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Article

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OPERATIONS, INFORMATION & TECHNOLOGY | RESEARCH ARTICLE

An inventory model for deteriorating items with partial backlogging using linear demand in fuzzy environment

Neeraj Kumar¹ and Sanjeey Kumar^{1*}

Abstract: This model presents the invention scheduling difficulty in industry with different operational constraints, as well as strategic plan of the business, revenue purpose, and limit on finishing. In this article, we have studied an inventory model for deteriorating items with shortages under partially backlogging. The aim of present study is to minimize the total cost function in fuzzy environment. Graded mean representation, signed distance, and centroid methods are used to defuzzify the total cost function over the planning horizon. Further, all costs are defuzzified with the help of triangular fuzzy numbers. Finally, sensitivity analysis is also given to show the effect of the costs.

Subjects: Mathematics & Statistics; Advanced Mathematics; Mathematics Education; Statistics & Probability

Keywords: inventory; linear demand rate; partial backlogging; triangular fuzzy number; graded mean representation method; signed distance method; centroid method

1. Introduction

Inventory is a physical stock or resource that stockiest keep in hand in order to promote the smooth and efficient running of the business. It is assumed that all products like volatile, liquid, medicines, materials, etc. have certain life period and after that deterioration will take place. In real-life inventory system, deterioration is a critical point. So it cannot be ignored. Generally, deterioration is defined as decay, damage, evaporation, spoilage, obsolescence, loss of utility, pilferage, or loss of



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PUBLIC INTEREST STATEMENT

The Article is to develop regarding invention inventory model. We assume that the invention scheduling difficulty in industry with different operational constraints, in addition to strategic plan of the business, profits purpose, and boundary on finishing. The aim of present study is to minimize the total cost. Here, three different methods have been used to increase the potential worth of the product. We have determined the optimal cycle time so that the total inventory cost is minimum.

marginal values of a commodity and the item cannot be used for its original purpose. This leads to decreases in the usefulness of the product. In our daily life, most of the physical goods like volatile liquids, medicine, food grains, alcohols, fresh products, flowers, fruits, vegetables, blood banks, and seafoods undergo decay or deterioration over time. There are various types of uncertainties mixed up in any inventory system. Still, these uncertainties cannot be treated by usual probabilistic model. For that reason, it becomes more suitable to deal such problems with fuzzy set theory rather than probability theory.

However, in certain situations, uncertainties are due to fuzziness, and such cases are dilated in the fuzzy set theory which was demonstrated by Zadeh (1965). Kaufmann and Gupta (1991) provided an introduction to fuzzy arithmetic operation and Zimmermann (1996) discussed the concept of the fuzzy set theory and its applications. Considering the fuzzy set theory in inventory modeling renders an authenticity to the model formulated since fuzziness is the closest possible approach to reality. As reality is imprecise and can only be approximated to a certain extent, same way, fuzzy theory helps one to incorporate uncertainties in the formulation of the model, thus bringing it closer to reality. Park (1987) applied the fuzzy set concepts to EOQ formula by representing the inventory carrying cost with a fuzzy number and solved the economic order quantity model using fuzzy number operations based on the extension principle. Vujosevic, Petrovic, and Petrovic (1996) used trapezoidal fuzzy number to fuzzify the order cost in the total cost of the inventory model without backorder, and got fuzzy total cost. Yao and Lee (1996) developed a backorder inventory model with fuzzy order quantity as triangular and trapezoidal fuzzy numbers and shortage cost as a crisp parameter. Gen, Tsujimura, and Zheng (1997) determined their input data as fuzzy numbers, and then the interval mean value concept was introduced to solve the inventory problem. Chang, Yao, and Lee (1998) determined the backorder inventory problem with fuzzy backorder such that the backorder quantity is a triangular fuzzy number. Chang (1999) developed the fuzzy production inventory model for fuzzify the product quantity as triangular fuzzy number. Lee and Yao (1999) refined the inventory without backorder models in the fuzzy sense, where the order quantity is fuzzified as the triangular fuzzy number. Yao, Chang, and Su (2000) assumed to be the order quantity and the total demand rate as triangular fuzzy numbers and obtained the fuzzy inventory model without shortages. Yao and Chiang (2003) considered the total cost of inventory without backorder. They fuzzified the total demand and cost of storing one unit per day into triangular fuzzy numbers and defuzzify by the centroid and the signed distance methods. Dutta, Chakraborty, and Roy (2005) developed a model in presence of fuzzy random variable demand where the optimum is achieved using a graded mean integration representation. Chang, Yao, and Ouyang (2006) determined the mixture inventory model involving variable lead-time with backorders and lost sales. First, they fuzzify the random lead-time demand to be a fuzzy random variable and then fuzzify the total demand to be the triangular fuzzy number and derive the fuzzy total cost. By the centroid method of defuzzification, they estimated the total cost in the fuzzy sense. Wee, Yu, and Chen (2007) introduced an optimal inventory model for items with imperfect quality and shortage backordering. Lin (2008) developed the inventory problem for a periodic review model with variable lead-time and fuzzified the expected demand shortage and backorder rate using signed distance method to defuzzify. Roy and Samanta (2009) discussed a fuzzy continuous review inventory model without backorder for deteriorating items in which the cycle time is taken as a symmetric fuzzy number. They used the signed distance method to fuzzify the total cost. Gani and Maheswari (2010) developed an EOQ model with imperfect quality items with shortages where defective rate, demand, holding cost, ordering cost and shortage cost are taken as triangular fuzzy numbers. Graded mean integration method is used for defuzzification of the total profit. Ameli, Mirzazadeh, and Shirazi (2011) developed a new inventory model to determine ordering policy for imperfect items with fuzzy defective percentage under fuzzy discounting and inflationary conditions. They used the signed distance method of defuzzification to estimate the value of total profit. Sadi-Nezhad, Memar Nahavandi, and Nazemi (2011) developed a periodic review model and a continuous review inventory model with fuzzy setup cost, holding cost and shortage cost. They also considered the lead-time demand and the lead-time plus one period's demand as random variables. They use two methods in the name of signed distance and possibility mean value to defuzzify. Uthayakumar and Valliathal (2011) developed an economic production model for

Weibull deteriorating items over an infinite horizon under fuzzy environment and considered some cost component as triangular fuzzy numbers and using the signed distance method to defuzzify the cost function. Kumar et al. developed a fuzzy inventory model with limited storage capacity. Kumari, Kumar, and Singh (2013) investigate a fuzzy two ware house inventory model with three component demand rate. Tayal, Singh, and Sharma (2014) determined an inventory model for deteriorating items with seasonal products and an option of an alternative market. Kumar and Rajput (2015) introduced Fuzzy Inventory Model for Deteriorating Items with Time Dependent Demand and Partial Backlogging. Kumar and Kumar (2016a) introduced an inventory model with stock-dependent demand rate for deterioration items. Kumar, Kumar, and Smarandache (2016) developed a Fuzzy Inventory Model with K-Release Rule. Recently Kumar and Kumar (2016b) presented an Inventory Model for deteriorating items stock dependent demand and partial backlogging.

