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**Working Paper**

## Poisson Process and Distribution-Free Statistics

Discussion Paper, No. 24

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*Suggested Citation:* Dwass, Meyer (1972) : Poisson Process and Distribution-Free Statistics, Discussion Paper, No. 24, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

This Version is available at:

<https://hdl.handle.net/10419/220384>

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Discussion Paper No. 24

POISSON PROCESS AND  
DISTRIBUTION-FREE STATISTICS, I

by

Meyer Dwass

November 20, 1972

## 1. INTRODUCTION

Suppose that  $Y(t)$ ,  $0 \leq t < \infty$ , is a Poisson process with stationary increments and parameter  $\lambda$ . Let  $T$  equal the time  $t$  at which  $Y(t) = t$  for the last time. That is

$$T = \sup (t: Y(t) = t)$$

The random variable  $T$  assumes the values  $0, 1, \dots, \infty$ . We have the following basic facts which are easily verified:

$$\begin{aligned} P(Y(t) < t, \text{ all } t > 0) \\ = P(T = 0) &= \begin{cases} 1 - \lambda, & \text{if } \lambda < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (1.1)$$

(See Appendix A)

$$P(T < \infty) = \begin{cases} 1 & \text{if } \lambda < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(T = k) = \frac{(\lambda k)^k e^{-\lambda k}}{k!} (1 - \lambda), \quad k=0, 1, \dots \quad (1.2)$$

if  $\lambda < 1$

Let  $U$  be a function of  $Y(\cdot)$  with the following property:

The value of  $U$  is completely determined by the function  $Y(t)$  for  $0 \leq t \leq T$ . (1.3)

Denote

$$G_n(t) = \frac{Y(tn)}{n}, \quad 0 \leq t \leq 1.$$

Subject to the condition that  $T = n$ , the behavior of  $G_n(t)$  is exactly that

to see that  $N_n$  is "distribution free". That is the distribution of  $N_n$  does not depend on the form of the cdf  $F$ , where  $F(u) = P(X_i < u)$ , as long as  $F$  is continuous. We assume here that  $F(u) = u, 0 \leq u \leq 1.$

It is obvious that  $N$  is geometrically distributed when  $\lambda < 1.$  Specifically,

$$P(N \geq k) = \lambda^k, \quad k = 0, 1, \dots \quad (1.7)$$

The counterpart of equation (1.4) is

$$P(N \geq k) = \sum_{n=0}^{\infty} P(N_n \geq k) \frac{n^n}{n!} (\lambda e^{-\lambda})^n (1 - \lambda)$$

It will be seen in Appendix C that

$$\frac{\lambda^k}{(1-\lambda)} = \sum_{n=0}^{\infty} \frac{n^{n-k}}{(n-k)!} (\lambda e^{-\lambda})^n, \quad k = 0, 1, \dots$$

(This follows from equation (C.1) with  $u = 0.$ )

Hence the counterpart of the relationship (1.4) is

$$P(N_n \geq k) = \frac{n^{n-1}}{(n-k)!} \frac{n!}{n^n} = \frac{n!}{(n-k)! n^k}$$

By a simple application of Stirling's formula, it follows that

$$\lim_{n \rightarrow \infty} P \left( \frac{N_n}{\sqrt{n}} \geq t \right) = e^{-t^2/2} \quad (1.8)$$

a result which was first proved by Smirnov [8].

We close this introduction by considering a second example.

Example 2 Let

$L =$  number of ladder points of  $Y(t) - t, t \geq 0.$  That is  $L$  equals the number of times that  $Y(t) - t$  achieves positive maxima which exceed all preceding maxima. (See Figure 1.)

Similarly we let  $L_n$  denote the number of ladder points in the empirical cdf  $F_n$ .<sup>1/</sup> The random variable  $L$  is also geometrically distributed with

$$P(L \geq k) = \lambda^k$$

Hence it follows that

$$P(N_n \geq k) = P(L_n \geq k) = \frac{n!}{(n-k)!n^k}$$

and

$$\lim_{n \rightarrow \infty} P\left(\frac{L_n}{\sqrt{n}} \geq t\right) = e^{-t^2/2}$$

