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Discussion Paper No. 41

EXTREMAL PROCESSES, III

by

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## 1. INTRODUCTION.

In this paper we show the relation between extremal processes and their "inverses." Suppose that  $Y(\cdot)$  is a real-valued, non-decreasing, right-continuous function defined on  $(a,b)$ ,  $(-\infty < a < b < \infty)$ . By the inverse of  $Y(\cdot)$  we mean the function  $Z(\cdot)$  defined by

$$Z(u) = \sup\{t \mid Y(t) \leq u\}$$

Thus,  $Z(\cdot)$  is also a real-valued, non-decreasing, right-continuous function. It is not hard to see that the inverse of  $Z(\cdot)$  is the original  $Y$ . We will show that the inverses of extremal processes are certain additive processes, that is processes with independent (but not stationary) increments.

## 2. DISCRETE TIME CASE.

We first consider the case of successive maxima of a sequence of independent and identically distributed random variables. This case serves as a motivation for the limiting case of extremal processes (see Sec. 2 of [1]) and also has some real interest of its own. Let  $X_1, X_2, \dots$  be a sequence of independent and identically distributed random variables with c.d.f.  $F(\cdot)$ . Define

$$M_n = \max(X_1, \dots, X_n)$$

and

$$M(t) = M_n \text{ for } n-1 \leq t < n, \quad n = 1, 2, \dots$$

Thus the random function  $M(\cdot)$  is with probability one a non-decreasing, right-continuous function. The structure of  $M(\cdot)$  is completely determined by the sequence  $M_1, M_2, \dots$  which is a Markov chain with non-stationary transitions. The evolution of  $M(\cdot)$  is as follows:

(a) If  $M(n) = x$  then  $M(\cdot)$  remains equal to  $x$  for a geometrically distributed period of time whose parameter depends on  $x$ . Specifically, if  $W(n, x)$  is the amount of time that the process remains equal to  $x$ , we have,

$$P(W(n, x) = k) = [F(x)]^{k-1} [1 - F(x)], \quad k = 1, 2, \dots$$

(b) The process then jumps to a new height whose c.d.f. is given by

$$F_x(y) = \frac{F(y) - F(x)}{1 - F(x)} \quad x < y < \infty$$

Notice that both the waiting time distributions in a state and the amount of the jump depend on  $x$ , the height of the process, but not on  $n$ , the time from which the evolution is being considered. Thus we can consider the inverse process  $Z(\cdot)$  evolving as follows:

(a') Let  $[A, B]$  be the smallest interval containing the total mass of the c.d.f. . That is

$$\lim_{x \rightarrow B} F(x) - \lim_{x \rightarrow A} F(x) = 1$$

and if  $[A', B']$  is strictly contained in  $[A, B]$  then the above limit, with  $A', B'$  replacing  $A, B$ , is less than one. (It may be that  $A = -\infty$  or  $B = \infty$ .) We define the inverse process  $Z(\cdot)$  by

$$Z(u) = \sup\{t \mid M(t) \leq u\}, \quad A < u < B \quad (2.1)$$

Thus, the values of  $Z(\cdot)$  are non-negative integers.

(b') If  $Z(u) = n$ , then  $Z(\cdot)$  remains equal to  $n$  for an additional amount of time  $R(u, n)$  whose distribution depends on  $u$ , but not on  $n$ . Specifically,

$$P(R(u, n) > s) = \frac{1 - F(s)}{1 - F(u)} \quad (2.2)$$

This follows from (b) in the description of  $M(\cdot)$  above since

$$P(Z(v) = n, u \leq v \leq s \mid Z(u) = n) = \frac{1 - F_x(s)}{1 - F_x(u)} = \frac{1 - F(s)}{1 - F(u)}$$

which does not depend on  $x$  whenever  $x \leq u \leq s$ .

(c') Given that  $Z(u) = n$  and  $Z(\cdot)$  remains equal to  $n$  until time  $s$  according to the distribution described in (b'), then  $Z(\cdot)$  jumps to a new height  $n + k$  with probability

$$[F(u)]^{k-1} [1 - F(u)], \quad k = 1, 2, \dots \quad (2.3)$$

It follows from the above description that  $Z(\cdot)$  is a Markov process. In the next section we give a different description of this process.