In the present work, we developed an inventory model for deteriorating items with linear demand rate with partially backlogging. Ordering cost, holding cost, deterioration rate, and shortage cost are assumed as a triangular fuzzy numbers. We used for defuzzification of the total cost function by graded mean representation, signed distance, and centroid methods. By comparing the results obtained by these methods, we get the better one as estimate of the total cost in the fuzzy sense. Numerical examples have been given to illustrate the model. Sensitivity analysis has also been carried out to observe the effects on the optimal solution. Rest of the paper is structured as follows. In Section 2, we at hand some basic of fuzzy theory, triangular fuzzy number, and defuzzication techniques. In Section 3, a numerical example is provided to show the feasibility of proposed model. In the final section, some conclusions are drawn from the obtained results and sensitive analyses with respect to different parameters are also made in this section.

2. Basic concepts of fuzzy set theory

In order to treat fuzzy inventory model using graded mean representation, signed distance and centroid to defuzzify, we need the following definitions (Figure 1).

Definition 2.1 (By Pu and Liu (1980), Definition 2.1). A fuzzy set a on $R = (-\infty, \infty)$ is called a fuzzy point if its membership function is

$$\mu_a(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases} \quad (1)$$

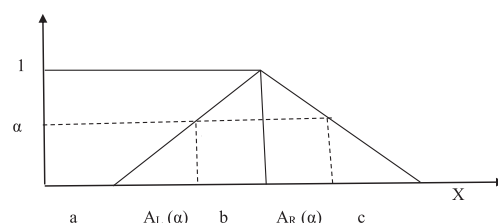
Where, the point a is called the support of fuzzy set a .

Definition 2.2 A fuzzy set $[a_\alpha, b_\alpha]$ where $0 \leq \alpha \leq 1$ and $a < b$ defined on R , is called a level of a fuzzy interval if its membership function is

$$\mu_{[a_\alpha, b_\alpha]}(x) = \begin{cases} \alpha, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Definition 2.3 A fuzzy number $\tilde{A} = (a, b, c)$, where $a < b < c$ and defined on R , is called a triangular fuzzy number if its membership function is

Figure 1. α -cut of a triangular fuzzy number.



$$\mu_A = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{Otherwise} \end{cases} \quad (3)$$

Definition 2.4 A fuzzy number $\tilde{A} = (a, b, c)$ is a triangular fuzzy number then the graded mean integration representation of \tilde{A} is defined as

$$P(\tilde{A}) = \frac{\int_0^{w_A} h \left(\frac{L^{-1}(h)+R^{-1}(h)}{2} \right) dh}{\int_0^{w_A} h dh},$$

With $0 < h \leq w_A$ and $0 < w_A \leq 1$.

$$P(\tilde{A}) = 1/2 \frac{\int_0^1 h[a + h(b-a) + c - h(c-a)] dh}{\int_0^1 h dh} = \frac{a + 4b + c}{6} \quad (4)$$

Definition 2.5 A fuzzy number $\tilde{A} = (a, b, c)$ is a triangular fuzzy number then the signed distance of \tilde{A} is defined as

$$P(\tilde{A}, \tilde{0}) = \int_0^1 d([A_L(\alpha), A_R(\alpha)], \tilde{0}) = \frac{1}{4}(a + 2b + c) \quad (5)$$

Definition 2.6 The centroid method on the triangular fuzzy number $\tilde{A} = (a, b, c)$ is defined as

$$C(\tilde{A}) = \frac{a + b + c}{3} \quad (6)$$

3. Assumptions and notations

The mathematical model in this paper is developed on the basis of the following assumptions and notations.

3.1. Notations

- (1) $D(t)$ is the demand rate at any time t per unit time.
- (2) A is the ordering cost per order.
- (3) θ is the deterioration rate, $0 < \theta \ll 1$.
- (4) T is the length of the Cycle.
- (5) Q is the ordering Quantity per unit.
- (6) h is the holding cost per unit per unit time.
- (7) S is the shortage Cost per unit time.
- (8) C is the unit Cost per unit time.
- (9) $K(t, T)$ is the total inventory cost per unit time.
- (10) \tilde{D} is the fuzzy demand.
- (11) $\tilde{\theta}$ is the fuzzy deterioration rate.
- (12) \tilde{h} is the fuzzy holding cost per unit per unit time.
- (13) \tilde{a} is the fuzzy shortage Cost per unit time.
- (14) \tilde{S} is the fuzzy unit Cost per unit time.
- (15) \tilde{C} is the total fuzzy inventory cost per unit time.

- (16) $K_{dG}(t_1, T)$ is the defuzzify value of $\tilde{K}_{dG}(t_1, T)$ by applying Graded mean integration method.
- (17) $K_{dS}(t_1, T)$ is the defuzzify value of $\tilde{K}_{dS}(t_1, T)$ by applying Signed distance method.
- (18) $K_{dC}(t_1, T)$ is the defuzzify value of $\tilde{K}_{dC}(t_1, T)$ by applying Centroid method.

3.2. Assumptions

- (1) Time horizon is finite.
- (2) Replenishment is instantaneous and lead-time is zero.
- (3) Shortages are allowed and partially backlogged. Unsatisfied demand is backlogged, and the fraction of shortages backordered is $1/(1 + \delta(T-t))$, where δ is a positive constant.
- (4) Linear demand rate $D(t) = a + bt$ is considered. Where a and b are positive constants and $a > 0, 0 < b < 1$.
- (5) There is no repair of deteriorated items occurring during the cycle.

4. Formulation of inventory model

It is considered that the $q(t)$ be the on-hand inventory at time t with initial inventory Q . Due to reason of market demand and deterioration of the items, the inventory level gradually diminishes during the time period $[0, t_1]$. The period $[t_1, T]$ is the period of shortages, which are partially backlogged. At any instant of time, the inventory level $q(t)$ is governed by the differential equations (Figure 2).

4.1. Crisp model

Let $q(t)$ be the inventory level at any time, which is governed by the following two differential equations

$$\frac{dq(t)}{dt} + \theta q(t) = -(a + bt); \quad 0 \leq t \leq t_1 \tag{7}$$

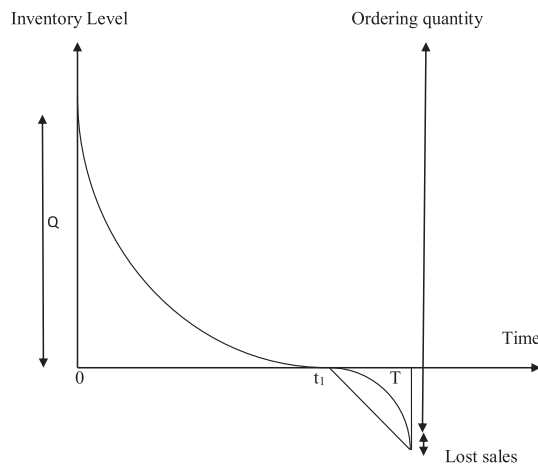
With $q(0) = Q$ and $q(t_1) = 0$.

$$\frac{dq(t)}{dt} = -\frac{(a + bt)}{1 + \delta(T - t)}; \quad t_1 \leq t \leq T \tag{8}$$

With $q(t_1) = 0$.

