focuses on analyzing locked/crossed quotes in financial markets, the framework constructed by this paper can be applied to study other phenomena (e.g. volatility and liquidity) in financial markets.

### PUBLIC INTEREST STATEMENT

Conventional views consider locked/crossed quotes (non-positive quote spreads) as detrimental occurrences. To ease the concerns that locked/crossed quotes may cause confusion or instability in financial markets, the Securities and Exchange Commission implemented rules that ban locked/crossed quotes, in the belief that these rules would promote market efficiency. However, this paper shows that, even in competitive financial markets that do not contain the inefficiencies that conventional views consider as the causes of non-positive quote spreads, locked/crossed quotes can still arise for purely informational reasons under efficient pricing. These locked/crossed quotes caused by information reasons reflect information transmitted in financial markets, facilitating price discovery. Thus, regulations banning locked/crossed quotes are inappropriate, as these regulations block information transmission in financial markets and then harm the efficiency of price discovery. These regulations should be replaced by those that eliminate only the locked/crossed quotes caused by inefficiencies in financial markets.

## 1. Introduction

The phenomena known as locked/crossed quotes have puzzled and concerned market professionals and regulators for years. A locked (crossed) quote is a situation in which the bid price of a financial asset equals (exceeds) its ask price. Though locked/crossed quotes seem counter-intuitive, they occur with surprising frequency in financial markets. For example, in the equity market, an average active stock on the New York Stock Exchange (NYSE) is locked or crossed about 4.05% of the time, and an average active stock on the Nasdaq is locked or crossed about 10.58% of the time (Shkilko, Van Ness, & Van Ness, 2008); in the foreign exchange market of seven major currency pairs,<sup>1</sup> the daily average frequency of locked quotes is between 3.8 and 6.1%, and the daily average frequency of crossed quotes is between 16.4 and 25.5% (Tran, 2013).

Market professionals and regulators attribute the occurrence of locked/crossed quotes to errors or inefficiency in pricing, inappropriate or illegal trading behavior, and faulty trading mechanisms,<sup>2</sup> arguing that these phenomena may impair price discovery and deteriorate the market quality (e.g. Clark, Dietrich, & Ng, 2009; Schmerken, 2003). Some researchers consider locked/crossed quotes to be inconsistent with an efficient market (e.g. Battalio, Hatch, & Jennings, 2004; Storkenmaier & Wagener, 2011). Researchers who use financial data often consider locked/crossed quotes as wrong or abnormal records and remove them from datasets (e.g. Chordia, Roll, & Subrahmanyam, 2001; Collver, 2009). To ease the concerns that locked/crossed quotes may cause confusion or even havoc in financial markets,<sup>3</sup> the Securities and Exchange Commission implemented a ban on locked/crossed quotes in 2007, in the belief that these rules would promote market efficiency.

However, the conventional perception of non-positive quote spreads as detrimental occurrences is challenged by some researchers. For example, Shkilko et al. (2008) suggest that market participants in the Nasdaq and NYSE engage in price discovery through locked and crossed quotes, and use locked and crossed quotes to avoid stale quotes and prevent liquidity shortages on electronic limit-order books. They do not find sufficient evidence to support the claim that locked/crossed quotes cause market quality deterioration in Nasdaq and NYSE. Tran (2013) studies locked/crossed quotes in foreign exchange markets of seven major currency pairs and finds that crossed quotes contribute more to price changes than locked quotes.

The main focus of the extant literature on locked/crossed quotes, which is all empirical, is the origin of these phenomena in different financial markets. We are interested, instead, in whether locked/crossed quotes appear in efficient financial markets that do not contain the factors considered by conventional views and extant studies as the causes of these phenomena. If the answer to this question is in the negative, regulations banning locked/crossed quotes may improve market efficiency; however, if the answer is in the positive, regulations banning locked/crossed quotes may harm market efficiency.

To investigate this question, we consider an abstract financial market lacking the factors that conventional views and extant studies consider as the origination for locked/crossed quotes. We find that, if there are at least three types of traders in this market, locked/crossed quotes can arise naturally due to adverse selection under efficient pricing. Since these locked/crossed quotes occur for informational reasons, they reflect information transmitted in financial markets and facilitate price discovery.

Hence, regulations *banning* locked/crossed quotes are inappropriate, and these regulations may actually harm the efficiency of price discovery.

Specifically, we consider a sequential trading model of dealership markets with asymmetric information, in the tradition of Glosten and Milgrom (1985). There is a risky asset in this market. At the beginning of each period, a market maker (*she*) posts a bid price and an ask price for this asset; subsequently, a trader (*he*) enters the market to trade with her at the posted prices. The market maker performs no brokerage services<sup>4</sup> and makes zero expected profit, subject to competition.<sup>5</sup>

These settings ensure efficient pricing and preclude inappropriate or illegal trading behavior, large volumes of excess buy orders, electronically unreachable quotes, and other factors that market professionals, regulators, and extant studies consider as the causes of locked/crossed quotes. As in Glosten and Milgrom (1985), we assume that both the market maker and traders observe all historical transactions, but only traders have private information about the value of the risky asset. Thus, the market maker faces an adverse selection problem: a trader agreeing to trade at the bid or ask price that she posts maybe because he has some information that she does not.

In contrast to the standard Glosten–Milgrom type model, which involves only two types of traders, *sophisticated* and *liquidity*, we introduce a third type of traders. We refer to the case in which there are (at least) three types of traders who use information differently in the market as *behavioral heterogeneity*. As in the standard Glosten–Milgrom type model, a sophisticated trader in our model correctly use both his private information and the information implicit in historical transactions to make decisions; a liquidity trader trades in accordance with his need for liquidation, acting as if all available information were irrelevant.

Behavioral heterogeneity is necessary for the occurrence of locked/crossed quotes in this financial market. To see this, note that quote spreads are always positive in financial markets without behaviorally heterogeneous traders, as Glosten and Milgrom (1985) point out. Consider the standard Glosten–Milgrom type model involving only two types of traders, sophisticated, and liquidity. In a financial market of this type, the market maker faces an expected loss from trading with a sophisticated trader, since she has a relative information disadvantage.<sup>6</sup> Under the zero-profit condition, she must set prices in such a way that she earns a positive expected profit from trading with a liquidity trader to compensate for her expected loss from trading with a sophisticated trader. The market maker's profit from selling one unit of the asset to a liquidity trader equals the ask price minus the market value of the asset, and her profit from purchasing one unit of the asset from a liquidity trader equals the market value of the asset minus the bid price (Glosten & Milgrom, 1985). Thus, that the market maker earns a positive profit from trading with a liquidity trader implies that she must set the ask price greater than the market value of the asset, and set the bid price lower than the market value of the asset.<sup>7</sup> Hence, the ask price exceeds the bid price, implying that the prices cannot be locked or crossed.<sup>8</sup>

However, once there is a third type of traders present in this financial market, locked/crossed quotes become possible: when the market maker's expected gain from trading with a trader of the third type exceeds her expected loss from trading with a sophisticated trader, the zero-profit condition requires that she must set prices in such a way that she receives a negative expected profit from trading with a liquidity trader. Thus, she must set the ask price lower than the market value of the asset and set the bid price greater than the market value of the asset. Hence, the bid price exceeds the ask price, that is, a locked or crossed quote occurs.

The above discussion shows that behavioral heterogeneity is a necessary condition for the occurrence of locked/crossed quotes. In Section 4, we provide sufficient conditions under which locked/crossed quotes appear in financial markets with behaviorally heterogeneous traders, given that the third type of traders is *naive* in the sense of only following their private information when making decisions.

In this paper, we assume that the third type of traders is naive traders for two reasons. First, the presence of naive traders in financial markets has empirical support (e.g. Cipriani & Guarino, 2005; Park & Sgroi, 2009). Second (and more importantly), changing trader types or introducing further types of traders does not fundamentally affect the results of this paper, but makes technical analysis more complicated. Thus, to better illustrate the intuition of this paper and keep technical analysis more tractable, we assume that the trader population consists of three particular types (sophisticated, liquidity, and naive).

The financial market that we construct in this paper does not contain the factors that conventional views and extant studies consider as the causes for locked/crossed quotes. Locked/crossed quotes occur in this financial market because of adverse selection under efficient pricing; that is, they occur for purely informational reasons and reflect information transmitted in the market. Thus, these locked/crossed quotes facilitate price discovery. These findings have an important policy implication: regulations aiming to promote market efficiency must be able to distinguish locked/crossed quotes caused by inefficiencies in financial markets (e.g. errors or inefficiency in pricing, inappropriate or illegal trading behavior, and faulty trading mechanisms) from those that occur for informational reasons. Though eliminating locked/crossed quotes caused by inefficiencies in financial markets, eliminating locked/crossed quotes that occur for informational reasons harms market efficiency, as these latter reflect information transmitted in financial markets, implying that blocking them would impair the efficiency of price discovery. Thus, the regulations banning all locked/crossed quotes are inappropriate, and should be replaced by regulations that only eliminate locked/crossed quotes caused by inefficiencies in financial markets; in other words, to remove those inefficiencies that cause these locked/crossed quotes.

Two important points should be clarified before proceeding. First, crossed quotes do not necessarily imply arbitrage opportunities. If crossed quotes disappear very quickly, traders cannot take advantage of them. This paper assumes that a trader can only take one action (buy or sell or hold) when trading with the market maker during a period; this mimics reality, in that a trader can take only one action at an exact moment or during a very short period. Thus, if crossed quotes disappear quickly, traders cannot profit from them. In each period, the bid price and ask price change in response to the information brought in by the trader entering the market at that period, which changes the quote spread and resolve crossed quotes. Thus, crossed quotes caused by informational factors disappear quickly.

While in real financial markets, some crossed quotes that last relatively long (up to a few minutes or a few hours), as Shkilko et al. (2008) point out, these long-lasting crossed quotes are also not real arbitrage opportunities. Shkilko et al. (2008) identify the origins of these long-lasting crossed quotes and explain why traders either do not intend or are unable to take advantage of these specious arbitrage opportunities.<sup>9</sup> Note that those long-lasting crossed quotes never appear in our model, because their causes are precluded by the setup of the model.

Second, besides providing an explanation for the occurrence of locked/crossed quotes, our model can also be applied to explain other interesting phenomena in financial markets. As an example, in Section 4, we provide sufficient conditions for the occurrence of “over-optimism,” a form of bubble, in which both the bid price and the ask price of an asset exceed the market value of this asset.

This paper is organized as follows. Section 2 reviews related literature. Section 3 details the basic setup of the model. Section 4 characterizes necessary and sufficient conditions for the occurrence of locked/crossed quotes; this section also provides sufficient conditions under which over-optimism occurs. Section 5 concludes and discusses potential extensions. All proofs are in Appendix 1.

## 2. Related literature

The issue of locked and crossed quotes is an area of concern, not only for market professionals and regulators, but also for academic researchers.

Much empirical literature using financial data considers locked and crossed quotes to arise from errors in pricing or anomalous records (e.g. Eleswarapu & Venkataraman, 2006; Hameed, Kang, & Viswanathan, 2010), and discards them from data (e.g. Chordia et al., 2001; Collver, 2009). Some other literature considers locked/crossed to be inconsistent with an efficient market (e.g. Battalio et al, 2004; Storckenmaier & Wagener, 2011).