## 2. ONE-SIDED MAXIMA. FIRST APPROACH

We define

$$M = \sup_{t > 0} Y(t) - t$$

which is the maximum exceedance of the Poisson process  $Y(t)$  above the straight line  $t$ . Suppose that  $\lambda < 1$ . If  $M \geq u$ , (where  $u$  is an arbitrary positive number not necessarily an integer) this means that  $Y(\cdot)$  intersects the straight line  $t + u$ , necessarily only a finite number of times. The probability that such an intersection last occurs at height  $n$  is

$$\frac{[\lambda(n-u)]^n}{n!} e^{-\lambda(n-u)} (1-\lambda).$$

Hence

$$P(M \geq u) = \sum_{n > u} \frac{(n-u)^n}{n!} (\lambda e^{-\lambda})^n e^{\lambda u} (1-\lambda) \quad (2.1)$$

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<sup>1/</sup> From now on throughout this paper it will be assumed that  $F_n$  is the empirical cdf of  $n$  independent random variables, each uniformly distributed on  $[0,1]$ .

$$P(M_n < u) = \sum_{\substack{i+j=n \\ i \leq u}} u \binom{n}{i} \frac{(i-u)^i (j+u)^{j-i}}{n^n} \quad (2.5)$$

Equations (2.4), (2.5) are well-known. See [1], [2], [3], [4]. It is easy to give a direct proof of (2.4) which is analogous to that of (2.1) using the relation (B.1) in Appendix B. The argument goes as follows:  $(M_n \geq u)$  means that  $F_n(t)$  crosses the line  $t + \frac{u}{n}$  somewhere. The probability of a last crossing at height  $i/n$  is the binomial probability

$$\binom{n}{i} \left(\frac{i}{n} - \frac{u}{n}\right)^i \left(1 - \frac{i}{n} + \frac{u}{n}\right)^{n-i}$$

times the conditional probability that the part of  $F_n(t)$  traversing between  $(\frac{i}{n} - \frac{u}{n}, \frac{i}{n})$  and  $(1,1)$  stays below  $t + \frac{u}{n}$ . By (B.1), Appendix B this equals

$$\frac{1 - (1 - \frac{u}{n})}{1 - (\frac{i}{n} - \frac{u}{n})} = \frac{u/n}{(n-i+u)/n}$$

Hence

$$P(M_n \geq u) = \sum_{i > u} \binom{n}{i} \left(\frac{i}{n} - \frac{u}{n}\right)^i \left(1 - \frac{i}{n} + \frac{u}{n}\right)^{n-i-1} \frac{u}{n}$$

which is the same as (2.4). In Section 3 we will give a more intuitive explanation of (2.1) and (2.4).

### 3. ONE SIDED MAXIMA. SECOND APPROACH

We first need a technical preliminary about the Poisson process which is interesting in its own right. Define

$$J = \begin{cases} \text{value of } Y(t) - t \text{ at the first instant that } Y(t) > t \text{ if} \\ \text{this takes place,} \\ 0, \text{ otherwise.} \end{cases}$$

