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**Article**

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## A NEW ESTIMATOR OF MEAN USING DOUBLE SAMPLING

Kalyan Rao Vadlamudi<sup>1</sup>, Stephen A. Sedory, Sarjinder Singh

### ABSTRACT

In this paper, we consider the problem of estimation of population mean of a study variable by making use of first-phase sample mean and first-phase sample median of the auxiliary variable at the estimation stage. The proposed new estimator of the population mean is compared to the sample mean estimator, ratio estimator and the difference type estimator for the fixed cost of the survey by using the concept of two-phase sampling. The magnitude of the relative efficiency of the proposed new estimator has been investigated through simulation study.

**Key words:** Two-phase sampling, relative efficiency, analytical and empirical comparison.

### 1. Introduction

Consider a population  $\Omega$  consisting of  $N$  units. Let  $(y_i, x_i)$ ,  $i = 1, 2, \dots, N$  be the values of the study variable  $Y$  and auxiliary variable  $X$  for the  $i$ th unit in the population.

Let

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^N y_i \quad (1.1)$$

and

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N x_i \quad (1.2)$$

be the population means of the study and auxiliary variables respectively. Survey statisticians are often interested in estimating the population mean  $\bar{Y}$  of the study variable. It is also well known that if the population mean  $\bar{X}$  of an auxiliary variable is known then it can be used to improve estimation strategies in survey sampling. Examples of such estimators are the ratio estimator due to Cochran

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(1940) and the linear regression estimator due to Hansen, Hurwitz and Madow (1953).

If such auxiliary information of the population mean  $\bar{X}$  is not known or complete auxiliary information is not known, then it can be relatively cheaper to obtain information on the auxiliary variable by taking a large preliminary sample for estimating population mean of the auxiliary variable to be used at the estimation stage. In other words, in the case of single auxiliary variable  $X$ , if the population mean  $\bar{X}$  of the auxiliary variable is unknown then we consider taking a preliminary large sample of  $m$  units by using simple random and without replacement sampling (SRSWOR) from the population of  $N$  units. In the sample  $s_1$  of  $m$  units, we observe only the auxiliary variable  $x_i$ ,  $i = 1, 2, \dots, m$ . From the given first-phase sample  $s_1$  of  $m$  units, we select another subsample  $s_2$  of  $n$  units by using SRSWOR sample. In the sample  $s_2$ , we measure the ordered pairs  $(y_i, x_i)$ ,  $i = 1, 2, \dots, n$  of the study and auxiliary variables. Then an unbiased estimator of the population mean  $\bar{X}$  based on the first-phase sample information as the sample mean is given by:

$$\bar{x}_m = \frac{1}{m} \sum_{i=1}^m x_i \quad (1.3)$$

The unbiased estimators of the population means  $\bar{Y}$  and  $\bar{X}$  of the study and auxiliary variables based on the second-phase sample information are, respectively, given by:

$$\bar{y}_n = \frac{1}{n} \sum_{i=1}^n y_i \quad (1.4)$$

and

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i \quad (1.5)$$

Neyman (1938) invented this sampling technique called double sampling or two-phase sampling, and later work related to this scheme is extensively reviewed in Singh (2003). It leads to ratio and regression type estimators of the population mean  $\bar{Y}$  in two-phase sampling as:

$$\bar{y}_{\text{rat(d)}} = \bar{y}_n \left( \frac{\bar{x}_m}{\bar{x}_n} \right) \quad (1.6)$$

and

$$\bar{y}_{\text{reg(d)}} = \bar{y}_n + \beta(\bar{x}_m - \bar{x}_n) \quad (1.7)$$

The variances of the sample mean, ratio and regression type estimators are, respectively, given by;

$$\begin{aligned}
 V(\bar{y}_n) &= \left(\frac{1}{n} - \frac{1}{N}\right) S_y^2 \\
 V(\bar{y}_{\text{rat(d)}}) &= \left(\frac{1}{m} - \frac{1}{N}\right) S_y^2 + \left(\frac{1}{n} - \frac{1}{m}\right) [S_y^2 + R^2 S_x^2 - 2RS_{xy}] \tag{1.8}
 \end{aligned}$$

and

$$V(\bar{y}_{\text{reg(d)}}) = \left(\frac{1}{m} - \frac{1}{N}\right) S_y^2 + \left(\frac{1}{n} - \frac{1}{m}\right) S_y^2 [1 - \rho_{xy}^2] \tag{1.9}$$

where

$$R = \frac{\bar{Y}}{\bar{X}}; \quad \rho_{xy} = \frac{S_{xy}}{S_x S_y}; \quad S_{xy} = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})(X_i - \bar{X});$$