A few researchers have a more positive attitude on locked/crossed quotes. Aggarwal and Conroy (2000) argue that locked/crossed quotes occur because of large volumes of excess buy orders. They find that a wholesaler, who has retail buy orders that exceed the sell orders, posts locked or crossed quotes to induce other market makers to change their quotes, enabling this wholesaler to execute the order flow that he received at the equilibrium price. Cao, Ghysels, and Hatheway (2000) find that, in the preopening period of the Nasdaq, dealers often lock and cross the market to signal the direction and magnitude of price movements. Shkilko et al. (2008) identify six origins of locked/crossed quotes in the Nasdaq and NYSE, based on which they suggest that locked/crossed quotes should be viewed as a mechanism that “allows market participants to cope with today’s increasingly competitive and fragmented trading environment.” Tran (2013) studies the causes and consequences of locked/crossed quotes in foreign exchange markets, and suggests that crossed quotes contribute more to price changes than locked quotes.

To our best knowledge, all extant studies analyzing locked/crossed quotes are empirical. We have not found other theoretical work examining locked/crossed quotes. The theoretical settings that are closest to ours are those of Glosten and Milgrom (1985), Avery and Zemsky (1998), and Park and Sabourian (2011).

Glosten and Milgrom (1985) construct an elegant setting to model the dynamics of matching buyers and sellers. In this seminal work, Glosten and Milgrom (1985) find that adverse selection leads to a positive bid-ask spread even when the market maker is risk-neutral and makes zero expected profits. The fundamental difference between their model and ours is that their model involves only two types of traders, whereas our model accommodates (at least) three types. As we show in later sections, this difference is the key reason for why locked/crossed quotes can occur in our model but not in theirs.

Avery and Zemsky (1998) and Park and Sabourian (2011) also adopt the Glosten–Milgrom framework. They focus on informational herding in competitive, efficient financial markets and provide necessary and sufficient conditions for herding behavior to occur. Our model is related to theirs, not only because we consider the Glosten–Milgrom type of financial markets, but also in the sense that the results in our paper are related to the herding behavior of sophisticated traders.<sup>10</sup> Despite that, locked/crossed quotes do not appear in their models, because they involve only two types of traders.

The introduction of naive traders has empirical support. In their experiments studying trading behavior in financial markets, Cipriani and Guarino (2005) observe that some participants only follow their private information even when information cascades have occurred. Park and Sgroi (2009) observe that a non-negligible proportion of the participants in their experiments adhere to their private information rather than following the developments of prices, while some other traders carefully follow these developments. Such “naive” actions have also been observed in experiments on other topics. For example, Guarnaschelli, McKelvey, and Palfrey (2000) observe that a positive proportion of participants in their experiment on collective decisions always only follow their private information.

### 3. The model

We consider a dealership financial market in the tradition of Glosten and Milgrom (1985), that is, a sequential trading model with asymmetric information between traders (*he*) and a market maker (*she*).

In this financial market, there is a single risky asset with value  $V$ .  $V$  is the true value, or liquidation value, of this asset, drawn by Nature at the very beginning of the game from the set of potential values  $\mathbb{V} = \{V_1, V_2, V_3\}$ . Let  $\Pr(\cdot)$  denote the prior distribution on  $\mathbb{V}$ . We assume that  $V_1 = 0$ ,  $V_2 = \mathcal{V}$ , and  $V_3 = 2\mathcal{V}$ , where  $\mathcal{V} > 0$ , and that  $\Pr(\cdot)$  is symmetric around  $V_2$ , that is,  $\Pr(V_1) = \Pr(V_3)$ .<sup>11</sup> When the game ends, each unit of the asset returns payoff  $V$  to its holder; but before the end of the game,  $V$  cannot be observed directly.



At the beginning of each period  $t$ ,<sup>12</sup> prior to the arrival of a trader, the market maker posts a bid price  $\text{bid}^t$  and an ask price  $\text{ask}^t$ . Subsequently, a trader enters the market to trade with her at the posted prices. Traders enter the market in an exogenous and random sequence. Each trader can only trade once when he is in the market. When a trader is in the market, he can either *buy* or *sell* one unit of the asset, or *hold*. These settings eliminate the possibility of excess demand or supply of the asset in each period; that is, there is no influence on prices caused by order volumes.

Let  $a^t$  denotes the action that the trader who enters the market at period  $t$  takes, and  $p^t$  denotes the transaction price of the asset at period  $t$ . Let  $H^t = ((a^1, p^1), (a^2, p^2), \dots, (a^{t-1}, p^{t-1}))$ ,  $t > 1$ , denote the history at period  $t$ . Historical transactions are public information to traders and the market maker. Moreover, traders have private information about the value of the asset: right before entering the market, each trader receives a private signal about the true value of the asset. Signals are noisy and follow a conditionally independent distribution. The set of possible signals is  $\mathbb{S} = \{S_1, S_2, S_3\}$ , with  $E[V|S_1] < E[V|S_2] < E[V|S_3]$ .  $\Pr(S_i|V_j)$  is the probability of signal  $S_i$  if the true value of the asset is  $V_j$ . We sometimes refer to  $S_1$  as the “low” signal,  $S_2$  as the “medium” signal, and  $S_3$  as the “high” signal. The market maker does not have private information about the value of the asset, but rather, the information implicit in historical transactions.

The risk-neutral market maker is subject to competition and thus makes zero expected profits. At period  $t$ , she must set the bid price and ask price as  $\text{bid}^t = E[V|a^t = \text{sell at bid}^t, H^t]$  and  $\text{ask}^t = E[V|a^t = \text{buy at ask}^t, H^t]$  in equilibrium, which means that the market maker makes zero profit both when she buys from and when she sells to a trader.<sup>13</sup> This assumption ensures that both bid price and ask price at each period are efficient. In the rest of this paper, when we say that the market maker earns zero profit from trading, we mean that she makes zero profit both when she sells to and when she buys from a trader.

The trader population is *behaviorally heterogeneous*, in the sense that there are three types of traders: *sophisticated*, *liquidity*, and *naive*. At each period, the prior probability that the entering trader is sophisticated is  $\mu > 0$ , that he is naive is  $\theta > 0$ , and that he is liquidity is  $1 - \mu - \theta > 0$ .

A sophisticated trader takes the action that maximizes his expected profit, conditioning on both his private information and the information inferred from the whole transaction history. That is, a sophisticated trader who enters the market at period  $t$  use both his private signal  $S^t$  and the transaction history  $H^t$  to update his belief about the value of the asset. Let  $E[V|H^t, S^t]$  denote this sophisticated trader’s expectation of  $V$ . Then he buys if and only if  $E[V|H^t, S^t] - \text{ask}^t > \max\{\text{bid}^t - E[V|H^t, S^t], 0\}$ , sells if and only if  $\text{bid}^t - E[V|H^t, S^t] > \max\{E[V|H^t, S^t] - \text{ask}^t, 0\}$ , and buys or sells with equal probability if  $E[V|H^t, S^t] - \text{ask}^t = \text{bid}^t - E[V|H^t, S^t] > 0$ . Otherwise, he holds. In other words, a sophisticated trader chooses to trade with the market maker only if he makes a strictly positive expected profit from trading.

Liquidity traders trade according to their needs for liquidation; thus, they act as if they ignore both their private information and the information implicit in historical transactions when trading. With equal probabilities, Nature randomly assigns a specific need for liquidation (purchase, sale, or hold) to a liquidity trader before he enters the market. Thus, at each period, the market maker meets a liquidity trader with one of these three different needs for liquidation with equal probability  $\frac{1-\mu-\theta}{3}$ .

The third type of traders, naive traders, only follow their own private information when making decisions. More specifically, a naive trader sells to the market maker if he receives the low signal  $S_1$ , holds if he receives the medium signal  $S_2$ , and buys from the market maker if he receives the high signal  $S_3$ .<sup>14</sup>

Our assumption on behavioral heterogeneity involves three types of traders (sophisticated, naive, and liquidity) in this financial market, which can be generalized to accommodate further types of traders, or three other types of traders. However, such generalization of our setting does not fundamentally affect the main results of this paper, but makes the technical analysis more complicated and tedious. Thus, we assume the three particular types of traders (sophisticated, naive, and liquidity) to render the technical analysis tractable. We illustrate the intuition for the generalization of our model in Section 4.

#### 4. Locked/crossed quotes

This section is organized as follows. We first show that behavioral heterogeneity is a necessary condition for the occurrence of locked/crossed quotes under efficient pricing in a competitive financial market (Theorem 1). Then we provide a sufficient condition under which locked/crossed quotes appear (Theorem 2). We conclude by discussing an application of our setup to explain the occurrence of another interesting phenomenon, over-optimism, in financial markets (Theorem 3).

**THEOREM 1** *In a competitive financial market with efficient pricing, locked/crossed quotes occur only if traders are behaviorally heterogeneous; that is, besides sophisticated traders and liquidity traders, there is at least a third type of traders who use information differently than sophisticated traders and liquidity traders.*

As Glosten and Milgrom (1985) point out, if there are only sophisticated traders and liquidity traders in the market, quote spreads are always positive: at each period  $t$ , the market maker needs to make a strictly positive profit from trading with a liquidity trader to compensate for her expected loss from trading with a sophisticated trader, which means that she must set the bid price and ask price as  $\text{ask}^t > E[V|H^t] > \text{bid}^t$ . To see this, recall that a liquidity trader's action reveals no information, which implies that when she is trading with a liquidity trader, the market maker can only rely on the information implicit in the transaction history to update her belief about the value of this asset. Thus, at period  $t$ , the market maker's expectation of the value of the asset when facing a liquidity trader is  $E[V|H^t]$ , which we refer to as the *market value* of the asset. Then the market maker's expected profit from selling to a liquidity trader is  $\text{ask}^t - E[V|H^t]$  and her expected profit from buying from him is  $E[V|H^t] - \text{bid}^t$ . Since the market maker always faces an expected loss both when she is buying from and when she is selling to a sophisticated trader, she must set prices in such a way that  $\text{ask}^t - E[V|H^t] > 0$  and  $E[V|H^t] - \text{bid}^t > 0$  to compensate her expected loss from trading with a sophisticated trader, as the zero-profit condition requires. Thus,  $\text{ask}^t > E[V|H^t] > \text{bid}^t$ ; that is, the quote spread is positive.

If there is only one type of traders in the market, prices never change. As we show in Appendix 1, if trader population consists only of sophisticated traders, then  $\text{ask}^t = E[V|S_3]$  and  $\text{bid}^t = E[V|S_1]$ , both of which are constant over time. Note that if there are sophisticated traders in the market, quote spreads are always positive.

If, instead, the trader population consists only of liquidity traders, then  $\text{ask}^t = \text{bid}^t = E[V]$  at every period, as shown in Appendix 1. Though  $\text{ask}^t = \text{bid}^t = E[V]$  means that quotes are locked at each period, we ignore this situation because unchanging prices conflict with the frequent price movements that we observe in reality.

From above, we can see that in a competitive financial market, if the trader population is not behaviorally heterogeneous, by which we mean that there are fewer than three types of traders present in the market, locked/crossed quotes never occur. Thus, if locked/crossed quotes appear, there must be (at least) a third type of traders.

When there is a third type of traders (naive traders) present in the market, it becomes possible that there is a period at which the market maker's expected gain from trading with a naive trader exceeds her expected loss from trading with a sophisticated trader. At this period, the market maker must set the bid price and the ask price in such a way that she faces a negative profit from trading



with a liquidity trader, as the zero-profit condition requires. As discussed earlier, the market maker's expected profit from selling to a liquidity trader is  $\text{ask}^t - E[V|H^t]$  and her expected profit from buying from him is  $E[V|H^t] - \text{bid}^t$ . Thus, to earn a negative profit from trading with a liquidity trader, the market maker must set the ask price lower than  $E[V|H^t]$ , and set the bid price greater than  $E[V|H^t]$ . Therefore,  $\text{ask}^t \leq E[V|H^t] \leq \text{bid}^t$ ; that is, a locked or crossed quote occurs. Theorem 1 provides a sufficient condition under which locked/crossed quotes occur with a positive probability.