The solution of Equations (7) and (8) is given by

Figure 2. Graphical representation inventory model.



$$q(t) = (1 - \theta t) \left[Q - a \left(t + \frac{t^2}{2} \right) - b \left(\frac{t^2}{2} + \theta \frac{t^3}{3} \right) \right] \quad (9)$$

$$q(t) = \left(\frac{a}{\delta} + \frac{b(1 + \delta T)}{\delta^2} \right) [\log \{1 + \delta(T - t)\} - \log \{1 + \delta(T - t_1)\}] + \frac{b}{\delta} (t - T_1) \quad (10)$$

By putting $q(t_1) = 0$, we have

$$Q = a \left(t_1 + \frac{t_1^2}{2} \right) - b \left(\frac{t_1^2}{2} + \theta \frac{t_1^3}{3} \right) \quad (11)$$

Now, Equation (9) becomes

$$q(t) = \left[a \left\{ \left(t_1 - \frac{t_1^2}{2} \right) - \left(t - \frac{t^2}{2} \right) \right\} + b \left\{ \left(\frac{t_1^2}{2} + \theta \frac{t_1^3}{3} \right) - \left(\frac{t^2}{2} + \theta \frac{t^3}{3} \right) \right\} \right. \\ \left. - a\theta t \left\{ \left(t_1 - \frac{t_1^2}{2} \right) - \left(t - \frac{t^2}{2} \right) \right\} - b\theta t \left(\frac{t_1^2}{2} - \theta \frac{t^2}{2} \right) \right] \quad (12)$$

(Neglecting higher powers of θ).

Total average No. of holding units (I_H) during period $[0, T]$ is given by

$$I_H = \int_0^{t_1} q(t) dt = \left[a \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + a\theta \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) - \frac{5}{24} b\theta t_1^4 + \frac{bt_1^3}{3} \right] \quad (13)$$

Total No. of deteriorated units (I_D) during period $[0, T]$ is given by

$$I_D = Q - \text{Total Demand}$$

$$I_D = Q - \int_0^{t_1} (a + bt) dt = \frac{at_1^2}{2} + \frac{b\theta t_1^3}{3} \quad (14)$$

Total average No. of shortage units (I_s) during period $[0, T]$ is given by

$$I_s = - \int_{t_1}^T \frac{(a + bt)}{1 + \delta(T - t)} dt = \left(\frac{a}{\delta} + \frac{b(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b}{2\delta} (t - t_1)^2 \quad (15)$$

Total cost of the system per unit time is given by

$$K(t_1, T) = \frac{1}{T} [A + hI_H + CI_D + SI_s] \\ K(t_1, T) = \frac{1}{T} \left[A + h \left\{ a \left(\frac{t_1^2}{2} - \frac{t_1^3}{3} \right) + a\theta \left(\frac{t_1^4}{8} - \frac{t_1^3}{6} \right) - \frac{5}{24} b\theta t_1^4 + \frac{bt_1^3}{3} \right\} \right. \\ \left. + C \left(\frac{at_1^2}{2} + \frac{b\theta t_1^3}{3} \right) + S \left\{ \left(\frac{a}{\delta} + \frac{b(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b}{2\delta} (t - t_1)^2 \right\} \right] \quad (16)$$

4.2. Fuzzy model

Throughout the development of EOQ models, previous authors have assumed that the deterioration rate is constant. In the above-developed crisp model, it was assumed that all the parameters were fixed or could be predicted with certainty; but in real life situations, due to uncertainty in the environment, it is not easy to define all the parameters specifically. Accordingly, we assume some of these parameters namely \tilde{a} , \tilde{b} , \tilde{C} , \tilde{S} , $\tilde{\theta}$, \tilde{h} may change within some limits.

Let

$\tilde{a} = (a_1, a_2, a_3)$, $\tilde{b} = (b_1, b_2, b_3)$, $\tilde{C} = (C_1, C_2, C_3)$, $\tilde{S} = (S_1, S_2, S_3)$, $\tilde{\theta} = (\theta_1, \theta_2, \theta_3)$, $\tilde{h} = (h_1, h_2, h_3)$, are as triangular fuzzy numbers.

Total cost of the system per unit time in fuzzy sense is given by

$$K(t_1, T) = \frac{1}{T} \left[A + \tilde{h}\tilde{a} \left(\frac{t_1^2}{2} - \frac{t_1^3}{3} \right) + \tilde{h}\tilde{a}\tilde{\theta} \left(\frac{t_1^4}{8} - \frac{t_1^3}{6} \right) - \frac{5}{24} \tilde{h}\tilde{b}\tilde{\theta} t_1^4 + \tilde{h}\tilde{b} \frac{t_1^3}{3} \right. \\ \left. + \tilde{C} \left(\frac{\tilde{a}t_1^2}{2} + \frac{\tilde{b}\tilde{\theta}t_1^3}{3} \right) + \tilde{S} \left\{ \left(\frac{\tilde{a}}{\delta} + \frac{\tilde{b}(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{\tilde{b}}{2\delta} (t - t_1)^2 \right\} \right] \quad (17)$$

We defuzzify the fuzzy total cost by graded mean representation, signed distance, and centroid method

(1) By *Graded Mean Representation Method*, Total Cost is given by

$$K_{dG}(t_1, T) = \frac{1}{6} \left[K_{dG_1}(t_1, T), K_{dG_2}(t_1, T), K_{dG_3}(t_1, T) \right]$$