$$S_x^2 = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})^2, \text{ and}$$

$$S_y^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2.$$

To our knowledge, the pioneer contributors, to the problem of estimating median, are Kuk and Mak (1989) by proposing very clear estimators of median in the presence of auxiliary information. Singh, Joarder and Tracy (2001) extended their idea to the situation of median estimation in two-phase sampling. The importance of double sampling and improvements on the estimation of population mean can also be seen in several publications by Vishwakarma and Kumar (2015), Vishwakarma and Gangele (2014), Vishwakarma and Singh (2011), Amin *et al.* (2016), and Sanaullah *et al.* (2014). However, none of these papers deal with the situation of making use of estimator of median of the auxiliary variable at the estimation stage of population mean of the study variable in two-phase sampling. This motivated the authors to think on these lines if some improvements can be seen by making the use of first-phase median of the auxiliary variable.

In the next section, we introduce a new estimator of the population mean in two-phase sampling which makes use of first-phase sample mean and sample median of the auxiliary variable.

## 2. Estimator

Let  $\hat{M}_x^*$  be the median of the auxiliary variable  $X$  based on the first phase sample  $s_1$  of  $m$  units. Let  $\hat{M}_x$  be the median of the auxiliary variable  $X$  based on the second phase sample  $s_2$  of  $n$  units.

Suppose  $x_{(1)}, x_{(2)}, \dots, x_{(m)}$  are the  $x$  values of first-phase sample units in the ascending order. Further, let  $t_1$  be an integer such that  $X_{(t_1)} \leq M_x \leq X_{(t_1+1)}$  and let  $p_1 = t_1/n$  be the proportion of  $X$  values in the first phase sample that are less than or equal to the median value  $M_x$ , an unknown population parameter. If  $\hat{p}_1$  is a predictor of  $p_1$ , the first-phase sample median  $\hat{M}_x^*$  can be written in terms of quintiles as  $\hat{Q}_x(\hat{p}_1)$ , where  $\hat{p}_1 = 0.5$ .

Suppose  $x_{(1)}, x_{(2)}, \dots, x_{(n)}$  are the  $x$  values of second-phase sample units in the ascending order. Further, let  $t_2$  be an integer such that  $X_{(t_2)} \leq M_x \leq X_{(t_2+1)}$  and let  $p_2 = t_2/n$  be the proportion of  $X$  values in the second phase sample that are less than or equal to the median value  $M_x$ , an unknown population parameter. If  $\hat{p}_2$  is a predictor of  $p_2$ , the first-phase sample median  $\hat{M}_x$  can be written in terms of quintiles as  $\hat{Q}_x(\hat{p}_2)$ , where  $\hat{p}_2 = 0.5$ .

Now we define a new estimator of the population mean  $\bar{Y}$  in two-phase sampling as:

$$\bar{y}_{kal} = \bar{y}_n + \beta_1^* (\bar{x}_m - \bar{x}_n) + \beta_2^* (\hat{M}_x^* - \hat{M}_x) \quad (2.1)$$

where  $\beta_1^*$  and  $\beta_2^*$  are unknown partial regression coefficients to be determined such that the variance of the estimator is minimum.

It may be worth pointing out that the proposed estimator  $\bar{y}_{kal}$  is an extension of the recent estimator of due to Lamichhane, Singh and Diawara (2015) from single-phase sampling to two-phase sampling.

To study the asymptotic properties of the proposed estimator,  $\bar{y}_{kal}$ , let us define the following error terms:

$$\varepsilon_0 = \frac{\bar{y}_n}{\bar{Y}} - 1; \quad \varepsilon_1 = \frac{\bar{x}_n}{\bar{X}} - 1; \quad \varepsilon_2 = \frac{\hat{M}_x}{M_x} - 1; \quad \varepsilon_3 = \frac{\bar{x}_m}{\bar{X}} - 1 \quad \text{and} \quad \varepsilon_4 = \frac{\hat{M}_x^*}{M_x} - 1$$

such that

$$E(\varepsilon_0) = E(\varepsilon_1) = E(\varepsilon_2) = 0; \quad E(\varepsilon_3) \approx E(\varepsilon_4) \approx 0$$

$$E(\varepsilon_0^2) = \left( \frac{1}{n} - \frac{1}{N} \right) C_y^2; \quad E(\varepsilon_1^2) = \left( \frac{1}{n} - \frac{1}{N} \right) C_x^2;$$