### 3. L. F. PROCESSES.

Consider the probability generating function

$$G(s, \theta) = \frac{1 - F(s)}{1 - \theta F(s)} = \sum_0^{\infty} [\theta F(s)]^k [1 - F(s)]$$

For  $s$  in  $(A, B)$ . We have that

$$H(s, t, \theta) = \frac{1 - F(t)}{1 - F(s)} \frac{1 - \theta F(s)}{1 - \theta F(t)} = G(s, \theta) / G(t, \theta) \quad (3.1)$$

is a probability generating function whenever  $A \leq s \leq t < B$  since it is easy to verify that

$$H(s, t, \theta) = a + (1 - a)G(t, \theta)$$

which displays  $H$  as a convex combination of two probability generating functions with

$$a = \frac{1 - F(t)}{1 - F(s)} \frac{F(s)}{F(t)}$$

The distribution described by  $H$  has been called a linear fractional distribution. We now describe an additive process which we call a linear fractional (L.F.) process,  $\hat{Z}(\cdot)$  as follows:

- 1) If  $A < t_1 < t_2 < \dots < t_k < B$  then the joint distribution of

$$\hat{Z}(t_1), \hat{Z}(t_2) - \hat{Z}(t_1), \dots, \hat{Z}(t_k) - \hat{Z}(t_{k-1})$$

is that of  $k$  independent, non-negative, integer-valued random variables whose probability generating functions are

$$G(t_1, \theta), H(t_1, t_2, \theta), \dots, H(t_{k-1}, t_k, \theta)$$

respectively.

- 2) Since for  $s < t < u$  we have that

$$H(s, t, \theta)H(t, u, \theta) = H(s, u, \theta)$$

it follows that a process with independent increments can be defined on  $(A,B)$  in accordance with the consistency requirements of Kolmogorov. The details are standard and are omitted. Notice that for any  $t$  in  $(A,B)$ ,  $\hat{Z}(t)$  has the generating function  $G(t,\theta)$ .

3) We suppose, as we can, that we have a version of  $\hat{Z}(\cdot)$  which is right-continuous.

Theorem 3.1 The process  $\hat{Z}(\cdot)$  defined in this section and the process  $Z(\cdot)$  defined in Section 2 agree. This means that  $\hat{Z}(t)$  and  $Z(t)$  have the same distribution for any  $t$  in  $(A,B)$  and the increments

$$\hat{Z}(t_2) - \hat{Z}(t_1), \dots, \hat{Z}(t_k) - \hat{Z}(t_{k-1}) \text{ and } Z(t_2) - Z(t_1), \dots, Z(t_k) - Z(t_{k-1})$$

have the same joint distributions for

$$A < t_1 < t_2 < \dots < t_k < B$$

Proof We see from the definition of  $\hat{Z}(\cdot)$  that  $P(\hat{Z}(t) = k) = [F(t)]^k [1 - F(t)]$ ,  $k = 0, 1, \dots$ . Let us check the distribution of  $Z(u)$ . From the definition of  $Z(\cdot)$  as the inverse of  $M(\cdot)$  we have that

$$P(Z(t) = 0) = P(X_1 > t) = 1 - F(t)$$

$$\begin{aligned} P(Z(t) = k) &= P(X_1 \leq t, \dots, X_k \leq t, X_{k+1} > t) \\ &= [F(t)]^k [1 - F(t)], \quad k = 1, 2, \dots \end{aligned}$$

Thus, the first requirement that  $\hat{Z}(t)$  and  $Z(t)$  have the same distributions, is satisfied. Next we check the second requirement that the increments of  $\hat{Z}(\cdot)$  and of  $Z(\cdot)$  have the same joint distributions. There are several different ways of doing this and it might be interesting to point out these different approaches.