Where

$$K_{dG_1}(t_1, T) = \frac{1}{T} \left[A + h_1 a_1 \left(\frac{t_1^2}{2} - \frac{t_1^3}{3} \right) + h_1 a_1 \theta_1 \left(\frac{t_1^4}{8} - \frac{t_1^3}{6} \right) + h_1 b_1 \frac{t_1^3}{3} - \frac{5}{24} h_3 b_3 \theta_3 t_1^4 \right. \\ \left. + C_1 \left(\frac{a_1 t_1^2}{2} + \frac{b_1 \theta_1 t_1^3}{3} \right) + S_1 \left\{ \left(\frac{a_1}{\delta} + \frac{b_1(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_1}{2\delta} (t - t_1)^2 \right\} \right] \quad (18)$$

$$K_{dG_2}(t_1, T) = \frac{1}{T} \left[A + h_2 a_2 \left(\frac{t_1^2}{2} - \frac{t_1^3}{3} \right) + h_2 a_2 \theta_2 \left(\frac{t_1^4}{8} - \frac{t_1^3}{6} \right) + h_2 b_2 \frac{t_1^3}{3} - \frac{5}{24} h_2 b_2 \theta_2 t_1^4 \right. \\ \left. + C_2 \left(\frac{a_2 t_1^2}{2} + \frac{b_2 \theta_2 t_1^3}{3} \right) + S_2 \left\{ \left(\frac{a_2}{\delta} + \frac{b_2(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_2}{2\delta} (t - t_1)^2 \right\} \right] \quad (19)$$

$$K_{dG_3}(t_1, T) = \frac{1}{T} \left[A + h_3 a_3 \left(\frac{t_1^2}{2} - \frac{t_1^3}{3} \right) + h_3 a_3 \theta_3 \left(\frac{t_1^4}{8} - \frac{t_1^3}{6} \right) + h_3 b_3 \frac{t_1^3}{3} - \frac{5}{24} h_1 b_1 \theta_1 t_1^4 \right. \\ \left. + C_3 \left(\frac{a_3 t_1^2}{2} + \frac{b_3 \theta_3 t_1^3}{3} \right) + S_3 \left\{ \left(\frac{a_3}{\delta} + \frac{b_3(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_3}{2\delta} (t - t_1)^2 \right\} \right] \quad (20)$$

$$K_{dG}(t_1, T) = \frac{1}{6} \left[K_{dG_1}(t_1, T) + 4K_{dG_2}(t_1, T) + K_{dG_3}(t_1, T) \right] \quad (21)$$

To minimize total cost function per unit time $K_{dG}(t_1, T)$, the optimal value of t_1 and T can be obtained by solving the following equations:

$$\frac{\partial K_{dG}(t_1, T)}{\partial t_1} = 0 \text{ and } \frac{\partial K_{dG}(t_1, T)}{\partial T} = 0; \tag{22}$$

Equation (22) is equivalent to

$$\begin{aligned} & \frac{1}{6T} \left[h_1 a_1 (t_1 - t_1^2) + h_1 a_1 \theta_1 \left(\frac{t_1^3}{2} - \frac{t_1^2}{2} \right) + h_1 b_1 t_1^2 - \frac{5}{6} h_3 b_3 \theta_3 t_1^3 + C_1 (a_1 t_1 + b_1 \theta_1 t_1^2) \right. \\ & - \frac{S_1 b_1}{\delta} (t - t_1) + 4 \left\{ h_2 a_2 (t_1 - t_1^2) + h_2 a_2 \theta_2 \left(\frac{t_1^3}{2} - \frac{t_1^2}{2} \right) + h_2 b_2 t_1^2 - \frac{5}{6} h_2 b_2 \theta_2 t_1^3 + C_2 (a_2 t_2 + b_2 \theta_2 t_1^2) \right. \\ & \left. \left. - \frac{S_2 b_2}{\delta} (t - t_1) \right\} + \left\{ h_3 a_3 (t_1 - t_1^2) + h_3 a_3 \theta_3 \left(\frac{t_1^3}{2} - \frac{t_1^2}{2} \right) + h_3 b_3 - \frac{5}{6} h_1 b_1 \theta_1 t_1^2 \right. \right. \\ & \left. \left. + C_3 (a_3 t_1 + b_3 \theta_3 t_1^2) - \frac{S_3 b_3}{\delta} (t - t_1) \right\} \right] = 0 \end{aligned} \tag{23}$$

And

$$\begin{aligned} & \frac{1}{6T} \left[\left\{ \frac{S_1 a_1}{\delta T} + \frac{S_1 b_1}{\delta} (1 + \log T) + \frac{S_1 b_1}{\delta^2} (2t_1 + b_1) + \frac{S_1}{\delta} (a_1 + 2b_1 T + b_1 t_1) - \frac{S_3}{\delta} \left(a_3 + \frac{b_3}{\delta} + 2b_3 T \right) \right\} \right. \\ & + 4 \left\{ \frac{S_2 a_2}{\delta T} + \frac{S_2 b_2}{\delta} (1 + \log T) + \frac{S_2 b_2}{\delta^2} (2t_1 + b_2) + \frac{S_2}{\delta} (a_2 + 2b_2 T + b_2 t_1) - \frac{S_2}{\delta} \left(a_2 + \frac{b_2}{\delta} + 2b_2 T \right) \right\} \\ & + \left\{ \frac{S_3 a_3}{\delta T} + \frac{S_3 b_3}{\delta} (1 + \log T) + \frac{S_3 b_3}{\delta^2} (2t_1 + b_3) + \frac{S_3}{\delta} (a_3 + 2b_3 T + b_3 t_1) - \frac{S_1}{\delta} \left(a_1 + \frac{b_1}{\delta} + 2b_1 T \right) \right\} \\ & - \frac{1}{6T^2} \left[6A + h_1 a_1 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_1 a_1 \theta_1 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_1 b_1 \frac{t_1^3}{3} - \frac{5}{24} h_3 b_3 \theta_3 t_1^4 \right. \\ & + C_1 \left(\frac{a_1 t_1^2}{2} + \frac{b_1 \theta_1 t_1^3}{3} \right) + S_1 \left[\left\{ \left(\frac{a}{\delta} + \frac{b(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{ 1 + \delta(T - t_1) \} \right] \right. \right. \\ & \left. \left. + \frac{b}{2\delta} (t - t_1)^2 \right\} \right] + 4 \left[h_2 a_2 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_2 a_2 \theta_2 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_2 b_2 \frac{t_1^3}{3} - \frac{5}{24} h_2 b_2 \theta_2 t_1^4 \right. \\ & + C_2 \left(\frac{a_2 t_1^2}{2} + \frac{b_2 \theta_2 t_1^3}{3} \right) + S_2 \left[\left\{ \left(\frac{a_2}{\delta} + \frac{b_2(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{ 1 + \delta(T - t_1) \} \right] \right. \right. \\ & \left. \left. + \frac{b_2}{2\delta} (t - t_1)^2 \right\} \right] + \left[h_3 a_3 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_3 a_3 \theta_3 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_3 b_3 \frac{t_1^3}{3} - \frac{5}{24} h_1 b_1 \theta_1 t_1^4 \right. \\ & + C_3 \left(\frac{a_3 t_1^2}{2} + \frac{b_3 \theta_3 t_1^3}{3} \right) + S_3 \left[\left\{ \left(\frac{a_3}{\delta} + \frac{b_3(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{ 1 + \delta(T - t_1) \} \right] \right. \right. \\ & \left. \left. + \frac{b_3}{2\delta} (t - t_1)^2 \right\} \right] \right] = 0 \end{aligned} \tag{24}$$

Further, for the total cost function $K_{dG}(t_1, T)$, to be convex, the following conditions must be satisfied

$$\frac{\partial^2 K_{dG}(t_1, T)}{\partial t_1^2} > 0, \frac{\partial^2 K_{dG}(t_1, T)}{\partial T^2} > 0; \tag{25}$$

$$\text{And } \left(\frac{\partial^2 K_{dG}(t_1, T)}{\partial t_1^2} \right) \left(\frac{\partial^2 K_{dG}(t_1, T)}{\partial T^2} \right) - \left(\frac{\partial^2 K_{dG}(t_1, T)}{\partial t_1 \partial T} \right)^2 > 0; \quad (26)$$

The second derivatives of the total cost function $K_{dG}(t_1, T)$ are complicated and it is very difficult to prove the convexity mathematically.