$$E(\varepsilon_2^2) = \left( \frac{1}{n} - \frac{1}{N} \right) \frac{1}{4\{f_x(M_x)\}^2 M_x^2}$$

$$\begin{aligned}
 E(\varepsilon_3^2) &= \left(\frac{1}{m} - \frac{1}{N}\right) C_x^2; & E(\varepsilon_4^2) &= \left(\frac{1}{m} - \frac{1}{N}\right) \frac{1}{4\{f_x(M_x)\}^2 M_x^2}; \\
 E(\varepsilon_0\varepsilon_1) &= \left(\frac{1}{n} - \frac{1}{N}\right) \rho_{xy} C_x C_y; & E(\varepsilon_1\varepsilon_3) &= \left(\frac{1}{m} - \frac{1}{N}\right) C_x^2; \\
 E(\varepsilon_0\varepsilon_2) &= -\left(\frac{1}{n} - \frac{1}{N}\right) \frac{S_{YM_x} \{f_x(M_x)\}^{-1}}{\bar{Y} M_x}; & E(\varepsilon_0\varepsilon_3) &= \left(\frac{1}{m} - \frac{1}{N}\right) \rho_{xy} C_x C_y; \\
 E(\varepsilon_0\varepsilon_4) &= -\left(\frac{1}{m} - \frac{1}{N}\right) \frac{S_{YM_x} \{f_x(M_x)\}^{-1}}{\bar{Y} M_x}; \\
 E(\varepsilon_1\varepsilon_2) &= -\left(\frac{1}{n} - \frac{1}{N}\right) \frac{S_{XM_x} \{f_x(M_x)\}^{-1}}{\bar{X} M_x} \\
 E(\varepsilon_1\varepsilon_4) &= -\left(\frac{1}{m} - \frac{1}{N}\right) \frac{S_{XM_x} \{f_x(M_x)\}^{-1}}{\bar{X} M_x}; \\
 E(\varepsilon_2\varepsilon_3) &= -\left(\frac{1}{m} - \frac{1}{N}\right) \frac{S_{XM_x} \{f_x(M_x)\}^{-1}}{\bar{X} M_x} \\
 E(\varepsilon_2\varepsilon_4) &= \left(\frac{1}{m} - \frac{1}{N}\right) \frac{1}{4\{f_x(M_x)\}^2 M_x^2}; \text{ and} \\
 E(\varepsilon_3\varepsilon_4) &= -\left(\frac{1}{m} - \frac{1}{N}\right) \frac{S_{XM_x} \{f_x(M_x)\}^{-1}}{\bar{X} M_x}
 \end{aligned}$$

where

$$S_{XM_x} = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})(I_{x_i} - 0.5),$$

$$f = n/N,$$

$$S_{YM_x} = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})(I_{x_i} - 0.5)$$

and

$$I_{x_i} = \begin{cases} 1, & \text{if } X_i \leq M_x \\ 0, & \text{otherwise} \end{cases}$$

Note that we used the following main result from Kuk and Mak (1989) in deriving the variance and co-variance expressions for the sample means and

sample median, that is, if  $F_x$  be the cumulative distribution function of  $X$ , then the sample median can be approximated as:

$$\hat{M}_x = M_x + (0.5 - p_x) f_x^{-1}(M_x) + \dots$$

where  $p_x$  be the proportion of  $I_{x_i}$  values taking a value of 1.

The new estimator  $\bar{y}_{kal}$  in terms of  $\varepsilon_i$ ,  $i = 0, 1, 2, 3, 4$  can be written as

$$\begin{aligned} \bar{y}_{kal} &= \bar{y}_n + \beta_1^* (\bar{x}_m - \bar{x}_n) + \beta_2^* (\hat{M}_x^l - \hat{M}_x) \\ &= \bar{Y}(1 + \varepsilon_0) + \beta_1^* [\bar{X}(1 + \varepsilon_3) - \bar{X}(1 + \varepsilon_1)] + \beta_2^* [M_x(1 + \varepsilon_4) - M_x(1 + \varepsilon_2)] \\ &= \bar{Y} + \bar{Y}\varepsilon_0 + \beta_1^* \bar{X}(\varepsilon_3 - \varepsilon_1) + \beta_2^* M_x(\varepsilon_4 - \varepsilon_2) \end{aligned} \quad (2.2)$$

Now we have the following theorems:

**Theorem 2.1.** The new proposed estimator  $\bar{y}_{kal}$  is an unbiased estimator of the population mean  $\bar{Y}$ .

