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Conference Paper

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Beiträge zur Jahrestagung des Vereins für Socialpolitik 2019: 30 Jahre Mauerfall - Demokratie und Marktwirtschaft - Session: Public Economics II, No. B16-V2

Provided in Cooperation with:

Verein für Socialpolitik / German Economic Association

Suggested Citation: Antony, Jürgen; Klarl, Torben (2019) : Resource depletion in a Ramsey economy with subsistence consumption, exogenous technical change and capital depreciation: A full characterization, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2019: 30 Jahre Mauerfall - Demokratie und Marktwirtschaft - Session: Public Economics II, No. B16-V2, ZBW - Leibniz-Informationszentrum Wirtschaft, Kiel, Hamburg

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Resource Depletion in a Ramsey Economy with Subsistence Consumption, Exogenous Technical Change and Capital Depreciation: A Full Characterization

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June 4, 2019

Abstract

This contribution is concerned with efficient use of a resources if households are characterized by Stone-Geary preferences with a minimum subsistence level of consumption. We provide a closed form solution for the case of Cobb-Douglas production with constant returns to scale in reproducible capital and effective resource input. The model allows for exogenous technical change and depreciation of reproducible capital. The closed form solution exists in terms of the Gaussian hypergeometric function. For some constellations of the model parameters, the economy asymptotically approaches the generalized Hartwick's investment rule. We calibrate our model to analyze the existence of a solution and the dynamics for resource-rich low-income countries.

JEL: Q30, E21, O44

Keywords:

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1 Introduction

This contribution reconsiders the continuous time Dasgupta-Heal-Solow-Stiglitz (Dasgupta and Heal 1974, Solow 1974 and Stiglitz 1974, DHSS from here on) model extended to include exogenous technical change and depreciation in reproducible man-made capital as has been proposed by Dixit et al. (1980, DHH from here on). We search for a closed form solution to the model that maximizes a utilitarian criterion that allows for subsistence consumption in the spirit of Stone (1954) and Geary (1950). Additionally, we investigate how results would be affected if the resource were subject to regenerating forces. We show that a closed form solution to the model exists for the entire adjustment along the saddle-path in terms of the Gaussian hypergeometric function. Our assumptions regarding production input are common in the related literature: reproducible man-made capital and resource extraction are essential input factors as in e.g. Mitra et al. (2013) and many others.

This contribution adds to the discussion on sustainability in the presence of scarce resources. A frequently cited concept of sustainability originating from the Brundtland Report (WCED 1987) puts forward the aim to “*make development sustainable to ensure that it meets the needs of the present without compromising the ability of future generations to meet their own needs*”. Stone-Geary preferences ensure that consumption never falls short of a minimum subsistence level. If one is willing to interpret *meeting needs* as to guarantee a particular minimum consumption or basic needs to which the Brundtland Report also refers, our results are helping to judge about whether sustainable development in the presence of necessary resources is feasible.

We contribute further to the existing literature in the following ways. First, we provide a technical contribution regarding closed form solutions for the full dynamics of an economy using special functions such as the Gaussian hypergeometric function. Second, we augment the DHSS model not only by technical change, depreciation and possibly regenerating resources which has been done elsewhere, but also by the introduction of minimum subsistence consumption. It will be shown that only under certain endowment situations, i.e. requirements for initial stocks of reproducible capital and the resource, a solution to the problem exists. Third, the ability to fully characterize the dynamics and initial conditions for an economy allows us to calibrate our model to situations relevant to e.g. low developed but resource-rich countries. We focus on resource-rich developing countries. As shown later within our calibration study, we find that resources seem to be most important for low-income countries. This allows us further to judge whether a particular endowment scenario is sufficient for a solution to economic problem to exist and how such a solution is characterized. Typical linearization techniques around a steady state wouldn't allow for this as the problem can not be solved for the initial values of the state variables. Finally, our approach is utilitarian by nature as we investigate households maximizing a discounted utility stream. As will be seen below, the presence of subsistence consumption leads the economy in some cases to asymptotically approach an egalitarian consumption path characterized by the generalized Hartwick (1977) investment rule formulated by Dixit et al. (1980). We therefore add to the literature initiated already by Solow (1974) on economic rules leading to egalitarian consumption paths motivated by utilitarian maximization problems.

Our findings are as follows. Solving the model in closed form allows us to pin down the initial co-state variables for the dynamic optimization problem. This in turn allows for a full characterization of conditions under which a solution to the economic problem exists. It is shown that any regeneration capacity for the resource essential to production violates the transversality condition associated with the resource stock. In case of regenerating forces, the economic problem is not meaningful as utility is unbounded in this case. Subsistence consumption implies particular minimum requirements for

initial endowments with reproducible man-made capital and resources. If these are not met, the economy is not able to cover subsistence consumption. Focusing on the steady state, we find that the equilibrium can be governed by zero or positive growth. The latter occurs if the rate of exogenous technical change exceeds the rate of time preference. In the former case, we are able to show that Hartwick's investment rule applies in steady state. Finally, we calibrate the model for developing but resource-rich countries and are able to trace the full dynamic development of the economy.

The plan of the paper is as follows. The next section reviews literature relevant to our contribution. Section 3 lays out the economic problem that we aim to solve and Section 4 presents the solution and elaborates on the solution's existence properties. We provide a calibration of our model in Section 5 and, finally, we discuss and conclude in Sections 6.

2 Review of Literature

The DHSS model has been subject to research until today by several authors. The Cobb Douglas constant returns to scale production structure with reproducible man-made capital and resource input has been employed by e.g. Antony and Klarl (2018), Benchekroun and Withagen (2011) and Asheim and Buchholz (2004) and others. Mitra et al (2013) employ a general constant returns to scale technology with reproducible man-made capital and resource input. Extensions of the model covering depreciation of capital and exogenous have been put forward first in the DHH model. Extension also considering endogenous resource augmenting technical change can be found in e.g. Groth (2007). Antony and Klarl (2018) introduce a minimum subsistence level of consumption in an utilitarian approach into the DHSS model without capital depreciation and technical change.

The DHSS model has been used in Hartwick's (1977) contribution and the resulting Hartwick's investment rule has been subject to ongoing research as well. It is well known, that this rule demands rents from resource extraction to be invested into a reproducible man-made stock of capital. Equivalently, the value of net investments or genuine savings equals zero at all points in time (see e.g. Hamilton and Atkinson 2006). The generalized Hartwick rule was first deduced from the DHH model and states that an egalitarian consumption path implies that the value of net investments or genuine savings are constant at all points in time but not equal to zero.

Egalitarian consumption paths and Hartwick's investment rule in its relation to utilitarian objectives has been put forward on the research agenda already in Solow (1974). Solow (1974) was showing that Rawls (1971) maximin criterion can be fulfilled by an egalitarian consumption path. Asheim and Buchholz (2004) analyzed undiscounted and discounted utility streams with exogenous restrictions to households' preferences as utilitarian objective functions. They work out under what conditions or restrictions maximization of the objective functions imply the economy to follow Hartwick's investment rule. Antony and Klarl (2018) show that maximizing discounted utility with Stone-Geary type preferences leads asymptotically to Hartwick's investment rule as consumption approaches its minimum subsistence level from above. We note that only a few contributions provide utilitarian criteria that justify the adoption of Hartwick's investment rule.

The asymptotic result in Antony and Klarl (2018) is to be expected as it can be shown that any competitive egalitarian path in a DHSS model with stationary technology must fulfill Hartwick's investment rule (Buchholtz et al. 2005, Withagen and Asheim 1998). Mitra (2002) already showed that Hartwick's investment rule is a necessary condition for such an egalitarian consumption path. Related to these paths, Mitra et al. (2013) provide necessary and sufficient conditions on the production technique to ensure that from any historical starting point a constant positive consumption stream results. Within a standard exhaustible resource model, Mitra (2015) proves the efficiency and

uniqueness of non-trivial maximin paths. Comparable results regarding the generalized Hartwick rule in the DHH model can be found in Heijnen (2008) and Sato and Kim (2002). These findings become relevant in the present contribution for some cases where the economy approaches an asymptotically egalitarian path.

Our contribution also touches upon the discussion on sustainability in the presence of non-renewable resources. It is beyond the scope of this section to fully review this strand of the literature. For a recent review regarding the relation between sustainability, genuine savings and Hartwick's investment rule see Hanley et al. (2015). Asheim and Buchholz (2004) regard any consumption path characterized by non-decreasing consumption as sustainable. Such paths can be derived from a discounted utilitarian maximization problem subject to restrictions exogenous to households' preferences. In contrast, Stone-Geary preferences guarantee a minimum level of consumption where the necessary restriction on consumption is already implied by preferences. As pointed out in the introduction, one might be willing to interpret consumption not falling below a minimum subsistence level to fulfill some sustainability concepts. More precisely, the question would be whether a non-monotonic consumption path that is strictly above a minimum subsistence level could be regarded as a sustainable development. Holden et al. (2014) provide a broad review on the concept of sustainability and sustainable development. They give an interpretation of the Brundtland Report that regards development sustainable if basic human needs are guaranteed in an intergenerational way. Any aspiration beyond these needs can only be regarded as sustainable if long-term ecological sustainability is respected. Focusing on the extraction and consumption of non-renewable resources, one is therefore left with the question whether the asymptotic exhaustion of such a resource combined with consumption asymptotically approaching basic needs from above is indeed satisfying this interpretation. It is not the aim of this contribution to fully answer this question. However, we would like contribute to the discussion on such an answer.

Although the analyzed DHSS/DHH setting extended by subsistence consumption is complex due to a non-homothetic instantaneous utility function, it allows for a closed-form solution if one indeed exists. The present model is a natural extension to the model presented by Antony and Klarl (2018) discussing a Ramsey economy for the case where the rate of technical progress as well as the depreciation rate on reproducible capital is zero. From a technical point of view, the contribution is related to recent publications also using special functions to solve dynamic economic problems. Hiraguchi (2014) solves a Ramsey problem with leisure as one argument of the utility function. The solution involves the Gaussian hypergeometric function. The same function appears in Boucekkine and Ruiz-Tamarit (2008), Boucekkine et al. (2008), Ruiz-Tamarit (2008), Hiraguchi (2009) solving Lucas types models. Guerrini (2010) uses the Gaussian hypergeometric function to solve the problem of an AK Ramsey economy with logistic population growth. Regarding problems related to environmental economics, Perez-Barahona (2011) use the Gaussian hypergeometric function to solve an AK Ramsey problem with scarce resources. The exponential Integral is found in the explicit solutions to a basic DHSS model without technical change and capital depreciation in Benchekroun and Withagen (2011). The formal representations involved in solving the problem we pose is most similar to the ones found in Boucekkine and Ruiz-Tamarit (2008).

Economically, our analysis is also related to the growth literature. Strulik (2010) and Steger (2000) solve a utility maximization problem with Stone-Geary preferences without considering natural resources necessary for production. They instead focus only on an AK type of production technology and present closed form solutions for the entire adjustment path of the economy. Their models are nested in ours if one is setting the output elasticity of the resource equal to zero. Mathematically, we implicitly also make use of what is known as the mathematical concept of a viability

kernel. As we can solve out for the entire dynamics of the model economy, we are also able to exactly solve for the conditions regarding the initial endowment of the economy for a solution to exist. These conditions form the viability kernel (for an economic application of this concept related to resource problems see e.g. Martinet and Doyen, 2007).

3 Subsistence Consumption in the DHSS Model

In this section, we lay out the intertemporal utilitarian problem that we aim to solve. Preliminary calculations are presented that are helpful in finding a solution to the problem provided one exists.

3.1 The Problem

The Economy is populated by a mass 1 of infinitively living representative households with the following Stone-Geary intertemporal utility function

$$U_t = \int_0^{\infty} \frac{(C_t - \underline{C})^{1-\eta} - 1}{1-\eta} e^{-\rho t} dt, \quad (1)$$

where C_t is consumption at time t , \underline{C} is the minimum subsistence level of consumption, $\eta > 0$ and $\rho > 0$ is the rate of time preference. We will refer to $C_t - \underline{C}$ as excess consumption in the sense that is taking place in excess of subsistence consumption.

We consider a social planner to maximize households' lifetime utility given the relevant budget constraints. These constraints are given, first, by the accumulation of reproducible capital, and second, by the use of a potentially renewable resource that is necessary for production.

We assume that production is given by the aggregate Cobb-Douglas production technology

$$Y_t = K_t^\alpha (A_t R_t)^{1-\alpha}, \quad (2)$$

where K_t denotes the accumulated level of reproducible capital, R_t is resource use and $0 < \alpha < 1$. A_t is the level of technology which we assume to grow at rate γ , i.e. $A_t = A_0 e^{\gamma t}$. Without loss of generality, we normalize A_0 at 1. With specification (2), we have constant returns to scale with respect to capital and resource input R_t as e.g. in Benckroun and Withagen (2011) or Asheim and Buchholz (2004). One might interpret K_t not only to represent physical but also any kind of reproducible capital like e.g. human capital. This last interpretation would allow also labor to participate in production.

Reproducible capital is produced from foregone final output with unit productivity and depreciates at a rate $\delta > 0$. The net increase in the stock of reproducible capital is therefore

$$\frac{\partial K_t}{\partial t} = \dot{K}_t = Y_t - C_t - \delta K_t. \quad (3)$$

Production requires the use of R_t units of a renewable resource at time t . The stock S_t of the resource develops according to

$$\dot{S}_t = -R_t + v. \quad (4)$$

$v \geq 0$ is a constant flow that adds to the resource' stock at every instance in time. This gain is independent of the stock of the resource.

The present value Hamiltonian for the representative household therefore reads as

$$H_t = \frac{(C_t - \underline{C})^{1-\eta} - 1}{1-\eta} e^{-\rho t} + \lambda_t [Y_t - C_t - \delta K_t] + \mu_t [-R_t + v]. \quad (5)$$

The first order conditions for a maximum read as

$$\frac{\partial H_t}{\partial C_t} = (C_t - \underline{C})^{-\eta} e^{-\rho t} - \lambda_t = 0, \quad (6)$$

$$-\frac{\partial H_t}{\partial K_t} = \dot{\lambda}_t = -\lambda_t \frac{\partial Y_t}{\partial K_t} + \lambda_t \delta, \quad (7)$$

$$\frac{\partial H_t}{\partial R_t} = \lambda_t \frac{\partial Y_t}{\partial R_t} - \mu_t = 0, \quad (8)$$

$$-\frac{\partial H_t}{\partial S_t} = \dot{\mu}_t = 0. \quad (9)$$

The corresponding transversality conditions are

$$\lim_{t \rightarrow \infty} \lambda_t K_t = 0, \quad (10)$$

$$\lim_{t \rightarrow \infty} \mu_t S_t = 0. \quad (11)$$

The production function and (8) imply

$$Y_t = K_t \left(\frac{A_t R_t}{K_t} \right)^{1-\alpha} = K_t \left(\frac{\lambda_t (1-\alpha)}{\mu_t} A_t \right)^{\frac{1-\alpha}{\alpha}}. \quad (12)$$

From (9) we can easily deduce that $\mu_t = \mu_0$ for all t . Using (7) and (8), we find λ_t to develop according to¹

$$\lambda_t = e^{\delta t} \left[\lambda_0^{\frac{\alpha-1}{\alpha}} + (1-\alpha)^{\frac{1-\alpha}{\alpha}} A_0^{\frac{1-\alpha}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{\gamma+\delta} \left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1 \right) \right]^{\frac{\alpha}{\alpha-1}}, \quad (13)$$

which gives the development of marginal utility of consumption in excess of subsistence consumption \underline{C} .