Method 1 A straightforward calculation shows that

$$\begin{aligned} P(\hat{Z}(v) = n, u \leq v \leq s \mid \hat{Z}(u) = n) &= \text{coefficient of } \theta^0 \text{ in } H(u, s, \theta) \\ &= \frac{1 - F(s)}{1 - F(u)} \end{aligned}$$

Thus the holding time distributions of  $\hat{Z}(\cdot)$  and  $Z(\cdot)$  coincide. Another straightforward computation shows that

$$\lim_{\epsilon \downarrow 0} P(Z(u + \epsilon) = k \mid Z(u + \epsilon) > Z(u)) = [F(u)]^{k-1} [1 - F(u)], k = 1, 2, \dots$$

Hence the jump distributions for  $\hat{Z}(\cdot)$  and  $Z(\cdot)$  coincide and the proof is complete.

Method 2 It is not too hard to show that the increments of  $\hat{Z}(\cdot)$  have the same joint distributions as do the increments of  $Z(\cdot)$  by direct computation. For typographical ease we illustrate the necessary computation in a special case, leaving the general case to the reader. Suppose that  $A < t_1 < t_2 < B$ . Let  $Z(t_1) = Z_1, Z(t_2) - Z(t_1) = Z_2$  and let  $r$  and  $k$  be positive integers. Then

$$\begin{aligned} P(Z_1 = r, Z_2 = k) &= P(X_1 \leq t_1, \dots, X_r \leq t_1, t_1 < X_{r+1} \leq t_2, X_{r+2} \leq t_2, \dots, X_{r+k} \leq t_2, X_{r+k+1} > t_2) \\ &= [F(t_1)]^r [F(t_2) - F(t_1)] [F(t_2)]^{k-1} [1 - F(t_2)] \end{aligned}$$

Hence, since we have determined the distribution of  $Z_1$  from the first part of the proof we have that

$$P(Z_2 = k \mid Z_1 = r) = \frac{1 - F(t_2)}{1 - F(t_1)} [F(t_2)]^{k-1} [F(t_2) - F(t_1)] \quad (3.2)$$

A similar computation shows that the same conditional probability applies when



$r = 0$ . Hence  $Z_1$  and  $Z_2$  are independent. The quantity on the right side of (3.2) is the coefficient of  $\theta^k$  in  $H(t_1, t_2, \theta)$ .

4. ANOTHER REPRESENTATION FOR  $\hat{Z}(\cdot)$ .

Let  $-\infty < C < D \leq \infty$ . We want to describe a process on  $(C, D)$  which is a generalization of compound Poisson processes. Let  $U(\cdot)$  be a Poisson process with stationary, independent increments over the time interval  $(C, D)$  and parameter  $\lambda$ . That is, if  $C \leq t_1 < t_2 < t_3 < D$ , then  $U(t_2) - U(t_1)$  and  $U(t_3) - U(t_2)$  are independent random variables which are Poisson distributed with parameters  $\lambda(t_2 - t_1)$  and  $\lambda(t_3 - t_2)$  respectively. Let  $K(\theta, t)$  be a characteristic function of a probability distribution which depends on  $t$  and is well-defined for every  $t$  in  $(C, D)$ . That is

$$K(\theta, t) = \int_{-\infty}^{\infty} (\exp i\theta x) dH_t(x)$$

where  $H$  is a c.d.f. for every  $t$  in  $(C, D)$ . Now define a process  $Y(\cdot)$  as follows:

If  $t$  is a jump point of  $U(\cdot)$  then  $Y(\cdot)$  jumps by an amount which has the c.d.f.  $H_t$ . That is, given that  $T_1 < T_2 < \dots$  are the jump points of  $U(\cdot)$  let  $J_1, J_2, \dots$  be a sequence of random variables which are independent and have c.d.f.'s  $H_{T_1}, H_{T_2}, \dots$  respectively. Now define  $Y(\cdot)$  as follows:

$$Y(t) = \begin{cases} 0, & C < t < T_1, \\ J_1, & T_1 \leq t < T_2, \\ J_1 + J_2, & T_2 \leq t < T_3 \\ \vdots & \vdots \\ \vdots & \vdots \end{cases}$$

If there are no jumps of  $U(\cdot)$  in  $(C,D)$  then  $Y(t) = 0$  for all  $t$  in  $(C,D)$ . If there are only a finite number  $N$  of jumps in  $(C,D)$  then  $Y(t) = J_1 + \dots + J_N$  for  $t$  in  $(T_N, D)$ .