**Proof.** Taking expected value on both sides of (2.2), we get

$$\begin{aligned} E(\bar{y}_{kal}) &= E\left[\bar{Y} + \bar{Y}\varepsilon_0 + \beta_1^* \bar{X}(\varepsilon_3 - \varepsilon_1) + \beta_2^* M_x(\varepsilon_4 - \varepsilon_2)\right] \\ &= \bar{Y} + \bar{Y} E(\varepsilon_0) + \beta_1^* \bar{X}(E(\varepsilon_3) - E(\varepsilon_1)) + \beta_2^* M_x(E(\varepsilon_4) - E(\varepsilon_2)) \\ &= \bar{Y} + 0 \\ &= \bar{Y} \end{aligned} \quad (2.3)$$

Thus the new estimator  $\bar{y}_{kal}$  is an unbiased estimator of  $\bar{Y}$  and it proves the theorem.

**Theorem 2.2.** The minimum variance, to the first order of approximation, of the new proposed estimator  $\bar{y}_{kal}$  is given by

$$\text{Min.}V(\bar{y}_{kal}) = \left(\frac{1}{m} - \frac{1}{N}\right) S_y^2 + \left(\frac{1}{n} - \frac{1}{m}\right) S_y^2 \left(1 - \rho_{xy}^2 - \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)}\right) \quad (2.4)$$

**Proof.** By the definition of variance, the variance of the unbiased estimator  $\bar{y}_{kal}$  is given by

$$V(\bar{y}_{kal}) = E[\bar{y}_{kal} - \bar{Y}]^2$$

$$\begin{aligned}
 &= E\left[\bar{Y}\varepsilon_0 + \beta_1^* \bar{X}(\varepsilon_3 - \varepsilon_1) + \beta_2^* M_x(\varepsilon_4 - \varepsilon_2)\right]^2 \\
 &= E\left[\bar{Y}^2 \varepsilon_0^2 + \beta_1^{*2} \bar{X}^2 (\varepsilon_3 - \varepsilon_1)^2 + \beta_2^{*2} M_x^2 (\varepsilon_4 - \varepsilon_2)^2 \right. \\
 &\quad \left. - 2\beta_1^* \bar{X}\bar{Y}\varepsilon_0(\varepsilon_3 - \varepsilon_1) - 2\beta_2^* M_x \bar{Y}\varepsilon_0(\varepsilon_4 - \varepsilon_2) + 2\beta_1^* \beta_2^* \bar{X}M_x(\varepsilon_3 - \varepsilon_1)(\varepsilon_4 - \varepsilon_2)\right] \tag{2.5}
 \end{aligned}$$

Further note that

$$E(\varepsilon_3 - \varepsilon_1)^2 = \left(\frac{1}{n} - \frac{1}{m}\right) C_x^2 \tag{2.6}$$

$$E(\varepsilon_4 - \varepsilon_2)^2 = \left(\frac{1}{n} - \frac{1}{m}\right) \frac{1}{4\{f_x(M_x)\}^2 M_x^2} \tag{2.7}$$

$$E(\varepsilon_0\varepsilon_3 - \varepsilon_0\varepsilon_1) = -\left(\frac{1}{n} - \frac{1}{m}\right) \rho_{xy} C_x C_y \tag{2.8}$$

$$E(\varepsilon_0\varepsilon_4 - \varepsilon_0\varepsilon_2) = \left(\frac{1}{n} - \frac{1}{m}\right) \frac{S_{YM_x}\{f_x(M_x)\}^{-1}}{\bar{Y} M_x} \tag{2.9}$$

and

$$E(\varepsilon_3\varepsilon_4 - \varepsilon_1\varepsilon_4 - \varepsilon_3\varepsilon_2 + \varepsilon_1\varepsilon_2) = -\left(\frac{1}{n} - \frac{1}{m}\right) \frac{S_{XM_x}\{f_x(M_x)\}^{-1}}{\bar{X} M_x} \tag{2.10}$$