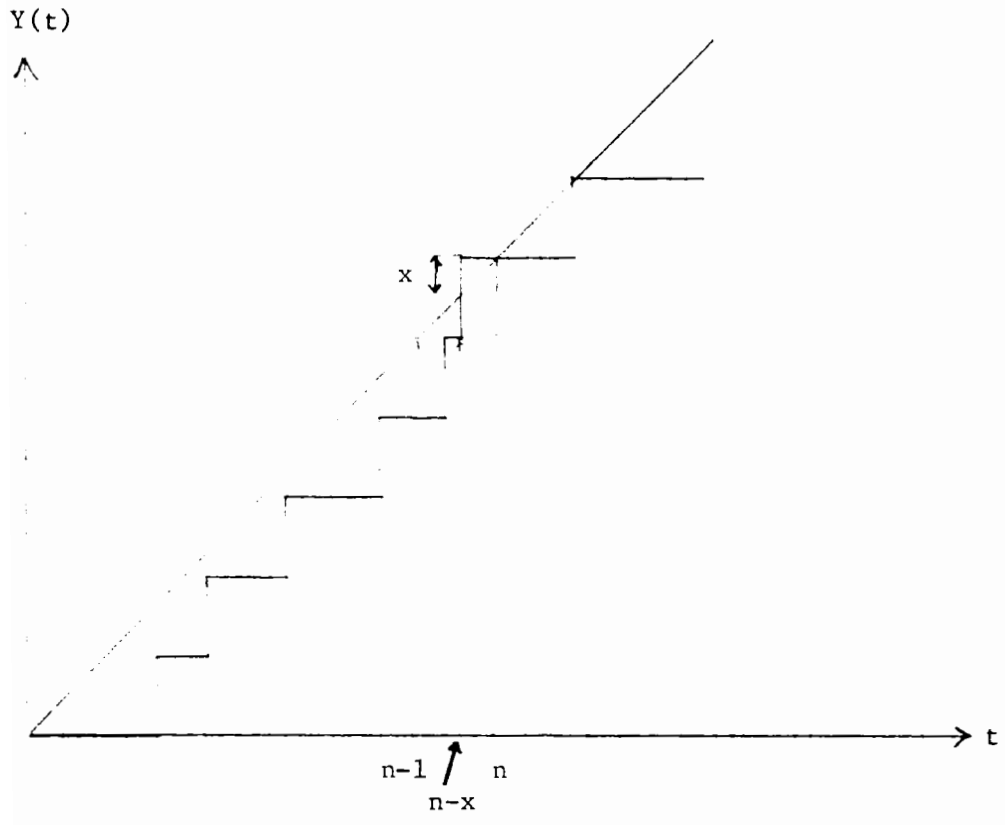


Figure 2

In other words,  $M$  is a sum of a geometric number of uniformly distributed random variables.

It is well-known that if  $U_1, U_2, \dots$  are independent random variables, each uniformly distributed on  $[0,1]$ , then

$$P\left(\sum_{i=1}^n U_i < u\right) = \sum_{i=0}^n \binom{n}{i} (-1)^i [c(u-i)]^n / n! \quad (3.3)$$

where

$$c(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Hence we can derive the distribution of  $M$  once again, using Theorem 3.3, by the calculation

$$\begin{aligned} P(M \leq u) &= \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^n \binom{n}{i} (-1)^i [c(u-i)]^n / n! \right\} \lambda^n (1-\lambda) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{n \geq i} \frac{[c(u-i)]^n}{(n-i)!} \lambda^n (1-\lambda) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} [(\lambda c(u-i))] e^{\lambda c(u-i)} (1-\lambda) \\ &= \sum_{0 \leq i < u} \frac{(i-u)^i}{i!} (\lambda e^{-\lambda})^i e^{\lambda u} (1-\lambda) \end{aligned}$$

which agrees with (2.2).

The Laplace transform of  $M$  is

$$\begin{aligned} Ee^{-\theta M} &= \sum_{r=0}^{\infty} \left( \frac{1-e^{-\theta}}{\theta} \right)^r \lambda^r (1-\lambda) \\ &= \frac{\theta(1-\lambda)}{\theta - \lambda(1-e^{-\theta})} \end{aligned}$$



conditionally distributed as  $n$  independent random variables, each uniformly distributed on  $[0,1]$ .

(b)  $M_n$  is distributed as

$$U_1 + \dots + U_{L_n}$$

where  $U_1, U_2, \dots$  is a sequence of independent, uniform  $[0,1]$  random variables, independent of  $L_n$ .

It is now easy to determine the asymptotic distribution of  $M_n$ .

Theorem 3.5

$$\lim P\left(\frac{M_n}{\sqrt{n}} \geq t\right) = e^{-2t^2}$$

Proof:

$\frac{M_n}{\sqrt{n}}$  is distributed as

$$\frac{U_1 + \dots + U_{L_n}}{\sqrt{n}} = \frac{U_1 + \dots + U_{L_n}}{L_n} \frac{L_n}{\sqrt{n}}$$

Since  $L_n$  converges in probability to  $\infty$  as  $n \rightarrow \infty$  (Example 2, Introduction) it follows that  $(U_1 + \dots + U_{L_n})/L_n$  converges in probability to  $\frac{1}{2}$ . Hence (by Example, 2, Introduction)

$$\lim P\left(\frac{M_n}{\sqrt{n}} \geq t\right) = \lim P\left(\frac{1}{2} \frac{L_n}{\sqrt{n}} \geq t\right) = e^{-(2t^2)/2}$$

which completes the proof. The above result was first proved by Kolmogorov [5].