3.2 Preliminary Calculations

The next step is now to solve for the development of the capital stock K_t . Readers not interested in the details behind the solution might skip the following and proceed directly with Proposition 1 further down below. Using (3) and (13) in (12), this stock develops as

$$\dot{K}_t = Y_t - C_t - \delta K_t = K_t \left[\left(\frac{\lambda_t (1-\alpha)}{\mu_0} A_t \right)^{\frac{1-\alpha}{\alpha}} - \delta \right] - (C_t - \underline{C}) - \underline{C}.$$

We rewrite this first order differential equation as

¹See Appendix A at the end of the paper for the details on the derivations.

$$\dot{K}_t + f(t)K_t = g(t), \quad (14)$$

with

$$f(t) = - \left[\left(\frac{\lambda_t(1-\alpha)}{\mu_0} A_t \right)^{\frac{1-\alpha}{\alpha}} - \delta \right],$$

$$g(t) = -(C_t - \underline{C}) - \underline{C} = -\lambda_t^{-\frac{1}{\eta}} e^{-\frac{\rho}{\eta}t} - \underline{C},$$

where $-f(t)$ is the net return on reproducible capital at time t . We denote the initial stock of capital at $t = 0$ by K_0 . The solution to the differential equation (14) is given by

$$K_t = K_0 e^{-\int_0^t f(z)dz} + \int_0^t g(z) e^{-\int_z^t f(s)ds} dz. \quad (15)$$

Appendix B at the end of the paper shows that

$$-\int_z^t f(s)ds = -\delta(t-z) + \frac{1}{1-\alpha} \ln \left[\frac{\varphi_1 + \varphi_2 (e^{\psi t} - 1)}{\varphi_1 + \varphi_2 (e^{\psi z} - 1)} \right] \quad (16)$$

with

$$\varphi_1 = \left[\frac{\lambda_0(1-\alpha)A_0}{\mu_0} \right]^{\frac{\alpha-1}{\alpha}},$$

$$\varphi_2 = \frac{\alpha}{\gamma + \delta},$$

$$\psi = \frac{1-\alpha}{\alpha}(\gamma + \delta).$$

At this point it is helpful to introduce the variable $x_t = e^{-\psi t}$. As time t develops from 0 to ∞ , the variable x_t ranges between 1 and 0. This makes it possible to solve the model analytically using the Gaussian hypergeometric function.

Using (16) in (15) gives the stock of capital K_t as²

$$K_t = K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \quad (17)$$

$$- e^{-\delta t} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \int_{x_t}^1 x_z^{-\frac{1}{\psi} \left(\frac{(\eta-1)\delta-\rho}{\eta} + \frac{\alpha-\eta}{1-\alpha} \frac{\psi}{\eta} \right) - 1} (1 - \zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z$$

$$- \underline{C} e^{-\delta t} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \int_{x_t}^1 x_z^{-\frac{1}{\psi} (\delta - \frac{\psi}{1-\alpha}) - 1} (1 - \zeta x_z)^{-\frac{1}{1-\alpha}} dx_z,$$

with

$$\zeta = \frac{\varphi_2 - \varphi_1}{\varphi_2} = \frac{\frac{1}{\varphi_1} - \frac{1}{\varphi_2}}{\frac{1}{\varphi_1}}.$$

By using (8), it can easily verified that $\frac{1}{\varphi_1}$ is equal to the initial capital productivity $\frac{Y_0}{K_0}$. As will become clear further down below, $\frac{1}{\varphi_2}$ is equal to the capital productivity in steady-state as $t \rightarrow \infty$, i.e. $\lim_{t \rightarrow \infty} \frac{Y_t}{K_t} = \frac{\gamma + \delta}{\alpha} = \frac{1}{\varphi_2}$. ζ therefore measures the relative distance of the limiting from the initial

²It is interesting to note that the integral in the second term of (17) simplifies very much in case $\alpha = \eta$. It is exactly this case that is discussed in Smith (2006) who presents a closed form solution to the Ramsey problem for $\alpha = \eta$.

capital productivity. If it were by chance that $\zeta = 0$, we would encounter an economy that starts in steady-state right away. Unsurprisingly, the expressions in (17) would simplify a great deal if this case prevails. As the first order conditions (6) and (8) imply $\lambda_0, \mu_0 > 0$, we necessarily find $\zeta < 1$.

Appendix B at the end of the paper demonstrates that the integrals in (17) - as long as they converge - can be computed using the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ which has in general the integral representation

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt. \quad (18)$$

This integral representation is valid for $R(c) > R(b) > 0$ where $R(\cdot)$ denotes the real part of the argument and $\Gamma(\cdot)$ the Gamma function (Abramowitz and Stegun, 1964, 15.3.1). In general, ${}_2F_1(a, b; c; z)$ defined as a Gauss series (Abramowitz and Stegun, 1964, 15.1.1) converges if $R(c - b - a) > 0$ for $|z| \leq 1$ and if $-1 < R(c - b - a) \leq 0$ for $|z| \leq 1$ but $z \neq 1$. Comparing the integral on the right hand side of (18) with the integrals in (17) reveals that the present case can be seen as a special case with $c - b - 1 = 0$ or equivalent $c = b + 1$.

If we apply the representation (18) to our problem, ζ will play the role of z . Admissible values for the initial co-state variables, i.e. $\lambda_0 > 0$ and $\mu_0 > 0$ (see the first order conditions (6) and (8)), imply $\zeta < 1$. If λ_0 is sufficiently small and/or μ_0 is sufficiently large, it might turn out that $\zeta \leq -1$. In this case, one has to take care about how to compute the integrals in (17) or other integrals of the same type that appear further down below. This is because the integral representation (18) is an analytic continuation of the Gaussian hypergeometric function defined by a Gauss series (Abramowitz and Stegun 1972, 15.3.1). Only for the restrictions on z and $R(c - b - a)$ laid out above, both are identical. In general, for $z \leq -1$ and $R(c) > R(b) > 0$, the integral (18) exists but the Gauss series that defines the hypergeometric function is not converging and, hence, it is not identical to the integrals that we aim to compute. In such cases, it is necessary to use analytic continuation formulas for ${}_2F_1(a, b; c; z)$ (see Abramowitz and Stegun 1972, 15.3.3 through 15.3.9). For a general discussion about this situation see Section 3.1 in Boucekkine and Ruiz-Tamarit (2008).

We therefore make notational use of

$$\begin{aligned} {}_2F_1(a, b; b+1; z) &= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \\ &= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)} \int_0^1 t^{b-1} (1-zt)^{-a} dt \\ &= b \int_0^1 t^{b-1} (1-zt)^{-a} dt, \end{aligned}$$

where we keep in mind that $z \leq -1$ needs special attention. Here we apply the continuation of the gamma function $\Gamma(b+1) = b\Gamma(b)$ and the fact that $\Gamma(1) = 1$ (Abramowitz and Stegun, 1964, 6.1.15). Inspecting (17) shows that we can apply this special case of the Gaussian hypergeometric function to both integrals. Through a suitable change in the variable of integration, the integrals ranging from x_t to 1 can be split up into two separate integrals each running from 0 to 1 and each representable by the hypergeometric function. We are ready to formulate the following proposition on the development of K_t .

Proposition 1: The optimal path for the capital stock K_t is given by

$$\begin{aligned}
K_t &= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \\
&- e^{-\delta t} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \frac{1}{\tilde{b}_1} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) \right. \\
&\quad \left. - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\
&- \underline{C} e^{-\delta t} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \frac{1}{\tilde{b}_2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right],
\end{aligned} \tag{19}$$

with

$$\begin{aligned}
\tilde{a}_1 &= \frac{\eta - \alpha}{\eta(1 - \alpha)}, \\
\tilde{b}_1 &= -\frac{1}{\psi} \left(\frac{(\eta - 1)\delta - \rho}{\eta} + \frac{\alpha - \eta}{1 - \alpha} \frac{\psi}{\eta} \right) = 1 + \frac{\alpha[(\eta - 1)\gamma + \rho]}{(1 - \alpha)(\delta + \gamma)\eta}, \\
\tilde{a}_2 &= \frac{1}{1 - \alpha}, \\
\tilde{b}_2 &= -\frac{1}{\psi} \left(\delta - \frac{\psi}{1 - \alpha} \right) = \frac{(1 - \alpha)\delta + \gamma}{(1 - \alpha)(\gamma + \delta)} = 1 + \frac{\alpha\gamma}{(1 - \alpha)(\gamma + \delta)} > 1.
\end{aligned}$$

Proof: Appendix B.

At this point, a view words on the admissible values for the model's parameters are in order. Later on, during inspecting the transversality conditions for the present optimization problem, it will become clear that $\tilde{b}_1, \tilde{b}_2 > 1$ need to be fulfilled for the transversality conditions to hold. It is obvious that this imposes the restriction $\gamma > 0$ in case of \tilde{b}_2 . $\tilde{b}_1 > 1$ implies further restrictions for the model's parameter. We will return to this issue further below where we deal with the transversality conditions in full detail. The variable ζ depends via φ_1 on the initial values of the co-state variables, i.e. λ_0 and μ_0 . The first order conditions (6) through (8) require $\lambda_0, \mu_0 > 0$ which in turn implies $\zeta < 1$.

With the development of the capital stock K_t at hand, we can now immediately proceed with the use of the resource R_t . This is particularly straightforward as $R_t = \left(\frac{\lambda_t(1-\alpha)}{\mu_0} A_t \right)^{\frac{1}{\alpha}} \frac{K_t}{A_t}$ which follows directly from equation (12).

Proposition 2: The optimal path for resource depletion R_t is given by

$$\begin{aligned}
R_t &= \frac{K_0}{A_0} \varphi_1^{-\frac{1}{1-\alpha}} x_t^{-1} \\
&- \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-1} \frac{1}{\tilde{b}_1} \frac{1}{A_0} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\
&- \frac{\underline{C}}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi} x_t^{-1} \frac{1}{\tilde{b}_2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right].
\end{aligned} \tag{20}$$

Proof: Appendix C.

As we now know the extend of resource extraction, it is natural to proceed with the development of the economy's resource stock S_t . By assumption, $S_t = S_0 - \int_0^t R_z dz$.

Proposition 3: The resource stock S_t follows the optimal path given by

$$\begin{aligned}
S_t &= S_0 - \int_0^t R_z dz, \\
&\text{with} \\
\int_0^t R_z dz &= -\frac{K_0}{A_0} \phi_1^{-\frac{1}{1-\alpha}} \frac{1}{\psi} [1 - x_t^{-1}] \\
&+ \phi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) [1 - x_t^{-1}] \\
&+ \phi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{\zeta^{1-\tilde{b}_1}}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} \left[\zeta^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta) \right. \\
&\quad \left. - (\zeta x_t)^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta x_t) \right] \\
&+ \frac{\underline{C}}{A_0} \phi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) [1 - x_t^{-1}] \\
&+ \frac{\underline{C}}{A_0} \phi_2^{-\frac{1}{1-\alpha}} \frac{\zeta^{1-\tilde{b}_2}}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} \left[\zeta^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta) \right. \\
&\quad \left. - (\zeta x_t)^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta x_t) \right].
\end{aligned} \tag{21}$$

Proof: Appendix D.

As production is Cobb-Douglas, (19) and (20) are determining the development of Y_t via (2). In order to complete our preliminary calculations, we look at consumption C_t .

Proposition 4: Consumption C_t follows the optimal path given by

$$C_t = \phi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} x_t^{\frac{\alpha(\rho-\gamma)}{(1-\alpha)(\gamma+\delta)\eta}} (1 - \zeta x_t)^{\frac{\alpha}{1-\alpha} \frac{1}{\eta}} + \underline{C} \tag{22}$$

Proof: The result in Proposition 4 follows from inserting (13) into (6) and using the definition of x_t .

4 Solving the DHSS Model

The preceding section developed the dynamics for all the important quantities in the economy under consideration. In order to solve the model and to trace out the full dynamics, we need to determine the initial values for the co-state variables λ_0 and μ_0 given the initial endowment of the economy, i.e. K_0 and S_0 .

4.1 Transversality

The transversality conditions (10) and (11) are necessary for the solution of the problem to characterize an optimum. For the capital stock, $\lim_{t \rightarrow \infty} \lambda_t K_t = 0$ needs to be fulfilled. Furthermore, the second transversality condition demands $\lim_{t \rightarrow \infty} \mu_t S_t = \mu_0 \lim_{t \rightarrow \infty} S_t = 0$. This gives rise to the following lemma.

Lemma 1: The initial stocks of capital, K_0 , and resources, S_0 , need to fulfill

$$K_0 = \varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}-1} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) \quad (23)$$

$$+ \underline{C} \varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{1}{\alpha-1}} \frac{1}{\psi} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta).$$

$$S_0 = \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta) \quad (24)$$

$$+ \frac{\underline{C}}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta).$$

Proof: Appendix E for (23) and F for (24).

At the outset of the paper, we considered the stock of the resource to be governed by $\dot{S}_t = -R_t + v$, where v is a constant (positive) flow that adds to the stock. The calculations in the Appendix show that any value for $v \neq 0$ will violate the transversality conditions involving S_t . This is why we here only consider the case $v = 0$.

Expression (24) implies several parameter restrictions. The first ones are about the parameter combinations \tilde{b}_1 and \tilde{b}_2 . For transversality (both conditions) to hold and to be meaningful, $\tilde{b}_1 - 1 > 0$ and $\tilde{b}_2 - 1 > 0$ must be satisfied. Only then the underlying integrals behind the hypergeometric functions in (24) converge and are finite. $\tilde{b}_1 - 1 > 0$ requires $(\eta - 1)\gamma + \rho > 0$ which demands, ceteris paribus, a high rate of time preference, a high η or a low elasticity of intertemporal substitution if the rate of technical progress γ is positive. That only $\gamma > 0$ is in accordance with the transversality condition (24) becomes obvious as only in this case $\tilde{b}_2 - 1 > 0$ is satisfied.

Expressions (23) and (24) are two nonlinear equations in the two still unknowns λ_0 and μ_0 . Both equations depend on these unknown via the definition of φ_1 and ζ , i.e. $\varphi_1 = \left(\frac{\lambda_0(1-\alpha)A_0}{\mu_0} \right)^{\frac{\alpha-1}{\alpha}}$ and $\zeta = \frac{\varphi_2 - \varphi_1}{\varphi_2}$.

4.2 Initial Co-State Variables

The non-linearity of (23) and (24) does not allow for an explicit solution for λ_0 and μ_0 . This subsection shows that the equilibrium, provided that it exists, is unique and that λ_0 and μ_0 characterize the optimum solution of our problem in terms of initial conditions for $t = 0$.

