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Discussion
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Daniel Huppmann and Sauleh Siddiqui

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An exact solution method for binary equilibrium problems with compensation and the power market uplift problem*

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We propose a novel method to find Nash equilibria in games with binary decision variables by including compensation payments and incentive-compatibility constraints from non-cooperative game theory directly into an optimization framework in lieu of using first order conditions of a linearization, or relaxation of integrality conditions. The reformulation offers a new approach to obtain and interpret dual variables to binary constraints using the benefit or loss from deviation rather than marginal relaxations. The method endogenizes the trade-off between overall (societal) efficiency and compensation payments necessary to align incentives of individual players. We provide existence results and conditions under which this problem can be solved as a mixed-binary linear program.

We apply the solution approach to a stylized nodal power-market equilibrium problem with binary on-off decisions. This illustrative example shows that our approach yields an exact solution to the binary Nash game with compensation. We compare different implementations of actual market rules within our model, in particular constraints ensuring non-negative profits (no-loss rule) and restrictions on the compensation payments to non-dispatched generators. We discuss the resulting equilibria in terms of overall welfare, efficiency, and allocational equity.

Keywords: binary Nash game, non-cooperative equilibrium, compensation, incentive compatibility, electricity market, power market, uplift payments

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1 Introduction

There are many environments and real-world situations where several players interact in a non-cooperative game with binary decisions, such as electricity markets (on-off decision for a power plant), transportation and facility location models (Caunhye et al., 2012), engineering (Rao, 1996), as well as agriculture and land-use planning (Tóth et al., 2011). Modeling Nash equilibria between players which face both binary and continuous decisions is a challenging problem (Scarf, 1990). Economists and game theorists usually apply brute-force methods by exploring all possible combinations and check every solution for deviation incentives of each player. When market-clearing prices to support a pure-strategy Nash equilibrium in the Walrasian sense do not exist, economists suggest to use multi-part pricing (Hotelling, 1938) or deviate from marginal-cost pricing to a “second-best” market outcome, such that no player should lose money from participating (Baumol and Bradford, 1970). However, a canonical approach to find pure-strategy Nash equilibria in binary games does not exist.

In many large-scale practical applications, exploring the entire solution space is not realistically possible. A common approach in such cases is to linearize the binary decisions; the Nash equilibrium can then be computed by solving the system of first order optimality conditions, a.k.a. equilibrium modeling using mixed complementarity problems or variational inequalities, if certain assumptions on convexity of the linearized problem hold. Recent work seeks a trade-off between relaxation of the complementarity (slackness) conditions or the integrality of discrete constraints to obtain stationary points that are presumed to be equilibria of the original problem (Gabriel et al., 2012, 2013; Fuller and Celebi, 2016).

In this work, we focus on applications where a relaxation or linearization of the binary decision variable is either not practical or yields incorrect results. Instead, we derive first order optimality conditions of the continuous variables for *both states of each binary variable* and include those in an overall equilibrium problem simultaneously. Our method then selects the state of the binary variable and corresponding continuous variable which provides the best response for each individual player. Hence, this set of best responses is an equilibrium.

However, due to the nature of a binary game, there are many instances where no set of strategies and no price vector exists that supports a Nash equilibrium in pure strategies; i.e., there is no outcome where the pay-offs to each stakeholder are such that no player has a profitable deviation. This is due to the non-convexity introduced by the binary decision variables and indivisibilities (O’Neill et al., 2005). For the remainder of this work, we abstract from players choosing mixed strategies and only consider Nash equilibria in pure strategies.

We introduce the notion of a “quasi-equilibrium” to describe situations where no equilibrium exists, but where a market operator or regulator can assign compensation payments in order to obtain an incentive-compatible outcome. These payments align the incentives of individual players with the objectives of the overall system, such as cost minimization or welfare maximization. A regulator may also choose to intervene when an equilibrium exists but its outcome is inferior to the solution that a benevolent planner might achieve. That is, the market operator may seek to minimize the deviation from the system optimum (i.e., all decisions by one planner) caused by the non-cooperative game among a number of decision makers, each seeking to optimize competing objectives. Our solution approach allows to endogenously consider the trade-off between regulatory intervention to improve market efficiency, and the distortions caused by these interventions.

Electricity markets are the real-world application of binary games which have received the most attention in the mathematical optimization literature (O’Neill et al., 2013; Liu and Hobbs, 2013; Wogrin et al., 2013; Liu and Ferris, 2013; Philpott et al., 2013; Bjorndal and Jörnsten, 2008; Hu and Ralph, 2007; Philpott and Schultz, 2006;

O’Neill et al., 2005). A challenging problem arises from the on-off decision of power plants, which usually incur substantial start-up or shut-down costs and, if operational, face minimum-generation constraints. Because power markets are usually based on marginal-cost, short-term pricing, the ramping or commitment costs (i.e., start-up costs) are not necessarily covered by resulting market prices.

As a consequence, many electricity systems have rules that generators must be “made whole” or have to be “in the money”; i.e., they receive compensation (also called “uplift payment”) to make sure that they do not lose money from participating in the market. This is commonly referred to as a “no-loss rule”. However, this may not be required from a game-theoretic point of view, and thereby lead to higher-than-necessary compensation payments. At the same time, there might exist regulations that only power plants that are actually generating electricity can receive compensation – the rationale being that it may create perverse incentives for market participants to be paid to not do something. We will discuss and illustrate in a numerical example how such market rules can actually overly restrict operational efficiency and thereby reduce welfare.

The rules governing market operations obviously differ across various fields. A topic that has received far less attention from the applied Operations Research community is agriculture. Yet it is also a form of binary game between independent farmers (or agro-commercial enterprises), which decide whether to plant a field or not. Interestingly, and in contrast to power markets, there are subsidies being paid in many countries to farmers that let some of their land lay idle. Hence, there is a payment for doing nothing! The key difference may be that farming is seen as a market less prone to exertion of market power and strategic behavior, and that letting a field lay idle is still seen to provide a benefit beyond mere price support, for instance by maintaining the landscape or providing breeding grounds for wildlife – public acceptance of support schemes may be higher in these cases.

The outline of this paper is as follows: in the next section, we summarize current approaches to solve binary Nash games and place our contribution in the context of methods applied to solve such problems in the power sector. In Section 3, we propose an exact solution method to solve binary equilibrium problems, which explicitly incorporates the trade-off between overall efficiency and compensation payments in cases where no equilibrium exists. Section 4 applies our method to a power market example from the literature to illustrate its advantages and flexibility to incorporate distinct market rules regarding uplift payments. Section 5 concludes with a discussion on methods, other possible applications, and future work.¹

2 Current approaches to solve binary games

In this section, we motivate our method by describing how current solution methods for binary games obtain equilibria, and we identify where our formulation can improve this process. While there exist brute-force methods that solve for an equilibrium considering all possible combinations of the binary variables and check ex-post for deviation incentives, we want to concentrate on mathematical programming techniques for obtaining equilibria. For large-scale applications such as those considered in this work, computational efficiency proves a hurdle in these brute-force methods. Solving a large number of equilibrium problems is not very elegant and suffers from a curse of dimensionality, because the number of equilibrium problems to be solved is 2^k , where k

¹The Appendix provides computational results for a numerical test case using a larger data set than the stylized example in Section 4. The GAMS codes for the stylized example, the numerical test case, as well as an additional example for a resource market application with multiple binary investment decisions in production and pipeline capacity for several player are available for download at https://github.com/danielhuppmann/binary_equilibrium under a *Creative Commons Attribution 4.0 International License*.

is the number of binary variables. Therefore, mathematicians and Operations Researchers are constantly looking for ways to apply advances in Variational Inequalities and Integer Programming to develop faster methods to solve such problems.

2.1 Optimization-based approaches to obtain equilibria

Game theory and equilibrium problems have been an integral part of the history of mathematical programming. Their formulations fit neatly in the application areas and computational techniques of optimization. Moreover, ideas and algorithms from equilibrium problems have benefited solution techniques for optimization as well. First order optimality (Karush-Kuhn-Tucker, KKT) conditions, derived from each individual player's optimization problem, can be solved simultaneously by stacking them to form an equilibrium problem. Interpretations from dual variables to constraints in a game theory analysis can prove very insightful; they provide essential information in equilibrium problems and are often interpreted as prices or marginal benefits for individual players (Facchinei and Pang, 2003; Ferris and Pang, 1997; Murphy et al., 1982).

However, this relationship between optimality conditions and equilibrium problems fails once a game includes binary decision variables. The reason is that optimality conditions cannot be directly derived for binary optimization problems. Thus, applied optimization researchers aim to relax these binary problems to derive optimality conditions, or obtain duals to constraints of individual players' optimization problems in other ways. Our main contribution is that we overcome both these hurdles to obtain exact equilibria of the binary games and dual variables to constraints that can be interpreted as prices or marginal benefits.

For games where the individual players' optimization problems are non-convex with continuous variables, Pang and Scutari (2013) introduce the notion of a "quasi-Nash equilibrium" to describe solutions that are stationary points derived from relaxed constraint qualifications. Their paper provides analysis into these equilibria and their interpretation. In our paper, we are looking at a similar but distinct concept of an equilibrium. While Pang and Scutari obtain "quasi-Nash equilibria" where standard Nash equilibria exist but are computationally unattainable, we focus on cases where standard Nash equilibria do not exist, and obtain reasonable solution points instead. Moreover, the analysis by Pang and Scutari is focused on continuous games while we focus exclusively on binary games.

A method based on a trade-off between relaxing the integrality and the complementarity constraints is developed by Gabriel et al. (2013). While relaxing integrality has been employed as a way to solve integer programs, relaxing complementarity – essentially the optimality conditions – was the novel idea of their contribution. By relaxing optimality conditions, the authors were able to obtain solutions that were close to being stationary points and presumed to be equilibria. By relaxing both integrality and complementarity, the authors were also able to obtain duals to constraints, though interpreting them was not straightforward. The drawback to this approach is that there was no clear evidence that the solutions obtained were guaranteed to be equilibria or the duals of the constraints were guaranteed to be the marginal benefit of relaxing those constraints. In our work, we prove that the solutions from our formulation are in fact binary equilibria and provide duals which can be interpreted as marginal benefits when relaxing the players' constraints.

The problem is tackled from a related angle by Fuller and Celebi (2016); they propose a *minimum disequilibrium* model. They define disequilibrium as the difference between the pay-off in the socially optimal outcome and the individually optimal decision, summed over all players. That is, they seek to minimize the aggregated opportunity costs for all market participants from following the instructions of a social planner. The authors relate the MD model both to the results obtained by a social

planner and to the model proposed by Gabriel et al. (2013).

One recent article focuses on solving integral Nash-Cournot games (Todd, 2014) and provides an efficient algorithm to obtain equilibria. The method does not employ a relaxation but still converges to integer equilibria under plausible conditions. However, this method works very well for a specific integer game with no constraints, and the algorithm is not applicable to the broad class of binary constrained games considered in this paper.

2.2 Dual variables in binary programs

As mentioned above, dual variables in constrained convex optimization contain useful information both for computational purposes and interpretation of the problem under consideration. In principle, dual variables can be interpreted as multipliers and provide the marginal improvement of the objective function given a relaxation of the associated primal constraint. In particular, dual variables can be interpreted as prices supporting a Walrasian equilibrium in market models derived from convex optimization problems. However, in mathematical programs with binary or discrete constraints, the interpretation as marginal relaxation is not valid any more because of the non-convex and disjoint feasible region. This is related to the difficulty of determining the value function of the original problem (Guzelsoy and Ralphs, 2007).

To overcome this caveat and obtain dual variables from binary programs, the following approach is often used (cf. O’Neill et al., 2005). Consider the general constrained problem:

$$\begin{aligned} \min_{x,y} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) \leq 0 \quad , \text{ where } x \in \{0, 1\}^n, y \in \mathbb{R}^m \end{aligned} \quad (1)$$

To obtain dual variables to the constraints $g(x, y)$, such problems are commonly solved in a two-step procedure: first, the original problem (1) is solved using integer programming techniques; then, the binary variables x are linearized, and constraints are added to fix these variables at the level determined to be optimal, x^* , in the first step:

$$\begin{aligned} \min_{x,y} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) \leq 0 \quad (\lambda) \\ & x = x^* \quad (\mu) \quad , \text{ where } (x, y) \in \mathbb{R}^{n+m} \end{aligned} \quad (2)$$

Solving the reformulated problem (2) allows to interpret the dual variables (λ, μ) in the sense of multipliers or shadow values; offering these prices as contracts to market participants yields a Nash equilibrium. Note that while the dual variables μ are not part of the original problem, they are obtained from the linearized problem and can be thought of as the “price [...] representing the integral activity for (each) agent” (O’Neill et al., 2005, p. 279).

These duals are also important for integer programs, so that most numerical solvers automatically report these values when solving mixed-integer programs. However, one must be careful when using this approach in practical applications, as the duals from the linearized model cannot be readily interpreted as marginal relaxations of the original binary model – that is, the marginal value λ of the linearized fixed program cannot be interpreted as dual to the constraint of the original, mixed-integer program (problem 1). This is, however, what many power markets are currently doing in practice: they use the dual variable to the energy balance constraint as locational marginal price and clear the market based on these pay-offs. The dual prices of the binary activities μ are neglected. Instead, market operators assign compensation payments to make whole individual generators after the fact.

2.3 Uplifts, compensation, and equilibria in power markets

There already exists a substantial breadth of Operations Research literature with regard to electricity markets and pricing in non-convex problems, and binary games are a prevalent concern in this area. The current practice in many centrally dispatched power markets is that, first, the welfare-optimal dispatch is computed by the Independent System Operator (ISO) and locational marginal prices (LMP) in the network are determined using the two-step approach outlined above (i.e., from the linearized model). Uplift payments (i.e., compensation to individual players) are then calculated after market-clearance, ex-post, to ensure that each market player is made whole based on these prices. This is usually called *bid cost recovery*, although actual implementations and rules differ across markets.

As discussed by Sioshansi (2014), system operators usually have non-confiscatory compensation rules. This means that they do not assign penalties for deviation, but only disburse positive compensation payments. In that respect, current market operation deviates from contracts \mathbf{T} proposed by O’Neill et al. (2005), which are derived from all duals (λ, μ) . Instead, standard compensation payments are based on the pay-offs from “LMPs” (the dual variable or vector λ only, in particular the duals to the nodal energy balance constraint). It is important to note that these two are not equivalent.

This approach does not actually guarantee that the incentives of all players are aligned in the resulting market outcome, because the nature of the non-cooperative binary game between market participants is side-stepped (cf. Sioshansi, 2014). Generators that are not dispatched by the ISO may have an incentive to enter the market, if they would earn positive profits given resulting market prices, or to deviate from the announced schedule. Some markets allow *self-scheduling*, which gives generators the option to determine their dispatch individually rather than surrendering their generation decision to the ISO (cf. Sioshansi et al., 2010).