On substituting (2.6) – (2.10) into (2.5), we have

$$\begin{aligned}
 V(\bar{y}_{kal}) &= \bar{Y}^2 \varepsilon_0^2 + \beta_1^{*2} \bar{X}^2 E(\varepsilon_3 - \varepsilon_1)^2 + \beta_2^{*2} M_x^2 E(\varepsilon_4 - \varepsilon_2)^2 \\
 &\quad - 2\beta_1^* \bar{X}\bar{Y}E(\varepsilon_0\varepsilon_3 - \varepsilon_0\varepsilon_1) - 2\beta_2^* M_x \bar{Y}E(\varepsilon_0\varepsilon_4 - \varepsilon_0\varepsilon_2) + 2\beta_1^* \beta_2^* \bar{X}M_x E(\varepsilon_3 - \varepsilon_1)(\varepsilon_4 - \varepsilon_2) \\
 &= \bar{Y}^2 \left(\frac{1}{n} - \frac{1}{N}\right) C_y^2 + \left(\frac{1}{n} - \frac{1}{m}\right) \beta_1^{*2} \bar{X}^2 C_x^2 + \beta_2^{*2} M_x^2 \left(\frac{1}{n} - \frac{1}{m}\right) \frac{1}{4\{f_x(M_x)\}^2 M_x^2} \\
 &\quad + 2\beta_1^* \bar{X} \bar{Y} \left(\left(\frac{1}{n} - \frac{1}{m}\right) \rho_{xy} C_x C_y\right) - 2\beta_2^* \bar{Y}M_x \left(\frac{1}{n} - \frac{1}{m}\right) \frac{S_{YM_x}\{f_x(M_x)\}^{-1}}{\bar{Y} M_x} \\
 &\quad - 2\beta_1^* \beta_2^* \bar{X}M_x \left(\frac{1}{n} - \frac{1}{m}\right) \frac{S_{XM_x}\{f_x(M_x)\}^{-1}}{\bar{X} M_x} \tag{2.11}
 \end{aligned}$$



On differentiating  $V(\bar{y}_{kal})$  with respect to  $\beta_1^*$  and  $\beta_2^*$  and each equating to zero, we have,

$$\frac{\partial V(\bar{y}_{kal})}{\partial \beta_1^*} = 0, \text{ and } \frac{\partial V(\bar{y}_{kal})}{\partial \beta_2^*} = 0$$

On solving the system of linear equations, the optimum values of  $\beta_1^*$  and  $\beta_2^*$  are given by

$$\beta_1^* = \frac{\{f_x(M_x)\}^{-2} S_{YM_x} S_{XM_x} - S_{xy} S_y^2}{\{f_x(M_x)\}^{-2} \left( \frac{1}{4} S_x^2 - S_{XM_x}^2 \right)} \quad (2.12)$$

and

$$\beta_2^* = \frac{S_x^2 S_{YM_x} - S_{xy} S_{XM_x}}{\{f_x(M_x)\}^{-1} \left( \frac{1}{4} S_x^2 - S_{XM_x}^2 \right)} \quad (2.13)$$

On substituting the optimum values of  $\beta_1^*$  and  $\beta_2^*$ , the minimum variance is given by

$$\begin{aligned} \text{Min.}V(\bar{y}_{kal}) &= \left( \frac{1}{m} - \frac{1}{N} \right) S_y^2 + \left( \frac{1}{n} - \frac{1}{m} \right) \left( S_y^2 - \frac{S_{xy}^2}{S_x^2} - \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right) \\ &= \left( \frac{1}{m} - \frac{1}{N} \right) S_y^2 + \left( \frac{1}{n} - \frac{1}{m} \right) S_y^2 \left( 1 - \rho_{xy}^2 - \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right) \end{aligned} \quad (2.14)$$

which proves the theorem.

**Remark:** It may be worth pointing out that the replacement of  $\beta_1^*$  and  $\beta_2^*$  with their consistent estimates lead to a new estimator of the population mean in two two-phase sampling which has same mean square error up to the first order of approximation as the minimum variance in (2.14). Such changes do not affect the results up to the first order of approximation. ((6.1) and (6.2) in Singh, Singh, and Upadhyaya (2007)).

### 3. Comparison of different estimators

Note that

$$0.25 S_x^2 - S_{XM_x}^2 > 0$$

$$\frac{S_{XM_x}^2}{S_x^2} < \frac{1}{4} \tag{3.1}$$

Thus the proposed new estimator  $\bar{y}_{kal}$  is always more efficient than the sample mean, ratio and the regression type estimator in two-phase sampling. It may be worth pointing out that the final minimum variance of the proposed estimator is free from the value of  $f_x(M_x)$ , hence from computational point.

In the next section, we focus on the cost analysis in two-phase sampling because it is considered as one among the list of cost effective sampling schemes in survey sampling.