To see this, we define the following additional quantities by decomposing the transversality conditions (23) and (24)

$$K_0^+ = \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{(1-\zeta)^{\frac{1}{1-\alpha}}}{\psi} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta), \quad (25)$$

$$\underline{K}_0 = \underline{C} \frac{(1-\zeta)^{\frac{1}{1-\alpha}}}{\psi} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) = \underline{C} \frac{1}{\psi} \frac{1}{\tilde{b}_2} (1-\zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta), \quad (26)$$

$$S_0^+ = \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta), \quad (27)$$

$$\underline{S}_0 = \frac{\underline{C}}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta). \quad (28)$$

In economic terms, these four quantities have the following interpretation. K_0^+ and S_0^+ are the

parts of the initial endowment with capital and resources that are necessary in optimum to allow for consumption in excess of the subsistence level, i.e. $C_t - \underline{C}$. \underline{K}_0 and \underline{S}_0 are the necessary endowments allowing for subsistence consumption \underline{C} given the households choice for the path of excess consumption $C_t - \underline{C}$. The equations just above give the optimum division of the initial endowments into the components necessary for subsistence and excess consumption. It is clear that both $K_0 - \underline{K}_0$ and $S_0 - \underline{S}_0$ need to be positive. Otherwise, initial endowments are simply insufficient to allow for subsistence consumption \underline{C} .

The four additional quantities are fully determined by the initial values λ_0 and μ_0 which pin down, first, their ratio via φ_1 and by that also ζ . Given ζ , the system can be solved for the levels of λ_0 and μ_0 as (25) and (27) also depend on μ_0 alone besides ζ .

From the above four definitions it follows that

$$\begin{aligned} \frac{K_0^+}{S_0^+} &= \psi(\tilde{b}_1 - 1)A_0\varphi_1^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1 - 1; \tilde{b}_1 + 1; \zeta)} \\ &= \psi(\tilde{b}_1 - 1)A_0\varphi_2^{\frac{1}{1-\alpha}} (1 - \zeta)^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1 - 1; \tilde{b}_1 + 1; \zeta)} \end{aligned} \quad (29)$$

and

$$\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0 - \underline{C} \frac{1}{\psi} \frac{1}{\tilde{b}_2} (1 - \zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{S_0 - \frac{\underline{C}}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2 - 1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta)}. \quad (30)$$

To find ζ_0 that solves for an equilibrium, we note that this ζ_0 needs to fulfill

$$\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0^+}{S_0^+}. \quad (31)$$

Proposition 5: If there exists a solution ζ_0 to the equilibrium condition $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0^+}{S_0^+}$ with $\zeta_0 < 1$, this solution is unique.

Proof: Appendix G making use of Lemma 1.

Appendix G proves that $\frac{K_0^+}{S_0^+}$ is decreasing and $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ is increasing (constant) for $\underline{C} > 0$ ($\underline{C} = 0$) in ζ for $\zeta < 1$.

The properties of the right hand side of (31) defined in (29) are $\lim_{\zeta \rightarrow -\infty} \frac{K_0^+}{S_0^+} \rightarrow \infty$ and $\lim_{\zeta \rightarrow 1} \frac{K_0^+}{S_0^+} = 0$. If subsistence consumption \underline{C} would be zero, the left hand side of (31) would be constant at $\frac{K_0}{S_0} > 0$. In this case, a solution obviously always exists. However, once $\underline{C} > 0$, there is the possibility that no solution exists. This happens whenever initial endowment with capital K_0 and the resource S_0 are too low. Sufficient initial endowments allow for $K_0 - \underline{K}_0 > 0$ and $S_0 - \underline{S}_0 > 0$ which imposes restrictions on admissible values for ζ . Define $\underline{\zeta}$ as the value for ζ that solves $K_0 - \underline{K}_0 = 0$, where \underline{K}_0 is given by (26). Appendix G shows that such $\underline{\zeta} < 1$ always exists for $\underline{C} > 0$. Only $\zeta \geq \underline{\zeta}$ are admissible candidates for a solution as otherwise K_0 is insufficient to guarantee subsistence consumption. On the other hand, ζ does not need to be too large as it is possible that a ζ too large turns $S_0 - \underline{S}_0$ negative. We define $\bar{\zeta}$ as the largest possible value for ζ that prevents $S_0 - \underline{S}_0$ from becoming non-positive (see the Appendix for the detailed derivations), i.e.

$$\bar{\zeta} = \operatorname{argmin}_{\zeta \leq 1} |S_0 - \underline{S}_0| = \operatorname{argmin}_{\zeta \leq 1} \left| S_0 - \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2 \tilde{b}_2 (\tilde{b}_2 - 1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta) \right|.$$

It can be shown that $S_0 - \underline{S}_0$ is decreasing in ζ for $C > 0$ and $\zeta \leq 1$. Hence, $\bar{\zeta}$ is bounded from above by one.

Whenever we find initial endowments and model parameters to imply $\underline{\zeta} \leq \bar{\zeta}$, we are faced with a situation that possesses a unique solution. Whenever $\underline{\zeta} > \bar{\zeta}$ prevails, there is no solution and initial endowments with K_0 and S_0 are too low. In case $C \rightarrow 0$, we find $\underline{\zeta} \rightarrow -\infty$ and $\bar{\zeta} \rightarrow 1$. $\underline{\zeta}$ together with $\bar{\zeta}$ define a viability kernel as in Martinet and Doyen (2007) as they decide on whether a solution to the problem exists or not.

4.3 Transitional Dynamics

Given the initial values for the co-state variables μ_0, λ_0 and ζ_0 which are unique if a solution exists, we are able to trace out the transitional dynamics of all model's variables. We discuss below the paths for C_t, K_t, R_t and S_t . This is done by using Proposition 4 and Lemma 1 in the Propositions 1 through 3. As they can be computed explicitly, we are able to fully characterize the model's stable arm towards the steady state.

Applying Proposition 4 to the solution ζ_0 gives the path of consumption as

$$C_t = \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} x_t^{\frac{\alpha(\rho-\gamma)}{(1-\alpha)(\gamma+\delta)\eta}} (1 - \zeta_0 x_t)^{\frac{\alpha}{1-\alpha} \frac{1}{\eta}} + \underline{C}. \quad (32)$$

Looking at (32) reveals that consumption can take quite different developments depending on the quantities ρ and γ . Remember that as t grows, x_t decreases from 1 towards 0. If the rate of time preference ρ exceeds the rate of technical change γ , households are too impatient to allow for steady positive consumption growth. Instead, consumption in general first grows and peaks at $t^* = \frac{\alpha}{1-\alpha} \frac{1}{\gamma+\delta} \ln \left(\zeta \frac{\rho+\gamma}{\rho-\gamma} \right)$. This can be easily affirmed by solving $\frac{\partial C_t}{\partial t} = 0$ and it is obvious that t^* only exists for the case $\rho > \gamma$.

If $\rho > \gamma$, consumption increases for $t < t^*$. After t^* , consumption declines and converges towards its subsistence level from above. If $\rho < \gamma$ we observe a monotonic behavior of consumption. Consumption steadily increases and converges against a growth path with the positive growth rate $\frac{\gamma-\rho}{\eta}$. In case it happens that $\rho = \gamma$ applies, consumption also behaves monotonic but converges against a constant value larger than subsistence consumption.

Using Lemma 1 (23) and Proposition 1 (19), the development of the capital stock K_t turns out to follow (see Appendix E)

$$K_t = x_t^{\frac{\alpha(\rho-\gamma)}{(1-\alpha)(\gamma+\delta)\eta}} (1 - \zeta_0 x_t)^{\frac{1}{1-\alpha}} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta_0 x_t) + \underline{C} (1 - \zeta_0 x_t)^{\frac{1}{1-\alpha}} \frac{1}{\psi} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta_0 x_t). \quad (33)$$

The stock of reproducible capital shares qualitatively the behavior of consumption. In case of $\rho > \gamma$ it is characterized by non-monotonic behavior with a peak at in general $t \neq t^*$. Whether the peak appears earlier compared with consumption, depends on the household's preferences. It can

be shown that capital peaks earlier in the (unlikely) case $\eta \leq \alpha$. In case $\eta > \alpha$, the peak can occur earlier or later.

Resource extraction can be calculated as $R_t = \left(\frac{\lambda_t(1-\alpha)}{\mu_0} A_t\right)^{\frac{1}{\alpha}} \frac{K_t}{A_t}$ by using (13) and (33). Resource extraction steadily declines as t passes by. Using (21) and (24), the resource stock S_t consequently develops according to (see Appendix F which uses Lemma 1 in Proposition 3)

$$S_t = \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} x_t^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta_0 x_t) \\ + \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} x_t^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta_0 x_t).$$

Of course, S_t steadily declines and approaches 0 asymptotically.

Output is given by $Y_t = K_t^\alpha (A_t R_t)^{1-\alpha}$ which can now easily be computed with the results at hand.

Using $R_t = \left(\frac{\lambda_t(1-\alpha)}{\mu_0} A_t\right)^{\frac{1}{\alpha}} \frac{K_t}{A_t}$, (13) and the definitions of $x_t = e^{-\psi t}$ and ζ gives

$$Y_t = \frac{K_t}{\varphi_2} (1 - \zeta_0 x_t)^{-1}. \quad (34)$$

Output shares qualitatively the dynamic properties of consumption and the stock of reproducible capital. Y_t displays monotonic dynamics in case $\gamma \geq \rho$. For $\rho > \gamma$ the behavior is again non-monotonic with a single peak before output continuously declines. From (34) it also becomes clear that ζ_0 measures the relative distance of the steady state capital productivity from its initial position at $t = 0$. Solving (34) for ζ_0 at $t = 0$, gives $\zeta_0 = \frac{y_0}{k_0} \frac{1}{\varphi_2}$ with $x_0 = 1$. Furthermore, (34) reveals that $\zeta_0 x_t$ measures the relative distance of the steady state capital productivity from its position at time t . As $x_t = e^{-\psi t}$ with $\psi = \frac{1-\alpha}{\alpha}(\gamma + \delta)$, ψ can be interpreted as a decay constant with an implied half life of $\frac{\ln(2)}{\psi} = \frac{\alpha}{1-\alpha} \frac{\ln(2)}{\gamma + \delta}$.

Given (34) and the definition of φ_2 , also the net rate of return for reproducible capital can easily be computed as

$$r_t = \alpha \frac{Y_t}{K_t} - \delta = \frac{\gamma + \delta}{1 - \zeta_0 x_t} - \delta. \quad (35)$$

The value of extracted resources at time t is given by $p_t R_t$ where the resource price p_t is due to (8) equal to $\frac{\mu_0}{\lambda_t}$. As production is Cobb-Douglas, we find $p_t R_t = (1-\alpha)Y_t$ for all t . We can follow genuine savings or net investments $I_t = Y_t - C_t - \delta K_t - p_t R_t$ and the net investment rate $i_t = 1 - \frac{C_t}{Y_t} - \delta \frac{K_t}{Y_t} - \frac{p_t R_t}{Y_t}$ numerically. Its interesting limiting properties are discussed in following subsection.

4.4 Limiting Behavior and Steady-State

Our economy approaches its steady-state as $t \rightarrow \infty$ or alternatively $x_t \rightarrow 0$. Considering the dynamic behavior of C_t , K_t and Y_t given by (32), (33) and (34) as $t \rightarrow \infty$ reveals that the growth rates of these three variables approach $\frac{\gamma-\rho}{\eta}$ in case $\gamma \geq \rho$. In this case, the three variables grow at a non-negative rate ad infinitum and the influence of subsistence consumption on the economy's growth rate vanishes asymptotically. If $\gamma \leq \rho$, however, the growth rates of C_t , K_t and Y_t tend to zero as the

influence of \underline{C} does not vanish³

The limiting behavior of C_t can easily be calculated by evaluating (32) for $x_t \rightarrow 0$. It follows that

$$\lim_{t \rightarrow \infty} C_t \begin{cases} = \underline{C}, \frac{\dot{C}_t}{C_t} \rightarrow 0 & \text{for } \rho > \gamma, \\ = \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} + \underline{C}, \frac{\dot{C}_t}{C_t} \rightarrow 0 & \text{for } \rho = \gamma, \\ \rightarrow \infty, \frac{\dot{C}_t}{C_t} \rightarrow \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma. \end{cases}$$

Using (33) for the capital stock as $t \rightarrow \infty$ ($x_t \rightarrow 0$) gives

$$\lim_{t \rightarrow \infty} K_t \begin{cases} = \underline{C} \frac{1}{\psi} \frac{1}{b_2} \equiv \underline{K}, \frac{\dot{K}_t}{K_t} \rightarrow 0 & \text{for } \rho > \gamma, \\ = \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{b_1} + \underline{C} \frac{1}{\psi} \frac{1}{b_2}, \frac{\dot{K}_t}{K_t} \rightarrow 0 & \text{for } \rho = \gamma, \\ \rightarrow \infty, \frac{\dot{K}_t}{K_t} \rightarrow \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma, \end{cases}$$

which, using (34), implies for output Y_t

$$\lim_{t \rightarrow \infty} Y_t \begin{cases} = \underline{C} \frac{1}{\psi \varphi_2} \frac{1}{b_2} \equiv \underline{Y}, \frac{\dot{Y}_t}{Y_t} \rightarrow 0 & \text{for } \rho > \gamma, \\ = \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}-1} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{b_1} + \underline{C} \frac{1}{\varphi_2} \frac{1}{\psi} \frac{1}{b_2}, \frac{\dot{Y}_t}{Y_t} \rightarrow 0 & \text{for } \rho = \gamma, \\ \rightarrow \infty, \frac{\dot{Y}_t}{Y_t} \rightarrow \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma. \end{cases}$$

Naturally, the transversality conditions imply that $\lim_{t \rightarrow \infty} S_t = 0$ and consequently also $\lim_{t \rightarrow \infty} R_t = 0$.

Finally, we turn to genuine savings or net investment I_t . Inspecting the limiting behavior of Y_t , C_t and K_t above reveals that we find rather different limiting characteristics for I_t depending on γ and ρ . Using the definitions of ψ , φ_2 and \tilde{b}_2 gives

$$\lim_{t \rightarrow \infty} I_t = \lim_{t \rightarrow \infty} (Y_t - C_t - \delta K_t - p_t R_t) \begin{cases} = -\frac{\underline{C}}{\tilde{b}_2}, \frac{\dot{I}_t}{I_t} \rightarrow 0 & \text{for } \rho \geq \gamma, \\ \rightarrow \infty, \frac{\dot{I}_t}{I_t} \rightarrow \frac{\gamma-\rho}{\eta} & \text{for } \rho < \gamma. \end{cases}$$

In case $\gamma \leq \rho$ we find $\lim_{t \rightarrow \infty} I_t = -\frac{\underline{C}}{\tilde{b}_2}$ which is constant.⁴ Hence, in the limit we find the generalized Hartwick rule fulfilled (Dixit et al. 1980). We note that genuine savings are asymptotically negative for $\underline{C} > 0$.⁵ In case $\gamma > \rho$ we find non-constant genuine savings which is an implication of the growing levels of consumption, output and reproducible capital. In the latter case, one can calculate the limiting rate of genuine savings as $\lim_{t \rightarrow \infty} \frac{\dot{I}_t}{I_t} = \alpha \frac{\gamma-\rho}{(\gamma+\delta)\eta} - (1-\alpha)$ which can be positive or negative depending on parameter values.