Before proceeding further, it is helpful to give some characterization of signals. Signals can have different conditional distributions:

**Increasing:**  $Pr(S|V_1) \leq Pr(S|V_2) \leq Pr(S|V_3)$ , with at least one strict inequality

**Uninformative:**  $Pr(S|V_1) = Pr(S|V_2) = Pr(S|V_3)$

**Decreasing:**  $Pr(S|V_1) \geq Pr(S|V_2) \geq Pr(S|V_3)$ , with at least one strict inequality

**U-shaped:**  $Pr(S|V_1) > Pr(S|V_2), Pr(S|V_3) > Pr(S|V_2)$

**Hill-shaped:**  $Pr(S|V_1) < Pr(S|V_2), Pr(S|V_3) < Pr(S|V_2)$

Bias of the conditional distribution of signal  $S$  is defined as  $Pr(S|V_3) - Pr(S|V_1)$ . A signal with a hill-shaped conditional distribution and a negative bias is called *nhill-shaped*, and a signal with a hill-shaped conditional distribution and a positive bias is called *phill-shaped*. Similar definitions apply to *nU-shaped* signals and *pU-shaped* signals. With these above notations and definitions, we can now characterize sufficient conditions under which locked/crossed quotes exist. Recall that  $\mu$  is the prior probability that the market maker meets a sophisticated trader, and  $\theta$  is the prior probability that she meets a naive trader at each period.

**THEOREM 2** Suppose that  $\mu \leq \theta$ ,  $S_1$  is *nU-shaped*,  $S_2$  is *uninformative*, and  $S_3$  is *phill-shaped*, then locked/crossed quotes exist.

The proof for Theorem 2 consists of two steps. In the first step, we characterize a sufficient condition under which locked/crossed quotes occur at a specific type of histories (Proposition 1). Note that this specific type of histories may not be the only type of histories at which locked/crossed quotes occur.

**Proposition 1** Suppose that  $\mu \leq \theta$ ,  $S_1$  is *nU-shaped*,  $S_2$  is *uninformative*, and  $S_3$  is *phill-shaped*. Then at any history  $H^t$  such that  $E[V|H^t, S_1] \geq E[V|H^t, S_2] \geq E[V|H^t, S_3]$ , we have  $\text{bid}^t \geq \text{ask}^t$ . That is, a locked or crossed quote occurs.

In the second step, we characterize a sufficient condition under which there exists at least one history  $H^t$  satisfying the requirement in Proposition 1, that is,  $E[V|H^t, S_1] \geq E[V|H^t] \geq E[V|H^t, S_3]$  at  $H^t$  (Proposition 2). These two steps together complete the proof for Theorem 2.

**Proposition 2** Suppose that  $\mu \leq \theta$ ,  $S_1$  is *nU-shaped*,  $S_2$  is *uninformative*, and  $S_3$  is *phill-shaped*. Then there exists some history  $H^t$  such that  $E[V|H^t, S_1] \geq E[V|H^t] \geq E[V|H^t, S_3]$ .

Note that our finding that locked/crossed quotes exist in markets with behaviorally heterogeneous traders continues to hold if we change trader types in this paper or introduce further types of traders into the market, given that the following two assumptions are maintained: (1) besides sophisticated traders, there are at least two other types of traders present in this market; and (2) some traders are "uninformative" in the sense that either they do not have private information, or they are liquidity traders who ignore their private information because of liquidation issues. In either case, an "uninformative" trader's action reveals nothing about his private information, that is, his action is uninformative, bringing no information into this market.

Note that an uninformative trader who does not have private information may still use “some” information inferred from historical transactions to make decisions, which is the key difference between this trader and a liquidity trader. Here “some” information means that an uninformative trader may use a different method than a sophisticated trader to glean information from the transaction history. For example, an uninformative trader may only infer information from transaction records of the last few periods (say, the last two or three periods) due to resource constraints (he can only observe transactions of the last few periods) or time constraints (he only has time to analyze a small number of transaction records), rather than from transaction records of all previous periods as a sophisticated trader does.

To illustrate the intuition for our claim that the ideas of this paper remain valid in the generalized version of our model, let us consider a financial market with  $N \geq 3$  types of traders who use information differently. Let Type 1 denotes uninformative traders, Type 2 denotes sophisticated traders, and the other  $N - 2$  types of traders denote those who use information differently than uninformative traders and sophisticated traders.<sup>15</sup>

Since an uninformative trader’s action brings no information into the market, if she is facing an uninformative trader, the market maker can only rely on the information implicit in historical transactions to update her belief about the value of the asset. That is, at period  $t$ , if the trader entering the market is an uninformative trader, the market maker’s expectation of the value of the asset is  $E[V|H^t]$ . Thus, her expected profit from selling to an uninformative trader is  $\text{ask}^t - E[V|H^t]$ , and her expected profit from purchasing from him is  $E[V|H^t] - \text{bid}^t$ .

Consider a period at which the market maker’s total expected profit from trading with the last  $N - 2$  types of traders (Type 3, ..., Type  $N$ ) exceeds her expected loss from trading with a sophisticated trader. Then at this period, the market maker must set prices in such a way that she makes an expected loss from trading with an uninformative trader, as the zero-profit condition requires. Thus, the prices she sets must satisfy  $\text{ask}^t - E[V|H^t] \leq 0$  and  $E[V|H^t] - \text{bid}^t \leq 0$ . Then  $\text{ask}^t \leq \text{bid}^t$ , which implies a locked or crossed quote.

The finding that locked/crossed quotes can arise due to adverse selection means that these phenomena can occur for purely informational reasons, which has an important policy implication: regulations targeting locked/crossed quotes must be able to distinguish locked/crossed quotes that occur for information reasons such as those in this paper from those that occur because of inefficiencies in financial markets (e.g. errors or inefficiency in pricing, inappropriate or illegal trading behavior, and faulty trading mechanisms). To promote market efficiency, the latter should be eliminated, while the former should not. Locked/crossed quotes that occur for informational reasons facilitate price discovery, as they reflect information transmitted in financial markets. Eliminating these locked/crossed quotes will harm the efficiency of price discovery. Thus, the regulations banning all locked/crossed quotes are inappropriate. Proper regulations should aim to eliminate only the locked/crossed quotes caused by inefficiencies in financial markets; in other words, to remove those inefficiencies that cause these locked/crossed quotes.

Behavioral heterogeneity, as discussed above, provides one explanation for the occurrence of locked/crossed quotes. It may also play an important role in the occurrence of other interesting phenomena in financial markets. For example, over-optimism, a form of bubble in which both the bid price and ask price of an asset exceed the market value of this asset, can also arise naturally under efficient pricing in financial markets with behaviorally heterogeneous traders, while it can never occur in financial markets without behaviorally heterogeneous traders. Theorem 3 provides a sufficient condition under which over-optimism can occur.

**THEOREM 3** *If  $S_1$  is nU-shaped,  $S_2$  is decreasing, and  $S_3$  is phill-shaped, then for any  $\xi \in (0, 1)$ , there exists  $\bar{\mu}_\xi \in (0, 1)$  such that for  $\mu, \theta \in (0, \xi)$  with  $\mu + \theta = \xi$  and  $\mu < \bar{\mu}_\xi$ , there is some history at which either a locked/crossed quote occurs, or over-optimism occurs.*

Over-optimism occurs if the market maker's expected gain from purchasing from a naive trader exceeds her expected loss from purchasing from a sophisticated trader, but her expected gain from selling to a naive trader cannot fully compensate for her expected loss from selling to a sophisticated trader. In this case, the zero-profit condition requires the market maker to set prices in such a way that she makes an expected loss if she buys from a liquidity trader but earns a positive profit if she sells to him. Thus, the market maker must set both the bid price and the ask price greater than the market value of the asset. Hence, over-optimism occurs.

## 5. Concluding remarks

In this paper, we consider a competitive financial market that does not contain the factors that conventional views and extant studies consider as the causes of locked/crossed quotes. We find that locked/crossed quotes arise naturally due to adverse selection under efficient pricing in competitive financial markets. Since these locked/crossed quotes occur for informational reasons, they reflect information transmitted in financial markets and facilitate price discovery. Thus, market participants should consider locked/crossed quotes that occur for informational reasons as a method of price discovery rather than detrimental occurrences.

Regulations banning locked/crossed quotes are inappropriate, as they block not only locked/crossed quotes caused by inefficiencies in financial markets but also those that occur for informational reasons. Blocking these latter would impair information transmission in financial markets, harming the efficiency of price discovery. Thus, proper regulations should aim to eliminate only locked/crossed quotes caused by inefficiencies in financial markets

There are multiple extensions of our model. One possible extension is to investigate how behavioral heterogeneity affects volatility and liquidity in financial markets: the presence of naive traders changes the richness of available information in the market, affecting traders' beliefs about market participants' (both traders' and the market maker's) beliefs about the value of the asset, and hence affecting prices. Specifically, the presence of naive traders imposes two opposite effects on the richness of available information in the market. On the one hand, as naive traders' actions truthfully reveal their private information, the presence of naive traders enriches available information in the market by enriching the information implicit in the transaction history. On the other hand, as the information implicit in historical transaction becomes richer, sophisticated traders put less weight on their private information, which means that their actions reveal less about their private information. Thus, the available information in the market becomes less rich. The exact influence of naive traders on the richness of available information in the market, and from there, on volatility and liquidity depends on which of these two effects dominates. We are currently working on this.

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### Notes

1. These seven currency pairs are EUR/USD, USD/JPY, GBP/USD, AUD/USD, USD/CHF, USD/CAD, and NZD/USD.
2. U.S. Securities and Exchange Commission (2005, p. 11158): ... this (a locked/crossed quote) indicates either that one or the other's quote is not valid, that brokers are not diligently representing their clients, or that inefficiencies exist that deter trading with the quoting market.
3. Fulfilling the Promise of the National Market System, STA's Perspective on U.S. Market Structure, Aug 2003.
4. Then all orders are market orders.
5. This zero-profit assumption can be relaxed to allow the market maker to earn a fixed amount of expected profit at each period without affecting the findings in this paper.
6. Though that both the market maker and sophisticated traders can use the information implicit in the transaction history, sophisticated traders have private information about the value of the asset, but the market maker does not.
7. The zero-profit condition in our model, as in other models that follow the tradition of Glosten and Milgrom (1985), requires the market maker to make zero profit both when she is selling to and when she is buying from a trader, which ensures that both the bid price and ask price are efficient.
8. As we show in Section 4, neither bid nor ask prices change in financial markets involving only a single type of traders. We ignore the cases in which there is only one type of traders in the market, because unchanging prices conflict with the frequent price movements that we observe in reality.
9. For example, traders in the NYSE inter-market are entitled to ignore 100-share NBBO (National Best Bid and Offer) quotes, as these quotes are not considered economic. Crossed quotes caused by these ignored 100-share NBBO quotes can last up to a few hours because traders do not want to take advantage of them (Shkilko et al., 2008).
10. Locked/crossed quotes occur in our model only if a sophisticated engages in herding behavior.
11. We assume symmetry on  $V$  and  $\Pr(\cdot)$  for technical simplicity. The ideas of this paper remain valid without these symmetry assumptions.
12. A period can be arbitrarily short, say one second or even one millisecond.
13. The assumption that the market maker makes zero profit under competition can be relaxed to that she earns fixed, positive expected profits in each period, that is, she sets  $\text{bid}^t = E[V|a^t = \text{sell at bid}^t, H^t] + \pi_b$  and  $\text{ask}^t = E[V|a^t = \text{buy at ask}^t, H^t] + \pi_o$ , where  $\pi_b > 0$  and  $\pi_o > 0$  are constant.  $\pi_b$  and  $\pi_o$  do not need to be equal.
14. This assumption can be changed to that a naive trader who enters the market at period  $t$  with signal  $S^t$  buys if and only if  $E[V|S^t] - \text{ask}^t > \max\{\text{bid}^t - E[V|S^t], 0\}$ , sells if and only if  $\text{bid}^t - E[V|S^t] > \max\{E[V|S^t] - \text{ask}^t, 0\}$ , and buys or sells with equal probability if  $E[V|S^t] - \text{ask}^t = \text{bid}^t - E[V|S^t] > 0$ . Otherwise, he holds. This change does not affect our results, but makes the technical analysis more complicated.
15. Note that we allow some traders of the last  $N - 2$  types (Type 3, Type N) to also be uninformative, if these uninformative traders are "uninformative" in different ways than Type 1 traders. For example, Type 1 traders are those who do not have private information but use

information implicit in the transaction history, while uninformative traders in the last  $N - 2$  types are liquidity traders.