An alternative to the current approach is the minimum uplift or convex hull pricing method, which relies on a convex approximation of the lower bound of the aggregate cost function to derive prices and the minimal uplifts to support the market outcome (Schiroy et al., 2015; Gribik et al., 2007; Hogan and Ring, 2003). This method acknowledges that compensation is required to deter generators from following profitable deviations from the dispatch chosen by the ISO. Alas, using the convex hull relaxes the integrality of the underlying problem, and therefore also does not solve for the exact binary equilibrium.

Another important problem of the two-step approach arises from the fact that the budget for necessary compensation payments is not considered when determining the dispatch, but only computed ex-post. This neglects the potential trade-off between efficient market operation and minimizing the budget required for compensation payments, which is usually funded from fees or levies on market participants. These fees may in turn cause distortions in the market. It is easy to conceive of situations where accepting a slight reduction in market efficiency (i.e., reduction in welfare, higher costs for dispatch) allows to significantly reduce the compensation payments required. Hence, there may exist opportunities for improvements, which the current two-step market operation fails to identify. The illustrative example in Section 4 shows just such a situation.

The method developed in this work tackles these caveats of current approaches and proposes an exact solution method for games in binary variables. Our method offers an important practical advantage: it allows to directly balance efficient market operation based on an exact method for finding solutions to binary equilibrium problems, on the one hand, with the amount of compensation payments to ensure that these outcomes stable against deviation by individual players, on the other.

2.4 Marginal relaxation vs. the loss from a binary deviation

There is a further caveat of using the duals of problem (2) for algorithms and (economic) interpretation of results: this approach introduces the dual μ as the marginal relaxation of the constraint that fixes x at its optimal value. However, it is more appropriate to ask not about a marginal relaxation, but a switch from one possible value of the binary variable to the other. We introduce the “switch value” κ as the loss incurred by switching from the optimal value of the binary problem $f(x^*, y^*)$ to the optimal value of the objective function given that the binary variable takes the other value, $x^\times = 1 - x^*$, where y^\times is chosen so as to minimize $f(x^\times, y)$, i.e., $y^\times = \arg \min_y f(x^\times, y)$. Then, κ can be determined by computing:

$$\kappa = -f(x^*, y^*) + f(x^\times, y^\times).$$

When $x \in \{0, 1\}^n$ is a binary vector rather than a one-dimensional variable, the switch value can be computed component-wise, i.e., for element $j \in [1, \dots, n]$ of the switch vector $\kappa \in \mathbb{R}_+^n$:

$$\kappa_j = -f(x^*, y^*) + f((x_1^*, \dots, x_{j-1}^*, x_j^\times, x_{j+1}^*, \dots, x_n^*), y^\times) \quad \forall j \in [1, \dots, n],$$

where y^\times is computed accordingly for each element of κ .

In the method proposed below, we use this notion of a switch value κ to choose between equilibria in binary games. This variable also serves as a selection mechanism in such cases where no binary equilibrium exists; it can be used as a solution strategy to find an appropriate quasi-equilibrium.

We believe that this approach holds substantial promise with regard to algorithmic advances of binary and integer programming, as well as allow a better representation of real-world problems in economics, engineering, and beyond. Given that most large-scale applied integer programs are not solved to optimality, a branch-and-bound algorithm or stopping criterion based on the switch value κ may yield better results than current methods; these are usually based on satisfying a tolerance criterion between the best integer solution found so far and the optimal objective value of the linearized problem.

3 An exact solution for binary equilibrium problems

We now turn to our exact solution method to solve an equilibrium problem with binary variables. In the following section, we assume that each player has exactly one binary decision variable; this simplification is made only in the interest of a concise and simple exposition of our approach. The number of continuous decision variables and constraints is arbitrary. In the electricity market example presented in the following section, we show that our methodology can easily be extended to larger problems with multiple binary decision variables for each player.

The core idea for our approach is intuitive: for each player, we derive the first order optimality conditions with respect to the continuous decision variables for both values that the binary variable can take. In addition, we formulate an explicit incentive-compatibility constraint to ensure that each player chooses the state of the binary variable that is most beneficial to her.

The game is defined by a set of players $i \in I = [1, \dots, n]$, where each player seeks to minimize an objective function $f_i(\cdot)$. In the following formulation, each player only has one binary decision variable $x_i \in \{0, 1\}$, a (vector of) continuous decision variable(s) $y_i \in \mathbb{R}^m$, and a set of k constraints $g_i : \mathbb{R}^m \times \{0, 1\} \rightarrow \mathbb{R}^k$ with a vector of length k of associated dual variables λ_i . As elaborated in the previous section, these dual variables are only meaningful for a fixed x_i . The feasible region of each player is denoted by $K_i = \{(x_i, y_i) \mid g_i(x_i, y_i) \leq 0\}$.

Each player's optimization problem reads as follows:

$$\min_{x_i \in \{0,1\}, y_i \in \mathbb{R}^m} f_i(x_i, y_i, y_{-i}(x_{-i})) \quad (3a)$$

$$\text{s.t. } g_i(x_i, y_i) \leq 0 \quad (\lambda_i) \quad (3b)$$

The vector $y_{-i} = (y_j)_{j \in I \setminus \{i\}}$ is the collection of all rivals' decisions in continuous variables, and thus is of dimension $m \times (n-1)$. The set of feasible strategies by the rivals is denoted by $K_{-i} = \prod_{j \in I \setminus \{i\}} (y_j(x_j))$. Because the continuous variables of the rivals' depend on their binary decisions, K_{-i} is usually disjoint and not convex.

The formulation of problem (3) implicitly assumes that each player's payoff is only affected by the continuous decision variables of her rivals, but not directly affected by their binary variables. This is a simplification only for notational convenience and can easily be relaxed.

A Nash equilibrium to this game is a set of strategies such that each player chooses her optimal strategy given the action by the rivals. This is equivalent to the notion that no player has an incentive to unilaterally change her decision; there exists no profitable deviation. This is formally defined below; we distinguish between deviation incentives in the binary and the continuous variables to facilitate the exposition of the methodology.

Definition 1 (Nash equilibrium in a binary game). *We define the binary game as a set of players $i \in I$, each seeking to solve an optimization problem as given by problem (3). A Nash equilibrium to this game is a vector $((x_i^*, y_i^*) \in K_i)_{i \in I}$ such that y_i^* is the optimal decision (i.e., best response) by player i given x_i^* and $y_{-i}^*(x_{-i}^*)$,*

$$f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) \leq f_i(x_i^*, y_i, y_{-i}^*(x_{-i}^*)) \quad \forall y_i \in \{y_i \mid g_i(x_i^*, y_i) \leq 0\} \quad \forall i \in I, \quad (4)$$

and such that there is no profitable deviation with regard to the binary variable,

$$f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) \leq f_i(x_i^\times, y_i^\times, y_{-i}^*(x_{-i}^*)) \quad \forall i \in I, \quad (5)$$

where x_i^\times is the alternative value of x_i , i.e., $x_i^\times = 1 - x_i^*$, and y_i^\times is a best response of player i under the assumption that $x_i = x_i^\times$, i.e.,

$$f_i(x_i^\times, y_i^\times, y_{-i}^*(x_{-i}^*)) \leq f_i(x_i^\times, y_i, y_{-i}^*(x_{-i}^*)) \quad \forall y_i \in \{y_i \mid g_i(x_i^\times, y_i) \leq 0\} \quad \forall i \in I. \quad (6)$$

Because existence or uniqueness of equilibria cannot be guaranteed in binary games, we need to devise a method to select among several possible outcomes, or to arrive at a desired point which is "almost" an equilibrium (or a quasi-equilibrium). For this purpose, we introduce another player, which we call *market operator*, as a coordination agent and equilibrium selection mechanism. This entity is modeled as the upper-level player within a hierarchical, two-stage setup, where the lower-level constraints represent the binary equilibrium problem. The market operator guides the players towards a binary (quasi-) equilibrium and assigns compensation payments if necessary.

This upper-level player ensures that the optimality conditions and incentive compatibility (i.e., no profitable unilateral deviation from the market outcome) for each player are satisfied. If there exists more than one equilibrium, the market operator can choose which solution will materialize according to an adequate objective function. The market operator can also assign compensation payments to individual players if no equilibrium is feasible without it, and her objective function can include the trade-off between efficient operation and the required level of compensation disbursements.

We now formally introduce the term *quasi-equilibrium* for solutions to the binary game that are not Nash equilibria according to Definition 1, but which can be made incentive-compatible with appropriate compensation payments by the market operator.

Definition 2 (Quasi-equilibrium in a binary game with compensation).

We define the binary game with compensation as a set of players $i \in I$, each seeking to solve an optimization problem as given by problem (3). A binary quasi-equilibrium to this game is a vector $((x_i^*, y_i^*) \in K_i)_{i \in I}$ and a compensation vector $(\zeta_i \in \mathbb{R}_+)_{i \in I}$ such that for each player:

1. y_i^* is the optimal feasible decision (i.e., best response) by player i given x_i^* and $y_{-i}^*(x_{-i}^*)$,

$$f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) \leq f_i(x_i^*, y_i, y_{-i}^*(x_{-i}^*)) \\ \forall y_i \in \{y_i \mid g_i(x_i^*, y_i) \leq 0\} \quad \forall i \in I, \quad (7)$$

2. no player can improve her own payoff by deviating from x_i^* by more than the compensation payment ζ_i ; i.e., the compensation is at least as great as the benefit from deviation with regard to the binary variable. Hence, there is no profitable deviation with regard to the binary variable given the compensation payment,

$$f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) - \zeta_i \leq f_i(x_i^\times, y_i^\times, y_{-i}^*(x_{-i}^*)) \quad \forall i \in I \quad (8)$$

where x_i^\times and y_i^\times are defined as in Definition 1,

3. and the compensation payments are minimal, i.e., if a compensation payment is required for a player, then the incentive-compatibility condition (8) holds with equality. That is,

$$\zeta_i = \min_{\zeta_i \in \mathbb{R}_+} \zeta_i \\ \text{s.t.} \quad f_i(x_i^*, y_i^*, y_{-i}^*(x_{-i}^*)) - \zeta_i \leq f_i(x_i^\times, y_i^\times, y_{-i}^*(x_{-i}^*)) \quad \forall i \in I. \quad (9)$$

Note that when $\sum_{i \in I} \zeta_i = 0$, the binary quasi-equilibrium is also a Nash equilibrium. In the definition of the quasi-equilibrium, we directly incorporate the notion that the compensation payments should be minimal. This is helpful because it eliminates those incentive-compatible solutions where the market operator “over-compensates” some players, and it allows to focus on a smaller set of candidate solutions to the binary game.

3.1 Determining each player’s best response

In the definitions above, we have simply stated that the continuous decision variables y_i^* are the optimal decision of player i given the binary variable and the rivals’ decisions. In order to efficiently compute the best response of each player given the decisions of her rivals, we use first order optimality conditions with regard to the continuous decision variables. Hence, we need to make sure these conditions are necessary and sufficient so that we can capture the entire equilibrium set. An assumption on compactness is also needed for the selection of certain parameters of our method.

- A1** We assume that for each player $i \in I$, problem (3) is such that the first order optimality (KKT) conditions are necessary and sufficient with respect to the variables y_i , and the feasible region defined by the constraints $g_i(x_i, y_i)$ is compact and non-empty, for a fixed realization of x_i .

As an example, the KKT conditions are necessary and sufficient for problem (3) if $f_i(\bar{\mathbf{x}}_i, \cdot, y_{-i})$ are convex and $g_i(\bar{\mathbf{x}}_i, \cdot)$ affine for any fixed value $\bar{\mathbf{x}}_i \in \{0, 1\}$ and any fixed vector $y_{-i} \in K_{-i}$.

Let the vector (x_i^*, y_i^*) denote the best response for each player within the overall problem, given the decision vector $(y_{-i}(x_{-i}))_{i \in I}$ by all rivals, and let $\tilde{y}_i^{(\bar{\mathbf{x}}_i)}$ denote the best response of player i for a fixed $x_i = \bar{\mathbf{x}}_i \in \{0, 1\}$. Then, the objective value $f_i(\bar{\mathbf{x}}_i, \tilde{y}_i^{(\bar{\mathbf{x}}_i)}, y_{-i}(x_{-i}))$ is the best pay-off that a player can do given $\bar{\mathbf{x}}_i$ and the rivals' strategies.

Under Assumption **A1**, if the value of x_i is fixed at $\bar{\mathbf{x}}_i$, the best response $\tilde{y}_i^{(\bar{\mathbf{x}}_i)}$ can be found by solving the respective first order optimality conditions:

$$0 = \nabla_{y_i} f_i(\bar{\mathbf{x}}_i, \tilde{y}_i^{(\bar{\mathbf{x}}_i)}, y_{-i}(x_{-i})) + \tilde{\lambda}_i^{(\bar{\mathbf{x}}_i)} \nabla_{y_i} g_i(\bar{\mathbf{x}}_i, \tilde{y}_i^{(\bar{\mathbf{x}}_i)}) \quad , \quad \tilde{y}_i^{(\bar{\mathbf{x}}_i)} \text{ (free)} \quad (10a)$$

$$0 \geq g_i(\bar{\mathbf{x}}_i, \tilde{y}_i^{(\bar{\mathbf{x}}_i)}) \quad \perp \quad \tilde{\lambda}_i^{(\bar{\mathbf{x}}_i)} \geq 0 \quad (10b)$$

Player i will choose the binary variable x_i such that its objective value is minimal given the decisions of the rivals $y_{-i}(x_{-i})$. Mathematically, the best response of player i regarding her binary variable x_i can be written as follows:

$$f_i(\mathbf{1}, \tilde{y}_i^{(\mathbf{1})}, y_{-i}(x_{-i})) < f_i(\mathbf{0}, \tilde{y}_i^{(\mathbf{0})}, y_{-i}(x_{-i})) \quad \Rightarrow \quad x_i^* = 1 \quad (11a)$$

$$f_i(\mathbf{1}, \tilde{y}_i^{(\mathbf{1})}, y_{-i}(x_{-i})) > f_i(\mathbf{0}, \tilde{y}_i^{(\mathbf{0})}, y_{-i}(x_{-i})) \quad \Rightarrow \quad x_i^* = 0 \quad (11b)$$

$$f_i(\mathbf{1}, \tilde{y}_i^{(\mathbf{1})}, y_{-i}(x_{-i})) = f_i(\mathbf{0}, \tilde{y}_i^{(\mathbf{0})}, y_{-i}(x_{-i})) \quad \Rightarrow \quad x_i^* = \{0, 1\} \quad (11c)$$

The logic of conditions (11) is similar to the notion of incentive compatibility in game theory, i.e., there exists no profitable deviation given the decisions of all rivals. Hence, a vector $(x_i^*, y_i^*(x_i^*))_{i \in I}$ that satisfies the incentive-compatibility constraints in Definition 1 for each player constitutes a Nash equilibrium. If the incentive-compatibility condition is not satisfied for any feasible strategy, it may be necessary to financially compensate a player to ensure that she doesn't deviate, as stated in Definition 2.