Lemma 4.1  $Y(\cdot)$  is a process with independent increments and

$$E(\exp i\theta Y(t)) = \exp -\lambda(t-C) - \lambda \int_C^t K(\theta, s) ds = \exp -\lambda \left( \int_C^t (1-K(\theta, s)) ds \right) \quad (4.1)$$

Proof That  $Y(\cdot)$  has independent increments is evident from the fact that  $U(\cdot)$  has independent increments. To evaluate the characteristic function of  $Y(t)$  we use the fact that given that there are  $n$  jump points in  $(C,t)$ , their locations are distributed the same as that of  $n$  points independently selected at random in that interval. Hence

$$E(\exp i\theta Y(t)) = \sum_{n=0}^{\infty} \left[ n! \int_{C < u_1 < \dots < u_n < t} \frac{K(\theta, u_1) \dots K(\theta, u_n) du_1 \dots du_n}{(t-C)^n} \right] [\lambda(t-C)]^n e^{-\lambda(t-C)} / n!$$

(The integral is understood to be 1 for  $n = 0$ .) We use the fact that

$$n! \int_{C < u_1 < \dots < u_n < t} K(\theta, u_1) \dots K(\theta, u_n) du_1 \dots du_n = \left[ \int_C^t K(\theta, u) du \right]^n$$

Hence the assertion about the characteristic function of  $Y(t)$  follows immediately.

If the jumps happen to be non-negative integer-valued random variables we will find it more convenient to deal with the probability generating function

Of  $Y(t)$ . In this case if we understand  $K(\theta, u)$  to be the probability generating function of a jump at time  $u$  then  $E\theta^{Y(t)}$  equals the right side of (4.1) with no further changes, ( $|\theta| \leq 1$ ). Similarly, if the jumps are positive it will be convenient for  $K(\theta, t)$  to be the Laplace transform of a jump at time  $u$ .

We point out that if  $K(\theta, s) = K(\theta)$  does not depend on  $s$  then  $Y(\cdot)$  is a conventional compound Poisson process. We shall refer to  $Y(\cdot)$  in general as an integrated compound Poisson (ICP process).

We now will show that under certain circumstances there exists a function  $D(\cdot)$  on  $(0, \infty)$  such that  $Z(D(u))$  is a process of the type described in Lemma 4.1, namely an ICP process.

Suppose that  $F$  is continuous in  $(A, B)$ . Then there exists a monotone non-decreasing function  $R$  on  $(0, 1)$  such that

$$F(R(u)) = u, \quad u \text{ in } (0, 1).$$

If  $F$  is strictly increasing in  $(A, B)$  then  $R = F^{-1}$ , the inverse function of  $F$ . If  $F$  is continuous but not strictly increasing then  $R$  is not uniquely determined but we suppose we choose one particular version of  $R$  once and for all. We now define a function  $D$  by

$$D(u) = R\left(\frac{u}{1+u}\right), \quad u \text{ in } (0, \infty).$$

Theorem 4.1 Suppose that  $F$  is continuous. Then  $Z(D(u))$  for  $u$  in  $(0, \infty)$  defines an ICP process with jump distribution determined by

$$\begin{aligned} K(\theta, s) &= E\theta^J = (s + (1-s)\theta)/(1 + s - \theta s) \\ &= \frac{s}{1+s} + \sum_{k=1}^{\infty} \frac{s^{k-1}}{(1+s)^k} \theta^k \end{aligned}$$

and  $\lambda = 1$ .

Proof We know that  $Z(\cdot)$  is a process with independent increments so the same is true of  $Z(D(\cdot))$ . We have that

$$E\theta^{Z(D(u))} = G(D(u), \theta) = \frac{1 - F(D(u))}{1 - \theta F(D(u))} = \frac{1}{1 + u - \theta u}$$

Let us now compute  $E\theta^{Y(u)}$  for the ICP process with  $\lambda = 1$  and the above indicated  $K$ . An easy computation shows that

$$\int_0^t K(s, \theta) ds = t - \log(1 + t(1-\theta))$$

Hence, for all positive  $u$ ,

$$E\theta^{Y(u)} = \frac{1}{1 + u - \theta u}$$

This completes the proof.