### 4. Cost Analysis

In this section we consider comparison of different estimators with cost aspects. Let  $C_0$  be the overhead cost,  $C_1$  be the cost of information from one unit in the first phase sample; and  $C_2$  be the cost of information from one unit in the second phase sample. Note that the value of  $C_1$  is always smaller than that of  $C_2$ . Thus the total cost function is given by

$$C = C_0 + mC_1 + nC_2 \tag{4.1}$$

Now we have the following results

For the fixed cost  $C$  of the survey, the minimum variance of the proposed estimator  $\bar{y}_{kal}$  is given by

$$Min.V(\bar{y}_{kal})_{fixed\_cost} =$$

$$= \frac{\left( \sqrt{C_1 S_y^2 \left( \rho_{xy}^2 + \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right)} + \sqrt{C_2 S_y^2 \left[ 1 - \rho_{xy}^2 - \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right]} \right)^2}{C - C_0} - \frac{S_y^2}{N} \tag{4.2}$$

The optimum values of  $m$  and  $n$  are given by

$$m = m_{kal} = \frac{(C - C_0) \sqrt{S_y^2 \left( \rho_{xy}^2 + \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right)}}{\sqrt{C_1} \left[ \sqrt{C_1 S_y^2 \left( \rho_{xy}^2 + \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right)} + \sqrt{C_2 S_y^2 \left( 1 - \rho_{xy}^2 - \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right)} \right]} \tag{4.3}$$

and

$$n = n_{kal} = \frac{(C - C_0) \sqrt{S_y^2 \left( 1 - \rho_{xy}^2 - \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right)}}{\sqrt{C_1} \left[ \sqrt{C_1 S_y^2 \left( \rho_{xy}^2 + \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right)} + \sqrt{C_2 S_y^2 \left( 1 - \rho_{xy}^2 - \frac{(S_x^2 S_{YM_x} - S_{xy} S_{XM_x})^2}{S_y^2 S_x^2 (0.25 S_x^2 - S_{XM_x}^2)} \right)} \right]} \quad (4.4)$$

which proves the theorem.

In case of single-phase sampling, the total cost function is given by

$$C = C_0 + nC_2 \quad (4.5)$$

From (4.11), we have the optimum sample size as:

$$n = \frac{C - C_0}{C_2} = n_u \text{ (say)} \quad (4.6)$$

The variance of the sample mean estimator is given by

$$V(\bar{y}_n)_{fixed\_cost} = \left( \frac{C_2}{C - C_0} - \frac{1}{N} \right) S_y^2 \quad (4.7)$$

The optimum values of  $m$  and  $n$  for the ratio estimator are given by

$$m = \frac{(C - C_0) \sqrt{2RS_{xy} - R^2 S_x^2}}{\sqrt{C_1} \left[ \sqrt{C_1 (2RS_{xy} - R^2 S_x^2)} + \sqrt{C_2 (S_y^2 + R^2 S_x^2 - 2RS_{xy})} \right]} = m_{rat} \quad (4.8)$$

and

$$n = \frac{(C - C_0) \sqrt{(S_y^2 + R^2 S_x^2 - 2RS_{xy})}}{\sqrt{C_1} \left[ \sqrt{C_1 (2RS_{xy} - R^2 S_x^2)} + \sqrt{C_2 (S_y^2 + R^2 S_x^2 - 2RS_{xy})} \right]} = n_{rat} \quad (4.9)$$

The minimum variance of the ratio estimator for the fixed cost is given by:

$$Min.V(\bar{y}_{rat})_{fixt\_cost} = \frac{\left[ \sqrt{C_1 (2RS_{xy} - R^2 S_x^2)} + \sqrt{C_2 (S_y^2 + R^2 S_x^2 - 2RS_{xy})} \right]^2}{C - C_0} - \frac{S_y^2}{N} \quad (4.10)$$

The optimum values of  $m$  and  $n$  for the regression type estimator  $\bar{y}_{reg(d)}$  are given by

$$m = \frac{(C - C_0)\sqrt{S_y^2 \rho_{xy}^2}}{\sqrt{C_1} \left[ \sqrt{C_1 S_y^2 \rho_{xy}^2} + \sqrt{C_2 S_y^2 (1 - \rho_{xy}^2)} \right]} = m_{reg} \tag{4.11}$$

and

$$n = \frac{(C - C_0)\sqrt{S_y^2 (1 - \rho_{xy}^2)}}{\sqrt{C_1} \left[ \sqrt{C_1 S_y^2 \rho_{xy}^2} + \sqrt{C_2 S_y^2 (1 - \rho_{xy}^2)} \right]} = n_{reg} \tag{4.12}$$

Note that for the fixed cost of the survey, the variances of the regression type estimator  $\bar{y}_{reg(d)}$  is given by

$$Min.V(\bar{y}_{reg})_{fixed\_cost} = \frac{\left( \sqrt{C_1 S_y^2 \rho_{xy}^2} + \sqrt{C_2 S_y^2 (1 - \rho_{xy}^2)} \right)^2}{C - C_0} - \frac{S_y^2}{N} \tag{4.13}$$