16. If  $\theta = 0$ , this lemma is the same as Lemma 1 in Park and Sabourian (2011).
17. If there signal  $S \in \mathbb{S}$  such that  $E[V|H^t, S] > E[V|H^t]$ , there must be another signal  $\hat{S} \in \mathbb{S}$  such that  $E[V|H^t, \hat{S}] < E[V|H^t]$ .

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## Appendix 1

*Proof of Theorem 1* Suppose that the trader population is not behaviorally heterogeneous, that is, the composition of trader population consists of only sophisticated traders, or liquidity traders, or both.

The detailed proof for the assertion that when trader population consists of only sophisticated traders and liquidity traders, the quote spread is always positive can be found in Glosten and Milgrom (1985), so we omit it here.

Suppose that the trader population consists only of liquidity traders. Given this specific trader population, under the zero-profit condition, the market maker must set both bid price and ask price to equal the market value of the asset for any given history; otherwise, she would make non-zero profits from trading. Thus, at any history  $H^t$ , she sets  $\text{ask}^t = \text{bid}^t = E[V|H^t]$ . Since liquidity traders bring no information into the market, the market value of the asset does not change over time, i.e.  $E[V|H^t] = E[V|H^1] = E[V]$ , which implies that  $\text{ask}^t = \text{bid}^t = E[V]$ . We ignore this case because non-changing prices conflict with the frequent price movements that we observe in reality.

Suppose that the trader population consists only of sophisticated traders. We ignore this case also because (1) given this trader population, no trade occurs; and (2) prices never change. To see this, recall that a sophisticated trader only trades if he makes a strictly positive profit from trading. When a sophisticated trader makes a strictly positive profit from trading, the market maker faces a loss. Thus, if the trader population consists only of sophisticated traders, the zero-profit condition forces the market maker to set the ask price at the highest of all the expected values of the asset, conditioning on the history and traders' private signals, that is,  $\text{ask}^t = \max\{E[V|H^t, S_i], i = 1, 2, 3\}$ . Otherwise, the market maker loses money when selling to a trader. Similarly, the market maker sets  $\text{bid}^t = \min\{E[V|H^t, S_i], i = 1, 2, 3\}$ . Since a sophisticated trader cannot make a strictly positive profit from trading with the market maker, he always holds. Thus, no trade occurs and no information is brought into the market. Hence  $\text{ask}^t = \max\{E[V|S_i], i = 1, 2, 3\} = E[V|S_3]$  and  $\text{bid}^t = \min\{E[V|S_i], i = 1, 2, 3\} = E[V|S_1]$  for all  $t$ , which implies that prices never change.

From above, we know that if the trader population is not behaviorally heterogeneous, then the quote spreads are always positive; in other words, locked/crossed quotes never occur. Thus, if locked/crossed quotes occur with a positive probability, there must be at least a third type of traders who use information differently than sophisticated traders and liquidity traders; that is, the trader population is behaviorally heterogeneous.  $\square$

$$\text{LEMMA 1 } E[V|H^t] = \sum_{i=1}^3 \Pr(S_i|H^t)E[V|H^t, S_i]$$

*Proof* As for any history  $H^t$  and signal  $S_i \in \mathbb{S}$ , we have

$$\begin{aligned} E[V|H^t, S_i] &= \sum_{j=1}^3 V_j \cdot \Pr(V_j|H^t, S_i) = \gamma \Pr(V_2|H^t, S_i) + 2\gamma \Pr(V_3|H^t, S_i) \\ &= \gamma \cdot \frac{\Pr(V_2, H^t) \cdot \Pr(S_i|V_2)}{\Pr(H^t) \cdot \Pr(S_i|H^t)} + 2\gamma \cdot \frac{\Pr(V_3, H^t) \cdot \Pr(S_i|V_3)}{\Pr(H^t) \cdot \Pr(S_i|H^t)}, \end{aligned}$$

then  $\Pr(S_i|H^t) \cdot E[V|H^t, S_i] = \gamma \cdot \Pr(V_2|H^t) \Pr(S_i|V_2) + 2\gamma \cdot \Pr(V_3|H^t) \cdot \Pr(S_i|V_3)$ , which implies that

$$\begin{aligned} \sum_{i=1}^3 \Pr(S_i|H^t)E[V|H^t, S_i] &= \gamma \cdot \Pr(V_2|H^t) \cdot \sum_{i=1}^3 \Pr(S_i|V_2) + 2\gamma \cdot \Pr(V_3|H^t) \cdot \sum_{i=1}^3 \Pr(S_i|V_3) \\ &= \gamma \cdot \Pr(V_2|H^t) + 2\gamma \cdot \Pr(V_3|H^t) = E[V|H^t]. \end{aligned}$$

Therefore, we have the result.  $\square$

Notation:  $q_i^t = \Pr(V_i|H^t)$ ,  $\beta_i^t = \Pr(\text{buy}|H^t, V_i)$ .

LEMMA 2 For any  $S \in \mathbb{S}$ ,  $\mu, \theta \in (0, 1)$ , and any history  $H^t$ ,  $E[V|H^t, S] - E[V|H^t]$  has the same sign as  $q_1^t q_2^t [\Pr(S|V_2) - \Pr(S|V_1)] + q_2^t q_3^t [\Pr(S|V_3) - \Pr(S|V_2)] + 2q_1^t q_3^t [\Pr(S|V_3) - \Pr(S|V_1)]$ .<sup>46</sup>

*Proof* For any history  $H^t$ , we have

$$\begin{aligned} E[V|H^t, S] - E[V|H^t] &= [\gamma \Pr(V_2|H^t, S) + 2\gamma \Pr(V_3|H^t, S)] - [\gamma \Pr(V_2|H^t) + 2\gamma \Pr(V_3|H^t)] \\ &= \gamma [\Pr(V_2|H^t, S) - q_2^t] + 2\gamma [\Pr(V_3|H^t, S) - q_3^t]. \end{aligned}$$

Since for any  $i = 1, 2, 3$ ,

$$\begin{aligned} \Pr(V_i|H^t, S) &= \frac{\Pr(V_i, H^t, S)}{\Pr(H^t, S)} = \frac{\Pr(H^t, S|V_i) \Pr(V_i)}{\sum_{j=1}^3 \Pr(H^t, S|V_j) \Pr(V_j)} \\ &= \frac{\Pr(S|V_i) \Pr(H^t|V_i) \Pr(V_i)}{\sum_{j=1}^3 \Pr(S|V_j) \Pr(H^t|V_j) \Pr(V_j)} = \frac{\Pr(S|V_i) \Pr(H^t, V_i)}{\sum_{j=1}^3 \Pr(S|V_j) \Pr(H^t, V_j)} \\ &= \frac{\Pr(S|V_i) \Pr(V_i|H^t) \Pr(H^t)}{\sum_{j=1}^3 \Pr(S|V_j) \Pr(V_j|H^t) \Pr(H^t)} \\ &= \frac{\Pr(S|V_i) \Pr(V_i|H^t)}{\sum_{j=1}^3 \Pr(S|V_j) \Pr(V_j|H^t)} = \frac{\Pr(S|V_i) q_i^t}{\sum_{j=1}^3 \Pr(S|V_j) q_j^t}, \end{aligned}$$

we have  $E[V|H^t, S] - E[V|H^t] = \gamma \left[ \frac{\Pr(S|V_2) q_2^t (\sum_{j=1}^3 q_j^t)}{\sum_{j=1}^3 \Pr(S|V_j) q_j^t} - q_2^t \right] + 2\gamma \left[ \frac{\Pr(S|V_3) q_3^t (\sum_{j=1}^3 q_j^t)}{\sum_{j=1}^3 \Pr(S|V_j) q_j^t} - q_3^t \right]$ , and simple computation leads to the conclusion.  $\square$

COROLLARY 1 If signal  $S \in \{S_1, S_2, S_3\}$  is uninformed, then for any history,  $E[V|H^t, S] = E[V|H^t]$  and  $E[V|S] = E[V]$ .

*Proof* If  $S$  is uninformed,

$$\begin{aligned} E[V|S] &= \Pr(V_1|S)V_1 + \Pr(V_2|S)V_2 + \Pr(V_3|S)V_3 = \gamma [\Pr(V_2|S) + 2\Pr(V_3|S)] \\ &= \gamma \cdot \frac{1}{\Pr(S)} \cdot [\Pr(V_2, S) + 2\Pr(V_3, S)] = \gamma \cdot \frac{1}{\Pr(S)} \cdot [\Pr(V_2, S) + 2\Pr(V_3) \Pr(S|V_3)] \\ &= \gamma \cdot \frac{1}{\Pr(S)} \cdot [\Pr(V_2, S) + \Pr(V_1) \Pr(S|V_1) + \Pr(V_3) \Pr(S|V_3)] \\ &= \gamma \cdot \frac{1}{\Pr(S)} \cdot [\Pr(V_2, S) + \Pr(V_1, S) + \Pr(V_3, S)] \\ &= \gamma \cdot \frac{1}{\Pr(S)} \cdot \Pr(S) = \gamma = E[V]. \end{aligned}$$

We can also derive this equality from the case  $H^t = H^1$ .  $\square$

Proposition 3 For any public history  $H^t$ , we must have

$$\begin{aligned} \max \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} &\geq \max \{ \text{bid}^t, \text{ask}^t \} \text{ and} \\ \min \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} &\leq \min \{ \text{bid}^t, \text{ask}^t \}. \end{aligned}$$

If one inequality is strict, so is the other.

*Proof* First, consider any history  $H^t$  at which  $E[V|H^t, S_1] = E[V|H^t, S_2] = E[V|H^t, S_3] = E[V|H^t]$ . In this case, we must have  $\text{ask}^t = \text{bid}^t = E[V|H^t]$ .

To see this, suppose  $\text{ask}^t > E[V|H^t]$ ; then a sophisticated trader does not buy from the market maker. As she makes strictly positive profit from selling to a naive trader or a liquidity trader at this history, the market maker's expected profit from selling one unit of the asset is strictly positive, which con-



tradicts the zero-profit condition. Therefore, we must have  $ask^t \leq E[V|H^t]$ . However,  $ask^t < E[V|H^t]$  implies a strictly negative profit for the market maker from selling one unit of the asset, which also contradicts the zero-profit condition. Thus, we must have  $ask^t = E[V|H^t]$  at the given history. Similarly, we must have  $bid^t = E[V|H^t]$  at this history.

