

Stokey, Nancy

**Working Paper**

## Self-Fulfilling Expectations

Discussion Paper, No. 408

**Provided in Cooperation with:**

Kellogg School of Management - Center for Mathematical Studies in Economics and Management Science, Northwestern University

*Suggested Citation:* Stokey, Nancy (1979) : Self-Fulfilling Expectations, Discussion Paper, No. 408, Northwestern University, Kellogg School of Management, Center for Mathematical Studies in Economics and Management Science, Evanston, IL

This Version is available at:

<https://hdl.handle.net/10419/220768>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

DISCUSSION PAPER NO. 408

Self-Fulfilling Expectations, Rational  
Expectations, and Durable Goods Pricing

by

Nancy L. Stokey

December 1979

Self-fulfilling Expectations, Rational Expectations,  
and Durable Goods Pricing

by

Nancy L. Stokey

November, 1979

Abstract

The market for a durable good sold by a monopolist is examined using both discrete-time and continuous-time versions of the same model. The assumption that buyers' expectations must be fulfilled along the realized path for production is shown to place no restrictions on the observed path. A stronger assumption is then adopted: that buyers' expectations also be fulfilled in the presence of any unexpected, exogenous perturbation to the stock. Under standard assumptions about demand there is then a unique equilibrium. However, if unit costs are constant and there are no capacity constraints, the equilibrium path for production is very sensitive to the time structure of the model. Thus, because of its effect on expectations, plant capacity is a crucial decision variable for a firm selling a durable good.

Self-fulfilling Expectations, Rational Expectations,  
and Durable Goods Pricing

by

Nancy L. Stokey\*

1. Introduction

In the market for a durable good, demand at any date depends not only on the current price but also on buyers' expectations about future prices, since these determine the expected user cost (implicit rental cost) of the services of the good. Thus, the market for a durable good provides a valuable opportunity for examining, on the micro level, the implications of various hypotheses about expectation formation.

Self-fulfilling expectations equilibria (SEE's) and rational expectations equilibria (REE's) are examined below in the context of a market for a durable good sold by a monopolist. An SEE in such a market is a pair of functions, one describing how buyers' expectations are formed and one describing the monopolist's sales strategy, that jointly have the following two properties:

- 1) the seller's strategy maximizes the present discounted value of profits, given the expectation function of buyers;  
and
- 2) buyers' expectations are fulfilled along the realized path of production.

(Note the distinction between buyers' expectation function--which describes how their beliefs depend on what they observe, and buyers' expectations--the

---

\* I am grateful to Norton Kamien, Roger Myerson and John Roberts, whose suggestions have improved both the content and the readability of this paper. This research was supported by the Center for Managerial Economics and Decision Sciences at Northwestern University.

values taken on by that function at various dates.)

Although in any SEE expectations are fulfilled along the observed (equilibrium) path, they would not necessarily be fulfilled if an unexpected exogenous shock were to cause a perturbation away from the equilibrium path. Therefore, a particular sales strategy might constitute part of an SEE only because expectations away from the observed path are incorrect.

An REE is an SEE that has the property that expectations would continue to be fulfilled after any such perturbation.<sup>1</sup> In the context here, an REE is a single function, describing both how buyers' expectations are formed and the monopolist's sales strategy in every contingency, that has the following properties:

- 1) each contingent sales strategy maximizes the present discounted value of profits given the relevant initial condition for the stock and given buyers' expectation function;  
and
- 2) given any initial level of the stock at any date, buyers' expectations are fulfilled along the realized path of production from that date on.

In any REE the firm is, in every contingency, maximizing profits given the (correctly perceived) expectations of buyers, and buyers are, in every contingency expecting (correctly) that the firm will so behave. I.e., each side of the market holds correct beliefs about how the other would behave in any situation, even in situations that may never be observed.

A simple model is used below to study SEE's and REE's. At time zero the (new) durable product is introduced. There are no production costs and no capacity constraints, and the product does not depreciate. There is a

stationary inverse demand function for the services of the good, and the monopolist uses an infinite planning horizon.