(2) By *Signed Distance Method*, Total cost is given by

$$K_{ds}(t_1, T) = \frac{1}{4} [K_{ds_1}(t_1, T), K_{ds_2}(t_1, T), K_{ds_3}(t_1, T)]$$

Where

$$\begin{aligned} K_{ds_1}(t_1, T) = & \frac{1}{T} \left[A + h_1 a_1 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_1 a_1 \theta_1 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_1 b_1 \frac{t_1^3}{3} - \frac{5}{24} h_3 b_3 \theta_3 t_1^4 \right. \\ & + C_1 \left(\frac{a_1 t_1^2}{2} + \frac{b_1 \theta_1 t_1^3}{3} \right) + S_1 \left[\left\{ \left(\frac{a}{\delta} + \frac{b(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] \right. \right. \\ & \left. \left. + \frac{b}{2\delta} (t - t_1)^2 \right\} \right] \end{aligned} \quad (27)$$

$$\begin{aligned} K_{ds_2}(t_1, T) = & \frac{1}{T} \left[A + h_2 a_2 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_2 a_2 \theta_2 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_2 b_2 \frac{t_1^3}{3} - \frac{5}{24} h_2 b_2 \theta_2 t_1^4 \right. \\ & + C_2 \left(\frac{a_2 t_1^2}{2} + \frac{b_2 \theta_2 t_1^3}{3} \right) + S_2 \left[\left\{ \left(\frac{a_2}{\delta} + \frac{b_2(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] \right. \right. \\ & \left. \left. + \frac{b_2}{2\delta} (t - t_1)^2 \right\} \right] \end{aligned} \quad (28)$$

$$\begin{aligned} K_{ds_3}(t_1, T) = & \frac{1}{T} \left[A + h_3 a_3 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_3 a_3 \theta_3 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_3 b_3 \frac{t_1^3}{3} - \frac{5}{24} h_1 b_1 \theta_1 t_1^4 \right. \\ & + C_3 \left(\frac{a_3 t_1^2}{2} + \frac{b_3 \theta_3 t_1^3}{3} \right) + S_3 \left[\left\{ \left(\frac{a_3}{\delta} + \frac{b_3(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] \right. \right. \\ & \left. \left. + \frac{b_3}{2\delta} (t - t_1)^2 \right\} \right] \end{aligned} \quad (29)$$

$$K_{ds}(t_1, T) = \frac{1}{4} [K_{ds_1}(t_1, T) + 2K_{ds_2}(t_1, T) + K_{ds_3}(t_1, T)] \quad (30)$$

The total cost function $K_{ds}(t_1, T)$ has been minimized following the same process as has been stated in case (i). To minimize total cost function per unit time $K_{ds}(t_1, T)$, the optimal value of t_1 and T can be obtained by solving the following equations:

$$\frac{\partial K_{ds}(t_1, T)}{\partial t_1} = 0 \text{ and } \frac{\partial K_{ds}(t_1, T)}{\partial T} = 0; \quad (31)$$

Equation (31) is equivalent to

$$\begin{aligned} & \frac{1}{4T} \left[h_1 a_1 (t_1 + t_1^2) + h_1 a_1 \theta_1 \left(\frac{t_1^3}{2} + \frac{t_1^2}{2} \right) + h_1 b_1 t_1^2 - \frac{5}{6} h_3 b_3 \theta_3 t_1^3 + C_1 (a_1 t_1 + b_1 \theta_1 t_1^2) - \frac{S_1 b_1}{\delta} (t - t_1) \right. \\ & + 2 \left\{ h_2 a_2 (t_1 + t_1^2) + h_2 a_2 \theta_2 \left(\frac{t_1^3}{2} + \frac{t_1^2}{2} \right) + h_2 b_2 t_1^2 - \frac{5}{6} h_2 b_2 \theta_2 t_1^3 + C_2 (a_2 t_2 + b_2 \theta_2 t_1^2) + \frac{S_2 b_2}{\delta} (t - t_1) \right. \\ & \left. \left. + \left\{ h_3 a_3 (t_1 + t_1^2) + h_3 a_3 \theta_3 \left(\frac{t_1^3}{2} + \frac{t_1^2}{2} \right) + h_3 b_3 - \frac{5}{6} h_1 b_1 \theta_1 t_1^2 + C_3 (a_3 t_1 + b_3 \theta_3 t_1^2) - \frac{S_3 b_3}{\delta} (t - t_1) \right\} \right] = 0 \end{aligned} \tag{32}$$

And

$$\begin{aligned} & \frac{1}{4T} \left[\left\{ \frac{S_1 a_1}{\delta T} + \frac{S_1 b_1}{\delta} (1 + \log T) + \frac{S_1 b_1}{\delta^2} (2t_1 + b_1) + \frac{S_1}{\delta} (a_1 + 2b_1 T + b_1 t_1) - \frac{S_3}{\delta} \left(a_3 + \frac{b_3}{\delta} + 2b_3 T \right) \right\} \right. \\ & + 2 \left\{ \frac{S_2 a_2}{\delta T} + \frac{S_2 b_2}{\delta} (1 + \log T) + \frac{S_2 b_2}{\delta^2} (2t_1 + b_2) + \frac{S_2}{\delta} (a_2 + 2b_2 T + b_2 t_1) - \frac{S_2}{\delta} \left(a_2 + \frac{b_2}{\delta} + 2b_2 T \right) \right\} \\ & + \left\{ \frac{S_3 a_3}{\delta T} + \frac{S_3 b_3}{\delta} (1 + \log T) + \frac{S_3 b_3}{\delta^2} (2t_1 + b_3) + \frac{S_3}{\delta} (a_3 + 2b_3 T + b_3 t_1) - \frac{S_1}{\delta} \left(a_1 + \frac{b_1}{\delta} + 2b_1 T \right) \right\} \\ & - \frac{1}{4T^2} \left[6A + h_1 a_1 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_1 a_1 \theta_1 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_1 b_1 \frac{t_1^3}{3} - \frac{5}{24} h_3 b_3 \theta_3 t_1^4 + C_1 \left(\frac{a_1 t_1^2}{2} + \frac{b_1 \theta_1 t_1^3}{3} \right) \right. \\ & + S_1 \left[\left\{ \left(\frac{a}{\delta} + \frac{b(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b}{2\delta} (t - t_1)^2 \right\} \right] \\ & + 2 \left[h_2 a_2 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_2 a_2 \theta_2 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_2 b_2 \frac{t_1^3}{3} - \frac{5}{24} h_2 b_2 \theta_2 t_1^4 + C_2 \left(\frac{a_2 t_1^2}{2} + \frac{b_2 \theta_2 t_1^3}{3} \right) \right. \\ & + S_2 \left[\left\{ \left(\frac{a_2}{\delta} + \frac{b_2(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_2}{2\delta} (t - t_1)^2 \right\} \right] \\ & + \left[h_3 a_3 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_3 a_3 \theta_3 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_3 b_3 \frac{t_1^3}{3} - \frac{5}{24} h_1 b_1 \theta_1 t_1^4 + C_3 \left(\frac{a_3 t_1^2}{2} + \frac{b_3 \theta_3 t_1^3}{3} \right) \right. \\ & \left. \left. + S_3 \left[\left\{ \left(\frac{a_3}{\delta} + \frac{b_3(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_3}{2\delta} (t - t_1)^2 \right\} \right] \right] = 0 \end{aligned} \tag{33}$$