We are now able to show that the number of times that  $M_n$  rises to new heights between height  $a$  and height  $b$  is Poisson distributed with parameter  $\log[(1 - F(a))/(1 - F(b))]$ . Let us describe this more carefully. We say that a record is set at time  $n$  and at height  $x$  if

$$x = M_n > \max(X_1, \dots, X_{n-1}), \quad n > 1$$

We agree that a record is automatically set at time  $n = 1$  at height  $X_1$ .

Theorem 4.2 Let  $V(t)$  equal the number of records set with heights no greater than  $t$ . If  $F$  is continuous then  $V(\cdot)$  is a process with independent increments which are Poisson distributed. Specifically, if  $A < a < b < B$  then  $V(b) - V(a)$  is Poisson distributed with parameter  $\log [(1 - F(a))/(1 - F(b))]$ .

Proof By Theorem 4.1, the process  $Z(D(\cdot))$  is an ICP process. The process  $U(\cdot)$  which counts the jumps of this ICP process is a Poisson process. Hence  $V(\cdot)$ , the process which counts the jumps of  $Z(\cdot)$  also has independent increments which are Poisson distributed. It remains to determine the parameters of these Poisson distributions. We have that

$$P(V(b) - V(a) = 0) = H(a,b,0) = \frac{1 - F(b)}{1 - F(a)} = \exp - \log [(1-F(a))/(1-F(b))]$$

which completes the proof. (For a direct proof, see Theorem 2.2 of [ 1 ].)

## 5. EXTREMAL PROCESS.

We now consider the counterpart of the above results for extremal processes. (See [ 1 ], [ 2 ] for the definition and a discussion of these processes.) These processes are defined in terms of a monotone, non-increasing, non-negative function  $Q$  whose total variation is concentrated on  $[a,b]$  where  $-\infty \leq a < b \leq \infty$ . A representation of an extremal process  $Y(t), 0 < t < \infty$  is given in terms of  $Q$  in Section 4 of [ 2 ] as follows: Under the condition that  $Y(t) = x$ , ( $x$  in  $(a,b)$ ),  $Y(\cdot)$  remains equal to  $x$  for a random amount of time  $W_1/Q(x)$ . The process then jumps to a height  $Z_1$  greater than  $x$  where  $Z_1$  has the c.d.f.  $R_x$  defined by

$$R_x(u) \begin{cases} 0, & u \leq y, \\ 1 - Q(u)/Q(x), & u > y \end{cases}$$

The process then remains at height  $Z_1$  a random amount of time  $W_2/Q(Z_1)$ , and so forth. The random variables  $W_1, W_2, \dots$  are independent, exponential (parameter 1), independent of all other random variables under consideration.

We now define the inverse  $Z(\cdot)$  to the extremal process  $Y(\cdot)$ , where

$$Z(u) = \sup\{t \mid Y(t) \leq u\}, \quad u \text{ in } (a, b), \quad (\text{c.f. } (2.1)) \quad (5.1)$$

In accordance with the above representation of  $Y(\cdot)$  we can give the following representation for  $Z(\cdot)$ : If  $Z(u) = z$  then  $Z(\cdot)$  remains equal to  $z$  for an additional amount of time  $R(u, z)$  where

$$P(R(u, z) > s) = Q(u)/Q(s) \quad (5.2)$$

Given that  $Z(u) = z$  and  $Z(\cdot)$  remains equal to  $z$  until time  $s$  according to the distribution described by (5.2), then  $Z(\cdot)$  jumps to a new height  $z + J$  where  $J$  is exponentially distributed with parameter  $Q(z)$ . That is

$$P(J > v) = \exp - vQ(z), \quad v > 0 \quad (5.3)$$

Notice the formal similarity between (5.1), (5.2), (5.3) and (2.1), (2.2), (2.3). We refer to the fact that for  $t > 0$ ,

$$P(Y(t) < u) = \exp - tQ(u) \quad (\text{see [ 2 ]})$$

Hence,

$$P(Z(u) > t) = P(Y(t) < u) = \exp - tQ(u)$$

and we have the Laplace transform,

$$E \exp - \theta Z(u) = \frac{Q(u)}{Q(u) + \theta}, \quad u \text{ in } (a,b), \quad \theta > 0 \quad (5.4)$$