The total stock of the good in the hands of consumers (cumulative sales to date) is viewed as the state variable of the system. At each date consumers form expectations about the total stock of the good that will have been sold at each date in the infinite future. These expectations (which are point expectations) are at each date conditioned on the current stock. It is assumed that buyers know the inverse demand function for the services of the good, so that their expectations about the path of the stock determine their expectations about the path of the implicit rental rate for the services of the good over the infinite future. The current selling price is then merely the present discounted value of this stream of implicit rents. (Expected future prices at any date can be calculated in the same way.)

The monopolist is assumed to know the function describing how buyers' expectations are formed. Given this function, it chooses a sales strategy that maximizes the present discounted value of its profit stream. If this sales strategy fulfills consumers' expectations, the result is an SEE.

It is shown below that SEE's exist but are not unique. On the contrary, the path of the stock in an SEE is arbitrary except that it must be increasing and must satisfy two boundary conditions. Hence the assumption that expectations are self-fulfilling has virtually no implications for the observed path of sales.

However, none of these SEE's is an REE; it is shown that no REE exists in the continuous-time model. The intuition behind this result comes through looking at a discrete-time version of the model. It is shown that in the discrete-time model, under standard assumptions about demand, there is

a unique REE. The function describing the REE is solved for in the case of a linear demand curve, with the length of the period explicitly included as a variable. The effect of shortening the period is examined, and it is shown that the equilibrium path for the stock over time is very sensitive to the length of the period. As the period becomes very short, virtually all sales occur in a short length of time. I.e., as the period becomes short, the seller saturates the market almost immediately. The limit of these REE's as the length of the period approaches zero is for the stock of the durable to jump immediately to the level that saturates the market (drives the price to zero).

The sensitivity of the REE path for the stock to the length of the period in the discrete-time model, and the non-existence of an REE in the continuous-time model, both depend crucially on two assumptions: that the marginal cost of production is constant (that it is zero is not important for the qualitative results) and that there are no capacity constraints. An implication of the model, then, is that plant size is a crucial decision variable of the firm, but the rate of production is not. Under rational expectations, any plant will be operated at full capacity until the market is saturated (until price equals marginal cost).

In section 2 the continuous-time model is described and the SEE's are characterized; in section 3 it is shown that no REE exists in the continuous-time model; in section 4 REE's of the discrete-time model are examined; and in section 5 the conclusions are discussed.

## 2. Self-fulfilling Expectations Equilibria

The product is introduced at time  $t = 0$ . Let  $\{Q(t), 0 \leq t\}$  denote the stock of the good in the hands of consumers at time  $t$ . There is a stationary inverse demand function for the services of the good,  $f(\cdot)$ . That is,  $f(X)$  is the market-clearing rental rate for the services of the good when the supply is  $X$ . Assume that the seller and all buyers know the demand curve.

At every date consumers form expectations about the stocks that will be available at all future dates. Although these expectations could be conditioned on whatever information is currently available to consumers, for simplicity it is assumed here that only the current stock affects expectations. Let  $\{H(s,t,X), 0 \leq t \leq s, 0 \leq X\}$  denote the stock of the good expected at time  $t$  to be available at time  $s$  given that  $Q(t) = X$ . (Consumers have point expectations.) Since buyers can observe the current stock, the function  $H(\cdot, \cdot, \cdot)$  must satisfy:

$$H(t,t,X) = X \quad , \quad 0 \leq t; 0 \leq X. \quad (1)$$

Moreover, it is reasonable to require that buyers' expectations be consistent in the following sense. If at date  $t$  with  $Q(t) = X_t$  buyers expect the stock at date  $s$  to be  $X_s$  and at date  $s' > s$  to be  $X_{s'}$ , then if expectations are fulfilled at date  $s$ , expectations about the stock at date  $s'$  should not be revised. Formally, the function  $\{H(s,t,X)\}$  should satisfy:

$$H(s',t,X) = H(s',s,H(s,t,X)) \quad , \quad 0 \leq t \leq s \leq s'; 0 \leq X. \quad (2)$$

If  $H(s,t,X)$  is continuous in  $s$ , the situation is represented by Figure 1. Each path represents the expectations of consumers about the



stock, given an initial condition. Along each path expectations about the future are fulfilled, and so remain unrevised. If at some date buyers' expectations were unfulfilled they would observe the current date and stock, and jump onto a new path.