The percent relative efficiency of the new proposed estimator  $\bar{y}_{kal}$  with respect to the sample mean estimator ( $\bar{y}_n$ ), ratio estimator ( $\bar{y}_{rat(d)}$ ), and regression type estimators ( $\bar{y}_{reg(d)}$ ):

$$RE(0) = RE(\text{sample mean}) = \frac{Min.V(\bar{y}_n)_{fixed\_cost}}{Min.V(\bar{y}_{kal})_{fixed\_cost}} \times 100\% \tag{4.14}$$

$$RE(1) = RE(\text{ratio}) = \frac{Min.V(\bar{y}_{rat})_{fixed\_cost}}{Min.V(\bar{y}_{kal})_{fixed\_cost}} \times 100\% \tag{4.15}$$

$$RE(2) = RE(\text{reg}) = \frac{Min.V(\bar{y}_{reg})_{fixed\_cost}}{Min.V(\bar{y}_{kal})_{fixed\_cost}} \times 100\% \tag{4.16}$$

In order to see the magnitude of the relative efficiency of the proposed estimator  $\bar{y}_{kal}$  over the mean, ratio and the regression type estimator, we did simulation study.

### 5. Simulation Study

From (3.1), it is clear that the maximum value of  $S_{XM_x} < \frac{S_x}{2}$ . We consider many populations of size  $N = 50,000$ ,  $\bar{X} = 50$ ,  $\bar{Y} = 20$ ,  $S_y^2 = 5$ ,  $S_x^2 = 10$  and

different values of the correlation coefficient  $\rho_{xy}$ . Note that it is very likely that the possible value of  $0 < S_{XM_x} < 0.5$  and  $0 < S_{YM_x} < 0.5$ . Consider a situation of having total cost  $C = \$5000$ , overhead cost  $C_0 = \$500$ , cost of collecting information from one unit in the first-phase sample  $C_1 = \$4$ , and the cost of collecting information from one unit in the sample  $C_2 = \$10$ . We wrote R-codes to compute the percent relative efficiency values and the optimum sample sizes for the four estimators. There are many situations where the proposed estimator performs better than the existing estimators, and in the simulation study we stored only those combinations of  $\rho_{xy}$ ,  $S_{XM_x}$  and  $S_{YM_x}$  where the value of  $RE(2)$  is greater than 105%. In other words, the proposed estimator is at least 105% more efficient than the linear regression type estimator in double sampling. The results so obtained are presented in Table 5.1

**Table 5.1.** Percent relative efficiency of the proposed new estimator over the three estimators and optimum sample sizes