Investments into reproducible capital, \dot{K}_t , behaves as follows. In case $\gamma \leq \rho$, we find $\lim_{t \rightarrow \infty} \dot{K}_t =$

³In this case, e.g. the variable $C_t - \underline{C}$ and the parts of K_t and Y_t not involved covering subsistence consumption grow at rate $\frac{\gamma-\rho}{\eta}$ which is negative here.

⁴Obviously, for $\underline{C} = 0$ we find $\lim_{t \rightarrow \infty} I_t = 0$ as well. In case $\gamma \leq \rho$ and zero subsistence consumption we find Hartwick's (1977) rule asymptotically at an egalitarian zero consumption path.

⁵We just note the result of constant genuine savings but do not want to interpret it with respect to some sustainability criterion. If this would be intended, one would have to follow the considerations in e.g. Pezzy (2004) taking account of exogenous technical change.

0. As Y_t and K_t become asymptotically stationary in this case, we find resource extraction in the limit to decline at the same rate as the level of technology grows. Technical change exactly compensates for the necessary reduction in resource use. For $\gamma > \rho$, $\lim_{t \rightarrow \infty} \frac{K_t}{Y_t} = \alpha \frac{\gamma - \rho}{(\gamma + \delta)\eta} > 0$. Now, technical change is that large that it leads asymptotically to positive growth in the reproducible capital stock.

Related to investments is the rate of return given in (35). From inspecting (35), one can see that the net rate of return for reproducible capital approaches the rate of technical change in the limit, i.e. $\lim_{t \rightarrow \infty} r_t = \lim_{t \rightarrow \infty} \frac{\partial Y_t}{\partial K_t} - \delta = \gamma$.

5 Implications for Resource-Rich Economies

This section uses the above findings to analyze the full adjustment path of the model economy calibrated to the situation of resource-rich low-income economies. Given that we can pin down the initial conditions for the solution of the problem, we can calibrate the model using recent World Bank data on endowments with produced and natural capital.

5.1 Calibration

Regarding households' preferences, ρ , η and \underline{C} need to be specified. The rate of time preference is a parameter that is frequently calibrated. We feel that an extensive discussion on this parameter's value is not necessary. We will chose $\rho = 0.03$ which seems to be a common choice also used in e.g. Benckroun and Withagen (2011).

There exist some contributions to the literature that calibrate the type of Stone-Geary utility function that is used in the present context. Achury et al. (2012) calibrate an intertemporal utility function identical to the present one in (1) for the US and use $\eta = \frac{1}{0.23}$ which is roughly equal to 4.3. They refer to their choice of η as a standard choice in the portfolio literature. Ogaki et al. (1996) provide estimates for $\frac{1}{\eta}$ ranging from 0.569 up to 0.646 corresponding to η decreasing from about 1.68 down to 1.55. Alavarez-Pelaez and Diaz (2005) are calibrating η in a range from 1.5 up to 2.5 in their application of Stone-Geary preferences. Ravn et al. (2006, 2008) analyze the influence of subsistence points such as subsistence consumption on the dynamics of macroeconomic development in general. Despite this, their specification for intertemporal utility is in accordance with the present situation. During calibration of their models they use a value of 2 for η . Regarding the choices for η , we follow Ravn et al. (2006, 2008) with a value of 2. This is an intermediate value that is in between what has been used in Alavarez-Pelaez and Diaz (2005) and Achury et al. (2012).

Regarding our model with a constant \underline{C} , we consider the absolute poverty used by the World Bank.⁶ As of today, the threshold for extreme absolute poverty is set at 1.90 US \$ at 2011 prices at purchasing power parity (PPP) a day available to an individual for covering basic needs (Ferreira et al. 2016). By now, this is considered to apply to low-income countries.⁷ We convert this numbers

⁶Values for subsistence consumption have also been proposed in Koulovatianos et al. (2007) and Atkeson and Ogaki (1996) which have been used also in Achury et al. (2012) and Ogaki et al. (1996). These numbers, however, reflect very specific countries which doesn't seem to be in accordance with our analysis. Additionally, investigating poverty lines in this context is interesting as they influences economic policy initiatives especially in low-income countries (see e.g. the United Nation's Sustainable Development Goal on poverty, <https://www.un.org/sustainabledevelopment/>).

⁷Additionally, the World Bank recently has introduced two additional poverty lines applying to lower- and upper- middle-income countries at 3.20 US \$ and 5.50 US \$ per day at 2011 prices and PPP. For the calculation behind these numbers see Joliffe and Prydz (2016) who furthermore provide an absolute poverty level for high-income countries at 21.70 US \$ per day at 2011 prices and PPP.

into yearly values at prices of 2014. We do so as we are using below the most recent numbers on resource endowments available for 2014. This gives a poverty line of 730.56 US\$ at PPP.⁸

The output elasticity of resource use R_t is, given the Cobb-Douglas production technology (2), equal to the share α_R of natural resource rents in GDP. Data on the share of non-renewable resource rents in GDP is available from the World Bank.⁹

	resource rents' share in GDP, α_R		labor income share in GDP, α_L	
	number countries	2010-2016	number countries	2014
Low-income	34	13.15	15	51.30
Lower-middle income	47	5.92	26	52.87
Upper-middle income	56	6.29	37	47.94
High-income	79	1.90	55	52.79
World	216	3.38	133	51.29

Table 1: Resource rents and labor income share in GDP in %

Note: Averages of resource rents are reported in percentages of GDP over indicated period of time. Data source <https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS>. Labor income share from the Penn World Tables 9.0 (variable labsh, <https://www.rug.nl/ggdc/productivity/pwt/>) for 2014. Country classification in accordance with the World Bank's classification scheme available at <https://datahelpdesk.worldbank.org/knowledgebase/articles/906519>.

Table 1 provides a summary of the data for different groups of countries classified according to the country's level of income. It is clearly visible that the resource dependence increases as income decreases. Resources seem to be most important for the low-income countries. We therefore focus on this particular group in the following.

As we will see below, the labor income share in GDP will be necessary as well for our calibration. Numbers for the labor income share in GDP in 2014 were taken from the Penn World Tables 9.0. For the labor share we cannot observe a clear pattern and observe values on average a little bit above 0.5 with only moderate variation.¹⁰ We therefore calibrate the labor income share α_L at 0.51, i.e. the world's average value found in Table 1.

In the course of calibration, reasonable numbers for the initial stocks of natural resources and reproducible capital have to be found. The World Bank (2018) provides estimates for stocks of produced, natural and human capital up to 2014 in US \$. This is part of a quite comprehensive cross country data base on what the World Bank terms "The Wealth of Nations". Although it is clear that such a data base provides estimates only, the data are the best available and can be of use for the present purpose Table 2 gives a summary of the data for 2014 in per capita terms. Values in int. \$ at 2014 prices were calculated by the implicit PPP exchange rate obtained from GDP data in US \$ at current prices and at PPP and current prices.¹¹

⁸Price changes are taken account by using the implicit GDP deflator obtained by dividing the time series for GDP at PPP valued at constant and current prices for low-income countries available at <https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.KD> and <https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD>. This results in a growth in prices of 5.34% between 2011 and 2014.

⁹Data are available from the World Bank Data Base at <https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS>. For the details on how the numbers are derived see World Bank (2011). Natural resources rents are the sum of oil, natural gas, coal (hard and soft), mineral, and forest rents.

¹⁰The labor shares reported in Table 1 are low compared with e.g. the traditional $\frac{2}{3}$ that is frequently used. See e.g. the discussion in Karabarbounis and Neiman (2014) on the recently decreasing development of the labor income share.

¹¹Data are available from the World Bank at <https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD> (US \$) and <https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD> (PPP). For 2014, this implies an exchange rate of 2.41656 int. \$ at PPP per US \$.

	2014 in US\$	2014 in int. \$ at PPP
produced capital	1,967	4,753
human capital	5,564	13,446
natural capital (incl. land)	6,421	15,517
natural capital (excl. land)	1,236	2,987
net foreign assets	-322	-778

Table 2: Capital/resource stock estimates 2014 per capita for low-income countries

Note: World Bank (2018, Appendix B) estimates for stocks of different types of capital per capita in 2014 US \$. High-income values are averaged values (weighted by population) for OECD and non-OECD high-income countries reported in World Bank (2018, p. 233). Produced capital: machinery, equipment, structures, urban land; natural capital (incl. land): energy resources (oil, natural gas, hard coal, lignite), mineral resources (bauxite, copper, gold, iron, lead, nickel, phosphate, silver, tin, zinc), timber resources, nontimber forest resources, crop land, pasture land, protected areas. natural capital (excl. land): natural capital (incl. land) less of crop land, pasture land, protected areas. Human capital estimated from expenditures on education.

For calibration, we use data on the stocks of human and produced capital along with the stock of natural resources excluding land. The latter was chosen as these type of resources correspond with the resources included in the World Bank data on resource rents' share in GDP. Ideally, we would like to exclude forest related resources as well as they are renewable. Unfortunately, resource rents are not published separately for this type of resource.

The data published in World Bank (2018) on produced capital originate largely from the Penn World Tables (PWT). Produced capital is estimated there by employing the perpetual inventory method applying country and capital good specific rates of depreciation. These rates vary between 3 and 8% per annum. We calibrate our model using $\delta = 0.05$ as an intermediate value in accordance with the PWT.¹²

From here on, \tilde{s} denotes the calibrated counterpart (measured in int. \$ at PPP) of the model's real variable s . For the initial resource stock \tilde{S}_0 the number on natural capital (excl. land) in Table 2 has been used. The model doesn't differentiate between physical and human capital, i.e. the only other input factor besides the resource is the stock of reproducible man-made capital with initial value \tilde{K}_0 . As pointed out in the beginning, this stock might very well be interpreted to contain human capital as well as physical or other produced capital.

In order to calibrate the initial value for \tilde{K}_0 , we explicitly take account of both produced and human capital in Table 2. Define

$$\tilde{K}_0 = \tilde{K}_{p,0}^{\alpha_1} \tilde{K}_{h,0}^{1-\alpha_1}, \quad (36)$$

with $0 \leq \alpha_1 \leq 1$ and denote by $\tilde{K}_{p,0}$ ($\tilde{K}_{h,0}$) produced (human) capital in the sense of Table 2. $\tilde{K}_{p,0}$ is produced capital plus net foreign assets as we are interested in the implications of the countries own resources. We set $(1 - \alpha_1)\alpha = \alpha_L$ where α_L is the labor income share in GDP in the sense of Table 1. The capital income share in GDP, α_K is given by the residual $1 - \alpha_L - \alpha_R$ with α_R the resource rents' share in GDP in the sense of Table 1. Hence, for calibrating α_1 , we can use $\alpha_1 = 1 - \frac{\alpha_L}{\alpha}$. Proceeding this way, we find the values for the output elasticities given in Table 3.

By using these output elasticities we arrive at the initial values for the reproducible capital stock \tilde{K}_0 of 8,131 int. \$ at PPP.

Before calculating the calibrated adjustment path implied by the model's dynamics, we have to make an assumption about the calibrated scenario. We chose to calibrate the model to reflect the situation prevailing during 2014. We therefore chose \tilde{Y}_0 to reflect 2014 gross national income (GNI)

¹²PWT country specific depreciation rates are available at <http://febpwt.webhosting.rug.nl/Home>; variable code for depreciation rates: delta.

	$1 - \alpha$	α	$\alpha_1 = 1 - \frac{\alpha_L}{\alpha}$	$1 - \alpha_1$
low-income	0.1315	0.8685	0.4128	0.5872
lower-middle-income	0.0592	0.9408	0.4579	0.5421
upper-middle-income	0.0629	0.9371	0.4558	0.5442
high-income	0.0190	0.9810	0.4801	0.5199

Table 3: Calibration values output elasticities

in int. \$ at PPP. GNI instead of GDP was chosen as the model considers a closed economy and we would like to match final output predicted by the model with what the real world economy is able to produce using its own resources.¹³ Consequently, we took account of net foreign assets in the stock of produced capital and focused on the countries' own produced capital stock. GNI per capita at current prices stood at 1,027 US \$ and, therefore, at 2,483 int. \$ at PPP.¹⁴

The only parameter left to be calibrated is the rate of technical change γ . We consider the cases $\gamma \in \{0.01, 0.02, 0.03, 0.04\}$. As we assume $\rho = 0.03$, the first three scenarios create a long-run growth rate of 0 for the economy, while the last scenario implies positive long-run growth. γ at 0.01 or 0.02 will be referred to as subsistence scenarios as consumption approaches \underline{C} in the long-run. Table 4 summarizes the calibrated scenario and the values for the model's variables.

var.	value	param.	value	param.	value
\tilde{K}_0	8,131	α	0.8685	δ	0.05
\tilde{Y}_0	2,483	ρ	0.03	γ	$\in \left\{ \begin{array}{l} 0.01, 0.02, \\ 0.03, 0.04 \end{array} \right\}$
\tilde{S}_0	2,986	η	2		

Table 4: Calibration values

Note: Calibration values as explained in the main text. Values for $\gamma < \rho$ reflect the subsistence scenarios. $\gamma > \rho$ is the growth scenario with positive long-run growth. All values corresponding to nominal variables are measured in int. \$ at PPP in 2014.

5.2 Calibration Results

Given the values discussed in the previous section, we are ready to solve the model and trace out its dynamics. We start with the scenario $\gamma = 0.01$ which we might term the lowest growth scenario. First, we need to find ζ_0 . This is straightforward as (13) and (33) imply $Y_0 = \frac{K_0}{(1-\zeta_0)\varphi_2}$ and therefore $\zeta_0 = 1 - \frac{\gamma + \delta}{\alpha} \frac{K_0}{Y_0}$. Given ζ_0 , we use (25) and (26) to solve for $\frac{\mu_0}{A_0}$ and (12) at $t = 0$ for λ_0 . Finally, μ_0 is identified by searching for the value for A_0 that implies \tilde{S}_0 being equal to the value in Table 4.

For larger values of γ , we could repeat the above steps. This would imply a different starting level of technology A_0 for each case considered. We believe that this would make scenarios less comparable. Therefore, we chose to fix \tilde{A}_0 at the level implied by $\gamma = 0.01$ and instead look for the starting value S_0 that would be required to match initial production with higher rates of technical change. Of course, these values are smaller than the number in Table 4. This exercise clearly demonstrates how technical change and initial resource stocks can be substituted against each other.

¹³See Asheim and Buchholz (2004) for a discussion of Net National Income in welfare measurement and its relation to Hartwick's investment rule and the DHSS model. As we consider the DHH model taking account of capital depreciation, we consider GNI.

¹⁴GNI is taken from the World Bank available at <https://data.worldbank.org/indicator/NY.GNP.ATLS.CD>, low-income countries' population is taken from World Bank (2018, p. 233).

Each of the scenarios needs to be viable, i.e. $\underline{\zeta} < \bar{\zeta}$ has to hold. We therefore compute $\underline{\zeta}$ and $\bar{\zeta}$ for all cases and find that a solution exists. Table 5 gives a summary of the computational calibration results.