Consumers' expectations about the stock of the durable good in turn determine, via the demand curve, the current market price. Since consumers know the demand function,  $R^e(s,t,X)$ , the implicit rental rate (user cost) expected at time  $t$  to prevail at time  $s$  if  $Q(t) = X$ , is given by:

$$R^e(s,t,X) = f(H(s,t,X)) \quad , \quad 0 \leq t \leq s; 0 \leq X.$$

The current selling price is then simply the present discounted value of the future expected implicit rents. Letting  $r$  denote the (constant) rate of interest, and  $p(t,X)$  the selling price at time  $t$  if  $Q(t) = X$ :

$$p(t,X) = \int_t^{\infty} e^{-r(t-s)} R^e(s,t,X) ds \quad 0 \leq t; 0 \leq X.$$

This price function is the only one having the following two properties. First, if expectations about the stock are fulfilled at every date  $s \geq t$ , the user cost (interest cost minus capital gains) for the durable at every date  $t$  is  $f(Q(t))$ . Second, whenever the stock is constant over time the price is constant over time. The latter condition rules out speculation.

Assume that the seller knows how buyers' expectations are formed, i.e., knows the function  $H(\cdot, \cdot, \cdot)$ . Given  $H(\cdot, \cdot, \cdot)$ , the seller's profits can be written as a function of his strategy for the stock,  $\{X(t), 0 \leq t\}$ . Assume that there are no capacity constraints but that the seller must choose  $\{X(t)\}$  to be continuous, with  $X(0) = 0$ . That is, although the rate

of sales is not constrained, the firm cannot sell a discrete quantity at any date. For simplicity, assume that there are no costs of production. Then profits are given by:

$$\begin{aligned} \pi(\{X(t)\}) &= \int_0^{\infty} e^{-rt} X'(t) p(t, X(t)) dt \\ &= \int_0^{\infty} X'(t) \int_t^{\infty} e^{-rs} f(H(s, t, X(t))) ds dt \quad ; \end{aligned}$$

and the profit-maximizing strategy  $\{Q(t)\}$  satisfies:

$$e^{-rt} f(H(t, T, Q(T))) + Q'(t) \int_t^{\infty} e^{-rs} f'(H(s, t, Q(t))) H_3(s, t, Q(t)) ds = 0 \quad , \quad (3a)$$

$$0 \leq T \leq t;$$

$$Q(0) = 0 \quad . \quad (3b)$$

Buyers' expectations are fulfilled along this path if:

$$H(s, t, Q(t)) = Q(s) \quad , \quad 0 \leq t \leq s. \quad (4)$$

Consequently, an SEE consists of a pair of functions  $H(\cdot, \cdot, \cdot)$  and  $Q(\cdot)$  satisfying (1)-(4). Note that when (2) and (3b) are satisfied, (4) is equivalent to the condition:

$$H(s, 0, 0) = Q(s) \quad , \quad 0 \leq s. \quad (4')$$

The first question to be asked is whether an SEE exists. It is clear that no SEE can be unique, since any SEE can be used to generate a whole family of equilibria simply by perturbing the expectation function in regions of the space away from the observed path. As long as the optimal strategy  $\{Q(\cdot)\}$  is unchanged, the equilibrium conditions continue to be satisfied. Hence, at best an equilibrium may be unique in the sense that  $\{Q(t)\}$  may be unique and  $\{H(\cdot, \cdot, \cdot)\}$  may be unique in the neighborhood of the observed path.

However, even in this sense there is not a unique SEE. It is shown in the following theorem that any increasing path for the stock that satisfies the appropriate boundary conditions is part of some SEE.