A direct implementation of the implicit "if-then"-logic requires additional binary variables and thereby considerably increases numerical complexity. We overcome this drawback by proposing a mathematically equivalent formulation to represent the incentive-compatibility logic using the original binary variable of each player (or vector of binary variables in larger applications). The resulting overall program will be shown in problem (14); but first, we will discuss the reformulation and introduce the equilibrium selection mechanism in more detail.

3.2 An efficient formulation of incentive compatibility

We introduce four non-negative variables $(\kappa_i^{(\mathbf{1})}, \kappa_i^{(\mathbf{0})}, \zeta_i^{(\mathbf{1})}, \zeta_i^{(\mathbf{0})})$ for each player, and a sufficiently large scalar (or vector of scalars) \bar{K} . In applied work, one can of course make the parameter \bar{K} specific to each player and constraint to improve computational efficiency. We omit this specification for notational convenience. The vector $\kappa_i^{(\bar{\mathbf{x}}_i)}$ is the switch value introduced in Section 2.4; it can be interpreted as the loss the player would incur by switching from her optimal value of the binary variable to the alternative.

The vector $\zeta_i^{(\bar{\mathbf{x}}_i)}$ denotes compensation payments to guarantee incentive compatibility; that is, in cases where the market operator requires a player to act against her own objective with regard to the binary variable so that a binary quasi-equilibrium can be obtained, a remuneration must be paid as compensation and to ensure that the player does not have a profitable deviation.

The vectors $\kappa_i^{(\bar{\mathbf{x}}_i)}$ and $\zeta_i^{(\bar{\mathbf{x}}_i)}$ are not dual variables in the original sense, but they do contain similar information regarding the solution. Hence, they are analogous in interpretation to a dual – but in terms of a binary deviation, not in the sense of a marginal relaxation. Alas, the term "shadow price" often used in economics as synonymous for dual variables could also be applied here.

We can now replace the incentive compatibility conditions (equations 11) by a more efficient formulation:

$$f_i(\mathbf{1}, \tilde{y}_i^{(1)}, y_{-i}) + \kappa_i^{(1)} - \zeta_i^{(1)} - \kappa_i^{(0)} + \zeta_i^{(0)} = f_i(\mathbf{0}, \tilde{y}_i^{(0)}, y_{-i}) \quad (12a)$$

$$\kappa_i^{(1)} + \zeta_i^{(1)} \leq x_i \tilde{K} \quad (12b)$$

$$\kappa_i^{(0)} + \zeta_i^{(0)} \leq (1 - x_i) \tilde{K} \quad (12c)$$

$$\kappa_i^{(1)}, \kappa_i^{(0)}, \zeta_i^{(1)}, \zeta_i^{(0)} \in \mathbb{R}_+$$

The market operator selects a solution such that the first order conditions and the incentive-compatibility constraints are satisfied for all players. In line with the definition of the quasi-equilibrium as the minimal compensation payment for each player i , the variables $\kappa_i^{(\bar{x}_i)}$ and $\zeta_i^{(\bar{x}_i)}$ cannot both be strictly greater than zero at a solution; this will be formally discussed when we introduce the problem of the market operator.

First, let us illustrate and discuss the interpretation of the variables $\kappa_i^{(\bar{x}_i)}$ and $\zeta_i^{(\bar{x}_i)}$ in more detail. The question is whether the solution for the overall equilibrium problem chosen by the market operator is aligned with the best response of each player. By this, we mean whether a player's individually optimal decision coincides with the (quasi-) equilibrium chosen by the market operator.

There are five possible outcomes regarding the incentive alignment of an individual player and the market operator; the cases are illustrated in Table 1. In cases I and II, the incentives are aligned, as the player would incur losses (a strictly worse payoff) by deviating from the outcome decided by the market operator. The respective switch value $\kappa_i^{(\bar{x}_i)}$ is strictly positive. There may be situations where the player has a strict preference regarding her binary decision, but the market operator is actually indifferent. In this case, we assume that the market operator will choose the solution so that it coincides with the player's preference; otherwise, the market operator would have to disburse compensation payments.

In cases III and IV, the solution chosen by the market operator is not in line with the player's individual best response; only by disbursing compensation payments can the market operator convince the player not to deviate to the individually optimal decision. As a consequence, the respective compensation payment $\zeta_i^{(\bar{x}_i)}$ is strictly positive, and a quasi-equilibrium is realized.

In the last case (no. V), the player is indifferent between her options, so the market operator is not restricted in selecting either outcome. The player does not have a positive switch value in either direction, and no compensation is required.

case	individually optimal x_i	equilibrium x_i chosen by market operator	$\kappa_i^{(1)}$	$\kappa_i^{(0)}$	$\zeta_i^{(1)}$	$\zeta_i^{(0)}$	incentives aligned
I	1	1	> 0	0	0	0	yes
II	0	0	0	> 0	0	0	yes
III	0	1	0	0	> 0	0	no
IV	1	0	0	0	0	> 0	no
V	indifferent	1 / 0	0	0	0	0	yes

Table 1: Incentive alignment between a player's individually optimal decision (her best response) and the (quasi-) equilibrium chosen by the market operator

3.3 Translating each player’s best response into the overall game

From equations (10), we have obtained two optimal decision vectors, $\tilde{y}_i^{(\bar{x}_i)}$, for each player for both values that the variable \bar{x}_i can take. We now need to translate which of these two decision variables is realized in the (quasi-) equilibrium and “seen” by the rivals in their own optimization problem, contingent on the equilibrium decision x_i . We do this by adding the following constraints:

$$\tilde{y}_i^{(0)} - x_i \tilde{K} \leq y_i \leq \tilde{y}_i^{(0)} + x_i \tilde{K} \quad (13a)$$

$$\tilde{y}_i^{(1)} - (1 - x_i) \tilde{K} \leq y_i \leq \tilde{y}_i^{(1)} + (1 - x_i) \tilde{K} \quad (13b)$$

The logic of constraints (13) is straightforward: the decision vector y_i , as it is considered by the rivals and the market operator in their optimization problems, must be equal to the optimal decision $\tilde{y}_i^{(\bar{x}_i)}$ for whichever value of x_i is the solution in the (quasi-) equilibrium, as stated by equations (11) and reformulated using equations (12), i.e., $x_i = 0 \Rightarrow y_i = \tilde{y}_i^{(0)}$ and $x_i = 1 \Rightarrow y_i = \tilde{y}_i^{(1)}$. The parameter \tilde{K} must be chosen suitably large so as not to constrain the continuous decision variable(s). Since this is a primal variable, it should be straightforward to find a suitable bound in any practical application.

We can now combine the incentive-compatibility constraints (12) with the equilibrium conditions (10) for the continuous decision variables into one set of constraints with binary variables. The non-linearity of the complementarity conditions (10) can be readily reformulated applying disjunctive constraints (Fortuny-Amat and McCarl, 1981) or using SOS type 1 variables (Siddiqui and Gabriel, 2013).

3.4 A multi-objective program subject to binary (quasi-) equilibria

So far, we have only replaced a number of equilibrium problems (for each possible combination of binary variables) by a set of integer constraints that exactly represent all binary (quasi-) equilibria. Next, we can apply multi-objective programming to direct the game towards desired solutions. To this end, we introduce the market operator, and we assume that she seeks to minimize an objective function consisting of two terms: a function $F(\cdot)$, which only depends on the actual market outcome (efficiency of the solution; cost-minimization or welfare-maximization may be a natural choice for this term) and a function $G(\cdot)$, which serves as a regularizer; in economic applications, it can be interpreted as a penalty term that seeks to minimize the compensation payments required to ensure incentive compatibility of the market solution.

As is common in multi-objective programming, one can easily explore the range of equilibria by solving the reformulated problem using different functional forms. Furthermore, one can sort binary (quasi-) equilibria from a “best” to a “worst” outcome by solving the model consecutively and in each step excluding the previous solutions; here, “best” is to be understood as given by the objective function $F(\cdot) + G(\cdot)$.

The overall problem is a mathematical program subject to a binary equilibrium problem, where $\bar{x}_i = \{0, 1\}$ are the binary variables in the lower-level problem.

$$\min_{\substack{x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)} \\ \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)}}} F\left((x_i, y_i)_{i \in I}\right) + G\left((\zeta_i^{(\bar{x}_i)})_{i \in I}\right) \quad (14a)$$

$$\text{s.t. } \nabla_{y_i} f_i\left(\mathbf{1}, \tilde{y}_i^{(1)}, y_{-i}\right) + (\tilde{\lambda}_i^{(1)})^T \nabla_{y_i} g_i\left(\mathbf{1}, \tilde{y}_i^{(1)}\right) = 0 \quad (14b)$$

$$0 \leq -g_i\left(\mathbf{1}, \tilde{y}_i^{(1)}\right) \perp \tilde{\lambda}_i^{(1)} \geq 0 \quad (14c)$$

$$\nabla_{y_i} f_i\left(\mathbf{0}, \tilde{y}_i^{(0)}, y_{-i}\right) + (\tilde{\lambda}_i^{(0)})^T \nabla_{y_i} g_i\left(\mathbf{0}, \tilde{y}_i^{(0)}\right) = 0 \quad (14d)$$

$$0 \leq -g_i\left(\mathbf{0}, \tilde{y}_i^{(0)}\right) \perp \tilde{\lambda}_i^{(0)} \geq 0 \quad (14e)$$

$$f_i\left(\mathbf{1}, y_i^{(1)}, y_{-i}\right) + \kappa_i^{(1)} - \zeta_i^{(1)} - \kappa_i^{(0)} + \zeta_i^{(0)} = f_i\left(\mathbf{0}, y_i^{(0)}, y_{-i}\right) \quad (14f)$$

$$\kappa_i^{(1)} + \zeta_i^{(1)} \leq x_i \tilde{K} \quad (14g)$$

$$\kappa_i^{(0)} + \zeta_i^{(0)} \leq (1 - x_i) \tilde{K} \quad (14h)$$

$$\tilde{y}_i^{(0)} - x_i \tilde{K} \leq y_i \leq \tilde{y}_i^{(0)} + x_i \tilde{K} \quad (14i)$$

$$\tilde{y}_i^{(1)} - (1 - x_i) \tilde{K} \leq y_i \leq \tilde{y}_i^{(1)} + (1 - x_i) \tilde{K} \quad (14j)$$

$$x_i \in \{0, 1\}, (y_i, \tilde{y}_i^{(\bar{x}_i)}) \in \mathbb{R}^{3m}, (\lambda_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)}) \in \mathbb{R}_+^{2k+4}$$

It is important to note that the binary variable of each player has an additional role in this formulation: it also controls which of the two potential states with regard to the continuous variables are active and “visible” to the rivals (constraints 14i and 14j). Furthermore, it ensures that the correct switch values and compensation payments are active (constraints 14g and 14h), in line with Table 1.

Theorem 1 (Exact solutions of the binary Nash game). *Under Assumption A1, any vector $(x_i, y_i)_{i \in I}$ is a solution to the binary game as stated in Definition 1 if and only if there exists a vector $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)})_{i \in I}$, with $\zeta_i^{(\bar{x}_i)} = 0 \forall i \in I$, such that $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)})_{i \in I}$ is a feasible point to problem (14).*

Proof. First, assume $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)})_{i \in I}$ with $\zeta_i^{(\bar{x}_i)} = 0 \forall i \in I$ is a feasible point to problem (14). Then, by Assumption A1, we know that the point $(x_i, y_i)_{i \in I}$ is an optimal solution for each player given fixed values of x_i and $y_{-i} \forall i \in I$. This satisfies the first part of Definition 1. Furthermore, we know that $\zeta_i^{(\bar{x}_i)} = 0 \forall i \in I$, and $(\kappa_i^{(\bar{x}_i)})_{i \in I}$ will be selected according to the constraints of problem (14). By these constraints, we know that $f_i(x_i, y_i, y_{-i}(x_{-i})) \leq f_i(x_i^\times, y_i^\times, y_{-i}(x_{-i})) \forall i \in I$, where x_i^\times is the alternative value of x_i (i.e., $x_i^\times = 1 - x_i$) and y_i^\times is a best response of player i , i.e.,

$$f_i(x_i^\times, y_i^\times, y_{-i}(x_{-i})) \leq f_i(x_i^\times, y_i, y_{-i}(x_{-i})) \quad \forall y_i \in \left\{y_i \mid g_i(x_i^\times, y_i) \leq 0\right\} \quad \forall i \in I.$$

This satisfies the second part of Definition 1 and thus we have shown that $(x_i, y_i)_{i \in I}$ is a solution to the binary game defined by Definition 1.

Now, we assume that $(x_i, y_i)_{i \in I}$ is a solution to the binary game defined by Definition 1. Choose \tilde{K} large enough so that it is greater than the difference between any upper and lower bounds on $y_i \forall i \in I$ and greater than the difference between the minimum and maximum value of $f_i(\cdot) \forall i \in I$. Such a value exists since by Assumption A1; the feasible set is compact and $f_i(\cdot)$ is continuous, so the maximum and minimum must exist. Then, by Definition 1, for any fixed value of x_i and y_{-i} , y_i is an optimal solution to the individual player’s optimization problem. Thus, you can find $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)})$ such that $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)})_{i \in I}$ satisfies the constraints (14b–14e). Take $\zeta_i^{(\bar{x}_i)} = 0 \forall i \in I$, and choose $\kappa_i^{(\bar{x}_i)}$ according to constraint (14f).

Thus, for any solution to the binary game in Definition 1 given by $(x_i, y_i)_{i \in I}$, we have shown that there exists a vector $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)})_{i \in I}$, with $\zeta_i^{(\bar{x}_i)} = 0 \forall i \in I$, such that $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)})_{i \in I}$ is a feasible point to problem (14). \square

An efficient technique to choose \tilde{K} is to linearize the binary variables in the individual optimization problems and minimize and maximize over $f_i(\cdot)$ to find the largest difference possible. The next theorem shows that the method can also be applied to obtain a quasi-equilibrium. Note that we need an assumption on the objective function before we can prove that our method can obtain a quasi-equilibrium.

A2 Assume that $F(\cdot)$ and $G(\cdot)$ are convex quadratic or linear functions for every fixed binary variable x_i and that $\partial G(\cdot)/\partial \zeta_i > 0 \forall i \in N$.