	$\gamma = 0.01$	$\gamma = 0.02$	$\gamma = 0.03$	$\gamma = 0.04$
ζ_0	0.7715	0.7335	0.6954	0.6573
$\underline{\zeta}$	0.5589	0.4732	0.3848	0.2935
$\bar{\zeta}$	0.8260	0.7980	0.7703	0.7427
\tilde{S}_0	2,986	2,904	2,837	2,778
\tilde{A}_0	0.005243			

Table 5: Calibration results

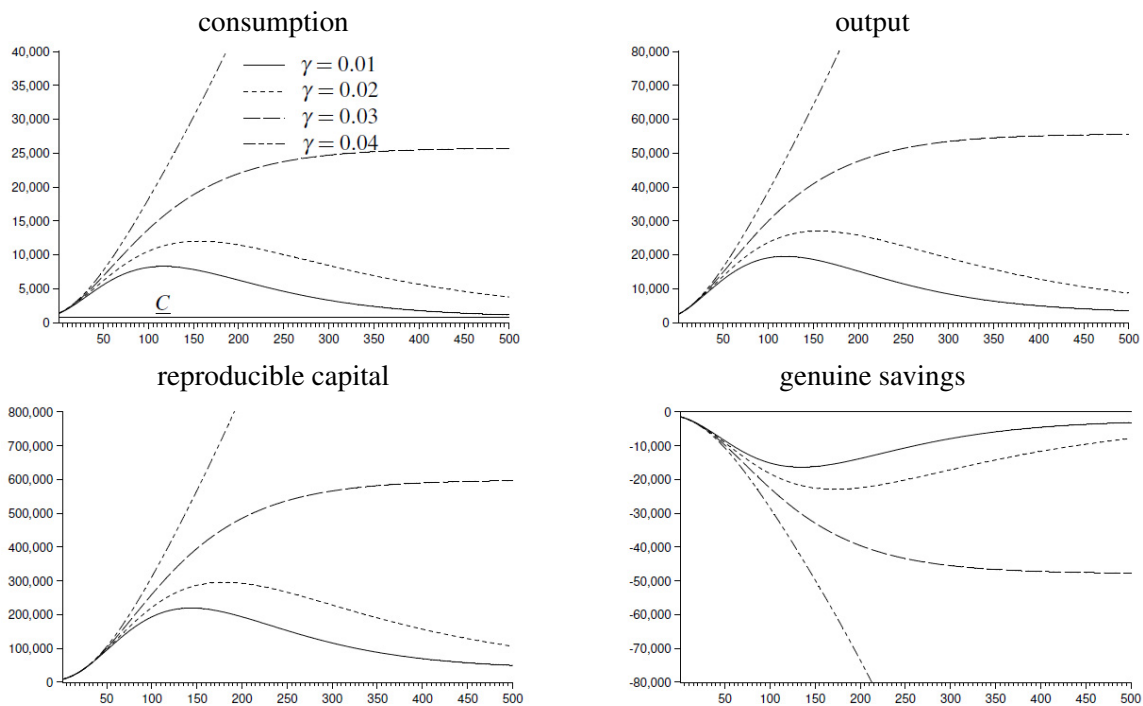
Figure 1 traces out the dynamics of consumption, output, the reproducible capital stock and genuine savings. We find that the endowment with natural and produced capital together with the implied level of technology allow consumption to go quite beyond the subsistence level during adjustment.

In the subsistence scenarios ($\gamma = 0.01$ and 0.02), annual consumption peaks at 8,271 and 11,994 2014 int. \$ at PPP respectively. Output realizes its peak at 19,509 and 26,978 2014 int. \$ at PPP. For the scenarios $\gamma = 0.03$ and 0.04 , both, consumption and output grow monotonically. In the first case, consumption and output converge to 25,768 and 55,714 2014 int. \$ at PPP. In the latter case, both quantities grow without bounds. The stock of reproducible capital shares qualitatively the pattern of consumption and output in its development.

Finally, we see that genuine savings are negative throughout. After correcting for the part in production due to resource depletion, investments into reproducible capital do not cover depreciation. This is only affordable in the long-run because technical change is substituting for these missing investments. For the cases $\gamma \leq \rho$, we find genuine savings to converge to a negative constant. For $\gamma > \rho$, genuine savings are negative as well but grow without bound.

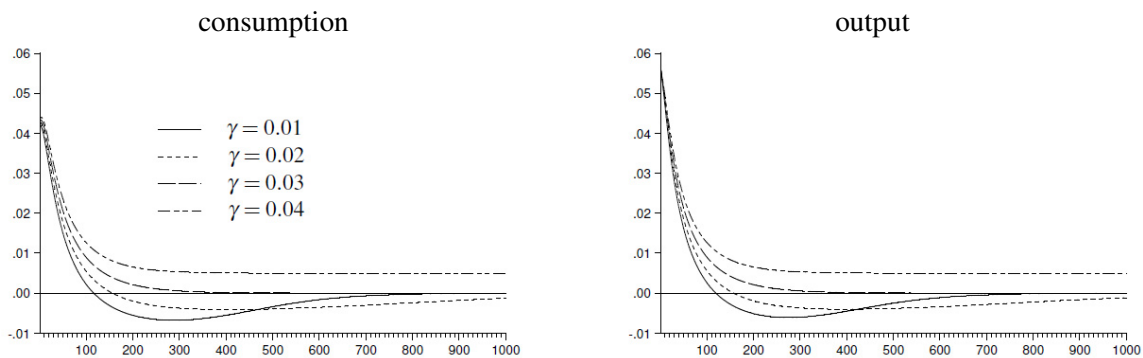
Figure 1 traces out the growth rates of consumption and output. For $\gamma < \rho$ we find growth rate to follow non monotonic paths. Initially, higher rates of technical change result in higher growth rates in consumption and output. However, this behavior changes later on as a higher γ translates into a higher stock of reproducible capital driving down the rate of return which is in turn decreasing growth.

The stock of natural capital and resource depletion behaves monotonically as can be seen from Figure 3. Due to the chosen calibration of the four scenarios, the initial stock of natural capital is highest and identical to World Bank estimates for $\gamma = 0.01$ only. As γ increases, the initial stock \tilde{S}_0 declines as less resources are needed to match initial output \tilde{Y}_0 given the initial stock of reproducible capital \tilde{K}_0 . Resource depletion declines monotonically as expected.



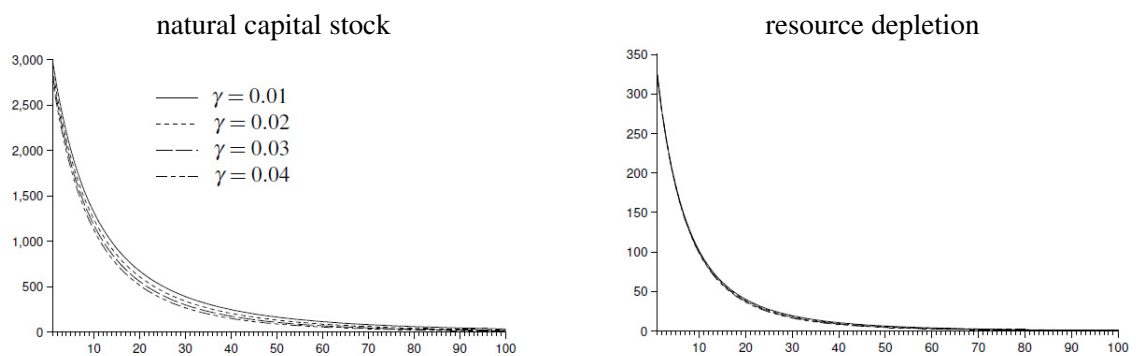
Note: Calibrated dynamics for \tilde{C}_t , \tilde{Y}_t , \tilde{K}_t and genuine savings $(1 - \alpha)\tilde{Y}_t - \tilde{C}_t - \delta\tilde{K}_t$ for different rates of technical change γ ; all quantities in 2014 int. \$ at PPP.

Figure 1: Dynamics level variables



Note: Growth rates of consumption \tilde{C}_t and final output \tilde{Y}_t across the calibrated scenarios.

Figure 2: Growth dynamics



Note: Stock of natural resources \tilde{S}_t and resource depletion \tilde{R}_t in 2014 int. \$ at PPP.

Figure 3: Resource dynamics

6 Discussion and Conclusion

It is interesting to discuss why the somewhat complex set-up of this version on the DHSS/DHH model allows for a closed form solution in terms of a special function. The key insight that can be gained from the results is that the assumption of constant returns to reproducible and natural capital leads to a particular production structure.

The reduced form for output in (34) is basically of the AK-type, i.e. $Y_t = \frac{K_t}{\phi_2} (1 - \zeta_0 x_t)^{-1}$. Capital productivity is, of course, not constant but is given by a bounded function of time via $x_t = e^{-\psi t}$ that is traceable. This implies also a quite simple closed for the rate of interest.

The issue of subsistence consumption is naturally tied to considerations involving Hartwick's (1977) investment rule. In the present model this could be addressed by setting depreciation and technical change equal to zero, i.e. $\delta = \gamma = 0$. The present model incorporates this special case although the formal representation changes drastically. This can be seen by looking at the development of λ_t in (13) which is repeated for convenience

$$\lambda_t = e^{\delta t} \left[\lambda_0^{\frac{\alpha-1}{\alpha}} + (1-\alpha)^{\frac{1-\alpha}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{\gamma+\delta} \left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1 \right) \right]^{\frac{\alpha}{\alpha-1}}.$$

The term affected by $\delta = \gamma = 0$ is

$$\frac{\alpha}{\gamma+\delta} \left(e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1 \right) = (1-\alpha) \frac{e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1}{\frac{1-\alpha}{\alpha}(\gamma+\delta)},$$

where $\frac{e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1}{\frac{1-\alpha}{\alpha}(\gamma+\delta)}$ is the Manly (1976) exponential transformation of t . For $\delta + \gamma \rightarrow 0$, $\frac{e^{\frac{1-\alpha}{\alpha}(\gamma+\delta)t} - 1}{\frac{1-\alpha}{\alpha}(\gamma+\delta)} \rightarrow t$. This changes the formal representation of the model dramatically but allows for a closed form solution of the dynamics. This case is treated with in Antony and Klarl (2018) where it is shown that the economy asymptotically fulfills Hartwick's investment rule.

In this paper, we are able to provide a closed form approach to a well known model in resource economics. The approach uses special functions as a series of more recent contributions applies to solve dynamic problems. The advantage over the usual linearization around the steady state is that we can pin down the initial conditions for the optimal path.

Availability of the initial conditions for the co-state variables and their relation to initial endowments allows us to calibrate the model on a scenario reflecting recent estimates for endowments with produced and natural capital by the World Bank. Given these numbers, we find that low-income but resource-rich countries on average can solve the dynamic problem implied by the poverty line reflecting subsistence consumption needs.

Appendix

A: Derivation of λ_t : From (8) we know that

$$\frac{K_t}{A_t R_t} = \left(\frac{\lambda_t (1 - \alpha)}{\mu_t} \right)^{-\frac{1}{\alpha}} A_t^{-\frac{1}{\alpha}}.$$

Inserting this into (7) together with $\mu_t = \mu_0$ and introducing $m_t = \lambda_t^{\frac{\alpha-1}{\alpha}}$ gives

$$\dot{m}_t = (1 - \alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} e^{\frac{1-\alpha}{\alpha} \gamma t} - \frac{1 - \alpha}{\alpha} \delta m_t.$$

The solution to this differential equation can be found quite straightforward as

$$\begin{aligned} m_t &= m_0 e^{-\frac{1-\alpha}{\alpha} \delta t} + \int_0^t (1 - \alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} e^{\frac{1-\alpha}{\alpha} \gamma z} e^{-\int_z^t \frac{1-\alpha}{\alpha} \delta ds} dz \\ &= e^{-\frac{1-\alpha}{\alpha} \delta t} \left[m_0 + (1 - \alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \int_0^t e^{\frac{1-\alpha}{\alpha} (\gamma + \delta) z} dz \right] \\ &= e^{-\frac{1-\alpha}{\alpha} \delta t} \left[m_0 + (1 - \alpha)^{\frac{1}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{1 - \alpha} \frac{1}{\gamma + \delta} \left(e^{\frac{1-\alpha}{\alpha} (\gamma + \delta) t} - 1 \right) \right]. \end{aligned}$$

This delivers

$$\lambda_t = e^{\delta t} \left[\lambda_0^{\frac{\alpha-1}{\alpha}} + (1 - \alpha)^{\frac{1-\alpha}{\alpha}} \mu_0^{\frac{\alpha-1}{\alpha}} \frac{\alpha}{\gamma + \delta} \left(e^{\frac{1-\alpha}{\alpha} (\gamma + \delta) t} - 1 \right) \right]^{\frac{\alpha}{\alpha-1}}.$$

It follows that

$$\begin{aligned} \frac{\lambda_t (1 - \alpha)}{\mu_0} A_t &= \left[\left(\frac{\lambda_0 (1 - \alpha)}{\mu_0} \right)^{\frac{\alpha-1}{\alpha}} e^{-\frac{1-\alpha}{\alpha} (\gamma + \delta) t} + \frac{\alpha}{\gamma + \delta} \left(1 - e^{-\frac{1-\alpha}{\alpha} (\gamma + \delta) t} \right) \right]^{\frac{\alpha}{\alpha-1}} \\ &= \varphi_2^{\frac{\alpha}{\alpha-1}} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{\alpha}{\alpha-1}}, \end{aligned} \tag{37}$$

with

$$\begin{aligned} \varphi_1 &= \left(\frac{\lambda_0 (1 - \alpha)}{\mu_0} \right)^{\frac{\alpha-1}{\alpha}}, \\ \varphi_2 &= \frac{\alpha}{\gamma + \delta}, \\ \psi &= \frac{1 - \alpha}{\alpha} (\gamma + \delta) \end{aligned}$$

B: The capital stock K_t :

To arrive at the solution for K_t , we need first to find $-\int_z^t f(s) ds$ with $f(s) = -\left(\frac{K_s}{A_s R_s} \right)^{\alpha-1} + \delta$.

$$-\int_z^t f(s) ds = \int_z^t \left(\frac{K_s}{A_s R_s} \right)^{\alpha-1} - \delta ds = \int_z^t \left(\frac{\lambda_s (1 - \alpha)}{\mu_0} A_s \right)^{\frac{1-\alpha}{\alpha}} - \delta ds.$$

Using (37) gives

$$\begin{aligned}
-\int_z^t f(s)ds &= \int_z^t \left(\frac{\lambda_s(1-\alpha)}{\mu_0} A_s \right)^{\frac{1-\alpha}{\alpha}} - \delta ds \\
&= -\delta(t-z) + \int_z^t [\varphi_1 e^{-\psi s} + \varphi_2 (1 - e^{-\psi s})]^{-1} ds \\
&= -\delta(t-z) + \frac{1}{1-\alpha} \ln \left[\frac{\varphi_1 + \varphi_2 (e^{\psi t} - 1)}{\varphi_1 + \varphi_2 (e^{\psi z} - 1)} \right]
\end{aligned}$$

with

$$\begin{aligned}
\varphi_1 &= \left(\frac{\lambda_0(1-\alpha)A_0}{\mu_0} \right)^{\frac{\alpha-1}{\alpha}}, \\
\varphi_2 &= \frac{\alpha}{\gamma + \delta}, \\
\psi &= \frac{1-\alpha}{\alpha} (\gamma + \delta).
\end{aligned}$$