| $\rho_{xy}$ | $S_{XM_x}$ | $S_{YM_x}$ | $n_u$ | $m_{rat}$ | $n_{rat}$ | $m_{reg}$ | $n_{reg}$ | $m_{kal}$ | $n_{kal}$ | $RE(0)$ | $RE(1)$ | $RE(2)$ |
|-------------|------------|------------|-------|-----------|-----------|-----------|-----------|-----------|-----------|---------|---------|---------|
| 0.80        | 0.05       | 0.40       | 450   | 483       | 406       | 515       | 386       | 540       | 370       | 107.7   | 108.3   | 105.4   |
| 0.80        | 0.05       | 0.45       | 450   | 483       | 406       | 515       | 386       | 550       | 364       | 110.1   | 110.6   | 107.8   |
| 0.80        | 0.05       | 0.50       | 450   | 483       | 406       | 515       | 386       | 565       | 354       | 113.4   | 114.0   | 111.0   |
| 0.80        | 0.10       | 0.45       | 450   | 483       | 406       | 515       | 386       | 544       | 368       | 108.7   | 109.2   | 106.4   |
| 0.80        | 0.10       | 0.50       | 450   | 483       | 406       | 515       | 386       | 556       | 360       | 111.4   | 112.0   | 109.1   |
| 0.80        | 0.15       | 0.45       | 450   | 483       | 406       | 515       | 386       | 539       | 371       | 107.5   | 108.1   | 105.3   |
| 0.80        | 0.15       | 0.50       | 450   | 483       | 406       | 515       | 386       | 549       | 364       | 109.8   | 110.4   | 107.5   |
| 0.80        | 0.20       | 0.50       | 450   | 483       | 406       | 515       | 386       | 543       | 368       | 108.5   | 109.0   | 106.2   |
| 0.80        | 0.25       | 0.50       | 450   | 483       | 406       | 515       | 386       | 538       | 371       | 107.4   | 107.9   | 105.1   |
| 0.85        | 0.05       | 0.35       | 450   | 516       | 385       | 568       | 352       | 600       | 332       | 115.4   | 112.8   | 106.8   |
| 0.85        | 0.05       | 0.40       | 450   | 516       | 385       | 568       | 352       | 616       | 322       | 118.9   | 116.3   | 110.1   |
| 0.85        | 0.05       | 0.45       | 450   | 516       | 385       | 568       | 352       | 637       | 309       | 123.9   | 121.2   | 114.8   |
| 0.85        | 0.05       | 0.50       | 450   | 516       | 385       | 568       | 352       | 670       | 288       | 131.7   | 128.8   | 122.0   |
| 0.85        | 0.10       | 0.35       | 450   | 516       | 385       | 568       | 352       | 594       | 336       | 113.8   | 111.3   | 105.4   |
| 0.85        | 0.10       | 0.40       | 450   | 516       | 385       | 568       | 352       | 606       | 328       | 116.6   | 114.1   | 108.0   |
| 0.85        | 0.10       | 0.45       | 450   | 516       | 385       | 568       | 352       | 623       | 317       | 120.7   | 118.1   | 111.8   |
| 0.85        | 0.10       | 0.50       | 450   | 516       | 385       | 568       | 352       | 649       | 301       | 126.7   | 123.9   | 117.3   |
| 0.85        | 0.15       | 0.40       | 450   | 516       | 385       | 568       | 352       | 598       | 333       | 114.8   | 112.3   | 106.4   |
| 0.85        | 0.15       | 0.45       | 450   | 516       | 385       | 568       | 352       | 613       | 324       | 118.2   | 115.6   | 109.4   |
| 0.85        | 0.15       | 0.50       | 450   | 516       | 385       | 568       | 352       | 633       | 311       | 122.9   | 120.2   | 113.8   |
| 0.85        | 0.20       | 0.45       | 450   | 516       | 385       | 568       | 352       | 604       | 330       | 116.1   | 113.6   | 107.5   |
| 0.85        | 0.20       | 0.50       | 450   | 516       | 385       | 568       | 352       | 620       | 319       | 120.0   | 117.3   | 111.1   |
| 0.85        | 0.25       | 0.45       | 450   | 516       | 385       | 568       | 352       | 596       | 334       | 114.4   | 111.9   | 106.0   |
| 0.85        | 0.25       | 0.50       | 450   | 516       | 385       | 568       | 352       | 610       | 326       | 117.6   | 115.0   | 108.9   |
| 0.85        | 0.30       | 0.50       | 450   | 516       | 385       | 568       | 352       | 602       | 331       | 115.7   | 113.2   | 107.1   |

**Discussion:** From the simulation study, it is observed that a high value of correlation coefficient  $\rho_{xy}$  and a high value of  $S_{YM_x}$  is required for the proposed new estimator to show percent relative efficiency of at least 105%. For example, if  $\rho_{xy} = 0.80$ ,  $S_{XM_x} = 0.05$ ,  $S_{YM_x} = 0.40$ , with a total cost of \$5000, if instead of we use only single phase sample mean estimator with optimum sample size  $n_u = 450$ , we should use the proposed estimator with optimum sample sizes  $m_{kal} = 540$  and  $n_{kal} = 370$ , then the percent relative efficiency value is  $RE(0) = 107.7\%$ . Instead of using the ratio estimator with optimum sample sizes  $m_{rat} = 483$  and second-phase sample size  $n_{rat} = 406$ , if one uses the proposed estimator with optimum sample sizes  $m_{kal} = 540$  and  $n_{kal} = 370$  then the percent relative efficiency value is  $RE(1) = 108.3\%$ . In the same way, if we use the proposed estimator with optimum sample sizes  $m_{kal} = 540$  and  $n_{kal} = 370$ , then the relative efficiency of the proposed estimator over the regression method of estimation with optimum sample sizes  $m_{reg} = 515$  and  $n_{reg} = 386$  is  $RE(2) = 105.4\%$ .

Note that  $n_{kal}(= 370) < n_{reg}(= 386) < n_{rat}(= 406) < n_u(= 450)$ . The optimum second-phase sample size remains lowest in case of the proposed estimator, thus the proposed estimator reduces efforts for collecting data on the second-phase of the sample for the fixed cost of the survey and provides efficient results. The rest of the results in Table 5.1 can also be interpreted in the same way. We conclude that there exist several situations where the proposed new estimator can be used more efficiently for a fixed cost of the survey.

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