Theorem 2 (Exact solutions of the binary Nash game with compensation). *Under Assumptions A1 and A2, if there exists a vector $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ that is an optimal solution to problem (14), then the vector $(x_i, y_i)_{i \in I}$ with compensation $(\zeta_i)_{i \in I}$ is a solution to the binary game as stated in Definition 2. Following the term introduced in Definition 2, we refer to this as a binary quasi-equilibrium.*

Furthermore, under Assumptions A1 and A2, if $(x_i, y_i)_{i \in I}$ is a solution to the binary game with compensation $(\zeta_i)_{i \in I}$ as stated in Definition 2, then there exists a vector $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$, such that $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ is a feasible point to problem (14).

Proof. First, assume $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ is an optimal solution to problem (14). Then, by A1, we know that the point $(x_i, y_i)_{i \in I}$ is an optimal solution for each player given fixed values of x_i and y_{-i} . This satisfies the first part of Definition 2. Furthermore, we know that $\kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)} \forall i \in I$ will be selected according to the constraints of problem (14). By these constraints, we know that $f_i(x_i, y_i, y_{-i}(x_{-i})) - \zeta_i \leq f_i(x_i^\times, y_i^\times, y_{-i}(x_{-i})) \forall i \in I$, where x_i^\times is the alternative value of x_i (i.e., $x_i^\times = 1 - x_i$) and y_i is a best response of player i with fixed x_i^\times , i.e.,

$$f_i(x_i^\times, y_i^\times, y_{-i}(x_{-i})) \leq f_i(x_i^\times, y_i, y_{-i}(x_{-i})) \quad \forall y_i \in \{y_i \mid g_i(x_i^\times, y_i) \leq 0\} \quad \forall i \in I.$$

This satisfies the second part of Definition 2.

By Assumption A2, we know that $\partial G(\cdot)/\partial \zeta_i > 0 \forall i \in I$ and, hence, for any optimal solution, for each player, $(\zeta_i)_{i \in I}$ is minimal. This satisfies the third part of Definition 2 and hence we have shown that if $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ is an optimal solution to problem (14), then $(x_i, y_i)_{i \in I}$ with compensation payments $(\zeta_i)_{i \in I}$ is a solution to the binary game defined by Definition 2.

Now, we assume that $(x_i, y_i)_{i \in I}$ with compensation payments $(\zeta_i)_{i \in I}$ is a quasi-equilibrium to the binary game with compensation defined by Definition 2. Choose \tilde{K} large enough so that it is greater than the difference between any upper and lower bounds on $y_i \forall i \in I$ and greater than the difference between the minimum and maximum value of $f_i \forall i \in I$. Such a value exists since by Assumption A1, the feasible set is compact and f_i is continuous, so the maximum and minimum must exist. Then, by Definition 2, for any fixed value of x_i and y_{-i} , y_i is an optimal solution to the individual player's optimization problem. Thus, you can find $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)})$ such that $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)})_{i \in I}$ satisfies the first two constraints in problem (14). Calculate $\zeta_i^{(\bar{x}_i)}$ from $\zeta_i \forall i \in I$ and choose $\kappa_i^{(\bar{x}_i)}$ according to the third constraint in problem (14). Thus, for any solution to the binary game in Definition 2 given by $(x_i, y_i)_{i \in I}$ and compensation payment $(\zeta_i)_{i \in I}$, we have shown that there exists a vector $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$, such that $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ is a feasible point to problem (14). \square

Corollary 1. *Under Assumptions A1 and A2, any vector $(x_i, y_i)_{i \in I}$ is a binary quasi-equilibrium for the Nash game in binary variables with compensation $(\zeta_i)_{i \in I}$ as defined in Definition 2 if there exists a vector $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$, such that $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ is a feasible solution to problem (14) and $\zeta_i^{(\bar{x}_i)}$ is minimal as defined in Definition 2.*

Proof. By the arguments in Theorem 2, for any point $(x_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ that is feasible to problem (14), the vectors $(x_i, y_i)_{i \in I}$ and $(\zeta_i)_{i \in I}$ satisfy the first two conditions of Definition 2. If, in addition, Condition 3 of Definition 2 is satisfied, i.e., ζ_i is minimal for each player $i \in I$, then $(x_i, y_i)_{i \in I}$ is a binary quasi-equilibrium with compensation $(\zeta_i)_{i \in I}$. \square

If a vector is a global minimum to the objective function (14a), this is the binary quasi-equilibrium with the lowest objective function value $F(\cdot) + G(\cdot)$. The following lemma and theorem provide conditions for the existence of a binary quasi-equilibrium that can be supported by appropriate compensation payments.

Lemma 1 (Existence of a Nash equilibrium in a game with fixed binary variables). *If for a fixed vector $(\bar{x}_i)_{i \in I}$, the objective function $f_i(\bar{x}_i, y_i, y_{-i}(\bar{x}_{-i}))$ of every player $i \in I$ is continuous with regard to y_i and $y_{-i}(\bar{x}_{-i})$, and quasi-convex in y_i , and the feasible region defined by the constraints $g_i(\bar{x}_i, y_i)$ is compact, convex and non-empty, then the resulting continuous game has a solution.*

Proof. The existence follows readily from Kakutani's fixed-point theorem (Glicksberg, 1952). \square

Relaxations of these conditions for the existence of a Nash equilibrium in continuous decision variables are also discussed in the literature (Facchinei and Pang, 2003; Nishimura and Friedman, 1981). The existence result for Nash equilibria in continuous games given fixed binary variables in Lemma 1 can be combined with Theorem 3 to provide reasonable conditions for the existence of binary quasi-equilibria.

Theorem 3 (Existence of a binary quasi-equilibrium). *Under Assumption A1 and A2, if for any fixed vector $(\bar{x}_i)_{i \in I}$, the resulting continuous game has a solution, then a corresponding binary quasi-equilibrium exists for the Nash game in binary variables.*

Proof. For any $(y_i)_{i \in I}$ that is a Nash equilibrium given the fixed vector $(\bar{x}_i)_{i \in I}$, we can find vectors $(\tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)})_{i \in I}$ such that $(\bar{x}_i, y_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)})_{i \in I}$ is a feasible point to constraints (14b–14e). Recall that our notation implies $\bar{x}_i = (\{\bar{x}_i, 1 - \bar{x}_i\})_{i \in I}$. Choose \tilde{K} as in the proof for Theorem 2.

We can then find values for $(\kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ such that $(\bar{x}_i, \tilde{y}_i^{(\bar{x}_i)}, \tilde{\lambda}_i^{(\bar{x}_i)}, \kappa_i^{(\bar{x}_i)}, \zeta_i^{(\bar{x}_i)})_{i \in I}$ satisfy constraints (14f–14j), and where either $\kappa_i^{(\bar{x}_i)} = 0$ or $\zeta_i^{(\bar{x}_i)} = 0$ for every player $i \in I$. By equation (14f), this implies that $\zeta_i^{(\bar{x}_i)}$ is minimal. By Corollary 1, $(\bar{x}_i, y_i)_{i \in I}$ with compensation payments $(\zeta_i)_{i \in I}$ is a binary quasi-equilibrium. \square

Theorem 3 implies that, if a Nash equilibrium solution to the continuous game exists for any fixed realization of the binary variables, then this solution can be supported as a quasi-equilibrium with appropriate compensation payments.

The reformulated multi-objective problem subject to a binary quasi-equilibrium (14) is a mixed-integer program. However, the incentive-compatibility constraint (condition 14f) can still cause numerical difficulties, because the players' objective function are often not linear and not even convex in terms of all variables, even when they are linear from the point of view of the player itself.

We will now introduce a special case, which allows to reduce problem (14) to a linear or quadratic convex mixed-integer program with linear constraints.

A3 Assume that the each lower-level player's objective function $f_i(x_i, y_i, y_{-i}(x_{-i}))$ can be separated into two functions, where the first part is linear with respect to x_i and y_{-i} , and the second part is linear only with respect to y_i ,

$$f_i(x_i, y_i, y_{-i}(x_{-i})) = f_i^{(x)}(x_i, y_{-i}(x_{-i})) + f_i^{(y)}(y_i | y_{-i}(x_{-i})),$$

and the partial derivative of the objective function with regard to the continuous variable y_i , $\nabla_{y_i} f_i^{(y)}(y_i^{(\bar{x}_i)} | y_{-i}(x_{-i}))$, is linear in all variables.

Furthermore, assume that all constraints $g_i(x_i, y_i)$ are affine and can therefore be written as:

$$g_i(x_i, y_i) \leq 0 \Rightarrow a_i x_i + A_i y_i \leq b_i$$

where a_i, b_i are vectors and A_i a matrix of suitable dimensions.

Note that the function $f_i^{(y)}(y_i | y_{-i}(x_{-i}))$ need not be linear with respect to all variables, only with regard to the player's own continuous decision variable y_i given a fixed value of y_{-i} .

Theorem 4 (Exact reformulation as a mixed-integer linear/quadratic program with linear constraints). *Under Assumptions A1, A2 and A3, the multi-objective program subject to a binary quasi-equilibrium (problem 14) can be reformulated as a quadratic integer program with linear constraints. Theorems (1) and (2) remain valid.*

Proof. By Assumptions A1, A2 and A3, the objective function is linear or convex quadratic, and the first order conditions and the players' constraints are linear. The complementarity conditions (14c) and (14e) can be reformulated using disjunctive constraints (Fortuny-Amat and McCarl, 1981).

The incentive-compatibility constraint (14f) can be reformulated as follows: since the function $f_i^{(y)}(y_i^{(\bar{x}_i)} | y_{-i}(x_{-i}))$ is linear with respect to $y_i^{(\bar{x}_i)}$, we know that

$$f_i^{(y)}(y_i^{(\bar{x}_i)} | y_{-i}(x_{-i})) = \left(\nabla_{y_i} f_i^{(y)}(y_i^{(\bar{x}_i)} | y_{-i}(x_{-i})) \right)^T y_i^{(\bar{x}_i)}.$$

By first order optimality, $\left(\nabla_{y_i} f_i^{(y)}(y_i^{(\bar{x}_i)} | y_{-i}(x_{-i})) \right) = - \left((\lambda_i^{(\bar{x}_i)})^T \nabla_{y_i} g_i(\bar{x}_i, y_i^{(\bar{x}_i)}) \right)$.

Then, applying the definition of $g_i(\cdot)$ in Assumption A3, $\nabla_{y_i} g_i(\bar{x}_i, y_i^{(\bar{x}_i)}) = A_i$, and using the complementarity condition of constraints (14c) and (14e), $(a_i x_i + A_i y_i - b_i)^T \lambda_i = 0$, the reformulation proceeds as follows:

$$\begin{aligned} f_i(\bar{x}_i, y_i, y_{-i}(x_{-i})) &= f_i^{(x)}(\bar{x}_i, y_{-i}(x_{-i})) + f_i^{(y)}(y_i^{(\bar{x}_i)} | y_{-i}(x_{-i})) \\ &= f_i^{(x)}(\bar{x}_i, y_{-i}(x_{-i})) + \left(\nabla_{y_i} f_i^{(y)}(y_i^{(\bar{x}_i)} | y_{-i}(x_{-i})) \right)^T y_i^{(\bar{x}_i)} \\ &= f_i^{(x)}(\bar{x}_i, y_{-i}(x_{-i})) - \left((\lambda_i^{(\bar{x}_i)})^T \nabla_{y_i} g_i(\bar{x}_i, y_i^{(\bar{x}_i)}) \right)^T y_i^{(\bar{x}_i)} \\ &= f_i^{(x)}(\bar{x}_i, y_{-i}(x_{-i})) - \left((\lambda_i^{(\bar{x}_i)})^T A_i \right)^T y_i^{(\bar{x}_i)} \\ &= f_i^{(x)}(\bar{x}_i, y_{-i}(x_{-i})) - (\lambda_i^{(\bar{x}_i)})^T (b_i - a_i \bar{x}_i) \end{aligned}$$

Therefore, problem (14) can be reformulated as a linear/quadratic convex integer program with linear constraints. \square

With this theorem, we show that our approach can be applied to a large number of problem classes and still be solved as a mixed-integer linear program. These include operational constraints such as capacity constraints or minimum generation constraints, and market forms including linear inverse demand functions.

3.5 Comparing the binary equilibrium to a social-planner model

Before illustrating the proposed method for identifying binary equilibria in non-cooperative games using a power market example, we discuss how the proposed method compares to a commonly used alternative approach for solving such problems. As a benchmark for this comparison, we use the following method: first, solve the welfare-maximizing problem assuming a central planner, then derive prices from the linearized model using the O'Neill et al. (2005) method, and finally compute compensation payments for each player that has a profitable deviation, to ensure that every market participant is at least indifferent and the outcome is stable in the Nash sense. This method is, in a way, a lexicographic solution approach to the over-all problem; mathematically, it can be seen as a hierarchical min-min problem. For illustration, we formulate this problem along the notation used in the derivation of the binary equilibrium method:

$$\begin{aligned} \min_{\zeta_i} \quad & G\left(\left(\zeta_i\right)_{i \in I}\right) + \left\{ \begin{array}{l} \min_{x_i, y_i} F\left(\left(x_i, y_i\right)_{i \in I}\right) \\ \text{s.t. } g_i(x_i, y_i) \leq 0 \quad \forall i \in I \end{array} \right\} \\ \text{s.t.} \quad & \left\{ \begin{array}{l} f_i(x_i, y_i, y_{-i}(x_{-i})) - \zeta_i \leq f_i(x_i^\times, y_i^\times, y_{-i}(x_{-i})) \\ \forall (x_i^\times, y_i^\times) \in \arg \min_{(x_i^\circ, y_i^\circ)} \left\{ f_i(x_i^\circ, y_i^\circ, y_{-i}(x_{-i})) \mid g_i(x_i^\circ, y_i^\circ) \leq 0 \right\} \end{array} \right\} \end{aligned} \quad (15)$$

$\forall i \in I$

Throughout the following discussion, we will denote the optimal solution of the inner welfare optimization of problem 15 as $(x_i^\circ, y_i^\circ)_{i \in I}$, while $(\zeta_i^\circ)_{i \in I}$ is the minimal compensation payment to guarantee incentive compatibility for all players. Similar as in the definition of the binary game introduced earlier, the objective function of player i evaluated at (x_i^\times, y_i^\times) is the best a player can do by deviating (cf. Definitions 1 and 2). In the context of this example, deviation is to be understood as not following the decision of the benevolent social planner.