K_t is consequently given by

$$\begin{aligned}
K_t &= K_0 e^{-\int_0^t f(s)ds} - \int_0^t (C_z - \underline{C}) e^{-\int_z^t f(s)ds} dz - \int_0^t \underline{C} e^{-\int_z^t f(s)ds} dz \\
&= K_0 e^{-\delta t} \left[\frac{\varphi_1 + \varphi_2 (e^{\psi t} - 1)}{\varphi_1} \right]^{\frac{1}{1-\alpha}} \\
&\quad - \int_0^t \lambda_z^{-\frac{1}{\eta}} e^{-\frac{\rho}{\eta} z} e^{-\delta(t-z)} \left[\frac{\varphi_1 + \varphi_2 (e^{\psi t} - 1)}{\varphi_1 + \varphi_2 (e^{\psi z} - 1)} \right]^{\frac{1}{1-\alpha}} dz \\
&\quad - \int_0^t \underline{C} e^{-\delta(t-z)} \left[\frac{\varphi_1 + \varphi_2 (e^{\psi t} - 1)}{\varphi_1 + \varphi_2 (e^{\psi z} - 1)} \right]^{\frac{1}{1-\alpha}} dz \\
&= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \\
&\quad - e^{-\delta t} \varphi_2^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t \varphi_2^{-\frac{1}{1-\alpha}} \lambda_z^{-\frac{1}{\eta}} e^{(-\frac{\rho}{\eta} + \delta - \frac{\psi}{1-\alpha})z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz \\
&\quad - \underline{C} \varphi_2^{\frac{1}{1-\alpha}} e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t \varphi_2^{-\frac{1}{1-\alpha}} e^{(\delta - \frac{\psi}{1-\alpha})z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz
\end{aligned}$$

$$\begin{aligned}
&= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \\
&\quad - e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t \left[\frac{\mu_0}{(1-\alpha)A_0} \varphi_2^{\frac{\alpha}{\alpha-1}} e^{(\delta - \frac{\alpha}{1-\alpha}\psi)z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{\frac{\alpha}{\alpha-1}} \right]^{-\frac{1}{\eta}} e^{(-\frac{\rho}{\eta} + \delta - \frac{\psi}{1-\alpha})z} \\
&\quad \times \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz \\
&\quad - \underline{C} e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t e^{(\delta - \frac{\psi}{1-\alpha})z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz \\
&= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \\
&\quad - e^{-\delta t} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t e^{(\frac{(\eta-1)\delta-\rho}{\eta} + \frac{\alpha-\eta}{1-\alpha} \frac{\psi}{\eta})z} \\
&\quad \times \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{1}{\eta}} dz \\
&\quad - \underline{C} e^{-\delta t} e^{\frac{\psi}{1-\alpha} t} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t} \right)^{\frac{1}{1-\alpha}} \int_0^t e^{(\delta - \frac{\psi}{1-\alpha})z} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi z} \right)^{-\frac{1}{1-\alpha}} dz
\end{aligned}$$

Introducing $x_t = e^{-\psi t}$ and $\zeta = \frac{\varphi_2 - \varphi_1}{\varphi_2}$ gives after simplification

$$\begin{aligned}
K_t &= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \tag{38} \\
&\quad - e^{-\delta t} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \int_0^t x_z^{-\frac{1}{\psi} \left(\frac{(\eta-1)\delta-\rho}{\eta} + \frac{\alpha-\eta}{1-\alpha} \frac{\psi}{\eta} \right)} (1 - \zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dz \\
&\quad - \underline{C} e^{-\delta t} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \int_0^t x_z^{-\frac{1}{\psi} (\delta - \frac{\psi}{1-\alpha})} (1 - \zeta x_z)^{-\frac{1}{1-\alpha}} dz
\end{aligned}$$

Using the variable x_z , we note that $-dx_z = \psi e^{-\psi z} dz = \psi x_z dz$ and consequently, $dz = -\frac{1}{\psi} x_z^{-1} dx_z$. The region of integration changes from $[0, t]$ to $[x_t, 1]$ with $x_t \leq 1$ if we integrate over $-x_z$ instead of z . Obviously, $x_z = 1$ for $z = 0$ and $\lim_{z \rightarrow \infty} x_z = 0$.

$$\begin{aligned}
K_t &= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \tag{39} \\
&\quad - e^{-\delta t} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \int_{x_t}^1 x_z^{-\frac{1}{\psi} \left(\frac{(\eta-1)\delta-\rho}{\eta} + \frac{\alpha-\eta}{1-\alpha} \frac{\psi}{\eta} \right) - 1} (1 - \zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z \\
&\quad - \underline{C} e^{-\delta t} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \int_{x_t}^1 x_z^{-\frac{1}{\psi} (\delta - \frac{\psi}{1-\alpha}) - 1} (1 - \zeta x_z)^{-\frac{1}{1-\alpha}} dx_z.
\end{aligned}$$

Next, we turn to the integrals in (39)

$$\begin{aligned}
& \int_{x_t}^1 x_z^{-\frac{1}{\psi} \left(\frac{(\eta-1)\delta-\rho}{\eta} + \frac{\alpha-\eta}{1-\alpha} \frac{\psi}{\eta} \right) - 1} (1 - \zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z = \\
& = \int_0^1 x_z^{-\frac{1}{\psi} \left(\frac{(\eta-1)\delta-\rho}{\eta} + \frac{\alpha-\eta}{1-\alpha} \frac{\psi}{\eta} \right) - 1} (1 - \zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z - \int_0^{x_t} x_z^{-\frac{1}{\psi} \left(\frac{(\eta-1)\delta-\rho}{\eta} + \frac{\alpha-\eta}{1-\alpha} \frac{\psi}{\eta} \right) - 1} (1 - \zeta x_z)^{\frac{\alpha-\eta}{\eta(1-\alpha)}} dx_z \\
& = \int_0^1 x_z^{\tilde{b}_1 - 1} (1 - \zeta x_z)^{-\tilde{a}_1} dx_z - \int_0^{x_t} x_z^{\tilde{b}_1 - 1} (1 - \zeta x_z)^{-\tilde{a}_1} dx_z \tag{40}
\end{aligned}$$

with

$$\begin{aligned}
\tilde{a}_1 &= \frac{\eta - \alpha}{\eta(1 - \alpha)} \\
\tilde{b}_1 &= -\frac{1}{\psi} \left(\frac{(\eta - 1)\delta - \rho}{\eta} + \frac{\alpha - \eta}{1 - \alpha} \frac{\psi}{\eta} \right) = 1 + \frac{\alpha[(\eta - 1)\gamma + \rho]}{(1 - \alpha)(\delta + \gamma)\eta}.
\end{aligned}$$

The integral representation of the Gaussian hypergeometric function is given by

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt,$$

for $R(c) > R(b) > 0$ where $R(\cdot)$ denotes the real part of the argument and $\Gamma(\cdot)$ the Gamma function (see Abramowitz and Stegun, 1972, 15.3.1) where the discussion in the main text applies. The first integral in (40) therefore equals in this case

$$\begin{aligned}
\int_0^1 x_z^{\tilde{b}_1 - 1} (1 - \zeta x_z)^{-\tilde{a}_1} dx_z &= \frac{\Gamma(\tilde{b}_1)\Gamma(1)}{\Gamma(\tilde{b}_1 + 1)} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) \\
&= \frac{\Gamma(\tilde{b}_1)\Gamma(1)}{\tilde{b}_1\Gamma(\tilde{b}_1)} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) \\
&= \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta).
\end{aligned}$$

The second integral in (40) can also be expressed in terms of the Gaussian hypergeometric function. We define $\tilde{x}_z = \frac{x_z}{x_t}$ which implies $dx_z = x_t d\tilde{x}_z$ and rewrite the integral as

$$\begin{aligned}
\int_0^{x_t} (x_t \tilde{x}_z)^{\tilde{b}_1 - 1} (1 - \zeta x_t \tilde{x}_z)^{-\tilde{a}_1} dx_z &= \int_0^1 (x_t \tilde{x}_z)^{\tilde{b}_1 - 1} (1 - \zeta x_t \tilde{x}_z)^{-\tilde{a}_1} x_t d\tilde{x}_z \\
&= x_t^{\tilde{b}_1} \int_0^1 \tilde{x}_z^{\tilde{b}_1 - 1} (1 - \zeta x_t \tilde{x}_z)^{-\tilde{a}_1} d\tilde{x}_z \\
&= x_t^{\tilde{b}_1} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t).
\end{aligned}$$

The second integral in (39) can also be formulated using the Gaussian hypergeometric function

$$\begin{aligned}
& \int_{x_t}^1 x z^{-\frac{1}{\psi}(\delta - \frac{\psi}{1-\alpha})-1} (1 - \zeta x_z)^{-\frac{1}{1-\alpha}} dx_z = \\
& = \int_0^1 x_z^{-\frac{1}{\psi}(\delta - \frac{\psi}{1-\alpha})-1} (1 - \zeta x_z)^{-\frac{1}{1-\alpha}} dx_z - \int_0^{x_t} x_z^{-\frac{1}{\psi}(\delta - \frac{\psi}{1-\alpha})-1} (1 - \zeta x_z)^{-\frac{1}{1-\alpha}} dx_z \\
& = \frac{1}{\tilde{b}_2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right], \\
& \text{with} \\
& \tilde{a}_2 = \frac{1}{1-\alpha}, \\
& \tilde{b}_2 = -\frac{1}{\psi} \left(\delta - \frac{\psi}{1-\alpha} \right) = \frac{(1-\alpha)\delta + \gamma}{(1-\alpha)(\gamma + \delta)} > 0,
\end{aligned} \tag{41}$$

where it is again required that $\tilde{b}_2 > 0$ which is here fulfilled in any case.

Using these results, the stock of capital K_t develops according to

$$\begin{aligned}
K_t &= K_0 e^{-\delta t} \left(\frac{\varphi_2}{\varphi_1} \right)^{\frac{1}{1-\alpha}} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \\
& - e^{-\delta t} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \frac{1}{\tilde{b}_1} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\
& - \underline{C} e^{-\delta t} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \frac{1}{\tilde{b}_2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right]
\end{aligned} \tag{42}$$

C: Resource use R_t : Using (8), (37), (38) and (42) gives

$$\begin{aligned}
R_t &= A_t^{-1} \left(\frac{\lambda_t(1-\alpha)}{\mu_0} A_t \right)^{\frac{1}{\alpha}} K_t \\
& = \varphi_2^{\frac{1}{\alpha-1}} (1 - \zeta x_t)^{\frac{1}{\alpha-1}} A_0^{-1} e^{-\gamma t} K_t \\
& = \frac{K_0}{A_0} \varphi_1^{-\frac{1}{1-\alpha}} x_t^{-1} \\
& - \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-1} \frac{1}{\tilde{b}_1} \frac{1}{A_0} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\
& - \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi} x_t^{-1} \frac{1}{\tilde{b}_2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right]
\end{aligned}$$

D: Resource stock S_t : The solution to (4) is given by

$$S_t = S_0 - \int_0^t R_z dz + \int_0^t v dz,$$

with

$$\begin{aligned}
\int_0^t R_z dz &= \int_0^t \frac{K_0}{A_0} \varphi_1^{-\frac{1}{1-\alpha}} x_z^{-1} dz \\
&- \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} \frac{1}{A_0} \int_0^t x_z^{-1} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) - x_z^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_z) \right] dz \\
&- \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi} \frac{1}{\tilde{b}_2} \int_0^t x_z^{-1} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) - x_z^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_z) \right] dz
\end{aligned}$$

Using $-d\zeta x_z = \zeta \psi x_z dz$ and obeying that we have to integrate from ζx_t to ζ gives

$$\begin{aligned}
\int_0^t R_z dz &= \frac{K_0}{A_0} \varphi_1^{-\frac{1}{1-\alpha}} \frac{\zeta}{\psi} \int_{\zeta x_t}^{\zeta} (\zeta x_z)^{-2} d\zeta x_z \\
&- \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} \frac{\zeta}{\psi^2} \frac{1}{\tilde{b}_1} \frac{1}{A_0} \int_{\zeta x_t}^{\zeta} (\zeta x_z)^{-2} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) - x_z^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_z) \right] d\zeta x_z \\
&- \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{\zeta}{\psi^2} \frac{1}{\tilde{b}_2} \int_{\zeta x_t}^{\zeta} (\zeta x_z)^{-2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) - x_z^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_z) \right] d\zeta x_z \\
&= -\frac{K_0}{A_0} \varphi_1^{-\frac{1}{1-\alpha}} \frac{1}{\psi} [1-x_t^{-1}] \\
&+ \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) [1-x_t^{-1}] \\
&+ \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0} \right)^{-\frac{1}{\eta}} \frac{\zeta^{1-\tilde{b}_1}}{\psi^2} \frac{1}{\tilde{b}_1} \frac{1}{A_0} \int_{\zeta x_t}^{\zeta} (\zeta x_z)^{\tilde{b}_1-2} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_z) d\zeta x_z \\
&+ \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) [1-x_t^{-1}] \\
&+ \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{\zeta^{1-\tilde{b}_2}}{\psi^2} \frac{1}{\tilde{b}_2} \int_{\zeta x_t}^{\zeta} (\zeta x_z)^{\tilde{b}_2-2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_z) d\zeta x_z.
\end{aligned}$$

Using

$$\int z^{b-2} {}_2F_1(a, b; c; z) dz = \frac{z^{b-1}}{b-1} {}_2F_1(a, b-1; c; z) + \text{constant}$$

gives

$$\begin{aligned}
\int_0^t R_z dz &= -\frac{K_0}{A_0} \varphi_1^{-\frac{1}{1-\alpha}} \frac{1}{\psi} [1-x_t^{-1}] \\
&+ \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) [1-x_t^{-1}] \\
&+ \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{\zeta^{1-\tilde{b}_1}}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} \left[\zeta^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta) - (\zeta x_t)^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta x_t) \right] \\
&+ \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) [1-x_t^{-1}] \\
&+ \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{\zeta^{1-\tilde{b}_2}}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} \left[\zeta^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta) - (\zeta x_t)^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta x_t) \right].
\end{aligned} \tag{43}$$

This gives S_t as

$$S_t = S_0 - \int_0^t R_z dz + vt,$$

where $\int_0^t R_z dz$ is given by (43) and the above parameter restrictions need to be fulfilled.