Theorem 1: Given any inverse demand function  $F: [0, \bar{Q}] \rightarrow [0, \bar{P}]$  satisfying:

$$\begin{aligned} f'(x) < 0 & \quad , \quad 0 \leq x \leq \bar{Q} ; \\ f(0) = \bar{P} & \quad ; \quad f(\bar{Q}) = 0 & \quad ; \end{aligned} \quad (5)$$

and path for the stock  $\{Q(t), 0 \leq t\}$  satisfying either:

$$Q'(t) > 0 \quad , \quad 0 \leq t \quad ; \quad (6a)$$

$$Q(0) = 0 \quad ; \quad \lim_{t \rightarrow \infty} Q(t) = \bar{Q} \quad ; \quad (6b)$$

or for some  $T > 0$ :

$$Q'(t) > 0 \quad , \quad 0 \leq t < T \quad ; \quad (7a)$$

$$Q'(t) = 0 \quad ; \quad T \leq t \quad ; \quad (7b)$$

$$Q(0) = 0 \quad ; \quad Q(T) = \bar{Q} \quad ; \quad (7c)$$

there exists a function  $H(\cdot, \cdot, \cdot)$  which, together with  $Q(\cdot)$ , satisfies (1)- (4) for the demand function  $f(\cdot)$ .

Proof: The proof consists of showing a method, given  $Q(\cdot)$ , of constructing an expectation function  $H(\cdot, \cdot, \cdot)$  satisfying the required conditions.

Suppose that  $Q(\cdot)$  satisfies (6), and let  $H(\cdot, \cdot, \cdot)$  be given by:

$$H(s, t, X) \equiv Q(s) + e^{r(s-t)} \frac{Q'(s)}{Q'(t)} [X - Q(t)] , \quad 0 \leq t \leq s: \quad (8)$$

$$0 \leq X \leq \bar{Q}.$$

Obviously  $H(\cdot, \cdot, \cdot)$  satisfies (1) and (4), and by hypothesis  $Q(\cdot)$  satisfies (3b). Since,

$$\begin{aligned} E(s', s, H(s, t, X)) &= Q(s') + e^{r(s'-s)} \frac{Q'(s')}{Q'(s)} \{Q(s) + e^{r(s-t)} \frac{Q'(s)}{Q'(t)} [X - Q(t)] - Q(s)\} \\ &= Q(s') + e^{r(s'-t)} \frac{Q'(s')}{Q'(t)} [X - Q(t)] \\ &= H(s', t, X) \quad ; \end{aligned}$$

(2) is also satisfied. Finally, it must be shown that (3a) is satisfied.

Differentiating (8),

$$H_3(s, t, X) = e^{r(s-t)} Q'(s)/Q'(t) . \quad (9)$$

Substituting into (3a) from (4) and (9) gives:

$$\begin{aligned} e^{-rt} f(Q(t)) + Q'(t) \int_t^{\infty} e^{-rs} f'(Q(s)) e^{r(s-t)} \frac{Q'(s)}{Q'(t)} ds \\ = e^{-rt} [f(Q(t)) + \int_t^{\infty} f'(Q(s)) Q'(s) ds] \end{aligned}$$

$$= e^{-rt} [f(Q(t)) + \int_{Q(t)}^{\bar{Q}} f(X) dX] = e^{-rt} f(\bar{Q}) = 0 .$$

Hence (3a) is also satisfied.

Suppose now that  $Q(\cdot)$  satisfies (7), and let  $H(\cdot, \cdot, \cdot)$  be given by:

$$H(s, t, X) \equiv \begin{cases} Q(s) + e^{r(s-t)} \frac{Q'(s)}{Q'(t)} [X - Q(t)], & 0 \leq t \leq T: \\ Q(s) + e^{r(s-T)} \frac{Q'(s)}{Q'(T)} [X - Q(t)], & T < t \quad ; \\ & 0 \leq X \leq \bar{Q}. \end{cases}$$

Clearly (1), (2), (3b) and (4) are satisfied as before. By the previous argument (3a) is satisfied for  $0 \leq t \leq T$ . For  $T < t$ ,  $f(Q(t)) = f(\bar{Q}) = 0$ , and  $Q'(t) = 0$ , so that (3a) is again satisfied. Q.E.D.

Theorem 1 shows that the assumption that expectations are self-fulfilling places few restrictions on the equilibrium path for the stock. Any path that begins at zero and increases monotonically to  $\bar{Q}$  can be an equilibrium path for the stock.