For comparison, we restate the multi-objective program subject to a binary (quasi-)equilibrium (problem 14) in simplified form:

$$\begin{aligned} \min_{\zeta_i} \quad & G\left(\left(\zeta_i\right)_{i \in I}\right) + F\left(\left(x_i, y_i\right)_{i \in I}\right) \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \nabla_{y_i} f_i\left(\bar{x}_i, \tilde{y}_i^{(\bar{x}_i)}, y_{-i}\right) + \left(\tilde{\lambda}_i^{(\bar{x}_i)}\right)^T \nabla_{y_i} g_i\left(\bar{x}_i, \tilde{y}_i^{(\bar{x}_i)}\right) = 0 \\ 0 \leq -g_i\left(\bar{x}_i, \tilde{y}_i^{(\bar{x}_i)}\right) \perp \tilde{\lambda}_i^{(\bar{x}_i)} \geq 0 \\ f_i\left(\mathbf{1}, y_i^{(1)}, y_{-i}\right) + \kappa_i^{(1)} - \zeta_i^{(1)} - \kappa_i^{(0)} + \zeta_i^{(0)} = f_i\left(\mathbf{0}, y_i^{(0)}, y_{-i}\right) \\ \text{“translation” constraints 14g-14j} \end{array} \right\} \end{aligned} \quad (14')$$

$\forall i \in I$

In the following discussion, the optimal values of each player's binary and continuous decision variables in equilibrium and the compensation payments are denoted by $(x_i^*, y_i^*, \zeta_i^*)_{i \in I}$.

The key difference between the above problems is that problem 15 solves an optimization problem while problem 14 solves for a non-cooperative equilibrium among all players. Thus, problem 15 is constrained by general power market constraints while problem 14 has additional equilibrium constraints. Clearly, problem 14 is a restricted version of problem 15 whenever there are no compensation payments or when the objective function does not include a penalty term (or regularizer) $G((\zeta_i)_{i \in I})$.

The following two theorems formalize the idea that the social planner problem is a less constrained version of the binary equilibrium problem. First, we show that if the socially optimal outcome $(x_i^\circ, y_i^\circ)_{i \in I}$ does not require any compensation payments, this solution (weakly) dominates the binary equilibrium outcome. Second, a similar argument can be made if profitable deviations exist in the socially optimal outcome,

but the social planner does not penalize those (i.e, compensation payments are assumed to incur no loss to overall welfare).

Theorem 5. *If any optimal solution obtained by the social planner (problem 15) does not require compensation payments to any player $i \in I$, then the objective value achieved by the social planner is at least as small as solution of the binary (quasi-) equilibrium model, i.e.,*

$$\begin{aligned} \sum_i \zeta_i^\circ = 0 &\Rightarrow f_i(x_i^\circ, y_i^\circ, y_{-i}^\circ(x_{-i}^\circ)) \leq f_i(x_i^\times, y_i^\times, y_{-i}^\circ(x_{-i}^\circ)) \quad \forall i \in I \\ &\Rightarrow F\left((x_i^\circ, y_i^\circ)_{i \in I}\right) \leq F\left((x_i^*, y_i^*)_{i \in I}\right). \end{aligned}$$

Proof. Assume $\sum_i \zeta_i^\circ = 0$ at any optimal point for problem (15). Whenever $\sum_i \zeta_i^\circ = 0$ in problem (15), the feasible region of problem (14) is a subset of the feasible region for problem (15). Moreover, since $G((\zeta_i)_{i \in I})$ is an increasing function of $(\zeta_i)_{i \in I}$, whenever $\sum_i \zeta_i^\circ = 0$, $G((\zeta_i)_{i \in I})$ is at its minimum. Thus, the optimal objective function value of problem (15) forms a lower-bound to any optimal objective function value of problem (14). These two facts combine to show that whenever $\sum_i \zeta_i^\circ = 0$ at any optimal point for problem (15), $F((x_i^\circ, y_i^\circ)_{i \in I}) \leq F((x_i^*, y_i^*)_{i \in I})$. \square

The reasoning for Theorem 5 is quite straightforward: if compensation payments are not required, the regularizer $G((\zeta_i)_{i \in I})$ (or penalty term for compensation payments) does not add anything to the objective value of problem (15) and the incentive-compatibility constraints are not relevant. Then, the binary-equilibrium model (problem 14) is a more constrained version of the inner problem of the social-welfare maximizing planner: the objective function $F((x_i, y_i)_{i \in I})$ and the constraints $g_i(x_i, y_i)$ are present in both programs, but the binary-equilibrium reformulation adds further constraints.

Theorem 6. *If compensation payments do not incur any cost to overall societal welfare in both problems (15) and (14), then the optimal objective value achieved by the social planner is at least as small as the solution of the binary (quasi-) equilibrium model, i.e.,*

$$G\left((\zeta_i)_{i \in I}\right) = 0 \Rightarrow F\left((x_i^\circ, y_i^\circ)_{i \in I}\right) \leq F\left((x_i^*, y_i^*)_{i \in I}\right).$$

Proof. Assume that $G((\zeta_i)_{i \in I}) = 0$ in both problems (15) and (14). Then, for any $(\zeta_i)_{i \in I}$, the feasible region of problem (14) is a subset of the feasible region of problem (15). Thus, for any optimal solution to problem (15), $F((x_i^\circ, y_i^\circ)_{i \in I}) \leq F((x_i^*, y_i^*)_{i \in I})$. \square

The reasoning is similar to Theorem 5: if compensation does not incur any penalty or negative effects, then redistribution of the biggest possible surplus (or least possible cost) is indeed the preferred strategy. The additional constraints of the binary-equilibrium method may restrict the solution space and could make the socially optimal outcome infeasible in problem (14).

So far, we have shown cases where the social planner is superior to the binary-equilibrium model. We now show that if compensation payments are to be used, there is a case where the opposite is true.

A4 Assume that the objective function of the overall problem is the aggregate of the pay-offs of each player, i.e.,

$$F(x_i, y_i) = \sum_i f_i(x_i, y_i, y_{-i}(x_{-i})).$$

This assumption means that the upper-level player acts as a social planner in the sense that it optimizes the joint pay-off of all market participants. It also implies that no player exerts market power (i.e., each firm is a price-taker). The other implication is that consumer welfare can be included in the objective function, if demand is modeled as an active player. Thus, this assumption does not restrict the inclusion of consumer welfare in more general settings.

Theorem 7. *Under Assumptions **A1**, **A2**, **A3** and **A4**, the optimal solution of the multi-objective problem subject to a binary quasi-equilibrium (problem 14) is at least as good as the solution to the social-welfare problem (15) subject to ex-post compensation to guarantee incentive compatibility, i.e.,*

$$F\left(\left(x_i^*, y_i^*\right)_{i \in I}\right) + G\left(\left(\zeta_i^*\right)_{i \in I}\right) \leq F\left(\left(x_i^\circ, y_i^\circ\right)_{i \in I}\right) + G\left(\left(\zeta_i^\circ\right)_{i \in I}\right).$$

Proof. Let $(x_i^\circ, y_i^\circ)_{i \in I}$ denote the optimal solution to the social-welfare problem, and $(\zeta_i^\circ)_{i \in I}$ is the vector of compensation payments necessary to align the incentives of all players. Following Assumption **A4**, we know that the first order optimality conditions of the social planner in problem (15) are identical to the stacked first order optimality conditions of each player, whenever the binary variables are fixed at x_i° , i.e.,

$$\nabla_{y_i} F_i(x_i^\circ, y_i) + (\lambda_i)^T \nabla_{y_i} g_i(x_i^\circ, y_i) = \left\{ \nabla_{y_i} f_i(x_i^\circ, y_i) + (\lambda_i)^T \nabla_{y_i} g_i(x_i^\circ, y_i) \right\}_{i \in I}$$

Therefore, the vector $(y_i^\circ)_{i \in I}$ is an equilibrium to the continuous game with binary variables fixed at $(x_i^\circ)_{i \in I}$, and following Theorem 3, the vector $(x_i^\circ, y_i^\circ)_{i \in I}$ can be implemented as a binary quasi-equilibrium. The outcome $(x_i^\circ, y_i^\circ)_{i \in I}$ with compensation vector $(\zeta_i^\circ)_{i \in I}$ and corresponding dual variables thus feasible to problem (14). \square

The above theorem has an important implication for the numerical implementation of the binary-equilibrium method: if Assumption **A4** is satisfied, the solution to the social-welfare optimization program (problem 15) can be used as a starting point for solving the multi-objective binary-equilibrium problem. As the social-welfare problem is less computationally challenging, this should help in developing more efficient solution methods for the binary equilibrium method.

Last, we formally state the conditions when the solutions of these two approaches coincide.

Corollary 2. *Under Assumptions **A1**, **A2**, **A3** and **A4**, whenever either $\sum_i \zeta_i^\circ = 0$ at optimality for problem (15) or $G((\zeta_i)_{i \in I}) = 0$ for both problems (15) and (14) the optimal objective function value of the multi-objective problem subject to a binary quasi-equilibrium (problem 14) is equal to the optimal objective function value of the social-welfare problem (15) subject to ex-post compensation to guarantee incentive compatibility, i.e.,*

$$F\left(\left(x_i^*, y_i^*\right)_{i \in I}\right) + G\left(\left(\zeta_i^*\right)_{i \in I}\right) = F\left(\left(x_i^\circ, y_i^\circ\right)_{i \in I}\right) + G\left(\left(\zeta_i^\circ\right)_{i \in I}\right).$$

Proof. The proof follows from combining the results of Theorems 5 and 6 with the results from Theorem 7. \square

In conjunction with Theorems 5 and 6, Theorem 7 has an important implications: if no compensation is necessary at the welfare-optimal solution and the assumptions for the theorem are satisfied, the two solutions coincide. However, if compensation is necessary to mitigate deviation incentives, then the multi-objective program subject to a binary equilibrium can yield a better over-all result. This is because the multi-objective program can incorporate the trade-off between welfare, on the one hand, and

the compensation payments necessary to guarantee a stable Nash equilibrium, on the other hand.

The standard approach in power market operation, as discussed above, neglects the nature of the binary, non-cooperative game between generators; because generators are usually only compensated for actual costs, they may have profitable deviations with regard to the binary decisions. We illustrate such a case in the application in Section 4. Convex-hull pricing, as introduced by Gribik et al. (2007), considers the deviation incentives of individual players, but this comes at the cost of relaxing integrality of the underlying problem. Neither approach directly considers the trade-off between market efficiency and compensation payments.

3.6 Computational complexity of the binary quasi-equilibrium

Last, let us compare the mathematical complexity of the social-welfare approach (problem 15) to our method in a linear problem setting, i.e., Assumption **A3** holds and the upper-level objective function is linear. Here, we focus on comparing the number of binary variables in each approach. The number of constraints and continuous variables also influence the computational complexity, and the number of incentive-compatibility constraints required for the reformulation increases in the number of alternative binary strategies (see constraint (14f) in the general formulation and constraint (21e) in the illustrative example in the next section). In our formulation of the binary equilibrium problem, the exponential increase of complexity in the number of binary variables is a more serious concern than additional constraints or continuous variables in a mixed-integer program, hence our focus on this aspect in the following discussion.

The two-stage approach following O’Neill et al. (2005) requires first solving a linear mixed-binary optimization problem with mn binary variables, where n is the number of players and m is the number of binary decision variables of each player, and then n optimization problems with m binary variables to determine the optimal alternative for each player. Because there is only one decision-maker in the first problem, it does not matter whether – in the interpretation of the model – there are many players with few variables, or a small number of players with numerous binary variables. In contrast, this distinction does make a difference in the binary-equilibrium formulation. As a consequence, the multi-objective program subject to a binary quasi-equilibrium (problem 14) requires to solve a mixed-binary optimization problem with $(1+k)2^m n$ binary variables, where k is the number of inequality constraints per player.

The approach requires 2^m binary variables to represent each permutation of binary decisions per player.² The term $k2^m$ in the number of binary variables is due to the disjunctive constraints reformulation to replace the complementarity conditions, which every player has to consider for all permutations of her options in binary variables. This is necessary to compute the optimal value of the continuous variables for each permutation. This curse of dimensionality is similar to the numerical caveats of applying brute-force enumeration strategies to solve binary games, which requires to solve 2^{mn} continuous problems. However, we illustrate in the numerical application presented in the following section that the number of binary variables in our method can be significantly reduced below this bound depending on the actual underlying problem.

In the power market uplift problem, the number of binary variables to obtain an exact solution is only $(m+k)n+l$, where l is the number of binary variables required for the ISO; this is, in principle, not a substantial increase in computational complexity compared to the mn binary variables in the social-planner problem plus an additional

²For an illustration of a more general application of the binary equilibrium method, we refer to the natural gas market investment and production game available for download at https://github.com/danielhuppmann/binary_equilibrium.

n optimization problems with m binary variables each to determine the compensation payments required for incentive-compatibility to be satisfied for each generator. In short, our approach scales well in the number of players, but not necessarily in the number of binary variables of each player. However, future research entails mitigating the effect of increase in the number of players. We discuss further practical aspects on the computational aspects of the problem in Section 4.5.

4 A binary game: The power market uplift problem

In order to illustrate that our approach is advantageous relative to other methods proposed in the literature for finding equilibria in binary games, we adapt the power market example from Gabriel et al. (2013). We model a nodal-pricing electricity market, in which a market operator collects bids and offers from all players and seeks to clear the market in a welfare-optimal manner. In contrast to the standard implementation of an Independent System Operator (ISO), the market operator in our setting explicitly takes into account the incentives of generators to deviate from the welfare-optimal dispatch schedule.

After formulating the model, we will compare numerical results under different sets of market rules regarding whether players are allowed to deviate from the schedule announced by the market operator, which players may receive compensation, and whether stakeholders may incur losses.

4.1 The power market model

In contrast to the notation introduced earlier, the market operator takes two roles in this example: first, taking the dispatch decision by the generators as given, she seeks to maximize short-run market efficiency. She assigns locational prices, which the generators consider in their individual optimization problems (cf. Hobbs, 2001). Second, she disburses compensation payments to align the incentives of generators with the overall (societally most beneficial) outcome. In the parlance of sequential games, the first function takes place at the same hierarchical level as the generators' decisions; the second function is equivalent to the market operator introduced in Section 3.4. To relate the illustrative model to the general formulation proposed in the previous section, the decisions of the demand-side-and-network player can be interpreted as additional constraints of the upper-level player; it serves to translate each generators' decision into the overall, market-clearing nodal prices and ensures that power flows are feasible.

The formulation of the power market model is related – but not quite identical – to the standard implementation of an ISO. Table 2 provides a summary of the notation used in the example.