E: Transversality condition for K_t :

$$\lim_{t \rightarrow \infty} \lambda_t K_t = 0 \quad (44)$$

With (37) we find

$$\begin{aligned} \lambda_t K_t &= e^{-\gamma t} \frac{\mu_0}{(1-\alpha)A_0} \varphi_2^{\frac{\alpha}{\alpha-1}} \left(1 - \frac{\varphi_2 - \varphi_1}{\varphi_2} e^{-\psi t}\right)^{\frac{\alpha}{\alpha-1}} K_t = e^{-\gamma t} \frac{\mu_0}{(1-\alpha)A_0} \varphi_2^{\frac{\alpha}{\alpha-1}} (1 - \zeta x_t)^{-\frac{\alpha}{1-\alpha}} K_t \\ &= K_0 e^{-(\gamma+\delta)t} \varphi_2 \varphi_1^{-\frac{1}{1-\alpha}} \frac{\mu_0}{(1-\alpha)A_0} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t) \\ &\quad - e^{-(\gamma+\delta)t} \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0}\right)^{1-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t) \frac{1}{\tilde{b}_1} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\ &\quad - \underline{C} e^{-(\gamma+\delta)t} \varphi_2^{\frac{\alpha}{\alpha-1}} \frac{\mu_0}{(1-\alpha)A_0} \frac{1}{\psi} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t) \frac{1}{\tilde{b}_2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right] \\ &= K_0 \varphi_2 \varphi_1^{-\frac{1}{1-\alpha}} \frac{\mu_0}{(1-\alpha)A_0} x_t^{-1} (1 - \zeta x_t) \\ &\quad - \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}} \left(\frac{\mu_0}{(1-\alpha)A_0}\right)^{1-\frac{1}{\eta}} \frac{1}{\psi} x_t^{-1} (1 - \zeta x_t) \frac{1}{\tilde{b}_1} \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\ &\quad - \underline{C} \varphi_2^{\frac{\alpha}{\alpha-1}} \frac{\mu_0}{(1-\alpha)A_0} \frac{1}{\psi} x_t^{-1} (1 - \zeta x_t) \frac{1}{\tilde{b}_2} \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right] \end{aligned}$$

As $t \rightarrow \infty$ we see $x_t \rightarrow 0$, and $x_t^{-1} \rightarrow \infty$. Rewriting $\lambda_t K_t$ as $\frac{x_t \lambda_t K_t}{x_t}$ and applying L'Hospital's rule as $x_t \rightarrow 0$ requires $\lim_{x_t \rightarrow 0} \frac{\partial x_t \lambda_t K_t}{\partial x_t} = 0$. $\frac{\partial x_t \lambda_t K_t}{\partial x_t}$ computed using (45) is given by

$$\begin{aligned} \frac{\partial x_t \lambda_t K_t}{\partial x_t} &= -K_0 \varphi_2 \varphi_1^{-\frac{1}{1-\alpha}} \frac{\mu_0}{(1-\alpha)A_0} \zeta \\ &\quad + \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{1-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} \zeta \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\ &\quad - \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{1-\frac{1}{\eta}} \frac{1}{\psi} (1 - \zeta x_t) \frac{1}{\tilde{b}_1} \left[-\tilde{b}_1 x_t^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \right] \\ &\quad - \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0}\right]^{1-\frac{1}{\eta}} \frac{1}{\psi} (1 - \zeta x_t) \frac{1}{\tilde{b}_1} \left[-x_t^{\tilde{b}_1} \frac{\partial {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t)}{\partial x_t} \right] \\ &\quad + \underline{C} \varphi_2^{\frac{\alpha}{\alpha-1}} \frac{\mu_0}{(1-\alpha)A_0} \frac{1}{\psi} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) \zeta \\ &\quad - \underline{C} \varphi_2^{\frac{\alpha}{\alpha-1}} \frac{\mu_0}{(1-\alpha)A_0} \frac{1}{\psi} \frac{1}{\tilde{b}_2} (1 - \zeta x_t) \left[-\tilde{b}_2 x_t^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \right] \\ &\quad - \underline{C} \varphi_2^{\frac{\alpha}{\alpha-1}} \frac{\mu_0}{(1-\alpha)A_0} \frac{1}{\psi} \frac{1}{\tilde{b}_2} (1 - \zeta x_t) \left[-x_t^{\tilde{b}_2} \frac{\partial {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t)}{\partial x_t} \right] \end{aligned}$$

Evaluating $\frac{\partial x_t \lambda_t K_t}{\partial x_t}$ at $x_t = 0$ gives as long as $\tilde{b}_1 - 1 > 0$ and $\tilde{b}_2 - 1 > 0$

$$\begin{aligned}
\left. \frac{\partial x_t \lambda_t K_t}{\partial x_t} \right|_{x_t=0} &= -K_0 \varphi_2 \varphi_1^{-\frac{1}{1-\alpha}} \frac{\mu_0}{(1-\alpha)A_0} \zeta \\
&+ \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{1-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) \zeta \\
&+ \underline{C} \varphi_2^{\frac{\alpha}{\alpha-1}} \frac{\mu_0}{(1-\alpha)A_0} \frac{1}{\psi} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) \zeta
\end{aligned}$$

Transversality therefore demands

$$\begin{aligned}
K_0 &= \varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}-1} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) \\
&+ \underline{C} \varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{1}{\alpha-1}} \frac{1}{\psi} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta).
\end{aligned} \tag{45}$$

If either $\tilde{b}_1 - 1 > 0$ or $\tilde{b}_2 - 1 > 0$ would not hold, the transversality condition for K_t would be violated because then $\lim_{x_t \rightarrow 0} \frac{\partial x_t \lambda_t K_t}{\partial x_t} \rightarrow \infty$.

Inserting the transversality condition (45) into (42) and using the definitions for x_t , \tilde{b}_1 and \tilde{b}_2 gives

$$\begin{aligned}
K_t &= e^{-\delta t} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \\
&+ \underline{C} e^{-\delta t} x_t^{-\frac{1}{1-\alpha}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \frac{1}{\psi} \frac{1}{\tilde{b}_2} x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) \\
&= x_t^{\frac{\alpha(\rho-\gamma)}{(1-\alpha)(\gamma+\delta)\eta}} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) \\
&+ \underline{C} (1 - \zeta x_t)^{\frac{1}{1-\alpha}} \frac{1}{\psi} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t)
\end{aligned}$$

As $t \rightarrow \infty$, $x_t \rightarrow 0$ and we observe for $\tilde{b}_1 - 1 > 0$ and $\tilde{b}_2 - 1 > 0$

$$\begin{aligned}
\lim_{x_t \rightarrow 0} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta x_t) &= \tilde{b}_1 \int_0^1 x_z^{\tilde{b}_1-1} dx_z = 1, \\
\lim_{x_t \rightarrow 0} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta x_t) &= \tilde{b}_2 \int_0^1 x_z^{\tilde{b}_2-1} dx_z = 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{t \rightarrow \infty} K_t &= \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} \lim_{x_t \rightarrow 0} x_t^{\frac{\alpha(\rho-\gamma)}{(1-\alpha)(\gamma+\delta)\eta}} \\
&+ \underline{C} \frac{1}{\psi} \frac{1}{\tilde{b}_2},
\end{aligned}$$

Depending on the parameter values, we find

$$\lim_{t \rightarrow \infty} K_t \begin{cases} = \underline{C} \frac{1}{\psi} \frac{1}{\tilde{b}_2} & \text{for } \rho > \gamma, \\ = \varphi_2^{\frac{\alpha}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi} \frac{1}{\tilde{b}_1} + \underline{C} \frac{1}{\psi} \frac{1}{\tilde{b}_2} & \text{for } \rho = \gamma, \\ \rightarrow \infty & \text{for } \rho < \gamma. \end{cases}$$

F: Transversality condition for S_t :

$$\lim_{t \rightarrow \infty} \mu_t S_t = \lim_{t \rightarrow \infty} \mu_0 S_t = \mu_0 \lim_{t \rightarrow \infty} S_t = 0$$

First, we compute resource extraction over the entire planing horizon. (43) reveals that $\int_0^\infty R_z dz$ only exists if $\tilde{b}_1 - 1 \geq 0$ and $\tilde{b}_2 - 1 \geq 0$ because $\lim_{t \rightarrow \infty} x_t = 0$. Rearranging (43) gives

$$\begin{aligned} \int_0^t R_z dz = & \left[-\frac{K_0}{A_0} \varphi_1^{-\frac{1}{1-\alpha}} \frac{1}{\psi} + \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) \right. \\ & \left. + \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) \right] [1 - x_t^{-1}] \\ & + \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{\zeta^{1-\tilde{b}_1}}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} \left[\zeta^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta) - (\zeta x_t)^{\tilde{b}_1-1} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta x_t) \right] \\ & + \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{\zeta^{1-\tilde{b}_2}}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} \left[\zeta^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta) - (\zeta x_t)^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta x_t) \right], \end{aligned} \quad (46)$$

where we note that the first term in brackets is zero due to the transversality condition (45) for K_t and, hence, we arrive for $t \rightarrow \infty$ at

$$\begin{aligned} \int_0^\infty R_z dz = & \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta) \\ & + \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta). \end{aligned} \quad (47)$$

Transversality demands

$$S_0 - \int_0^\infty R_z dz + vt = 0, \quad (48)$$

which is only possible if $v = 0$ and then implies

$$\begin{aligned} S_0 = & \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta) \\ & + \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2} \frac{1}{\tilde{b}_2(\tilde{b}_2-1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta). \end{aligned} \quad (49)$$

Inserting (49) into (46) and using (45) gives

$$S_t = \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2 \tilde{b}_1(\tilde{b}_1-1)} x_t^{\tilde{b}_1-1} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta x_t) \\ + \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2 \tilde{b}_2(\tilde{b}_2-1)} x_t^{\tilde{b}_2-1} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta x_t).$$

As long as $\tilde{b}_1 - 1 > 0$ and $\tilde{b}_2 - 1 > 0$, obviously $\lim_{t \rightarrow \infty} S_t = \lim_{x_t \rightarrow 0} S_t = 0$.

G: Co-state variables: We prove that the transversality conditions (45) and (47) uniquely pin down the initial value of ζ if a solution to the problem exists.

Define

$$K_0^+ = \varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{\alpha(1-\eta)}{(1-\alpha)\eta}-1} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi \tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta), \\ \underline{K}_0 = \underline{C} \varphi_1^{\frac{1}{1-\alpha}} \varphi_2^{\frac{1}{\alpha-1}} \frac{1}{\psi \tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) = \underline{C} \frac{1}{\psi \tilde{b}_2} (1-\zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta), \\ S_0^+ = \varphi_2^{\frac{\alpha-\eta}{(1-\alpha)\eta}} \left[\frac{\mu_0}{(1-\alpha)A_0} \right]^{-\frac{1}{\eta}} \frac{1}{\psi^2 \tilde{b}_1(\tilde{b}_1-1)} \frac{1}{A_0} {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta), \\ \underline{S}_0 = \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2 \tilde{b}_2(\tilde{b}_2-1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta).$$

Equilibrium demands

$$\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0^+}{S_0^+}, \quad (50)$$

with

$$\frac{K_0^+}{S_0^+} = \psi(\tilde{b}_1-1)A_0\varphi_1^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta)} \\ = \psi(\tilde{b}_1-1)A_0\varphi_2^{\frac{1}{1-\alpha}} (1-\zeta)^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta)}, \quad (51)$$

$$\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = \frac{K_0 - \underline{C} \frac{1}{\psi \tilde{b}_2} (1-\zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta)}{S_0 - \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2 \tilde{b}_2(\tilde{b}_2-1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta)}. \quad (52)$$

We first notice that (51) and (52) demand $\zeta < 1$.

We show first that $\frac{K_0^+}{S_0^+}$ given by (51) is decreasing in ζ . Second, we show that $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ given by (52) is increasing in ζ . This implies that there can be at most one solution to (50).

Investigating $\frac{K_0^+}{S_0^+}$, we have to distinguish three cases, i.e. $\tilde{a}_1 < 0, \tilde{a}_1 = 0, \tilde{a}_1 > 0$.

Case 1: $\tilde{a}_1 < 0$: Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that $\frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta)}$ is decreasing in ζ in case $\tilde{a}_1 < 0$. It is obvious that $(1-\zeta)^{\tilde{a}_2}$ is decreasing in ζ as well because $\tilde{a}_2 = \frac{1}{1-\alpha} > 0$. Therefore, $\frac{K_0^+}{S_0^+}$ is in this case decreasing in ζ .

Case 2: $\tilde{a}_1 = 0$: This case prevails if it happens to be that $\eta = \alpha$. Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that in this case $\frac{\partial} {\partial \zeta} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta)} = 0$ applies. As $(1-\zeta)^{\tilde{a}_2}$ is decreasing

in ζ , $\frac{K_0^+}{S_0^+}$ is in this case again decreasing in ζ .

Case 3: $\tilde{a}_1 > 0$: The denominator in $\frac{K_0^+}{S_0^+}$ is increasing in ζ as $\frac{\partial {}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta)}{\partial \zeta} = \frac{\tilde{a}_1(\tilde{b}_1-1)}{\tilde{b}_1+1} {}_2F_1((\tilde{a}_1+1, \tilde{b}_1; \tilde{b}_1+2; \zeta) > 0$ (Abramowitz and Stegun 1972, 15.2.1) because $\tilde{b}_1 - 1 > 0$ is required by the transversality conditions (45) and (49). There are opposing forces at work in the nominator as ${}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)$ increases and $(1-\zeta)^{\tilde{a}_2}$ decreases in ζ . To find out which is stronger, we define $h(\zeta)$ as

$$h(\zeta) = (1-\zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) = (1-\zeta)^{\tilde{a}_2-\tilde{a}_1} (1-\zeta)^{\tilde{a}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)$$

with

$$\tilde{a}_2 - \tilde{a}_1 = \frac{1}{1-\alpha} - \frac{\eta - \alpha}{\eta(1-\alpha)} = \frac{\alpha}{\eta(1-\alpha)} > 0,$$

$${}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) = \tilde{b}_1 \int_0^1 x^{\tilde{b}_1-1} (1-zx)^{-\tilde{a}_1} dx.$$

Therefore,

$$\begin{aligned} \frac{\partial h(\zeta)}{\partial \zeta} &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} - \tilde{a}_1 \frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1} (1-\zeta)^{\tilde{a}_1} \frac{\partial {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{\partial \zeta} \\ &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} - \tilde{a}_1 \frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1} (1-\zeta)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 x^{\tilde{b}_1} (1-\zeta x)^{-\tilde{a}_1-1} dx \\ &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1} (1-\zeta)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1} (1-\zeta x)^{-\tilde{a}_1-1} - x^{\tilde{b}_1-1} \frac{(1-\zeta x)^{-\tilde{a}_1}}{1-\zeta} \right) dx \\ &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1} (1-\zeta)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1} (1-\zeta x)^{-\tilde{a}_1-1} - x^{\tilde{b}_1-1} \frac{(1-\zeta x)^{-\tilde{a}_1}}{1-\zeta} \right) dx \\ &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1} (1-\zeta)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1-1} (1-\zeta x)^{-\tilde{a}_1-1} \left(x - \frac{1-\zeta x}{1-\zeta} \right) \right) dx \\ &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} + (1-\zeta)^{\tilde{a}_2-\tilde{a}_1} (1-\zeta)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1-1} (1-\zeta x)^{-\tilde{a}_1-1} \frac{x-1}{1-\zeta} \right) dx \\ &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} - \tilde{a}_1 (1-\zeta)^{\tilde{a}_2-1} \tilde{b}_1 \int_0^1 x^{\tilde{b}_1-1} (1-x) (1-\zeta x)^{-\tilde{a}_1-1} dx \\ &= -(\tilde{a}_2 - \tilde{a}_1) \frac{h(\zeta)}{1-\zeta} - \tilde{a}_1 (1-\zeta)^{\tilde{a}_2-1} \frac{{}_2F_1(\tilde{a}_1+1, \tilde{b}_1; \tilde{b}_1+2; \zeta)}{\tilde{b}_1+1}. \end{aligned}$$

As $\tilde{a}_2 - \tilde{a}_1 > 0$ and $\tilde{a}_1 > 0$ in this case, we find $\frac{\partial h(\zeta)}{\partial \zeta} < 0$. Summing up case 3, the denominator in $\frac{K_0^+}{S_0^+}$ is increasing while the nominator is decreasing in ζ . Hence, $\frac{K_0^+}{S_0^+}$ is again decreasing in ζ .