Let us consider now the random variables  $I_1, I_2, \dots$  defined earlier. This is the succession of excesses of  $Y(t)$  over the line  $t$ . The counterparts for empirical cdf's is  $I_1^{(n)}, I_2^{(n)}, \dots$  the succession of excesses of  $n(F_n(t) - t)$

$$N(1) = N, L(1) = L, N_n(1) = N_n, L_n(1) = L_n.$$

It should be emphasized, however, that the random variable  $T$  still refers to last crossing of the line  $t$  (c.f. Section 1.). Equivalent to (1.1) is the relationship

$$P(Y(t) < \rho t \text{ all } t > 0) = \begin{cases} 1 - \frac{\lambda}{\rho} & \text{if } \lambda < \rho \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $N(\rho)$  is geometrically distributed with

$$P(N(\rho) \geq k) = \left(\frac{\lambda}{\rho}\right)^k, \quad k = 0, 1, \dots$$

Since

$$P(N(\rho) \geq k) = P(N \geq k) / \rho^k$$

it follows that for  $\rho > 1$ ,

$$P(N_n(\rho) \geq k) = P(N_n \geq k) / \rho^k = \frac{1}{\rho} \frac{n!}{(n-k)! n^k} \quad k = 0, 1, 2, \dots \quad (4.1)$$

Before going further, let us point out one interesting consequence of (4.1); namely,

$$P(N_n(\rho) \geq 1) = \frac{1}{\rho} \quad (4.2)$$

But this is just

$$1 - P(F_n(t) < \rho t, \text{ all } 0 < t \leq 1)$$

which we know equals  $1/\rho$  by (B.1), Appendix B. Thus (4.1) generalizes that assertion.

We now see an asymptotic result for  $N_n(\rho)$  by letting  $\rho$  vary with  $n$ , as follows.

To obtain a version of Theorem 3.4 for  $M_n(\rho)$  we consider the random variables

$$I_1, I_2, \dots, \quad J_1, J_2, \dots$$

$$I_1^{(n)}, I_2^{(n)}, \dots, \quad J_1^{(n)}, J_2^{(n)}, \dots$$

where the role of the line  $t$  is now played by  $\rho t$ . In other words,  $I_1, I_2, \dots$  is the succession of excesses of  $Y(t)$  over the line  $\rho t$ ;  $J_1, J_2, \dots$  is the succession of increases of  $Y(t) - \rho t$  at the succession of ladder points of  $Y(t) - \rho t$ . (Strictly speaking the notation for the  $I_i$ 's and  $J_i$ 's should indicate their dependence on  $\rho$  but we suppress the  $\rho$  in the interest of typography.) Since when  $\rho = 1$ , the conditional distributions of the jumps as described in Theorems 3.2 and 3.4 do not depend on the actual value of  $\lambda$  as long as  $\lambda < 1$ , it follows similarly that the following is true:

Theorem 4.1

$$P((I_1, \dots, I_k) \in A \mid N(\rho) \geq k) \tag{a}$$

$$= P((J_1, \dots, J_k) \in A \mid L(\rho) \geq k) \tag{b}$$

$$= P((I_1^{(n)}, \dots, I_k^{(n)}) \in A \mid N_n(\rho) \geq k) \tag{c}$$

$$= P((J_1^{(n)}, \dots, J_k^{(n)}) \in A \mid L_n(\rho) \geq k) \tag{d}$$

$$= P((U_1, \dots, U_k) \in A)$$

where  $U_1, \dots, U_k$  are independent random variables each uniformly distributed on  $[0,1]$ . For (a) and (b) we need to assume that  $\rho \geq \lambda$ . For (c) and (d) we need  $\rho \geq 1$ .

It follows now, as in Theorem 3.4, that

Lemma 5.1 Suppose  $\lambda < 1$ .

$$\begin{aligned} & P((-r < Y(t) - t < s; 0 \leq t \leq T) \cup (T = 0)) \\ &= \frac{P(M < s) P(M < r)}{P(M < r + s)} \end{aligned}$$

Proof: Define

$$R_1(r, s) = P(Y(t) \text{ hits } t-r \text{ and avoids } t+s \text{ en route})$$

$$R_2(r, s) = P(Y(t) \text{ hits } t+s \text{ and avoids } t-r \text{ en route})$$

Since  $\lambda < 1$ ,

$$P(Y(t) \text{ hits } t - r) = 1 = R_1(r, s) + R_2(r, s)$$

By the regenerative nature of the Poisson process,

$$\begin{aligned} & P((Y(t) \text{ hits either } t - r \text{ or } t + s \quad 0 \leq t \leq T) \cap (T > 0)) \\ &= R_2(r, s) + R_1(r, s) (1 - P(M < r)). \end{aligned}$$