The generators

Each generator $i \in I$ seeks to maximize her profits from generating and selling electricity over the time horizon $t \in T$. For consistency with the previous chapter, the generator's optimization problem is written in minimization form:

$$\min_{x_{ti}, y_{ti}, z_{ti}^{\text{on}}, z_{ti}^{\text{off}}} - p_{tn(i)} y_{ti} + c_i^G y_{ti} + c_i^{\text{on}} z_{ti}^{\text{on}} + c_i^{\text{off}} z_{ti}^{\text{off}} \quad (16a)$$

$$\text{s.t. } x_{ti} g_i^{\text{min}} \leq y_{ti} \leq x_{ti} g_i^{\text{max}} \quad (\alpha_{ti}^{\text{on}}, \beta_{ti}^{\text{on}}) \quad (16b)$$

$$x_{ti} - x_{(t-1)i} + z_{ti}^{\text{on}} - z_{ti}^{\text{off}} = 0 \quad (16c)$$

$$x_{ti} \in \{0, 1\}, \quad y_{ti}, z_{ti}^{\text{on}}, z_{ti}^{\text{off}} \in \mathbb{R}_+$$

Sets & Mappings	
$n, m \in N$... nodes
$t \in T$... time step, hours
$i \in I$... generators, power plant units
$j \in J$... load, demand units
$l \in L$... power lines
$i \in I_n, j \in J_n$... generator/load unit mapping to node n
$n(i), n(j)$... node mapping to generator i /load unit j
$\phi \in \Phi$... set of dispatch options (schedules) for each generator
$t \in T_\phi$... hours in which a generator is active in dispatch option ϕ
Primal variables	
x_{ti}	... on/off decision for generator i in hour t
$z_{ti}^{\text{on}}, z_{ti}^{\text{off}}$... inter-temporal start-up/shut-down decision
y_{ti}	... actual generation by generator i in hour t
y_{ti}^{on}	... generation if binary variable is fixed at \bar{x}_i
d_{tj}	... demand by unit j in hour t
δ_{tn}	... voltage angle
Dual variables	
$\alpha_{ti}^{\text{on}}, \beta_{ti}^{\text{on}}$... dual to minimum activity/maximum generation capacity
p_{tn}	... locational marginal price
ν_{tj}	... dual to maximum load constraint
μ_{tl}^+, μ_{tl}^-	... dual to voltage angle band constraints
ξ_{tn}^+, ξ_{tn}^-	... dual to thermal line capacity constraints
γ_t	... dual to slack bus constraints
Switch and compensation variables	
$\kappa_{ti}^{\text{on}}, \kappa_{ti}^{\text{off}}$... switch value (defined per time step)
ζ_i	... compensation payment (defined over entire time horizon)
Parameters	
c_i^G	... linear generation costs
$c_i^{\text{on}}, c_i^{\text{off}}$... start-up/shut-down costs
c_ϕ^D	... total commitment costs in dispatch option ϕ (start-up, shut-down)
g_i^{min}	... minimum activity level if power plant is online
g_i^{max}	... maximum generation capacity
x_i^{init}	... power plant status at start of model horizon ($t = 0$)
u_{tj}^D	... utility of demand unit j for using electricity
d_{tj}^{max}	... maximum load of unit j
f_l^{max}	... thermal capacity of power line l
B_{nk}, H_{lk}	... line/node susceptance/network transfer matrices

Table 2: Notation for the nodal power market problem

where the linear generation costs are given by c_i^G , the (binary) start-up costs are given by c_i^{on} , and the (binary) shut-down costs are given by c_i^{off} . The operation schedule is denoted by x_{ti} , the decision how much electricity to generate and sell to the grid is y_{ti} . The ramping decisions in a particular period are denoted by z_{ti}^{on} (start-up) and z_{ti}^{off} (shut-down), respectively. The status of a power plant at the beginning of the model horizon (i.e., x_{0i}) is given by the parameter x_i^{init} .

The first set of constraints (16b) concerns the maximum generation capacity and the minimum activity level ($g_i^{\text{min}}, g_i^{\text{max}}$), if the power plant is operating ($x_{ti} = 1$). The shadow variables ($\alpha_{ti}^{\text{on}}, \beta_{ti}^{\text{on}}$) are only meaningful given a fixed operation schedule x_{ti} , and we only compute them if the power plant is operational. If the power plant is not switched on, generation is equal to zero; however, due to costs incurred by shutting down the plant (assuming it was operational in the previous period or at the beginning of the model horizon), total profits may be negative even when the plant is not generating electricity.

The second constraint (16c) concerns the inter-temporal consideration, i.e., the decision in which time periods the power plant is operational: while the start-up and shut-down variables z_{ti}^{on} and z_{ti}^{off} are binary in nature, they can be relaxed to positive real numbers without loss of information. Integrality of these variables is automatically enforced by the on/off variables x_{ti} . Because these variables are determined by constraint (16c) for any given commitment schedule, we do not explicitly include the first order conditions for these variables.

The market clearing price p_{tn} is the vector of the locational marginal prices over time, where $n(i)$ denotes the node at which generator i is located; the set $n, m \in N$ denotes the nodes in the network. The price arises as the dual of the market clearing condition introduced below, and each generator takes the price at her node as given.

Assuming that the power plant i is operating in period t , the optimal amount of power generated y_{ti}^{on} and the dual variables associated with the constraints can be determined by solving the generator's first order optimality (KKT) conditions:

$$0 = c_i^G - p_{tn(i)} + \beta_{ti}^{\text{on}} - \alpha_{ti}^{\text{on}} \quad , \quad y_{ti}^{\text{on}} \text{ (free)} \quad (17a)$$

$$0 \leq -g_{ti}^{\text{min}} + y_{ti}^{\text{on}} \quad \perp \quad \alpha_{ti}^{\text{on}} \geq 0 \quad (17b)$$

$$0 \leq g_{ti}^{\text{max}} - y_{ti}^{\text{on}} \quad \perp \quad \beta_{ti}^{\text{on}} \geq 0 \quad (17c)$$

Otherwise, the amount generated is zero, and we do not require the dual variables in that case. Hence, in contrast to the general formulation in Section 3, we can omit the KKT conditions in this example for the case that the power plant is not operating in period t .

Demand for electricity and network feasibility (market clearing)

The spot market is cleared by a player seeking to maximize the welfare (utility) of consumers while guaranteeing feasibility of the transmission system. She takes the unit commitment at time t and the dispatch of each power plant as given, written here as y_{ti} , and she assigns locational prices p_{tn} , which the generators consider in their optimization problems.

There are a set of units that consume electricity $j \in J$ (load d_{tj}), each located at a specific node $n(j)$. The sets I_n and J_n are the generators and load units located at node n , respectively. There are a set of power lines $l \in L$ connecting the nodes; the direct-current load flow (DCLF) characteristics are captured using the susceptance matrix B_{nm} (node-to-node) and network transfer matrix H_{nl} (node-to-line mapping, cf. Leuthold et al., 2012). This formulation is equivalent to using a power transfer distribution factor (PTDF) matrix.

This player maximizes the utility of demand from using electricity d_{tj} , where the per-unit utility is given by u_{tj}^D . The first constraint is the energy balance constraint (18b), balancing load, generation and net injections into the network at each

node n . The second constraint is the maximum demand, as a load unit cannot use more than d_{tj}^{max} units of electricity.

The next constraints (18d,18e) ensure that network flows are feasible and the thermal capacity f_l^{max} of each power line is observed. Constraints (18f,18g) guarantee that the voltage angle δ_{tn} is within the range $[-\pi, \pi]$. The $B-H$ -formulation requires to define one arbitrary node as *slack bus* \hat{n} , at which the voltage angle $\delta_{t\hat{n}}$ is zero by assumption (constraint 18h). In line with the previous notation, we write the objective function as a minimization problem:

$$\min_{d_{tj}, \delta_{tn}} \sum_{j \in J} -u_{tj}^D d_{tj} \quad (18a)$$

$$\text{s.t.} \quad \sum_{j \in J_n} d_{tj} - \sum_{i \in I_n} y_{ti} + \sum_{m \in N} B_{nm} \delta_{tm} = 0 \quad (p_{tn}) \quad (18b)$$

$$d_{tj}^{max} - d_{tj} \geq 0 \quad (\nu_{tj}) \quad (18c)$$

$$f_l^{max} - \sum_{n \in N} H_{ln} \delta_{tn} \geq 0 \quad (\mu_{tl}^+) \quad (18d)$$

$$f_l^{max} + \sum_{n \in N} H_{ln} \delta_{tn} \geq 0 \quad (\mu_{tl}^-) \quad (18e)$$

$$\pi - \delta_{tn} \geq 0 \quad (\xi_{tn}^+) \quad (18f)$$

$$\pi + \delta_{tn} \geq 0 \quad (\xi_{tn}^-) \quad (18g)$$

$$\delta_{t\hat{n}} = 0 \quad (\gamma_t) \quad (18h)$$

Because the decision variables of demand and voltage angle are continuous, this problem can be solved simultaneously with the generators' problems using first order optimality conditions:

$$0 \leq -u_{tj}^D + p_{tn(i)} + \nu_{tj} \quad \perp \quad d_{tj} \geq 0 \quad (19a)$$

$$0 = \sum_{m \in N} B_{mn} p_{tn} + \sum_{l \in L} H_{ln} (\mu_{tl}^+ + \mu_{tl}^-) + \xi_{tn}^+ - \xi_{tn}^- - \begin{cases} \gamma_t & \text{if } n = \hat{n} \\ 0 & \text{else} \end{cases}, \quad \delta_{tn} \text{ (free)} \quad (19b)$$

$$0 = \sum_{j \in J_n} d_{tj} - \sum_{i \in I_n} y_{ti} + \sum_{m \in N} B_{nm} \delta_{tm}, \quad p_{tn} \text{ (free)} \quad (19c)$$

$$0 \leq d_{tj}^{max} - d_{tj} \quad \perp \quad \nu_{tj} \geq 0 \quad (19d)$$

$$0 \leq f_l^{max} - \sum_{n \in N} H_{ln} \delta_{tn} \quad \perp \quad \mu_{tl}^+ \geq 0 \quad (19e)$$

$$0 \leq f_l^{max} + \sum_{n \in N} H_{ln} \delta_{tn} \quad \perp \quad \mu_{tl}^- \geq 0 \quad (19f)$$

$$0 \leq \pi - \delta_{tn} \quad \perp \quad \xi_{tn}^+ \geq 0 \quad (19g)$$

$$0 \leq \pi + \delta_{tn} \quad \perp \quad \xi_{tn}^- \geq 0 \quad (19h)$$

$$0 = \delta_{t\hat{n}}, \quad \gamma_t \text{ (free)} \quad (19i)$$

The upper-level market operator

As stated above, we distinguish between the short-term role of the market operator, which clears the market taking the dispatch of the generators as given, and the upper-level player as introduced in Section 3, which takes the role of an equilibrium selection mechanism. She assigns compensation payments to align the incentives of market participants and ensure that no player has a profitable deviation. Mathematically,

this player forms the upper level of a two-stage, hierarchical game; the lower level is the binary quasi-equilibrium between the generators and demand, with compensation payments if necessary.

It is obvious from the upper-level player's objective function that this is closely related to the generators and the market-clearing player, but not identical – the market operator does not only consider short-term market efficiency, but includes the welfare loss from the disbursement of compensation payments ζ_i . The objective function satisfies Assumptions **A2** and **A4**, and the individual player's problems satisfy Assumptions **A1** and **A3**.

$$\min \sum_{t \in T} \left[\sum_{i \in I} c_i^G y_{ti} + c_i^{\text{on}} z_{ti}^{\text{on}} + c_i^{\text{off}} z_{ti}^{\text{off}} - \sum_{j \in J} u_{tj}^D d_{tj} \right] + \sum_{i \in I} \zeta_i \quad (20)$$

s.t. KKT conditions of demand and network feasibility (equations 19)

KKT conditions of the generators (equations 17)

binary equilibrium between generators (equations 21) specified below

The first part of the objective function is the sum of the generators' incurred costs and the utility of load units from using electricity; this is equivalent to $F(\cdot)$ in the theoretical formulation (problem 14). The second part is the regularizer $G(\cdot)$, although it has a distinct interpretation in this example: because the compensation payments to generators have to be funded through fees on market participants or from general taxation, they usually involve some efficiency loss from market distortions. Here, we assume that every dollar paid in compensation is a 100% loss to overall welfare. One could assume that the loss is even greater because of the aforementioned distortions; alternatively, one could argue that compensation does not incur a loss at all, as it is just a redistribution of rents between stakeholders. We leave it to future, more policy-applied work to tackle this question.

Let us now turn to the equations necessary to guarantee the binary equilibrium between the generators. If $x_{ti} = 0$ (i.e., the power plant of generator i is switched off in period t), the first order conditions can be omitted altogether; both the generation level and the short-term profits in this case are zero, and the fixed costs from starting up or shutting down will be included in the incentive-compatibility constraint. This leaves the KKT conditions of the generators (equations 17) to determine the optimal, short-term dispatch in the case that the generator is operating in this period ($x_{ti} = 1$).

Next, the inter-temporal constraint of the power plant operation status has to be considered (constraint 21a); as discussed above, the start-up and shut-down variables ($z_{ti}^{\text{on}}, z_{ti}^{\text{off}}$) are determined by the unit commitment variables x_i and we therefore do not need to include KKT conditions for them. They only serve to translate the change in the unit commitment level into realized dispatch costs.

Next, let us turn to the incentive-compatibility constraint: we now have multiple time periods and we formulate the incentive-compatibility constraint in a different way than in the general formulation in Section 3 (equation 14f). This is due to the problem that it is not obvious how to allocate start-up costs over multiple time periods. It is therefore preferable to define the compensation payments $\zeta_i \in \mathbb{R}_+$ over the entire model horizon and also write the incentive-compatibility constraint in this way, rather than as a period-by-period constraint. As a consequence, we also change the interpretation of $(\kappa_{ti}^{\text{on}}, \kappa_{ti}^{\text{off}}) \in \mathbb{R}$: it now represents the short-term profits or losses (revenue less generation costs), and it is not any more restricted to positive values, in contrast to the switch value in the previous section and the overview in Table 1.

We directly use the linearization proposed in Theorem 4 to write the value of the bilinear objective function ($p_{tn(i)} y_{ti} - c_i^G y_{ti}$) as a linear function ($\beta_{ti}^{\text{on}} g_{ti}^{\text{max}} - \alpha_{ti}^{\text{on}} g_{ti}^{\text{min}}$). Constraints (21b–21d) succinctly describe the short-term profits or losses to the vector $(\kappa_{ti}^{\text{on}}, \kappa_{ti}^{\text{off}})$ – where κ_{ti}^{on} can only be different from zero if the power plant

is operating in time period t in equilibrium ($x_{ti} = 1$), and κ_{ti}^{off} is free iff it is shut down ($x_{ti} = 0$).

$$x_{ti} - x_{(t-1)i} + z_{ti}^{\text{on}} - z_{ti}^{\text{off}} = 0 \quad (21a)$$

$$\beta_{ti}^{\text{on}} g_{ti}^{\text{max}} - \alpha_{ti}^{\text{on}} g_{ti}^{\text{min}} - \kappa_{ti}^{\text{on}} + \kappa_{ti}^{\text{off}} = 0 \quad (21b)$$

$$|\kappa_{ti}^{\text{on}}| \leq x_{ti} \tilde{K} \quad (21c)$$

$$|\kappa_{ti}^{\text{off}}| \leq (1 - x_{ti}) \tilde{K} \quad (21d)$$

$$\sum_{t \in T} \left[\kappa_{ti}^{\text{on}} - c_i^{\text{on}} z_{ti}^{\text{on}} - c_i^{\text{off}} z_{ti}^{\text{off}} \right] + \zeta_i \geq \sum_{t \in T_\phi} \left[\beta_{ti}^{\text{on}} g_{ti}^{\text{max}} - \alpha_{ti}^{\text{on}} g_{ti}^{\text{min}} \right] - c_{\phi i}^D \quad \forall \phi \in \Phi \quad (21e)$$

The last constraint (equation 21e) is the incentive-compatibility condition: it ensures that the profits for each generator (per period, short-term profits or losses from generating less the commitment costs) in the actual market outcome plus the compensation (left-hand side) are greater than the profits which that player could earn in any other dispatch schedule ϕ (right-hand side). The revenues for each dispatch schedule can be computed from the duals to the maximum generation and minimum activity constraints, summing over the periods in which the generator is operational according to schedule ϕ ; these periods are collected in the set $T_\phi \subseteq T$. The total commitment costs for each dispatch schedule are denoted by $c_{\phi i}^D$.