Further, for the total cost function $K_{ds}(t_1, T)$, to be convex, the following conditions must be satisfied

$$\frac{\partial^2 K_{ds}(t_1, T)}{\partial t_1^2} > 0, \quad \frac{\partial^2 K_{ds}(t_1, T)}{\partial T^2} > 0; \tag{34}$$

$$\text{And } \left(\frac{\partial^2 K_{ds}(t_1, T)}{\partial t_1^2} \right) \left(\frac{\partial^2 K_{ds}(t_1, T)}{\partial T^2} \right) - \left(\frac{\partial^2 K_{ds}(t_1, T)}{\partial t_1 \partial T} \right)^2 > 0; \tag{35}$$

The second derivatives of the total cost function $K_{ds}(t_1, T)$ are complicated and it is very difficult to prove the convexity mathematically.

(3) By *Centroid Method*, Total cost is given by

$$K_{dc}(t_1, T) = \frac{1}{3} [K_{dc_1}(t_1, T), K_{dc_2}(t_1, T), K_{dc_3}(t_1, T)]$$

Where

$$K_{dc_1}(t_1, T) = \frac{1}{T} \left[A + h_1 a_1 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_1 a_1 \theta_1 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_1 b_1 \frac{t_1^3}{3} - \frac{5}{24} h_3 b_3 \theta_3 t_1^4 + C_1 \left(\frac{a_1 t_1^2}{2} + \frac{b_1 \theta_1 t_1^3}{3} \right) + S_1 \left[\left\{ \left(\frac{a}{\delta} + \frac{b(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b}{2\delta} (t - t_1)^2 \right\} \right] \right] \quad (36)$$

$$K_{dc_2}(t_1, T) = \frac{1}{T} \left[A + h_2 a_2 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_2 a_2 \theta_2 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_2 b_2 \frac{t_1^3}{3} - \frac{5}{24} h_2 b_2 \theta_2 t_1^4 + C_2 \left(\frac{a_2 t_1^2}{2} + \frac{b_2 \theta_2 t_1^3}{3} \right) + S_2 \left[\left\{ \left(\frac{a_2}{\delta} + \frac{b_2(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_2}{2\delta} (t - t_1)^2 \right\} \right] \right] \quad (37)$$

$$K_{dc_3}(t_1, T) = \frac{1}{T} \left[A + h_3 a_3 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_3 a_3 \theta_3 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_3 b_3 \frac{t_1^3}{3} - \frac{5}{24} h_1 b_1 \theta_1 t_1^4 + C_3 \left(\frac{a_3 t_1^2}{2} + \frac{b_3 \theta_3 t_1^3}{3} \right) + S_3 \left[\left\{ \left(\frac{a_3}{\delta} + \frac{b_3(1+\delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_3}{2\delta} (t - t_1)^2 \right\} \right] \right] \quad (38)$$

$$K_{dc}(t_1, T) = \frac{1}{3} [K_{dc_1}(t_1, T) + K_{dc_2}(t_1, T) + K_{dc_3}(t_1, T)] \quad (39)$$

The total cost function $K_{dc}(t_1, T)$ has been minimized following the same process as has been stated in case (i). To minimize total cost function per unit time $K_{dc}(t_1, T)$, the optimal value of t_1 and T can be obtained by solving the following equations:

$$\frac{\partial K_{dc}(t_1, T)}{\partial t_1} = 0 \text{ and } \frac{\partial K_{dc}(t_1, T)}{\partial T} = 0; \quad (40)$$

Equation (40) is equivalent to

$$\begin{aligned} & \frac{1}{3T} \left[h_1 a_1 (t_1 - t_1^2) + h_1 a_1 \theta_1 \left(\frac{t_1^3}{2} + \frac{t_1^2}{2} \right) + h_1 b_1 t_1^2 - \frac{5}{6} h_3 b_3 \theta_3 t_1^3 + C_1 (a_1 t_1 + b_1 \theta_1 t_1^2) - \frac{S_1 b_1}{\delta} (t - t_1) \right. \\ & + \left. \left\{ h_2 a_2 (t_1 - t_1^2) + h_2 a_2 \theta_2 \left(\frac{t_1^3}{2} + \frac{t_1^2}{2} \right) + h_2 b_2 t_1^2 - \frac{5}{6} h_2 b_2 \theta_2 t_1^3 + C_2 (a_2 t_2 + b_2 \theta_2 t_1^2) + \frac{S_2 b_2}{\delta} (t - t_1) \right. \right. \\ & \left. \left. + \left\{ h_3 a_3 (t_1 - t_1^2) + h_3 a_3 \theta_3 \left(\frac{t_1^3}{2} + \frac{t_1^2}{2} \right) + h_3 b_3 - \frac{5}{6} h_1 b_1 \theta_1 t_1^2 + C_3 (a_3 t_1 + b_3 \theta_3 t_1^2) - \frac{S_3 b_3}{\delta} (t - t_1) \right\} \right] = 0 \end{aligned} \quad (41)$$