We now turn to any history  $H^t$  at which at least one of the three expectations  $E[V|H^t, S_1]$ ,  $E[V|H^t, S_2]$ , and  $E[V|H^t, S_3]$  is strictly greater than  $E[V|H^t]$ . Then at this history, at least one of these three expectations is strictly less than  $E[V|H^t]$ . In other words, we are checking a history at which

$$\max \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > E[V|H^t]$$

and

$$\min \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} < E[V|H^t].$$

Suppose that  $bid^t \geq E[V|H^t, S]$ ,  $\forall S \in \{S_1, S_2, S_3\}$ . Then  $bid^t \geq E[V|H^t, S_1]$ ,  $bid^t > E[V|H^t]$ , and a sophisticated trader sells to the market maker regardless of what signal he receives.

When purchasing one unit of the asset at  $bid^t$  from a naive trader, the market maker's expected profit is  $\Pr(S_1|H^t) \{E[V|H^t, S_1] - bid^t\} \leq 0$ . Since she also makes a strict negative profit if she trades with a sophisticated trader, the market maker must earn a strictly positive gain from trading with a liquidity trader at  $bid^t$ . As her expected profit from purchasing one unit of the asset from a liquidity trader is  $\frac{1-\theta-\mu}{3} \{E[V|H^t] - bid^t\}$ , we have  $E[V|H^t] > bid^t$ , contradicting  $bid^t > E[V|H^t]$ . Thus there exists at least one signal  $S \in \{S_1, S_2, S_3\}$  such that  $bid^t < E[V|H^t, S]$ . Then,  $\max \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > bid^t$ .

Similarly, if  $ask^t \geq \max \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\}$ , then  $ask^t > E[V|H^t, S]$ , which implies that only a naive trader who receives signal  $S_3$  or a liquidity trader will buy from the market maker. Since the market maker makes positive profit  $ask^t - E[V|H^t, S_3]$  from selling one unit of the asset to a naive trader, she must make negative profit from selling one unit of the asset to a liquidity trader, which implies  $ask^t \leq E[V|H^t, S]$ , contradicting  $ask^t > E[V|H^t, S]$ .

So there exists at least one signal  $S \in \{S_1, S_2, S_3\}$  such that  $ask^t < E[V|H^t, S]$ .

Therefore,  $\max \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} > \max \{bid^t, ask^t\}$ .

Similarly we have  $\min \{E[V|H^t, S_1], E[V|H^t, S_2], E[V|H^t, S_3]\} < \min \{bid^t, ask^t\}$  at this given history.  $\square$

**Proposition 4** If  $\mu > \theta$ , then for any history  $H^t$ , this financial market is not crossed.

*Proof* Consider any history  $H^t$  with  $t \geq 2$ . If  $E[V|H^t, S_1] = E[V|H^t, S_2] = E[V|H^t, S_3]$ , we have  $ask^t = E[V|H^t] = bid^t$ ; that is, the quote is locked.

Suppose  $E[V|H^t, S] > E[V|H^t] > E[V|H^t, \hat{S}]$ .<sup>17</sup> In this case, if  $bid^t \geq E[V|H^t]$ , the market maker earns a non-positive profit from buying from a liquidity trader. Since the market maker always earn a strictly negative profit from buying from a sophisticated trader, she must set the bid price in such a way that she makes a strictly positive profit from buying from a naive trader, which implies that  $bid^t < E[V|H^t, S_1]$ . Thus, a sophisticated trader who receives signal  $S_1$  does not sell. Moreover, since the market maker makes non-positive profit from buying from a liquidity trader, her total expected profit from buying from a sophisticated trader or a naive trader is positive.

Assume  $S'$  and  $S''$  are the other two signals, and  $S'$  sell. If a sophisticated trader who receives signal  $S''$  sells as well, the total expected profit of the market maker makes from buying one unit from a naive

trader or a sophisticated trader is

$$\begin{aligned} & \mu \Pr(S'|H^t) \{E[V|H^t, S'] - \text{bid}^t\} + \mu \Pr(S''|H^t) \{E[V|H^t, S''] - \text{bid}^t\} \\ & + \theta \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ < & \mu \Pr(S'|H^t) \{E[V|H^t, S'] - \text{bid}^t\} + \mu \Pr(S''|H^t) \{E[V|H^t, S''] - \text{bid}^t\} \\ & + \mu \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ = & \mu \{E[V|H^t] - \text{bid}^t\} \leq 0, \end{aligned}$$

contradicting with the assertion that the market maker's total expected profit from buying one unit from a naive trader or a sophisticated trader is positive.

If a sophisticated trader who receives signal  $S''$  does not sell, that is,  $E[V|H^t] < E[V|H^t, S'']$ , the total expected profit of the market maker from buying from a sophisticated trader or a naive trader is

$$\begin{aligned} & \mu \Pr(S'|H^t) \{E[V|H^t, S'] - \text{bid}^t\} + \theta \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ < & \mu \Pr(S'|H^t) \{E[V|H^t, S'] - \text{bid}^t\} + \mu \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ \leq & \mu \Pr(S'|H^t) \{E[V|H^t, S'] - \text{bid}^t\} + \mu \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ & + \mu \Pr(S''|H^t) \{E[V|H^t, S''] - \text{bid}^t\} \\ = & \mu \{E[V|H^t] - \text{bid}^t\} \leq 0, \end{aligned}$$

which contradicts that the market maker's total expected profit from buying one unit from a naive trader or a sophisticated trader is positive.

Therefore,  $\text{bid}^t$  must be strictly less than  $E[V|H^t]$ . Similarly, we have  $\text{ask}^t > E[V|H^t]$ . □

LEMMA 3  $\frac{q_m^{t+1}}{q_t^{t+1}} = \frac{\Pr(\sigma^t|V_m, H_t)q_m^t}{\Pr(\sigma^t|V_t, H_t)q_t^t}$ .

*Proof*  $\frac{q_m^{t+1}}{q_t^{t+1}} = \frac{\Pr(V_m|H_{t+1})}{\Pr(V_t|H_{t+1})} = \frac{\Pr(V_m|H_t, \sigma^t)}{\Pr(V_t|H_t, \sigma^t)} = \frac{\Pr(\sigma^t|V_m, H_t)\Pr(V_m, H_t)}{\Pr(\sigma^t|V_t, H_t)\Pr(V_t, H_t)} = \frac{\Pr(\sigma^t|V_m, H_t)q_m^t}{\Pr(\sigma^t|V_t, H_t)q_t^t}$ . □

*Proof of Proposition 1* If  $E[V|H^t, S_3] = E[V|H^t, S_2] = E[V|H^t, S_1]$ , then by Proposition 3, we have  $\text{ask}^t = \text{bid}^t = E[V|H^t]$ .

If  $E[V|H^t, S_1] > E[V|H^t, S_2] > E[V|H^t, S_3]$ , we have  $E[V|H^t, S_1] > \text{ask}^t$  and  $E[V|H^t, S_3] < \text{bid}^t$ .

Suppose that  $\text{bid}^t < E[V|H^t]$ . Then a sophisticated trader sells to the market maker only if he receives signal  $S_3$ . As the market maker makes zero profit from buying one unit of the asset from a trader, we have

$$\begin{aligned} & \mu \Pr(S_3|H^t) \{E[V|H^t, S_3] - \text{bid}^t\} + \theta \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ & + \frac{1 - \mu - \theta}{3} \{E[V|H^t] - \text{bid}^t\} = 0. \end{aligned}$$

As  $E[V|H^t, S_1] > E[V|H^t] \geq \text{bid}^t$  and  $\mu \leq \theta$ , we have

$$\begin{aligned} & \mu \Pr(S_3|H^t) \{E[V|H^t, S_3] - \text{bid}^t\} + \mu \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ & \leq \mu \Pr(S_3|H^t) \{E[V|H^t, S_3] - \text{bid}^t\} + \theta \Pr(S_1|H^t) \{E[V|H^t, S_1] - \text{bid}^t\} \\ & \leq 0 \end{aligned}$$

Then

$$\Pr(S_3|H^t)E[V|H^t, S_3] + \Pr(S_1|H^t)E[V|H^t, S_1] \leq \Pr(S_3|H^t) \text{bid}^t + \Pr(S_1|H^t) \text{bid}^t,$$

which is equivalent to

$$E[V|H^t] - \Pr(S_2|H^t)E[V|H^t, S_2] = [1 - \Pr(S_2|H^t)]E[V|H^t] \leq [1 - \Pr(S_2|H^t)] \text{bid}^t.$$

Thus,  $E[V|H^t] \leq \text{bid}^t$ , contradicting  $E[V|H^t] > \text{bid}^t$ . Therefore,  $\text{bid}^t \geq E[V|H^t]$ .

Similarly, suppose that  $\text{ask}^t > E[V|H^t]$ , then a sophisticated trader buys from the market maker only if he receives signal  $S_1$ . By the zero-profit condition, we have

$$\begin{aligned} & \mu \Pr(S_1|H^t) \{ \text{ask}^t - E[V|H^t, S_1] \} + \theta \Pr(S_3|H^t) \{ \text{ask}^t - E[V|H^t, S_3] \} \\ & + \frac{1 - \mu - \theta}{3} \{ \text{ask}^t - E[V|H^t] \} = 0. \end{aligned}$$

As  $\text{ask}^t > E[V|H^t] > E[V|H^t, S_3]$  and  $\mu \leq \theta$ , we know that

$$\begin{aligned} & \mu \Pr(S_1|H^t) \{ \text{ask}^t - E[V|H^t, S_1] \} + \mu \Pr(S_3|H^t) \{ \text{ask}^t - E[V|H^t, S_3] \} \\ & \leq \mu \Pr(S_1|H^t) \{ \text{ask}^t - E[V|H^t, S_1] \} + \theta \Pr(S_3|H^t) \{ \text{ask}^t - E[V|H^t, S_3] \} \\ & \leq 0, \end{aligned}$$

Then

$$\Pr(S_1|H^t)E[V|H^t, S_1] + \Pr(S_3|H^t)E[V|H^t, S_3] > \Pr(S_1|H^t) \text{ask}^t + \Pr(S_3|H^t) \text{ask}^t,$$

which is equivalent to

$$E[V|H^t] - \Pr(S_2|H^t)E[V|H^t, S_2] = [1 - \Pr(S_2|H^t)]E[V|H^t] \geq [1 - \Pr(S_2|H^t)] \text{ask}^t.$$

Therefore,  $E[V|H^t] \geq \text{ask}^t$ , contradicting  $E[V|H^t] < \text{ask}^t$ . Thus,  $\text{ask}^t \leq E[V|H^t] \leq \text{bid}^t$ . □

*Proof of Proposition 2* Suppose at any history  $H^t$ , we always have  $E[V|H^t, S_3] > E[V|H^t, S_2] > E[V|H^t, S_1]$ . Since  $S_2$  is uninformed, by Corollary 1, we have  $E[V|H^t, S_2] = E[V|H^t]$ . Then by Lemma 2, we know

$$q_1^t q_2^t [\Pr(S_1|V_2) - \Pr(S_1|V_1)] + q_2^t q_3^t [\Pr(S_1|V_3) - \Pr(S_1|V_2)] + 2q_1^t q_3^t [\Pr(S_1|V_3) - \Pr(S_1|V_1)] < 0 \tag{A1}$$

and

$$q_1^t q_2^t [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + q_2^t q_3^t [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + 2q_1^t q_3^t [\Pr(S_3|V_3) - \Pr(S_3|V_1)] > 0, \tag{A2}$$

where (A2) is equivalent to

$$\frac{q_1^t}{q_2^t} [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + 2 \frac{q_1^t}{q_2^t} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] > 0.$$

Consider the history  $H^\infty$  with  $a^t = \text{buy}$ ,  $\forall t$ . At any period  $t$  of this history, a sophisticated trader buys from the market maker only if he receives signal  $S_3$ , so that

$$\begin{aligned} \beta_i^t &= \Pr(\text{buy}|H^t, V_i) \\ &= \mu \Pr(S_3|H^t, V_i) + \theta \Pr(S_3|H^t, V_i) + \frac{1 - \mu - \theta}{3} \\ &= (\mu + \theta) \Pr(S_3|V_i) + \frac{1 - \mu - \theta}{3}. \end{aligned}$$

Since  $S_3$  is phill-shaped, we have

$$1 > \frac{(\mu + \theta) \Pr(S_3|V_1) + \frac{1-\mu-\theta}{3}}{(\mu + \theta) \Pr(S_3|V_i) + \frac{1-\mu-\theta}{3}} = \frac{\beta_1^t}{\beta_i^t} > 0, \quad i = 2, 3$$

then by Lemma 3 we know that for any  $t$ ,

$$1 > \left[ \frac{(\mu + \theta) \Pr(S_3|V_1) + \frac{1-\mu-\theta}{3}}{(\mu + \theta) \Pr(S_3|V_i) + \frac{1-\mu-\theta}{3}} \right]^t = \frac{q_1^t}{q_i^t} > 0, \quad i = 2, 3.$$

As  $q_3^t > q_1^t$ , we know that at any period  $t$  of this history,  $E[V|H^t] > E[V]$ .