The final set of constraints of the binary quasi-equilibrium “translates” the optimal generation decision for both states of the binary variable ($y_{ti}^{\text{on}}, 0$) into the generation level which is actually realized in equilibrium, y_{ti} .

$$0 \leq y_{ti} \leq x_{ti} g_{ti}^{\text{max}} \quad (21f)$$

$$y_{ti}^{\text{on}} - (1 - x_{ti}) g_{ti}^{\text{max}} \leq y_{ti} \leq y_{ti}^{\text{on}} + (1 - x_{ti}) g_{ti}^{\text{max}} \quad (21g)$$

As stated in the previous section, the binary variables have an additional role in this formulation relative to a standard unit-commitment model: rather than simply stating whether a plant is operating or not, they control which of the two potential states with regard to the continuous variables are active and realized in equilibrium ($y_{ti} = y_{ti}^{\text{on}}$ if $x_{ti} = 1$, $y_{ti} = 0$ otherwise). Furthermore, it ensures that the variables capturing the short-term profit ($\kappa_{ti}^{\text{on}}, \kappa_{ti}^{\text{off}}$) are correctly assigned.

4.2 Myopic behavior vs. consistent deviation incentives

There is one subtle difference between the general binary Nash game formulation in Section 3 and the present market power example: in the general formulation of the incentive-compatibility constraint, each player considers the impact of its own binary decision on the market outcome, comparing the respective pay-offs $f_i(\mathbf{1}, \tilde{y}_i^{(1)}, y_{-i}(x_{-i}))$ and $f_i(\mathbf{0}, \tilde{y}_i^{(0)}, y_{-i}(x_{-i}))$, as specified in equations (11). In the example formulated here, each generator is assumed to act in a perfectly competitive manner in the sense of revealing her true cost structure to the market operator and taking the price as given, irrespective of her own decision. One could say that the generators are assumed to be myopic; i.e., they do not consider that their decision to deviate will impact the resulting market price. That is in line with Scarf’s definition that introducing a new activity must be profitable at the “old” prices (Scarf, 1990).

From a game-theoretic point of view, this is inconsistent; however, from the point of view of market operation in the electricity context, our formulation is valid. Of course, this argument does not hold in general, because a market operator (or coordination agent) in our sense usually does not exist for commodities markets or other real-world applications of binary games. To model such situations, each player’s own impact on prices contingent on its entry or exit should be captured. The method proposed

here allows to do this, although the power market model would have to be formulated differently.

Even in the electricity context, it may be appropriate to compute compensation payments based on the alternative prices (i.e., prices resulting after a deviation), rather than those in the actual market outcome. It is not a priori obvious whether compensation payments to guarantee a binary quasi-equilibrium would be larger or smaller in either formulation. We leave this analysis for future research.

4.3 Alternative rules for compensation payments

The incentive-compatibility constraint as stated above (equation 21e) is the direct extension of constraint (12) in a multi-period setting. The short-term profits or losses are succinctly captured by the vector $(\kappa_{ti}^{\text{on}}, \kappa_{ti}^{\text{off}}) \in \mathbb{R}$, and the start-up and shut-down costs are linear terms. As a consequence, this method is flexible and allows to easily implement a wide range of market rules regarding compensation disbursements. To illustrate the versatility of the approach and its applicability to different market designs, we formulate two alternative versions of the model.

First, we implement a *no-loss* rule to replace the incentive-compatibility constraints: no generator may earn negative profits (i.e., lose money out of pocket):

$$\sum_{t \in T} [\kappa_{ti}^{\text{on}} - c_i^{\text{on}} z_{ti}^{\text{on}} - c_i^{\text{off}} z_{ti}^{\text{off}}] + \zeta_i \geq 0 \quad (21e')$$

In this setting, there is no constraint stating that the dispatch selected by the ISO has to be incentive-compatible for each generator. Instead, every market participant is forced to follow the schedule selected by the market operator. Hence, this setting omits the game-theoretic considerations. Furthermore, the power plant is compensated even if it would incur losses irrespective of the selected dispatch, so there may be over-compensation. This can happen in our setting because there are shut-down costs and some generators are operational at the beginning of the model horizon.

The second rule stipulates that only power plants can receive compensation payments if they were active at least once over the model horizon; this is to reflect the potential concern that no generator should receive compensation for doing nothing.

$$\zeta_i \leq \sum_{t \in T} x_{ti} \tilde{K} \quad (21e'')$$

4.4 Illustrative results

The power system adapted from Gabriel et al. (2013) consists of 6 nodes, with 9 generators and 4 load units (see Figure 1). Each generator has a maximum generation capacity of 100 MW and a minimum generation level, if operating, of 50 MW. Because we assume that one time period t lasts for one hour, capacity (MW) and energy (MWh) are equivalent.

Generators $g3$ to $g6$ are operational at the beginning of the model horizon and the power plants differ with regard to start-up, shut-down, and marginal generation costs. Demand for electricity varies over time, with a high utility for energy (or large willingness-to-pay, WTP) in the first hour and lower WTP in the second hour. All data for generators and load units are provided in Table 3. Regarding the multi-objective function representing the market operator, we assume that each monetary unit paid in compensation is a one-for-one loss of welfare (sum of consumer utility, generator profits, and congestion rents).

All lines have a thermal capacity of 300 MW, except for the two inter-connector lines $n2 - n4$ and $n3 - n6$, which have a reduced thermal capacity of 20 MW. Due to these bottlenecks, the standard, welfare-optimal unit commitment model yields losses for some generators.

We compare three different *market rule cases*; the game-theoretic considerations and regulations (constraints) concerning losses and compensations are repeated here for clarity:

Game-theoretic: This is the binary quasi-equilibrium solution, as selected by the market operator; every generator receives compensation such that she has no profitable deviation. The constraint is repeated here for easier comparison with the other market designs.

$$\sum_{t \in T} \left[\kappa_{ti}^{\text{on}} - c_i^{\text{on}} z_{ti}^{\text{on}} - c_i^{\text{off}} z_{ti}^{\text{off}} \right] + \zeta_i \geq \sum_{t \in T_\phi} \left[\beta_{ti}^{\text{on}} g_{ti}^{\text{max}} - \alpha_{ti}^{\text{on}} g_{ti}^{\text{min}} \right] - c_{\phi i}^D \quad \forall i \in I, \phi \in \Phi \quad (\text{cf. 21e})$$

No-loss rule: We solve the multi-objective program (problem 20) subject to the constraint that no player earns negative profits (instead of constraint 21e). Players may have profitable deviations, for which they are not compensated.

$$\sum_{t \in T} \left[\kappa_{ti}^{\text{on}} - c_i^{\text{on}} z_{ti}^{\text{on}} - c_i^{\text{off}} z_{ti}^{\text{off}} \right] + \zeta_i \geq 0 \quad \forall i \in I \quad (\text{cf. 21e}')$$

Incidentally, this case yields identical results as the *standard approach*, in which the power market model is solved according to the two-stage procedure following O'Neill et al. (2005), i.e., pay-offs are calculated based on prices from the dual to the energy-balance constraint from the linearized problem, and non-confiscatory make-whole payments are computed ex-post (cf. Sioshansi, 2014). However, the observation that the two-stage procedure and the integrated multi-objective yield an identical result is specific to this stylized example, and not a general property of the multi-objective program under a no-loss rule.

No-loss & active: We add a constraint to the previous case *No-loss rule* stating that only active generators can receive compensation.

$$\zeta_i \leq \sum_{t \in T} x_{ti} \tilde{K} \quad \forall i \in I \quad (\text{cf. 21e}'')$$

The rationale for this rule may be to prevent gaming, or in response to the public perception that there should not be payments for *not* providing a service.

Table 4 summarizes the rents earned by the generators in the three market rule cases, as well as the required compensation. Table 5 shows the resulting nodal prices as well as the amount generated and consumed by each unit. Generators $g1$ and $g2$ are not operational at the beginning of the time horizon and are not switched on in

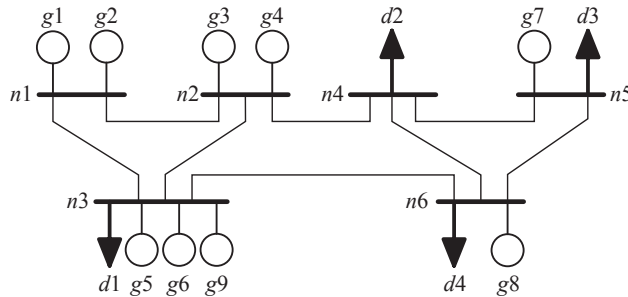


Figure 1: 6-node network, adapted from Gabriel et al. (2013), p. 18

any of these cases; therefore, they are omitted from the tables wherever all entries are zero.

For reasons of illustration, we first discuss the results for the *No-loss* case; as stated before, this yields the same outcome as the method proposed by O'Neill et al. (2005) with ex-post compensation in this example. Generator $g3$ is switched off immediately and therefore incurs shut-down costs of \$300, while generator $g4$ is operating, but the nodal price at node $n2$ is below her marginal costs. Both of these generators are made whole such that they do not incur losses. The deviation incentives of each player are shown in Table 6; here, one can see that generator $g9$ has a profitable deviation given market prices in the no-loss case. As a consequence, this generator has an incentive to switch on the power plant in spite of the schedule announced by the ISO, as she would earn positive profits given the prevailing market prices. Hence, if self-scheduling is an option for the generator, this outcome would not be a Nash equilibrium and the solution would not be stable against deviations.

In the *game-theoretic* case, the market operator could choose to compensate generator $g9$ to counteract self-scheduling by that player, and then obtain the same dispatch and nodal prices as in the no-loss case. This would require to compensate generator $g9$ to the tune of \$95 and generator $g3$ with \$40. Generator $g4$ does not have a profitable deviation, because her losses are at least as great under all alternative dispatch schedules; hence, she does not receive compensation. The objective value of the market operator would be \$2965 (total welfare of \$3100 less \$135 disbursed as compensation).

However, because of the integrated consideration of market efficiency and compensation payments, the market operator realizes that it is preferable to dispatch generator $g9$ and instead shut down generator $g4$, realizing an objective value of \$2975. Generator $g9$ now incurs losses, because the resulting locational marginal prices given the new dispatch are lower than her marginal costs; for these losses, she is compensated by the market operator to prevent her from leaving the market. Overall, market efficiency is slightly reduced, but the compensation required to maintain incentive compatibility in this binary quasi-equilibrium is significantly lower than the payments necessary to guarantee incentive compatibility of the welfare-optimal solution.

The last case, *No-loss & active*, illustrates how strict market rules can hamper efficient market operation, even when they are intended to mitigate strategic behavior or increase public acceptance of compensation payments. Generator $g3$ would incur losses from shutting down at the beginning of the period, but cannot receive compensation if she doesn't generate at least in one period; therefore, the market operator dispatches this plant throughout the model horizon. Because this would result in in-

	c_i^G	c_i^{on}	c_i^{off}	x^{init}		u_{1j}^D	u_{2j}^D	d_{1j}^{max}	d_{2j}^{max}
$g1$	24	100	500	0	$d1$	25	20	100	50
$g2$	22	140	350	0	$d2$	26	20	100	50
$g3$	20	180	300	1	$d3$	26	21	100	50
$g4$	18	220	250	1	$d4$	17	21	100	50
$g5$	16	250	220	1					
$g6$	14	300	180	1					
$g7$	12	350	140	0					
$g8$	10	500	100	0					
$g9$	14	105	100	0					

(a) Cost structure (in \$/MWh and \$, respectively) and initial operational status by generator

(b) Utility and maximum demand by load unit and time period (in \$/MWh and MW, respectively)

Table 3: Data for generators and load

	x_i^{init}	No-loss rule			Game-theoretic			No-loss & active		
		(x_{ti})	π_i	ζ_i	(x_{ti})	π_i	ζ_i	(x_{ti})	π_i	ζ_i
$g3$	1	(0, 0)	-300	300	(0, 0)	-300	15	(1, 1)	-525	525
$g4$	1	(1, 1)	-160	160	(0, 0)	-250	65	(1, 0)	-335	335
$g5$	1	(1, 1)	50		(1, 1)	-50		(1, 1)	-50	50
$g6$	1	(1, 1)	200		(1, 1)	100		(1, 1)	100	
$g7$	0	(1, 1)	690		(1, 1)	715		(1, 1)	690	
$g8$	0	(1, 1)	680		(1, 1)	730		(1, 1)	550	
$g9$	0	(0, 0)			(1, 1)	-5	5	(0, 0)		
Generator profit			1160	460		940	85		435	910
Consumer surplus			1380			1480			1830	
Congestion rent			560			640			740	
Total welfare			3100	460		3060	85		3005	910

Table 4: Profits by generator (π_i) before compensation is disbursed, and rents by stakeholder group for each market rule case; on-off status and initial operational status by generator

feasible flows on the network in the second, low-demand period, the market operator shuts down generator $g4$ at the end of the first period. Now, this generator incurs losses from generating at a nodal price below marginal costs in the first period and the shutting-down costs, and she is compensated accordingly.

In this case, welfare is reduced by 3% compared to the *No-loss* case, while the required compensation payments are almost twice as high. Assuming that every dollar disbursed in compensation payments is a 100% loss of welfare, the overall utility and profits in the system is 30% lower than in the best-possible outcome, the game-theoretic case. At the same time, it is interesting to note that in the *No-loss & active* case, consumer surplus and congestion rents are substantially larger than in the other cases – before accounting for the transfers necessary to finance the compensation payments.

Of course, the question is why some generators were already operational at the beginning of the model horizon. If this is because the market operator/ISO dispatched them on the previous day, and then allows them to incur losses on the following day, this may not seem equitable. If, on the other hand, this was a decision taken by the generators themselves based on faulty projections or bad planning, it could be argued that letting them lose money from such myopic decisions is a good penalty to incentivize better planning. We do not venture further in this discussion here; our intention here is only to illustrate that a no-loss rule and compensation may not always be necessary from a game-theoretic point of view.