And

$$\begin{aligned}
 & \frac{1}{3T} \left\{ \left[\frac{S_1 a_1}{\delta T} + \frac{S_1 b_1}{\delta} (1 + \log T) + \frac{S_1 b_1}{\delta^2} (2t_1 + b_1) + \frac{S_1}{\delta} (a_1 + 2b_1 T + b_1 t_1) - \frac{S_3}{\delta} \left(a_3 + \frac{b_3}{\delta} + 2b_3 T \right) \right] \right\} \\
 & + \left\{ \frac{S_2 a_2}{\delta T} + \frac{S_2 b_2}{\delta} (1 + \log T) + \frac{S_2 b_2}{\delta^2} (2t_1 + b_2) + \frac{S_2}{\delta} (a_2 + 2b_2 T + b_2 t_1) - \frac{S_2}{\delta} \left(a_2 + \frac{b_2}{\delta} + 2b_2 T \right) \right\} \\
 & + \left\{ \frac{S_3 a_3}{\delta T} + \frac{S_3 b_3}{\delta} (1 + \log T) + \frac{S_3 b_3}{\delta^2} (2t_1 + b_3) + \frac{S_3}{\delta} (a_3 + 2b_3 T + b_3 t_1) - \frac{S_1}{\delta} \left(a_1 + \frac{b_1}{\delta} + 2b_1 T \right) \right\} \\
 & - \frac{1}{3T^2} \left[3A + h_1 a_1 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_1 a_1 \theta_1 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_1 b_1 \frac{t_1^3}{3} - \frac{5}{24} h_3 b_3 \theta_3 t_1^4 + C_1 \left(\frac{a_1 t_1^2}{2} + \frac{b_1 \theta_1 t_1^3}{3} \right) \right] \\
 & + S_1 \left[\left\{ \left(\frac{a}{\delta} + \frac{b(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b}{2\delta} (t - t_1)^2 \right\} \right] \\
 & + \left[h_2 a_2 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_2 a_2 \theta_2 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_2 b_2 \frac{t_1^3}{3} - \frac{5}{24} h_2 b_2 \theta_2 t_1^4 + C_2 \left(\frac{a_2 t_1^2}{2} + \frac{b_2 \theta_2 t_1^3}{3} \right) \right] \\
 & + S_2 \left[\left\{ \left(\frac{a_2}{\delta} + \frac{b_2(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_2}{2\delta} (t - t_1)^2 \right\} \right] \\
 & + \left[h_3 a_3 \left(\frac{t_1^2}{2} + \frac{t_1^3}{3} \right) + h_3 a_3 \theta_3 \left(\frac{t_1^4}{8} + \frac{t_1^3}{6} \right) + h_3 b_3 \frac{t_1^3}{3} - \frac{5}{24} h_1 b_1 \theta_1 t_1^4 + C_3 \left(\frac{a_3 t_1^2}{2} + \frac{b_3 \theta_3 t_1^3}{3} \right) \right] \\
 & + S_3 \left[\left\{ \left(\frac{a_3}{\delta} + \frac{b_3(1 + \delta T)}{\delta^2} \right) \left[\log T - (T - t_1) + \frac{1}{\delta} \log \{1 + \delta(T - t_1)\} \right] + \frac{b_3}{2\delta} (t - t_1)^2 \right\} \right] = 0
 \end{aligned} \tag{42}$$

Further, for the total cost function $K_{dc}(t_1, T)$, to be convex, the following conditions must be satisfied

$$\frac{\partial^2 K_{dc}(t_1, T)}{\partial t_1^2} > 0, \quad \frac{\partial^2 K_{dc}(t_1, T)}{\partial T^2} > 0; \tag{43}$$

$$\text{And } \left(\frac{\partial^2 K_{dc}(t_1, T)}{\partial t_1^2} \right) \left(\frac{\partial^2 K_{dc}(t_1, T)}{\partial T^2} \right) - \left(\frac{\partial^2 K_{dc}(t_1, T)}{\partial t_1 \partial T} \right)^2 > 0; \tag{44}$$

The second derivatives of the total cost function $K_{dc}(t_1, T)$ are complicated and it is very difficult to prove the convexity mathematically.

5. Numerical example

Consider an inventory system with following parametric values.

5.1. Crisp model

$A = \text{Rs. } 180/\text{order}$, $C = \text{Rs. } 18/\text{unit}$, $h = \text{Rs. } 4/\text{unit/year}$, $a = 100 \text{ units/year}$, $b = .1 \text{ units/year}$, $\theta = .01/\text{year}$, $S = \text{Rs. } 13/\text{unit/year}$.

The solution of crisp model is $K(t_1, T) = \text{Rs. } 394.2524$, $t_1 = .6547 \text{ year}$, $T = .8695 \text{ year}$.

5.2. Fuzzy model

$\tilde{a} = (48, 80, 112)$, $\tilde{b} = (.04, .8, .10)$, $\tilde{C} = (15, 20, 22)$, $\tilde{S} = (12, 14, 16)$, $\tilde{\theta} = (.006, .010, .012)$, $\tilde{h} = (3, 4, 6)$

The solution of fuzzy model can be determined by following three methods.

By *Graded Mean Representation Method*, we have

- (1) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{S}, \tilde{\theta}, \tilde{h}$ all are triangular fuzzy numbers $K_{dG}(t_1, T) = \text{Rs. } 394.2084, t_1 = .5807 \text{ year}, T = .9123 \text{ year}.$
- (2) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{S}, \tilde{\theta}$ are triangular fuzzy numbers $K_{dG}(t_1, T) = \text{Rs. } 386.7852, t_1 = .7245 \text{ year}, T = .9460 \text{ year}.$
- (3) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{\theta}$ are triangular fuzzy numbers $K_{dG}(t_1, T) = \text{Rs. } 384.5274, t_1 = .7017 \text{ year}, T = .8496 \text{ year}.$
- (4) When \tilde{a}, \tilde{b} and $\tilde{\theta}$ are triangular fuzzy numbers $K_{dG}(t_1, T) = \text{Rs. } 384.2250, t_1 = .6920 \text{ year}, T = .8603 \text{ year}.$
- (5) When \tilde{a} and \tilde{b} are triangular fuzzy numbers $K_{dG}(t_1, T) = \text{Rs. } 381.8978, t_1 = .6135 \text{ year}, T = .8612 \text{ year}.$

By *Signed Distance Method*, we have

- (1) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{S}, \tilde{\theta}, \tilde{h}$ all are triangular fuzzy numbers $K_{dS}(t_1, T) = \text{Rs. } 414.6096, t_1 = .6508 \text{ year}, T = .8385 \text{ year}.$
- (2) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{S}, \tilde{\theta}$ are triangular fuzzy numbers $K_{dS}(t_1, T) = \text{Rs. } 401.7852, t_1 = .6835 \text{ year}, T = .8460 \text{ year}.$
- (3) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{\theta}$ are triangular fuzzy numbers $K_{dS}(t_1, T) = \text{Rs. } 398.5274, t_1 = .7015 \text{ year}, T = .8596 \text{ year}.$
- (4) When \tilde{a}, \tilde{b} and $\tilde{\theta}$ are triangular fuzzy numbers $K_{dS}(t_1, T) = \text{Rs. } 398.5254, t_1 = .7120 \text{ year}, T = .8605 \text{ year}.$
- (5) When \tilde{a} and \tilde{b} are triangular fuzzy numbers $K_{dS}(t_1, T) = \text{Rs. } 397.8978, t_1 = .7131 \text{ year}, T = .8561 \text{ year}.$