Since  $\Pr(S_3|V_2) - \Pr(S_3|V_1) > 0$  and  $\Pr(S_3|V_3) - \Pr(S_3|V_1) > 0$ , we have

$$\lim_{t \rightarrow \infty} \left\{ \frac{q_1^t}{q_3^t} [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + 2 \frac{q_1^t}{q_2^t} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] \right\}$$

$$= \Pr(S_3|V_3) - \Pr(S_3|V_2) < 0,$$

which implies that there exists some  $t' > 1$  such that for any  $t \geq t'$ , we have

$$\frac{q_1^t}{q_3^t} [\Pr(S_3|V_2) - \Pr(S_3|V_1)] + [\Pr(S_3|V_3) - \Pr(S_3|V_2)] + 2 \frac{q_1^t}{q_2^t} [\Pr(S_3|V_3) - \Pr(S_3|V_1)] \leq 0.$$

Then  $E[V|H^{t'}, S_3] \leq E[V|H^{t'}]$ , contradicting  $E[V|H^{t'}, S_3] > E[V|H^{t'}, S_2] = E[V|H^{t'}]$ .

Thus, at the given history  $H^\infty$ , we must have  $E[V|H^t, S_3] \leq E[V|H^t, S_2] \leq E[V|H^t, S_1]$  at some period  $t$ .  $\square$

The proof of Theorem 2 comes straightforward from Propositions 2 and 1, so we omit it.

*Proof of Theorem 2* Consider a history  $H^\infty$  such that at each period  $t$ ,  $a^t = \text{buy}$ . By Lemma 2, we know that for any given history  $H^t$ ,  $E[V|H^t, S_2] < E[V|H^t]$ , as  $S_2$  is decreasing.

**Claim T.3.1:**  $E[V|H^t, S_2] \geq \text{ask}^t$  only if  $E[V|H^t, S_1] \geq \text{ask}^t$ .

Suppose that at some history  $H^t$ , we have  $E[V|H^t, S_2] \geq \text{ask}^t$  but  $E[V|H^t, S_1] < \text{ask}^t$ . As  $E[V|H^t, S_1] < \text{ask}^t \leq E[V|H^t, S_2] < E[V|H^t]$ , we know that  $E[V|H^t, S_3] > E[V|H^t] > \text{ask}^t$ , which implies that the market maker earns a strictly negative expected profit from selling one unit of the asset to a naive trader. Since the market maker always faces an expected loss when trading with a sophisticated trader, she must earn a strictly positive expected profit from selling one unit of the asset to a liquidity trader, which requires  $\text{ask}^t > E[V|H^t]$ , contradicting with  $\text{ask}^t < E[V|H^t]$ . Thus  $E[V|H^t, S_1] \geq \text{ask}^t$ .

**Claim T.3.2:** If  $E[V|H^t, S] \geq \text{ask}^t, \forall S \in \{S_1, S_2\}$ , then  $E[V|H^t, S_1] > E[V|H^t]$ .

By Proposition 3, we know that, given any history and any period, there always exists a signal  $S' \in \{S_1, S_2, S_3\}$  such that  $E[V|H^t, S'] \leq \text{ask}^t$ . Thus if  $E[V|H^t, S_1] \geq \text{ask}^t$  and  $E[V|H^t, S_2] \geq \text{ask}^t$ , we must have  $E[V|H^t, S_3] \leq \text{ask}^t \leq E[V|H^t, S_2] < E[V|H^t]$ . Thus  $E[V|H^t, S_1] > E[V|H^t]$ .

**Claim T.3.3:** If  $E[V|H^t, S] \geq \text{ask}^t, \forall S \in \{S_1, S_3\}$ , then  $E[V|H^t, S_1] \geq E[V|H^t]$ .

In this case, the market maker receives non-positive expected profit from selling one unit of the asset to a sophisticated trader or a naive trader. Then the market maker must make non-negative profit from selling one unit of the asset to a liquidity trader, i.e.  $\text{ask}^t - E[V|H^t] \geq 0$ . Thus,  $E[V|H^t, S_1] \geq \text{ask}^t \geq E[V|H^t]$ .

**Claim T.3.4:** There exists some  $t_{\varepsilon 1}$  such that  $E[V|H^{t_{\varepsilon 1}}, S_1] \geq E[V|H^{t_{\varepsilon 1}}]$ .

Suppose at any period of this history,  $E[V|H^t, S_1] < E[V|H^t]$ . By Claim T.3.2, we know that  $E[V|H^t, S_2] < ask^t$ . Since  $E[V|H^t, S_1] < E[V|H^t]$  and  $E[V|H^t, S_2] < E[V|H^t]$ , we have  $E[V|H^t, S_3] > E[V|H^t] > E[V|H^t, S]$ ,  $\forall S \in \{S_1, S_2\}$ , which implies that  $E[V|H^t, S_3] > ask^t$ . Thus, all the buys in the previous history are made or by sophisticated traders with signal  $S_3$ , or by naive traders with signal  $S_3$ , or by liquidity traders. Then at each period  $t$ ,  $\beta_i^t = (\theta + \mu) \Pr(S_3|V_i) + \frac{1-\theta-\mu}{3} = \xi \Pr(S_3|V_i) + \frac{1-\xi}{3}$ , which implies

$$\frac{q_{\xi 1}^t}{q_{\xi i}^t} = \frac{\Pr(V_1|H^t)}{\Pr(V_i|H^t)} = \frac{\Pr(H^t|V_1) \Pr(V_1)}{\Pr(H^t|V_i) \Pr(V_i)} = \left[ \frac{(\theta + \mu) \Pr(S_3|V_1) + \frac{1-\theta-\mu}{3}}{(\theta + \mu) \Pr(S_3|V_i) + \frac{1-\theta-\mu}{3}} \right]^{t-1} \frac{\Pr(V_1)}{\Pr(V_i)}, \quad i = 2, 3.$$

As  $S_3$  is phill-shaped, we have  $\frac{(\theta+\mu)\Pr(S_3|V_1)+\frac{1-\theta-\mu}{3}}{(\theta+\mu)\Pr(S_3|V_i)+\frac{1-\theta-\mu}{3}} < 1$ , then  $\lim_{t \rightarrow \infty} \frac{q_{\xi 1}^t}{q_{\xi i}^t} = 0$ .

Recall that for each signal  $S$ ,  $E[V|H^t, S] - E[V|H^t]$  has the same sign as

$$q_{\xi 1}^t q_{\xi 2}^t [\Pr(S|V_2) - \Pr(S|V_1)] + q_{\xi 2}^t q_{\xi 3}^t [\Pr(S|V_3) - \Pr(S|V_2)] + 2q_{\xi 1}^t q_{\xi 3}^t [\Pr(S|V_3) - \Pr(S|V_1)],$$

which is equivalent to

$$\frac{q_{\xi 1}^t}{q_{\xi 3}^t} [\Pr(S|V_2) - \Pr(S|V_1)] + [\Pr(S|V_3) - \Pr(S|V_2)] + 2 \frac{q_{\xi 1}^t}{q_{\xi 2}^t} [\Pr(S|V_3) - \Pr(S|V_1)].$$

Then

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{q_{\xi 1}^t}{q_{\xi 3}^t} [\Pr(S_1|V_2) - \Pr(S_1|V_1)] + [\Pr(S_1|V_3) - \Pr(S_1|V_2)] + 2 \frac{q_{\xi 1}^t}{q_{\xi 2}^t} [\Pr(S_1|V_3) - \Pr(S_1|V_1)] \\ & = \Pr(S_1|V_3) - \Pr(S_1|V_2) > 0, \end{aligned}$$

which implies that there exists some  $t'$  such that at any period  $t \geq t'$ ,

$$\frac{q_{\xi 1}^t}{q_{\xi 3}^t} [\Pr(S_1|V_2) - \Pr(S_1|V_1)] + [\Pr(S_1|V_3) - \Pr(S_1|V_2)] + 2 \frac{q_{\xi 1}^t}{q_{\xi 2}^t} [\Pr(S_1|V_3) - \Pr(S_1|V_1)] \geq 0.$$

Therefore  $E[V|H^t, S_1] \geq E[V|H^t]$ ,  $\forall t \geq t'$ , contradicting the assumption that  $E[V|H^t, S_1] < E[V|H^t]$  at every period. Thus, there exists some period(s)  $t$  of this history at which  $E[V|H^t, S_1] \geq E[V|H^t]$ . Let  $t_{\xi 1}$  denote the earliest of these periods.

Note that Claim T.3.4 implies that given other parameters unchanged,  $t_{\xi 1}$  depends on the value of  $\xi$  only, and it does not change if the ratio between  $\mu$  and  $\theta$  changes. Thus, given  $\xi$  and other parameters unchanged,  $E[V|H^{t_{\xi 1}}]$ ,  $E[V|H^{t_{\xi 1}}, S]$ ,  $\Pr(S|H^{t_{\xi 1}})$ , and  $\Pr(S|H^{t_{\xi 1}})$ ,  $\forall S \in \{S_1, S_2, S_3\}$ , are constant, regardless of the ratio between  $\mu$  and  $\theta$ .

**Claim T.3.5:** At any  $t < t_{\xi 1}$ , we must have  $E[V|H^t, S_3] > E[V|H^t]$ .

By the definition of  $t_{\xi 1}$ , we know that at any period  $t < t_{\xi 1}$ , we have  $E[V|H^t, S_1] < E[V|H^t]$ . If  $E[V|H^t, S_3] \leq E[V|H^t]$ , then we must have  $E[V|H^t, S_2] > E[V|H^t]$ , contradicting the assumption that  $S_2$  is decreasing.

**Claim T.3.6:** There exists  $\bar{\mu}_{\xi} \in (0, \xi)$  such that for any given  $\mu < \bar{\mu}_{\xi}$ , either  $bid^{t_{\xi 1}} \geq E[V|H^{t_{\xi 1}}]$  and  $ask^{t_{\xi 1}} \geq E[V|H^{t_{\xi 1}}]$ , or  $bid^{t_{\xi 1}} \geq E[V|H^{t_{\xi 1}}] \geq ask^{t_{\xi 1}}$ .