In a manner which is wholly analogous to that of Section 3, we can now construct an additive process  $\hat{Z}(\cdot)$  with the following properties:

- (a)  $Z(u)$  is exponentially distributed with Laplace transform given by (5.4).
- (b) If  $a < t_1 < t_2 < \dots < t_k < b$  then the increments  $\hat{Z}(t_1), \hat{Z}(t_2) - \hat{Z}(t_1), \dots, \hat{Z}(t_k) - \hat{Z}(t_{k-1})$  are mutually independent and  $\hat{Z}(t_i) - \hat{Z}(t_{i+1})$  has Laplace transform

$$\frac{Q(t_2)}{Q(t_1)} \left[ \frac{Q(t_1) + \theta}{Q(t_2) + \theta} \right]$$

(This transform is analogous to the generating function (3.1).)

The following theorem is comparable to Theorem 3.1.

Theorem 5.1 The processes  $\hat{Z}(\cdot)$  and  $Z(\cdot)$  agree, in the sense that  $\hat{Z}(t_1), \hat{Z}(t_2) - \hat{Z}(t_1), \dots, \hat{Z}(t_k) - \hat{Z}(t_{k-1})$  and  $Z(t_1), Z(t_2) - Z(t_1), \dots, Z(t_k) - Z(t_{k-1})$  have the same joint distributions.

Proof A proof can be modelled after the proofs of Theorem 3.1 and we leave the details in that direction to the reader. For another proof we offer the following: According to Theorem 9.2 of [ 2 ] it is evident that

that  $\hat{Z}(\cdot)$  has independent increments, as does  $Z(\cdot)$ . Hence to prove the theorem it need only be shown that for any  $t$  in  $(a,b)$   $\hat{Z}(t)$  and  $Z(t)$  are identically distributed. But this is true from the definition of  $\hat{Z}(\cdot)$  and from (5.4).

We now prove a counterpart of Theorem 4.1. First we point out that if  $Q$  is continuous on  $(a,b)$  then there exists a function  $R$  such that

$$Q(R(u)) = u, \quad u \text{ in } (0,\infty).$$

We now define a function  $D$  by

$$D(u) = R\left(\frac{1}{u+1}\right)$$

Note that  $D$  is a monotone non-decreasing function of  $u$ .

Theorem 5.2 Suppose that  $Q$  is continuous in  $(a,b)$ . Then the process  $Z(D(\cdot))$  is distributed like  $Y_0 + Y(\cdot)$  where  $Y(\cdot)$  is an ICP process on the time interval  $(0,\infty)$  with jump distribution whose Laplace transform is

$$K(\theta, s) = E \exp -\theta J = \frac{1 + \theta u}{1 + \theta(u+1)}, \quad \theta \text{ and } u \text{ in } (0,\infty)$$

and  $\lambda = 1$ , and  $Y_0$  is an exponential (parameter 1) random variable independent of the process  $Y(\cdot)$ .

Proof Since  $Z(\cdot)$  has independent increments the same is true of  $Z(D(\cdot))$ . We have that



$$E \exp -\theta Z(D(u)) = \frac{Q(R(\frac{1}{u+1}))}{Q(R(\frac{1}{u+1})) + \theta} = \frac{1}{1 + (u+1)\theta}$$

Let us now compute  $E \exp -\theta Y(u)$  for the ICP process  $Y(\cdot)$  with  $\lambda = 1$  and the above  $K$ .

$$\int_0^u (1 - K(\theta, s)) ds = \log \frac{1 + \theta(u+1)}{1 + \theta}$$

Hence

$$E \exp -\theta Y(u) = \frac{1 + \theta}{1 + \theta(u+1)}$$

and

$$E \exp -\theta (Y_0 + Y(u)) = \frac{1}{1 + \theta(u+1)}$$

Since  $Z(D(\cdot))$  and  $Y_0 + Y(\cdot)$  are both processes with independent increments, this completes the proof.

From Theorem 5.2 one can easily prove the counterpart of Theorem 4.2. Since this appears in [ 2 ] as Theorem 5.1, (b), (c) we leave the details to the reader.