By *Centroid Method*, we have

- (1) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{S}, \tilde{\theta}, \tilde{h}$ all are triangular fuzzy numbers $K_{dC}(t_1, T) = \text{Rs. } 417.6576, t_1 = .6245 \text{ year}, T = .8987 \text{ year}.$
- (2) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{S}, \tilde{\theta}$ are triangular fuzzy numbers $K_{dC}(t_1, T) = \text{Rs. } 408.9852, t_1 = .7155 \text{ year}, T = .9166 \text{ year}.$
- (3) When $\tilde{a}, \tilde{b}, \tilde{C}, \tilde{\theta}$ are triangular fuzzy numbers $K_{dC}(t_1, T) = \text{Rs. } 405.6284, t_1 = .6815 \text{ year}, T = .8596 \text{ year}.$
- (4) When \tilde{a}, \tilde{b} and $\tilde{\theta}$ are triangular fuzzy numbers $K_{dC}(t_1, T) = \text{Rs. } 404.2651, t_1 = .6720 \text{ year}, T = .8603 \text{ year}.$
- (5) When \tilde{a} and \tilde{b} are triangular fuzzy numbers $K_{dC}(t_1, T) = \text{Rs. } 403.6478, t_1 = .6834 \text{ year}, T = .8623 \text{ year}.$

5.3. Sensitivity analysis

A sensitivity analysis is performed to study the effects of changes in fuzzy parameters \tilde{a}, \tilde{b} and $\tilde{\theta}$ on the optimal solution by taking the defuzzify values of these parameters (Figures 3-5). The results are shown in tables.

Figure 3. Effect of the demand parameter a on total fuzzy cost.

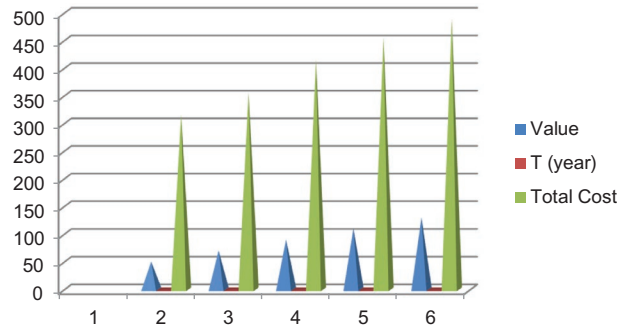


Figure 4. Effect of the demand parameter b on total fuzzy cost.

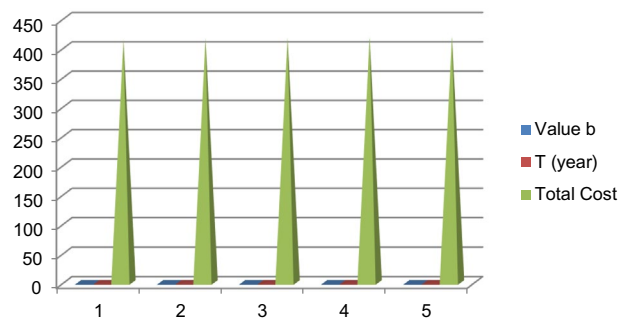
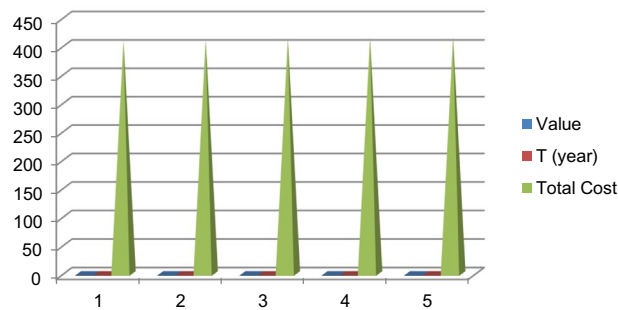


Figure 5. Effect of the demand parameter θ on total fuzzy cost.



5.4. Observations

- (1) From Table 1, as we increase the parameter a , the optimum values of t_1 and T decrease. By this effect, the total fuzzy cost $K_{dg}(t_1, T)$ increases.
- (2) From Table 2, as we decrease the parameter b , the optimum values of t_1 decrease and T decrease. By this effect, the total fuzzy cost $K_{dg}(t_1, T)$ increases.
- (3) From Table 3, as we increase the parameter θ , the optimum values of t_1 decrease and T decrease. By this effect, the total fuzzy cost $K_{dg}(t_1, T)$ increases.

Table 1. Sensitivity analysis on parameter a

a (units/year)	t_1 (year)	T (year)	$K_{dg}(t_1, T)$ (Rs)
48	.8412	1.1642	315.6482
67	.7326	1.0249	356.1471
80	.6402	.9182	415.7485
96	.6146	.8514	454.7462
112	.5498	.7646	489.3586

Table 2. Sensitivity analysis on parameter b

b (units/year)	t_1 (year)	T (year)	$K_{dg}(t_1, T)$ (Rs)
.05	.6413	.9542	414.3487
.07	.6526	.9247	416.4472
.09	.6505	.9124	417.6482
.11	.6845	.9214	418.6462
.13	.6798	.9241	419.5581

Table 3. Sensitivity analysis on parameter θ

θ	t_1 (year)	T (year)	$K_{dg}(t_1, T)$ (Rs)
.005	.6873	.9342	410.2483
.007	.6846	.9347	411.4172
.009	.6705	.9184	413.6182
.011	.6644	.9154	414.6862
.013	.6696	.9121	415.9582

6. Conclusion

In this present model we developed an EOQ model for deteriorating items with linear demand rate in fuzzy environment. Shortages are allowed and partially backlogged. The deterioration cost, ordering cost, holding cost, and shortage cost are represented by triangular fuzzy numbers. For defuzzification, we used graded mean, signed distance, and centroid method to evaluate the optimal time period of optimistic stock t_1 and total cycle length T which minimizes the total cost. The numerical example shows that graded mean representation method offer minimum cost as compared to signed distance method and centroid method. A sensitivity analysis is also conducted on the parameters a , b , and θ to investigate the effects of fuzziness. Finding suggest that the change in parameters a , b , and θ will result the change in fuzzy cost with some changes in t_1 and T . The increase in values of these parameters will result in increase in fuzzy cost, but decreases t_1 and T . Similarly the decrease in values of these parameters will result in decrease in fuzzy cost, but increases t_1 and T .

The proposed model can be extended for stock-dependent demand, price-dependent demand, two warehouse systems, and the effect of inflations in fuzzy environment and many more.

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