Consider any  $\xi \in (0, 1)$ . Let  $ask_{\xi, \mu}^{t_{\xi 1}}$  and  $bid_{\xi, \mu}^{t_{\xi 1}}$  denote the ask price and bid price at period  $t_{\xi 1}$  for a given  $\mu$ .  $E\pi_s^{t_{\xi 1}}(\mu, \xi)$  denote the expected profit the market maker receives when selling one unit of the asset to a trader, and  $E\pi_b^{t_{\xi 1}}(\mu, \xi)$  denote the expected profit she receives from buying one unit of the asset from a trader.

**Case T.3.6.1:**  $E[V|H^{t_{\xi 1}}, S_3] > E[V|H^{t_{\xi 1}}]$ .

In this case, at period

$t_{\xi 1}$ , we have  $E[V|H^{t_{\xi 1}}, S_1] \geq E[V|H^{t_{\xi 1}}]$  and  $E[V|H^{t_{\xi 1}}, S_3] > E[V|H^{t_{\xi 1}}]$ . Then a sophisticated trader at period  $t_{\xi 1}$  sells if he receives signal  $S_2$ .

**Subcase T.3.6.1.i:** A sophisticated trader buys at period  $t_{\xi 1}$  if he receives signal  $S_3$ .

In this case,  $E[V|H^{t_{\xi 1}}, S_3] > \text{ask}_{\xi, \mu}^{t_{\xi 1}}$ , then the market maker receives strictly negative expected profit from selling one unit of the asset to a sophisticated or naive trader. Then the market maker must make strictly positive profit from selling one unit of the asset to a liquidity trader; that is,  $\text{ask}_{\xi, \mu}^{t_{\xi 1}} > E[V|H^{t_{\xi 1}}]$ .

If  $\text{bid}_{\xi, \mu}^{t_{\xi 1}} \geq E[V|H^{t_{\xi 1}}, S_1]$ , then  $\text{bid}_{\xi, \mu}^{t_{\xi 1}} > E[V|H^{t_{\xi 1}}]$ .

If  $\text{bid}_{\xi, \mu}^{t_{\xi 1}} < E[V|H^{t_{\xi 1}}, S_1]$ , then a sophisticated trader sells only if he receives signal  $S_2$ . Thus the market maker's expected profit from buying one unit of the asset from a trader is

$$\begin{aligned} E\pi_b^{t_{\xi 1}}(\mu, \xi) &= \mu \Pr(S_2|H^{t_{\xi 1}}) \{E[V|H^{t_{\xi 1}}, S_2] - \text{bid}_{\xi, \mu}^{t_{\xi 1}}\} + \frac{1-\theta-\mu}{3} \{E[V|H^{t_{\xi 1}}] - \text{bid}_{\xi, \mu}^{t_{\xi 1}}\} \\ &\quad + \theta \Pr(S_1|H^{t_{\xi 1}}) \{E[V|H^{t_{\xi 1}}, S_1] - \text{bid}_{\xi, \mu}^{t_{\xi 1}}\} \\ &= \mu \Pr(S_2|H^{t_{\xi 1}}) \{E[V|H^{t_{\xi 1}}, S_2] - \text{bid}_{\xi, \mu}^{t_{\xi 1}}\} + \frac{1-\xi}{3} \{E[V|H^{t_{\xi 1}}] - \text{bid}_{\xi, \mu}^{t_{\xi 1}}\} \\ &\quad + (\xi - \mu) \Pr(S_1|H^{t_{\xi 1}}) \{E[V|H^{t_{\xi 1}}, S_1] - \text{bid}_{\xi, \mu}^{t_{\xi 1}}\}. \end{aligned}$$

Then as  $E\pi_b^{t_{\xi 1}}(\mu, \xi) = 0$ , we have

$$\text{bid}_{\xi, \mu}^{t_{\xi 1}} = \frac{\mu \Pr(S_2|H^{t_{\xi 1}})E[V|H^{t_{\xi 1}}, S_2] + (\xi - \mu) \Pr(S_1|H^{t_{\xi 1}})E[V|H^{t_{\xi 1}}, S_1] + \frac{1-\xi}{3}E[V|H^{t_{\xi 1}}]}{\mu \Pr(S_2|H^{t_{\xi 1}}) + (\xi - \mu) \Pr(S_1|H^{t_{\xi 1}}) + \frac{1-\xi}{3}}.$$

As  $\frac{\partial \text{bid}_{\xi, \mu}^{t_{\xi 1}}}{\partial \mu}$  has the same sign as

$$\begin{aligned} &\{ \Pr(S_2|H^{t_{\xi 1}})E[V|H^{t_{\xi 1}}, S_2] - \Pr(S_1|H^{t_{\xi 1}})E[V|H^{t_{\xi 1}}, S_1] \} \\ &\quad \times \left[ \mu \Pr(S_2|H^{t_{\xi 1}}) + (\xi - \mu) \Pr(S_1|H^{t_{\xi 1}}) + \frac{1-\xi}{3} \right] \\ &\quad - \left\{ \mu \Pr(S_2|H^{t_{\xi 1}})E[V|H^{t_{\xi 1}}, S_2] + (\xi - \mu) \Pr(S_1|H^{t_{\xi 1}})E[V|H^{t_{\xi 1}}, S_1] + \frac{1-\xi}{3}E[V|H^{t_{\xi 1}}] \right\} \\ &\quad \times [\Pr(S_2|H^{t_{\xi 1}}) - \Pr(S_1|H^{t_{\xi 1}})] \\ &= \xi \Pr(S_1|H^{t_{\xi 1}}) \Pr(S_2|H^{t_{\xi 1}}) \{E[V|H^{t_{\xi 1}}, S_2] - E[V|H^{t_{\xi 1}}, S_1]\} \\ &\quad + \frac{1-\xi}{3} \Pr(S_2|H^{t_{\xi 1}}) \{E[V|H^{t_{\xi 1}}, S_2] - E[V|H^{t_{\xi 1}}]\} \\ &\quad + \frac{1-\xi}{3} \Pr(S_1|H^{t_{\xi 1}}) \{E[V|H^{t_{\xi 1}}] - E[V|H^{t_{\xi 1}}, S_1]\}, \end{aligned}$$

which is strictly negative as  $E[V|H^{t_{\xi 1}}, S_2] < E[V|H^{t_{\xi 1}}] \leq E[V|H^{t_{\xi 1}}, S_1]$ , we know that  $\text{bid}_{\xi, \mu}^{t_{\xi 1}}$  is strictly decreasing with  $\mu$  when  $\xi$  is given. As for any given  $\xi \in (0, 1)$ ,

$$\lim_{\mu \rightarrow 0} \text{bid}_{\xi, \mu}^{t_{\xi 1}} = \frac{\xi \Pr(S_1|H^{t_{\xi 1}})E[V|H^{t_{\xi 1}}, S_1] + \frac{1-\xi}{3}E[V|H^{t_{\xi 1}}]}{\xi \Pr(S_1|H^{t_{\xi 1}}) + \frac{1-\xi}{3}} > E[V|H^{t_{\xi 1}}],$$

which implies that for the given  $\xi$ , there exists  $\bar{\mu}_{\xi b} \in (0, \xi)$  such that for any  $\mu < \bar{\mu}_{\xi b}$ ,  $\text{bid}_{\xi, \mu}^{t_{\xi 1}} > E[V|H^{t_{\xi 1}}]$ . Together with  $\text{ask}_{\xi, \mu}^{t_{\xi 1}} > E[V|H^{t_{\xi 1}}]$ , we know that over-optimism occurs.  $\square$



**Subcase T.3.6.1.ii:** A sophisticated trader holds at period  $t_{\xi}^1$  if he receives signal  $S_3$ .

In this case,  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} \geq E[V|H^{t_{\xi}^1}, S_3] \geq \text{bid}_{\xi, \mu}^{t_{\xi}^1}$ , then at this period, a sophisticated trader buys only if he receives signal  $S_1$ , and sells only if he receives signal  $S_2$ . Thus the market maker's expected profit from buying one unit of the asset from a trader is

$$\begin{aligned} E\pi_b^{t_{\xi}^1}(\mu, \xi) &= \mu \Pr(S_2|H^{t_{\xi}^1}) \{E[V|H^{t_{\xi}^1}, S_2] - \text{bid}_{\xi, \mu}^{t_{\xi}^1}\} + \frac{1-\theta-\mu}{3} \{E[V|H^{t_{\xi}^1}] - \text{bid}_{\xi, \mu}^{t_{\xi}^1}\} \\ &\quad + \theta \Pr(S_1|H^{t_{\xi}^1}) \{E[V|H^{t_{\xi}^1}, S_1] - \text{bid}_{\xi, \mu}^{t_{\xi}^1}\} \\ &= \mu \Pr(S_2|H^{t_{\xi}^1}) \{E[V|H^{t_{\xi}^1}, S_2] - \text{bid}_{\xi, \mu}^{t_{\xi}^1}\} + \frac{1-\xi}{3} \{E[V|H^{t_{\xi}^1}] - \text{bid}_{\xi, \mu}^{t_{\xi}^1}\} \\ &\quad + (\xi - \mu) \Pr(S_1|H^{t_{\xi}^1}) \{E[V|H^{t_{\xi}^1}, S_1] - \text{bid}_{\xi, \mu}^{t_{\xi}^1}\}, \end{aligned}$$

then as  $E\pi_b^{t_{\xi}^1}(\mu, \xi) = 0$ , we have

$$\text{bid}_{\xi, \mu}^{t_{\xi}^1} = \frac{\mu \Pr(S_2|H^{t_{\xi}^1})E[V|H^{t_{\xi}^1}, S_2] + (\xi - \mu) \Pr(S_1|H^{t_{\xi}^1})E[V|H^{t_{\xi}^1}, S_1] + \frac{1-\xi}{3}E[V|H^{t_{\xi}^1}]}{\mu \Pr(S_2|H^{t_{\xi}^1}) + (\xi - \mu) \Pr(S_1|H^{t_{\xi}^1}) + \frac{1-\xi}{3}},$$

which is decreasing in  $\mu$  when  $\xi$  is given. As for any given  $\xi \in (0, 1)$ ,

$$\lim_{\mu \rightarrow 0} \text{bid}_{\xi, \mu}^{t_{\xi}^1} = \frac{\xi \Pr(S_1|H^{t_{\xi}^1})E[V|H^{t_{\xi}^1}, S_1] + \frac{1-\xi}{3}E[V|H^{t_{\xi}^1}]}{\xi \Pr(S_1|H^{t_{\xi}^1}) + \frac{1-\xi}{3}} > E[V|H^{t_{\xi}^1}],$$

which implies that for the given  $\xi$ , there exists  $\bar{\mu}_{\xi h} \in (0, \xi)$  such that for any  $\mu < \bar{\mu}_{\xi h}$ ,  $\text{bid}_{\xi, \mu}^{t_{\xi}^1} > E[V|H^{t_{\xi}^1}]$ .

As  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} \geq E[V|H^{t_{\xi}^1}, S_3] > E[V|H^{t_{\xi}^1}]$ , we know over-optimism occurs. □

**Subcase T.3.6.1.iii:** A sophisticated trader sells at period  $t_{\xi}^1$  if he receives signal  $S_3$ .

In this case, a sophisticated trader buys from the market maker only if he receives signal  $S_1$ . Suppose  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} \leq E[V|H^{t_{\xi}^1}]$ , then  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} < E[V|H^{t_{\xi}^1}, S_1]$  and  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} < E[V|H^{t_{\xi}^1}, S_3]$ , which implies that the market maker makes strictly negative profit from selling one unit of the asset to a sophisticated trader or a naive trader. Thus, the market maker must make strictly positive profit from selling one unit of the asset to a liquidity trader, which requires  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} > E[V|H^{t_{\xi}^1}]$ , contradicting  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} \leq E[V|H^{t_{\xi}^1}]$ . Therefore,  $\text{ask}_{\xi, \mu}^{t_{\xi}^1}$  must be strictly greater than  $E[V|H^{t_{\xi}^1}]$ .