We also have

$$P(M < s) = R_1(r, s) P(M < r + s)$$

Hence,

$$\begin{aligned} & P((r < Y(t) - t < s, 0 \leq t \leq T) \cup (T = 0)) \\ &= 1 - P((Y(t) \text{ hits either } t - r \text{ or } t + s, 0 \leq t \leq T) \cap (T > 0)) \\ &= 1 - R_2(r, s) + R_1(r, s)(1 - P(M < r)) \\ &= R_2(r, s) P(M < r) \\ &= \frac{P(M < r) P(M < s)}{P(M < r + s)} \end{aligned}$$

which completes the proof.

Substituting in the right side of (5.3) and cancelling out the  $(1-\lambda)$ 's gives (5.1) with  $\lambda e^{-\lambda} = e^{-1}u$ . Since  $\lambda e^{-\lambda}/e^{-1}$  defines a 1-1 map of  $[0,1]$  onto itself this gives (5.1). Similarly, if we use (2.2) and substitute in (5.3) this gives (5.2).

From (5.2) it follows that the  $Q_n(r,s)$  terms satisfy the following difference equations:

$$\sum_{\substack{i+j=n \\ j \leq r+s}} Q_i(r,s) \frac{i^i}{i!} \frac{(j-r-s)^j}{j!} = \begin{cases} \sum_{\substack{i+j=n \\ i \leq r \\ j \leq s}} \frac{(i-r)^i}{i!} \frac{(j-s)^j}{j!} & \text{if } n \leq r+s \\ 0 & \text{if } n > r+s \end{cases}$$

These can be used for iterative numerical computation of the  $Q_n(r,s)$ . For  $r = s$  the above difference equations are similar to ones developed by other methods by Massey [6].

APPENDIX B

Let  $F_n(t)$ ,  $0 \leq t \leq 1$  be an empirical cdf of  $n$  independent random variables, each uniformly distributed over  $[0,1]$ . Then if  $\gamma > 1$

$$P(F_n(t) < \gamma t \text{ all } t \text{ in } [0,1]) = 1 - \frac{1}{\gamma} \tag{B.1}$$

This is easily proved by induction on  $n$  as follows. Let  $U$  be the largest of the  $n$  points selected in  $[0,1]$ . Then  $U$  has the integrating density  $nu^{n-1}$  in  $[0,1]$ . By the induction hypothesis

$$P(F_n(t) < \gamma t \text{ all } t \mid U = u) = \begin{cases} [u - \frac{(n-1)}{n\gamma}] / u & \text{if } u > 1/\gamma \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$P(F_n(t) < \gamma t \text{ all } t) = \int_0^1 (u - \frac{(n-1)}{n\gamma}) (1/u) nu^{n-1} du = 1 - \frac{1}{\gamma}$$

Since (B.1) clearly holds for  $n = 1$ , this completes the proof. Another proof of (B.1) appears in Section 4 (See 4.1).

$$(n + u)^{n-k} = (n + u)^{n-k-1} (n - k + k + u)$$

Hence, from (C.1)

$$\lambda^k e^{\lambda u} / (1-\lambda) = \sum_{n \geq k+1} \frac{(n+u)^{n-k-1}}{(n-k-1)!} (\lambda e^{-\lambda})^n + (k+u) \sum_{n \geq k} \frac{(n+u)^{n-k-1}}{(n-k)!} (\lambda e^{-\lambda})^n$$

The first expression on the right can be evaluated by (C.1) to equal

$$(\lambda e^{-\lambda}) \lambda^k e^{\lambda(u+1)} / (1 - \lambda)$$

Hence considering the second expression on the right,

$$\begin{aligned} (k+u) \sum_{n \geq k} \frac{(n+u)^{n-k-1}}{(n-k)!} (\lambda e^{-\lambda})^n &= (\lambda^k e^{\lambda u} - (\lambda e^{-\lambda}) \lambda^k e^{\lambda(u+1)}) / (1-\lambda) \\ &= \lambda^k e^{\lambda u} \end{aligned}$$

which proves (C.2) also in the case that  $u \geq 0$ . It follows that the left sides of (C.1) and (C.2) have absolutely convergent power series expansions in  $u$  for all  $u$ . Hence from considerations of analyticity both sides of (C.1) and (C.2) are equal for all  $u$ .