4.5 Numerical implementation and a note on computation

Reformulating the complementarity conditions of the demand-side player (equations 19) and the generators (equations 17b and 17c) using disjunctive constraints yields a mixed-integer linear program (Fortuny-Amat and McCarl, 1981). This approach to determine a binary quasi-equilibrium requires $3|T||I| = 54$ binary variables for the generators and $2|T|(|J| + |L| + |N| - 1) = 68$ binary variables for the disjunctive-constraints reformulation of the ISO. The total number of binary variables is therefore $|T|(2(|I| + |J| + |L| + |N| - 1) + |I|) = 122$. The large scalars \bar{K} for the disjunctive constraints reformulation and the constraints on assigning the correct values $\kappa_{ti}^{on}, \kappa_{ti}^{off}$ (equations 21c and 21d) were set to 1000.

		No-loss rule		Game-theoretic		No-loss & active	
		t1	t2	t1	t2	t1	t2
Price	n1	18	12.8	16.5	12.8	13.5	12.8
	n2	18	11.6	17	11.6	11	11.6
	n3	18	14	16	14	16	14
	n4	26	20	26	20	28.5	20
	n5	26	18.8	26	18	26	18.8
	n6	26	17.6	27	17	23.5	17.6
Generation	g3					25	25
	g4	40	25			25	
	g5	50	25	40	25	37.5	25
	g6	50	30	50	25	50	30
	g7	50	50	50	50	50	50
	g8	50	50	50	50	50	50
Load	d1	100	50	100	50	100	50
	d2	10	30	65	40		30
	d3	30	50		50	37.5	50
	d4	100	50	75	50	100	50
Total dispatch		240	180	240	190	237.5	180

Table 5: Dispatch and price for each market structure (prices in \$/MWh, dispatch in MWh)

This compares to $|T| \cdot |I| = 18$ binary variables to compute the welfare-optimal dispatch and integer pricing following the approach proposed by O’Neill et al. (2005). Our method is therefore more computationally expensive, but the number of binary variables increases only linearly in the number of time periods, and sublinear in the number of generators, load units, nodes and lines. Solving the resulting equilibrium problem for every permutation of binary variables instead and checking for incentive compatibility and profitable deviations ex-post (i.e., the brute-force approach) grows exponentially in complexity and requires solving $2^{|T| \cdot |I|} > 262,000$ (linear) equilibrium problems.

The numerical model presented in this section is implemented in GAMS and solved using the GUROBI solver. The code includes the possibility of multiple bidding blocks for each generator and demand unit, as in the model proposed by Gabriel et al. (2013), even though this option is not used here. The GAMS code is published under a *Creative Commons Attribution 4.0 International License* and is available for download at https://github.com/danielhuppmann/binary_equilibrium. An algorithm for enumerating all permutations and checking for deviation incentives ex-post was also implemented to verify the accuracy of our methodology.

5 Conclusions and outlook

Non-cooperative games with binary decision variables are often encountered in real-world applications, from engineering to economics. It is well known that equilibria in such problems do not necessarily exist, and even if they do exist, finding them is mathematically challenging. The most frequently studied example is the power market uplift problem, which seeks to reconcile the difficulty of finding market-clearing prices – based on the short-term, efficient dispatch – with obtaining incentive-compatible outcomes in decentralized, non-cooperative markets. To date, no approach to exactly

	No-loss rule			Game-theoretic			No-loss & active		
	(0,0)	(1,0)	(1,1)	(0,0)	(0,1)	(1,1)	(0,0)	(0,1)	(1,1)
<i>g1</i>	0	-750	-530	0	-787	-380	0	-862	-642
<i>g2</i>	0	-590	-470	0	-627	-370	0	-702	-582
<i>g3</i>	-300	-350	-260	-300	-375	-690	15	-525	-435
<i>g4</i>	-250	-250	-160	-250	-275	-630	65	-425	-335
<i>g5</i>	-220	-120	50	-220	-220	-520	-50	-220	-50
<i>g6</i>	-180	20	200	-180	-80	-480	100	-80	100
<i>g7</i>		210	690		235	-10		210	690
<i>g8</i>		200	680		250	-120		75	555
<i>g9</i>	0	-5	95		-105	-105	0	-105	-105
			ζ_i						ζ_i

Table 6: Profits by generator for each on/off option, and compensation (in \$); market outcome in bold

solve such games exists.

In this work, we propose an exact solution method for binary equilibrium problems based on computing optimal responses for each player for both values of the binary variable (or vector), rather than assuming a linearization of the binary variable or relaxing the optimality conditions. We then add an explicit incentive-compatibility constraint to ensure that no player has a profitable deviation. We define the notion of a binary quasi-equilibrium to describe situations where no equilibrium exists for the original problem, but in which compensation payments can align the incentives of players such that no player has a profitable deviation and a stable equilibrium is realized.

To this end, we introduce a market operator that acts as an equilibrium selection mechanism according to an upper-level objective function. By recasting the binary equilibrium problem as a hierarchical multi-objective program subject to a binary quasi-equilibrium, our method allows to explicitly incorporate the trade-off between market efficiency and the budget required for compensation payments to obtain an incentive-compatible market outcome. With regard to the power market, this can be interpreted as striking a balance between maximizing short-run welfare (consumer utility less generation costs) and the amount of uplift payments, which are usually funded through taxation, usage fees, or price mark-ups.

Instead of a shadow price from a marginal relaxation of the integral constraint, our method yields a “switch value,” which is the loss a player would incur if she were to deviate from her individually optimal decision. The switch value can also be readily interpreted as the compensation payment a player should receive if the market operator requires her to deviate from the individually optimal strategy. As elaborated earlier, we also believe that the switch value can be used for algorithmic improvements and new approaches to solve binary problems (e.g., act as a stopping criterion, guide a branch-and-bound algorithm).

Most importantly, we show that this method can be reformulated and solved as a mixed-binary linear program under general conditions, and that the solution is at least as good as the current practice under certain assumptions. Hence, the approach can be applied to a wide range of real-world problems, including Nash equilibria in energy and natural resource markets. The approach allows to include a variety of market regulations, such as “no-loss” rules common in power markets. These rules can be formulated as linear constraints and therefore do not substantially increase the numerical complexity of obtaining a binary quasi-equilibrium.

The solution method for binary equilibrium problems proposed in this work can be extended to include Generalized Nash games (cf. Harker, 1991) or games with individual joint constraints (Nabetani et al., 2011), as well as equilibrium problems in discrete rather than binary variables. Furthermore, more general non-cooperative games can be solved in this framework, such as games based on conjectural variations (Wogrin et al., 2013). Extending the mathematical approach to stochastic applications is also straightforward.

In economic applications, the binary variables can be interpreted as on/off decisions, or as market-entrance or investment decisions in a dynamic, two-stage setting. Lumpy investment in the European power grid capacity by national regulators, each seeking to shift rents towards their domestic constituents, is a natural next application of the binary equilibrium method (cf. Huppmann and Egerer, 2015). The European regulatory agency ACER and the inter-TSO compensation mechanism under its purview are very similar to the structure of the structure of our proposed method, where an upper-level coordination player guides the non-cooperative players towards a beneficial equilibrium.

Because most current power markets are based on a welfare-optimal dispatch with ex-post compensation payments, there may exist numerous gaming opportunities for large utilities or other market participants in the current setting; the settlement in 2013

between the Federal Energy Regulatory Commission (FERC) and JP Morgan regarding manipulative bidding strategies is a case in point³. It will be the subject of future research to analyze whether our solution method for binary equilibrium problems will be more or less prone to market power exertion and strategic behavior. Our method explicitly incorporates game-theoretic considerations (i.e., deviation incentives) rather than simplistic no-loss rules, and it includes the trade-off between maximizing market efficiency and minimizing compensation payments. As a consequence, we believe that our approach has significant potential to improve the current practice in power market operation. In particular, our approach allows to include the incentives of non-active players to enter the market; hence, the behavior of non-dispatched generators entering the market through self-commitment (or “self-scheduling”) can be more effectively addressed (cf. Sioshansi et al., 2010).

As a numerical application, we solve the power market uplift problem analyzed by Gabriel et al. (2013). We illustrate that the current practice in power market operation can lead to situations where players have profitable deviations. In particular, when considering the nature of the non-cooperative game in our stylized example, the market operator prefers to deviate from the welfare-optimal dispatch, because a slight reduction in market efficiency is traded for a strong reduction in compensation payments necessary to maintain incentive compatibility of the market outcome.

To illustrate the flexibility of our approach, we also solve the model under a hypothetical market regulation stating that a) no generator may lose money, and b) only active generators may receive compensation. This yields a welfare loss of 3% relative to the optimal solution. At the same time, compensation payments are almost twice as high as in the currently used approach, so that compensation payments eat up almost a third of total welfare in the market. We take this as a warning that market rules may have rather counter-intuitive effects, even when they are implemented with the aim of preventing strategic behavior or mitigating other inefficiencies.

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³See FERC’s news release from July 30, 2013, Docket Nos. IN11-8-000 and IN13-5-000. <http://www.ferc.gov/media/news-releases/2013/2013-3/07-30-13.asp>

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Appendix

GAMS codes, a numerical test case & additional examples

The GAMS code for the binary-equilibrium model presented in Chapter 4 is published under a *Creative Commons Attribution 4.0 International License* and is available for download at https://github.com/danielhuppmann/binary_equilibrium. The repository also includes an application of this method to a natural gas market investment and production game to illustrate how the binary equilibrium approach can be applied to more general cases where players interact in settings with multiple binary variables.

A numerical test case

To illustrate the numerical properties of the binary-equilibrium method, we apply the power market model presented in Chapter 4 to the open-source model and data set provided by H. Pandzic, Y. Dvorkin, T. Qiu, Y. Wang, and D. Kirschen at the Renewable Energy Analysis Lab (REAL), University of Washington, Seattle, USA (Online, http://www.ee.washington.edu/research/real/gams_code.html). This open-source data and model were specifically adapted to provide an easily scalable test case, using demand and wind timeseries over multiple days and apply different settings to the dataset (differentiated by favourable or unfavourable wind situations, wind penetration, power plant cost characteristics, grid congestion, etc.).

The test cases were run on a Dell Optiplex 9020 workstation (Windows 7, 8-core 64bit Intel i7-4790 CPU@3.6GHz, 16GB RAM) using the GUROBI solver in GAMS 24.4.6, and setting the GUROBI options to 3 threads on 6 cores.

For the present purpose, we use 7 consecutive days, each day represented by 12 two-hour timesteps, 48 nodes (two out of three regions in the dataset), 64 power plants, and 79 lines (with transmission constraints). Then, we apply two different scenarios: first, a case with favourable wind conditions, in which not much congestion occurs in the network. In this case, the welfare-optimal solution (768 binary variables) was found within 0.5-2 seconds, except for *day6*, where it took 92 seconds (optimality threshold 0.01%). Using the respective starting point, GUROBI took between 102 and 150 seconds to determine that this was indeed the optimal outcome, and that no welfare improvement (considering the trade-off between maximum welfare and compensation payments) is possible (9,552 binary variables, at a 0.1% optimality threshold). For details, see the parameter “report_summary” in the results file in the github repository `output/report_MOPBQE_region_1.2_wind_fav.gdx`.

As a second case, we set the switch in the original dataset to “unfavourable wind conditions”. As before, computation times are up to ten seconds for the social-welfare problem (*WF-max*), and between 102 and 150 seconds for the binary equilibrium problem (*Bin Eq*), except for Day 3, where the solver encountered numerical problems and did not find any solution.

Over the entire week, congestion rents are much higher than in the “favourable wind case”, and the binary equilibrium method identified reductions of compensation payments of up to 35%, at no discernible loss of aggregate, or gross, welfare (i.e. before compensation payments). In Table 7, we show both the compensation required to guarantee no losses as well as the incentive-compatible compensation payments (“IC comp”), in line with the discussion in Section 3.5 (Problem 15). Note that the compensation payments are around 10% of generators profits – they may seem small compared to the overall welfare, but this is because the consumer surplus is computed as the difference between the actual price and the load-shedding price bound.

Over the entire week, congestion rents are much higher than in the case with favorable wind conditions, and the binary equilibrium method identified reductions of compensation payments of up to 35%, at no discernible loss of aggregate welfare. For further details, please refer to `output/report_MOPBQE_region_1.2_wind_unfav.gdx` in the github repository.

Last, let us point out that there are almost 10,000 dispatch options for the 64 generators, already accounting for eliminating all dispatch schedules that are not possible due to minimum up- or downtime constraints (this is shown in the report “dispatch options” in the gdx report file). If one were to try brute-force enumeration of all permutations of options to determine stable Nash equilibria, this would require to solve 10^{128} equilibrium problems for these test instances. In all cases, we also tried to solve the binary-equilibrium model directly without using the starting point derived from the socially optimal outcome. In no case was a solution found within the specified time limit of one hour.

	Generator profit	Wind rent	Consumer surplus	Congestion rent	Gross welfare	No-loss comp	Net welfare	IC comp	Net welfare	Time (sec)
Day 1	WF-max Bin Eq	0.092 0.111	0.277 0.207	11.677 11.615	0.026 0.139	12.071 12.071	0.023 12.048	0.016 0.011	12.1 12.061	1.9 146.9
								<i>33.54%</i>		
Day 2	WF-max Bin Eq	0.102 0.120	0.295 0.223	12.520 12.475	0.029 0.128	12.946 12.946	0.021 12.925	0.014 0.010	12.931 12.936	1.0 136.1
								<i>32.64%</i>		
Day 3	WF-max Bin Eq	0.129	0.290	12.240	0.037	12.696	0.007	0.012	12.684	1.4
Day 4	WF-max Bin Eq	0.085 0.099	0.262 0.131	12.066 12.022	0.033 0.195	12.446 12.446	0.026 12.420	0.019 0.014	12.428 12.432	1.9 115.2
								<i>25.41%</i>		
Day 5	WF-max Bin Eq	0.079 0.096	0.248 0.187	11.838 11.764	0.031 0.150	12.196 12.196	0.026 12.170	0.017 0.014	12.179 12.183	1.9 102.4
								<i>20.93%</i>		
Day 6	WF-max Bin Eq	0.054 0.076	0.187 0.300	9.873 9.750	0.020 0.008	10.133 10.133	0.023 10.110	0.014 0.014	10.119 10.119	9.7 115.2
								<i>3.47%</i>		
Day 7	WF-max Bin Eq	0.048 0.066	0.179 0.280	9.631 9.527	0.020 0.006	9.878 9.878	0.024 9.854	0.015 0.015	9.863 9.863	7.9 102.3
								<i>0.68%</i>		

Table 7: Summary of numerical tests under unfavorable wind conditions (welfare and rents in m\$, computation time in seconds) relative reduction of payments required to guarantee incentive compatibility given in italics