That a sophisticated trader sells when he receives signal  $S_3$  implies  $E[V|H^{t_{\xi}^1}, S_3] < \text{bid}_{\xi, \mu}^{t_{\xi}^1}$ , then  $\text{bid}_{\xi, \mu}^{t_{\xi}^1} > E[V|H^{t_{\xi}^1}]$ . As  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} > E[V|H^{t_{\xi}^1}]$ , we know that over-optimism occurs. □

Given  $\xi \in (0, 1)$ , let  $\bar{\mu}_{\xi} = \min\{\bar{\mu}_{\xi b}, \bar{\mu}_{\xi h}, \frac{1}{2}\xi\}$ , then (T.3.6.1.i), (T.3.6.1.ii), and (T.3.6.1.iii) imply that if  $E[V|H^{t_{\xi}^1}, S_3] > E[V|H^{t_{\xi}^1}]$ , for any  $\mu \in (0, \bar{\mu}_{\xi})$ ,  $\text{bid}_{\xi, \mu}^{t_{\xi}^1} > E[V|H^{t_{\xi}^1}]$  and  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} > E[V|H^{t_{\xi}^1}]$ , which implies over-optimism.

Note that we require  $\mu < \frac{1}{2}\xi$  here, as  $\mu > \frac{1}{2}\xi$  implies  $\mu > \theta$ . Therefore,  $\text{ask}_{\xi, \mu}^{t_{\xi}^1} > E[V|H^{t_{\xi}^1}] > \text{bid}_{\xi, \mu}^{t_{\xi}^1}$  by Proposition 4.

**Case T.3.6.2:**  $E[V|H^{t_{\xi}^1}, S_3] \leq E[V|H^{t_{\xi}^1}]$ .

Since  $S_2$  is decreasing, we have  $E[V|H^{t_{\xi}^1}, S_2] < E[V|H^{t_{\xi}^1}]$ . Then a sophisticated trader buys when he receives signal  $S_1$ . That is,  $E[V|H^{t_{\xi}^1}, S_1] > \text{ask}_{\xi, \mu}^{t_{\xi}^1}$ . We also have  $E[V|H^{t_{\xi}^1}, S_1] > \text{bid}_{\xi, \mu}^{t_{\xi}^1}$ .

Note that if  $E[V|H^{t_{\xi}^1}, S_3] \geq \text{ask}_{\xi, \mu}^{t_{\xi}^1}$ , the market maker receives strictly negative expected profit from selling one unit of the asset to a sophisticated trader or a naive trader. Then the market maker must make

strictly positive profit from selling one unit of the asset to a liquidity trader, that is,  $\text{ask}^{t_{21}} > E[V|H^{t_{21}}]$ , which implies that  $\text{ask}^{t_{21}} > E[V|H^{t_{21}}, S_3]$ , contradicting  $E[V|H^{t_{21}}, S_3] \geq \text{ask}^{t_{21}}$ . Thus we must have  $E[V|H^{t_{21}}, S_3] < \text{ask}^{t_{21}}$ , that is, a sophisticated trader does not buy when he receives signal  $S_3$ .

Suppose that  $E[V|H^{t_{21}}, S_3] > \text{bid}_{\xi, \mu}^{t_{21}}$ . From above we have  $\text{ask}^{t_{21}} > E[V|H^{t_{21}}, S_3] > \text{bid}_{\xi, \mu}^{t_{21}}$ . Then at this period, a sophisticated trader buys only if he receives signal  $S_1$ , and sells only if he receives signal  $S_2$ .

Note that  $E[V|H^{t_{21}}, S_3] > \text{bid}_{\xi, \mu}^{t_{21}}$  implies  $\text{bid}_{\xi, \mu}^{t_{21}} > E[V|H^{t_{21}}]$ . Then the market maker's expected profit from purchasing one unit of the asset of a trader is

$$E\pi_b^{t_{21}}(\mu, \xi) = \mu \Pr(S_2|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_2] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \frac{1-\theta-\mu}{3} \left\{ E[V|H^{t_{21}}] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \theta \Pr(S_1|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_1] - \text{bid}_{\xi, \mu}^{t_{21}} \right\}$$

Since  $E\pi_b^{t_{21}}(\mu, \xi) = 0$ , we know that

$$\mu \Pr(S_2|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_2] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \theta \Pr(S_1|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_1] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} < 0,$$

then  $\mu \Pr(S_2|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_2] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \mu \Pr(S_1|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_1] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} < 0$  as  $\mu \leq \theta$ , which implies that

$$E[V|H^{t_{21}}] - \Pr(S_3|H^{t_{21}})E[V|H^{t_{21}}, S_3] < [1 - \Pr(S_1|H^{t_{21}})] \text{bid}_{\xi, \mu}^{t_{21}} \leq E[V|H^{t_{21}}] - \Pr(S_3|H^{t_{21}})E[V|H^{t_{21}}, S_3]$$

This contradicts with  $E[V|H^{t_{21}}] \geq E[V|H^{t_{21}}, S_3]$ . Thus  $\text{bid}_{\xi, \mu}^{t_{21}} > E[V|H^{t_{21}}, S_3]$ .

Suppose  $\text{bid}_{\xi, \mu}^{t_{21}} < E[V|H^{t_{21}}]$ . Then if a sophisticated trader sells when he receives signal  $S_2$ , that is,  $E[V|H^{t_{21}}, S_2] < \text{bid}_{\xi, \mu}^{t_{21}}$ , as  $E\pi_b^{t_{21}}(\mu, \xi) = 0$ , we have

$$\mu \Pr(S_2|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_2] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \mu \Pr(S_3|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_3] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \theta \Pr(S_1|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_1] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} < 0.$$

Since  $\mu \leq \theta$  and  $E[V|H^{t_{21}}, S_1] > \text{bid}_{\xi, \mu}^{t_{21}}$ , we have

$$\mu \Pr(S_2|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_2] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \mu \Pr(S_3|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_3] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \mu \Pr(S_1|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_1] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} < 0.$$

Then

$$\Pr(S_2|H^{t_{21}})E[V|H^{t_{21}}, S_2] + \Pr(S_3|H^{t_{21}})E[V|H^{t_{21}}, S_3] + \Pr(S_1|H^{t_{21}})E[V|H^{t_{21}}, S_1] = E[V|H^{t_{21}}] < \Pr(S_2|H^{t_{21}}) \text{bid}_{\xi, \mu}^{t_{21}} + \Pr(S_3|H^{t_{21}}) \text{bid}_{\xi, \mu}^{t_{21}} + \Pr(S_1|H^{t_{21}}) \text{bid}_{\xi, \mu}^{t_{21}} = \text{bid}_{\xi, \mu}^{t_{21}},$$

contradicting  $\text{bid}_{\xi, \mu}^{t_{21}} \leq E[V|H^{t_{21}}]$ .

If  $E[V|H^{t_{21}}, S_2] \geq \text{bid}_{\xi, \mu}^{t_{21}}$ , we have  $E[V|H^{t_{21}}] > \text{bid}_{\xi, \mu}^{t_{21}}$ . Then

$$\mu \Pr(S_3|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_3] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} + \theta \Pr(S_1|H^{t_{21}}) \left\{ E[V|H^{t_{21}}, S_1] - \text{bid}_{\xi, \mu}^{t_{21}} \right\} < 0.$$

Since  $\mu \leq \theta$  and  $E[V|H^{t_{\xi}}, S_1] > \text{bid}_{\xi, \mu}^{t_{\xi}}$ , we have

$$\mu \Pr(S_3|H^{t_{\xi}}) \{E[V|H^{t_{\xi}}, S_3] - \text{bid}_{\xi, \mu}^{t_{\xi}}\} + \mu \Pr(S_1|H^{t_{\xi}}) \{E[V|H^{t_{\xi}}, S_1] - \text{bid}_{\xi, \mu}^{t_{\xi}}\} < 0.$$

Then

$$E[V|H^{t_{\xi}}] - \Pr(S_2|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_2] < [1 - \Pr(S_2|H^{t_{\xi}})] \text{bid}_{\xi, \mu}^{t_{\xi}} < [1 - \Pr(S_2|H^{t_{\xi}})]E[V|H^{t_{\xi}}],$$

contradicting  $E[V|H^{t_{\xi}}, S_2] \leq E[V|H^{t_{\xi}}]$ . Thus,  $\text{bid}_{\xi, \mu}^{t_{\xi}} \geq E[V|H^{t_{\xi}}]$ .

Note that when  $\text{ask}_{\xi, \mu}^{t_{\xi}} > E[V|H^{t_{\xi}}]$ , a sophisticated trader buys only if he receives signal  $S_1$ . Then

$$\text{ask}_{\xi, \mu}^{t_{\xi}} = \frac{\mu \Pr(S_1|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_1] + (\xi - \mu) \Pr(S_3|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_3] + \frac{1-\xi}{3}E[V|H^{t_{\xi}}]}{\mu \Pr(S_1|H^{t_{\xi}}) + (\xi - \mu) \Pr(S_3|H^{t_{\xi}}) + \frac{1-\xi}{3}},$$

which strictly increases with  $\mu$ . Thus,  $\text{ask}_{\xi, \mu}^{t_{\xi}} > E[V|H^{t_{\xi}}]$  if and only if

$$\frac{\mu \Pr(S_1|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_1] + (\xi - \mu) \Pr(S_3|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_3] + \frac{1-\xi}{3}E[V|H^{t_{\xi}}]}{\mu \Pr(S_1|H^{t_{\xi}}) + (\xi - \mu) \Pr(S_3|H^{t_{\xi}}) + \frac{1-\xi}{3}} > E[V|H^{t_{\xi}}].$$

As for any  $\mu < \frac{1}{2}\xi$ , we know

$$\begin{aligned} \lim_{\mu \rightarrow \frac{1}{2}\xi} \text{ask}_{\xi, \mu}^{t_{\xi}} &= \frac{\frac{1}{2}\xi \Pr(S_1|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_1] + \frac{1}{2}\xi \Pr(S_3|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_3] + \frac{1-\xi}{3}E[V|H^{t_{\xi}}]}{\frac{1}{2}\xi \Pr(S_1|H^{t_{\xi}}) + \frac{1}{2}\xi \Pr(S_3|H^{t_{\xi}}) + \frac{1-\xi}{3}} \\ &= \frac{\frac{1}{2}\xi \{E[V|H^{t_{\xi}}] - \Pr(S_2|H^{t_{\xi}})E[V|H^{t_{\xi}}, S_2]\} + \frac{1-\xi}{3}E[V|H^{t_{\xi}}]}{\frac{1}{2}\xi [1 - \Pr(S_2|H^{t_{\xi}})] + \frac{1-\xi}{3}} \\ &> E[V|H^{t_{\xi}}], \end{aligned}$$

then for any given  $\xi \in (0, 1)$ , there exists  $\bar{\mu}_{\xi} \in (0, \frac{1}{2}\xi)$  such that if and only if  $\mu \in (\bar{\mu}_{\xi}, \frac{1}{2}\xi)$ , we have  $\text{ask}_{\xi, \mu}^{t_{\xi}} > E[V|H^{t_{\xi}}]$ . Otherwise, when  $\mu \in (0, \bar{\mu}_{\xi})$ ,  $\text{ask}_{\xi, \mu}^{t_{\xi}} \leq E[V|H^{t_{\xi}}] \leq \text{bid}_{\xi, \mu}^{t_{\xi}}$ ; that is, a locked or crossed quote occurs.  $\square$



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