We turn to $\frac{K_0 - K_0}{S_0 - S_0}$ given by (52). Its denominator is obviously decreasing in ζ as $\tilde{a}_2 = \frac{1}{1-\alpha} > 0$ and $\frac{\partial {}_2F_1(\tilde{a}_2, \tilde{b}_2-1; \tilde{b}_2+1; \zeta)}{\partial \zeta} = \frac{\tilde{a}_2(\tilde{b}_2-1)}{\tilde{b}_2+1} {}_2F_1(\tilde{a}_2+1, \tilde{b}_2; \tilde{b}_2+2; \zeta)$ with $\tilde{b}_2 - 1 > 0$ due to the transversality condition (49).

The nominator in $\frac{K_0 - K_0}{S_0 - S_0}$ is increasing in ζ . To see this, define

$$k(\zeta) = (1-\zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) = (1-\zeta)^{\tilde{a}_2} \tilde{b}_2 \int_0^1 x^{\tilde{b}_2-1} (1-\zeta x)^{-\tilde{a}_2} dx.$$

Therefore,

$$\begin{aligned}
\frac{\partial k(\zeta)}{\partial \zeta} &= \tilde{a}_2(1-\zeta)^{\tilde{a}_2} \tilde{b}_2 \left[\int_0^1 x^{\tilde{b}_2} (1-\zeta x)^{-\tilde{a}_2-1} dx - \int_0^1 x^{\tilde{b}_2-1} \frac{(1-\zeta x)^{-\tilde{a}_2}}{1-\zeta} dx \right] \\
&= \tilde{a}_2(1-\zeta)^{\tilde{a}_2} \tilde{b}_2 \int_0^1 x^{\tilde{b}_2-1} (1-\zeta x)^{-\tilde{a}_2-1} \left[x - \frac{1-\zeta x}{1-\zeta} \right] dx \\
&= -\tilde{a}_2(1-\zeta)^{\tilde{a}_2-1} \tilde{b}_2 \int_0^1 x^{\tilde{b}_2-1} (1-x)(1-\zeta x)^{-\tilde{a}_2-1} dx \\
&= -\tilde{a}_2(1-\zeta)^{\tilde{a}_2-1} \frac{{}_2F_1(\tilde{a}_2+1, \tilde{b}_2; \tilde{b}_2+2; \zeta)}{\tilde{b}_2+1}
\end{aligned}$$

which is negative for $\zeta < 1$.

Summing up, we have shown that the left hand side of (50) is increasing while the right hand side is decreasing in ζ . If an equilibrium fulfilling (50) exists, it is unique.

Properties of $\frac{K_0^+}{S_0^+}$: To work out conditions for existence, we focus first on $\frac{K_0^+}{S_0^+}$ given by (51). Any solution ζ needs to fulfill $\zeta < 1$; we know that $\frac{K_0^+}{S_0^+}$ is decreasing in ζ . We show first that $\frac{K_0^+}{S_0^+}$ is unbounded from above for $\zeta \rightarrow -\infty$. Let ε_1 be an arbitrarily large but finite real number. The critical term in $\frac{K_0^+}{S_0^+}$ is $(1-\zeta)^{\tilde{a}_2} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}$. Now suppose that

$$\lim_{\zeta \rightarrow -\infty} (1-\zeta)^{\tilde{a}_2} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta)}{{}_2F_1(\tilde{a}_1, \tilde{b}_1-1; \tilde{b}_1+1; \zeta)} < \varepsilon_1 \quad (53)$$

would be true. As $\frac{K_0^+}{S_0^+}$ decreases with ζ . This would imply that for any finite $\zeta < 1$ and for $\zeta \rightarrow -\infty$ it would be true that

$$\begin{aligned}
&{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) < \varepsilon_1 (1-\zeta)^{-\tilde{a}_2} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta), \\
&\tilde{b}_1 \int_0^1 x^{\tilde{b}_1-1} (1-\zeta x)^{-\tilde{a}_1} dx - \varepsilon_1 (1-\zeta)^{-\tilde{a}_2} \tilde{b}_1 (\tilde{b}_1-1) \int_0^1 x^{\tilde{b}_1-2} (1-x)(1-\zeta x)^{-\tilde{a}_1} dx < 0, \\
&\int_0^1 x^{\tilde{b}_1-2} (1-\zeta x)^{-\tilde{a}_1} [x - \varepsilon_1 (\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2} (1-x)] dx < 0, \\
&\int_0^1 x^{\tilde{b}_1-2} (1-\zeta x)^{-\tilde{a}_1} \kappa(x; \varepsilon_1) dx < 0, \quad (54)
\end{aligned}$$

with

$$\kappa(x; \varepsilon_1) = [x - \varepsilon_1 (\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2} (1-x)],$$

where $\kappa(x; \varepsilon_1)$ is an affine function of x . $\kappa(x; \varepsilon_1)$ is zero for $x = \bar{x}_0(\varepsilon_1)$ with

$$\bar{x}_0(\varepsilon_1) = \frac{\varepsilon_1 (\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2}}{1 + \varepsilon_1 (\tilde{b}_1-1)(1-\zeta)^{-\tilde{a}_2}}. \quad (55)$$

Therefore, $\kappa(x; \varepsilon_1) < 0$ for $x < \bar{x}_0(\varepsilon_1)$ and $\kappa(x; \varepsilon_1) > 0$ for $x > \bar{x}_0(\varepsilon_1)$. For any finite ε_1 , $\bar{x}_0(\varepsilon_1) \rightarrow 0$ for $\zeta \rightarrow -\infty$ as $\tilde{a}_2 = \frac{1}{1-\alpha} > 0$. As we integrate from 0 to 1, $\kappa(x; \varepsilon_1)$ becomes positive for $0 \leq x \leq 1$ as $\zeta \rightarrow -\infty$ and inequality (54) cannot be fulfilled. Hence, $\frac{K_0^+}{S_0^+}$ cannot be bounded from above as $\zeta \rightarrow -\infty$ and $\lim_{\zeta \rightarrow -\infty} \frac{K_0^+}{S_0^+} = \infty$.

Next, turn to the case $\zeta \rightarrow 1$. Suppose that $\frac{K_0^+}{S_0^+}$ would be bounded from below by some $\varepsilon_2 > 0$. By the same logic as above, this would imply for any $\zeta < 1$ and $\zeta \rightarrow 1$ that

$$\begin{aligned} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta) &> \varepsilon_2(1 - \zeta)^{-\tilde{a}_2} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta), \\ \int_0^1 x^{\tilde{b}_1-2}(1 - \zeta x)^{-\tilde{a}_1} \kappa(x; \varepsilon_2) dx &> 0. \end{aligned} \quad (56)$$

For any finite $\varepsilon_2 > 0$, $\bar{x}_0(\varepsilon_2) \rightarrow 1$ for $\zeta \rightarrow 1$ as $\tilde{a}_2 = \frac{1}{1-\alpha} > 0$. As we integrate from 0 to 1, $\kappa(x; \varepsilon_2)$ becomes negative for $0 \leq x \leq 1$ as $\zeta \rightarrow 1$ and inequality (56) cannot be fulfilled. Hence, $\frac{K_0^+}{S_0^+}$ cannot be bounded from below and $\lim_{\zeta \rightarrow 1} \frac{K_0^+}{S_0^+} = 0$.

Properties of $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ as $\zeta \rightarrow 1$: We turn to $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ which we know is increasing in ζ for $\zeta < 1$. If a maximum exists, it must be reached as $\zeta \rightarrow 1$. The critical term in the nominator is $(1 - \zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)$ which can be written as $(1 - \zeta) \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{(1 - \zeta)^{1 - \tilde{a}_2}}$. We are interested in

$$\lim_{\zeta \rightarrow 1} (1 - \zeta) \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{(1 - \zeta)^{1 - \tilde{a}_2}} \quad (57)$$

as $\lim_{\zeta \rightarrow 1} (1 - \zeta)$ is finite and equal to zero, we can rewrite this expression as

$$\lim_{\zeta \rightarrow 1} (1 - \zeta) \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{(1 - \zeta)^{1 - \tilde{a}_2}} = \left[\lim_{\zeta \rightarrow 1} (1 - \zeta) \right] \left[\lim_{\zeta \rightarrow 1} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{(1 - \zeta)^{1 - \tilde{a}_2}} \right] \quad (58)$$

if the second limit on the right hand side in the above equation is finite. 15.4.23 in DLMF (URL) states that

$$\lim_{\zeta \rightarrow 1} \frac{{}_2F_1(a, b; c; z)}{(1 - z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \quad (59)$$

if $R(c - a - b) < 0$. Applied to our case, $c - a - b = 1 + \tilde{b}_2 - \tilde{a}_2 - \tilde{b}_2 = 1 - \tilde{a}_2 = -\frac{\alpha}{1-\alpha} < 0$. Furthermore, $\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} = \frac{\Gamma(\tilde{b}_2+1)\Gamma(\tilde{a}_2-1)}{\Gamma(\tilde{a}_2)\Gamma(\tilde{b}_2)} = \frac{\tilde{b}_2}{\tilde{a}_2-1}$ which is finite. Hence, $\lim_{\zeta \rightarrow 1} (1 - \zeta) \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)}{(1 - \zeta)^{1 - \tilde{a}_2}} = 0$ and $\lim_{\zeta \rightarrow 1} K_0 - \underline{K}_0 = K_0$.

The critical term in the denominator of $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ is ${}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta)$. As $\frac{\partial {}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta)}{\partial \zeta} = \frac{\tilde{a}_2(\tilde{b}_2 - 1)}{\tilde{b}_2 + 1} {}_2F_1(\tilde{a}_2 + 1, \tilde{b}_2; \tilde{b}_2 + 2; \zeta) > 0$ for $\zeta < 1$, $S_0 - \underline{S}_0$ declines with ζ in this range. 15.3.6 in Abramowitz and Stegun (1972) implies that $\lim_{\zeta \rightarrow 1} {}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta) = \frac{\Gamma(\tilde{b}_2+1)\Gamma(2-\tilde{a}_2)}{\Gamma(\tilde{b}_2+1-\tilde{a}_2)\Gamma(2)}$ if $2 - \tilde{a}_2 = \frac{1-2\alpha}{1-\alpha} > 0$ which is the case for $\alpha < \frac{1}{2}$. In case $\alpha > \frac{1}{2}$ we find ${}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta) \rightarrow \infty$ as $\zeta \rightarrow 1$. In both cases, it is possible that $S_0 - \underline{S}_0$ turns negative as ζ grows for $\zeta < 1$. Define $\bar{\zeta}$ as

$$\bar{\zeta} = \operatorname{argmin}_{\zeta \leq 1} \left| S_0 - \frac{C}{A_0} \varphi_2^{-\frac{1}{1-\alpha}} \frac{1}{\psi^2 \tilde{b}_2(\tilde{b}_2 - 1)} {}_2F_1(\tilde{a}_2, \tilde{b}_2 - 1; \tilde{b}_2 + 1; \zeta) \right|, \quad (60)$$

As $S_0 - \underline{S}_0$ is decreasing in ζ for $\zeta < 1$, the admissible range for a solution to the present problem has the upper bound $\bar{\zeta}$. Therefore, if $\bar{\zeta} < 1$ ($\bar{\zeta} = 1$) we find $S_0 - \underline{S}_0|_{\zeta=\bar{\zeta}} = 0$ ($S_0 - \underline{S}_0|_{\zeta=\bar{\zeta}} \geq 0$).

Lastly, we turn to $\frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ as $\zeta \rightarrow -\infty$. Again, we start with the nominator $K_0 - \underline{K}_0 = K_0 - C \frac{1}{\psi} (1 - \zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)$. We know already that $(1 - \zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta)$ is decreasing in ζ for $\zeta < 1$. Obviously, $K_0 - \underline{K}_0$ then declines as $\zeta \rightarrow -\infty$. 15.3.4 in Abramowitz and Stegun (1972) states that

$${}_2F_1(a, b; c; z) = (1 - z)^{-a} {}_2F_1(a, c - b; c; \frac{z}{z-1}) \quad (61)$$

which implies for the present case

$$(1 - \zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) = {}_2F_1(\tilde{a}_2, 1; \tilde{b}_2 + 1; \frac{\zeta}{\zeta - 1}). \quad (62)$$

As $\tilde{a}_2, \tilde{b}_2 + 1 > 0$ and $\lim_{\zeta \rightarrow -\infty} \frac{\zeta}{\zeta - 1} = 1$, $\lim_{\zeta \rightarrow -\infty} (1 - \zeta)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta) = \infty$. This implies that $K_0 - \underline{K}_0$ becomes necessarily negative if ζ becomes too small. The range for admissible values for ζ is therefore bounded from below at $\underline{\zeta}$ which satisfies the condition

$$K_0 = \underline{C} \frac{1}{\psi} \frac{1}{\tilde{b}_2} (1 - \underline{\zeta})^{\tilde{a}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \underline{\zeta}). \quad (63)$$

We observe $\lim_{\zeta \rightarrow \underline{\zeta}} \frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = 0$.

Taken together, if $\underline{\zeta} < \bar{\zeta}$ and $\bar{\zeta} < 1$, $\lim_{\zeta \rightarrow \bar{\zeta}} \frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} \rightarrow \infty$. If if $\underline{\zeta} < \bar{\zeta}$, $\bar{\zeta} = 1$, $\lim_{\zeta \rightarrow \bar{\zeta}} \frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0}$ either diverges to infinity or a strictly positive constant. The latter occurs if $S_0 - \underline{S}_0 \neq 0$ for $\zeta \leq 1$ or if $\alpha < \frac{1}{2}$ and $S_0 - \underline{S}_0 = 0$ for $\zeta = 1$. In all possible cases we therefore observe $\lim_{\zeta \rightarrow \bar{\zeta}} \frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} > \lim_{\zeta \rightarrow \bar{\zeta}} \frac{K_0^+}{S_0^+}$.

Furthermore, if $\underline{\zeta} < \bar{\zeta}$ we know that $\lim_{\zeta \rightarrow \underline{\zeta}} \frac{K_0 - \underline{K}_0}{S_0 - \underline{S}_0} = 0$ and $\lim_{\zeta \rightarrow \underline{\zeta}} \frac{K_0^+}{S_0^+} > 0$ as $\frac{K_0^+}{S_0^+}$ is decreasing in ζ for $\zeta < 1$ and approaches 0 as $\zeta \rightarrow 1$.

If it happens that $\underline{\zeta} = \bar{\zeta}$, this value is the unique solution to the initial value problem. If we find $\underline{\zeta} > \bar{\zeta}$, there is no solution to the initial value problem because initial endowments K_0, S_0 are too low to allow for subsistence consumption \underline{C} .

This proves that a unique solution always exists if and only if $\underline{\zeta} \leq \bar{\zeta}$.

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