Note too that the demand curve restricts the set of equilibrium paths for the stock only through the endpoint condition: the stock must either reach  $\bar{Q}$  or else approach  $\bar{Q}$  asymptotically. That is, only the point  $f^{-1}(0)$ , the point where the demand curve cuts the horizontal axis, affects the set of SEE's.

In any SEE the implicit rental rate at time  $t$ ,  $f(Q(t))$ , starts at  $\bar{P}$  at time  $t = 0$  and falls monotonically, reaching zero at time  $T$  if  $Q(T) = \bar{Q}$  or approaching zero asymptotically if  $Q(t)$  approaches  $\bar{Q}$  asymptotically. The selling price  $p(t, Q(t))$  is given by:

$$p(t) = \int_t^{\infty} e^{-r(s-t)} f(Q(s)) ds, \quad 0 \leq t;$$

so that:

$$p'(t) = rp(t) - f(Q(t)), \quad 0 \leq t$$

Since  $Q(t)$  is strictly increasing if  $Q(t) < \bar{Q}$ , and since  $f' < 0$ :

$$p(t) < f(Q(t))/r, \quad \text{if } Q(t) < \bar{Q};$$

so that:

$$p'(t) < 0, \quad \text{if } Q(t) < \bar{Q}.$$

Hence the price is strictly decreasing, either reaching zero (if  $Q(T) = \bar{Q}$  for some  $T$ ) or approaching zero asymptotically (if  $Q(t)$  approaches  $\bar{Q}$  asymptotically).

Note that any two SEE's in general lead to different values for total (discounted) consumers' and producer's surpluses. Consumers' surplus is largest in SEE's where the stock increases to  $\bar{Q}$  very rapidly. In these cases  $f(Q(t))$  falls to zero very rapidly, so that  $p(0,0)$ , although positive, is very small, and  $p(t, Q(t))$  falls to zero very rapidly. Profits are virtually zero in these cases. Profit is largest in SEE's where the stock increases very rapidly to  $Q^M \in \arg \max_X Xf(X)$ , and increase very slowly thereafter. In these cases the implicit rents,  $Q(t)f(Q(t))$ , are very close to their maximal value except at dates very far in the future. Hence total discounted profit is approximately equal to  $Q^M f(Q^M)/r$ .

### 3. The Non-existence of an REE in the Continuous-time Model

Although any path for production that saturates the market (either in finite time or asymptotically) is consistent with self-fulfilling expectations, none of these equilibria need be an REE.

For example, let  $H(\cdot, \cdot, \cdot)$  and  $Q(\cdot)$  describe an SEE, and suppose that at time  $T > 0$  there is an exogenous perturbation that reduces the stock by  $\Delta$ . At time  $T$ , buyers' expectations about the stock at dates  $t > T$  are no longer described by the path  $H(s, T, Q(T)) = H(s, 0, 0)$ . Instead, as shown in Figure 2, they are described by the path  $\{H(s, T, Q(T) - \Delta), s > T\}$ . Furthermore, at time  $T$  the seller will want to revise his production strategy. By assumption he knows the expectation function of buyers. Hence he will choose the new production strategy  $\{\hat{Q}(t), t > T\}$  that maximizes the present discounted value of future profits, given buyers' expectations (see Figure 2). In general, buyers' expectations will not be fulfilled along the revised path for production.

Suppose, however, that the function  $H(\cdot, \cdot, \cdot)$  has the property that, in addition to describing buyers' expectations, starting at any date  $T \geq 0$  and any initial stock  $X \geq 0$ , the path  $\{H(s, T, X), s \geq T\}$  describes the sales strategy that maximizes the present discounted value of future profits, given buyers' expectations. Then it is as if buyers realize that the seller knows how expectations are formed, and realize that the seller always chooses the sales strategy that maximizes the present discounted value of future profits. This is the sense in which buyers' expectations are "rational."

Formally, an REE is a function  $\{H(s, t, X), 0 \leq t \leq s; 0 \leq X \leq Q\}$ , continuous in  $s$ , satisfying (1) and (2), and having the property that:

$$\{H(s,T,X), s \geq T\} \in \arg \max_{\{Q(s), s \geq T\}} \int_T^{\infty} Q'(t) \int_t^{\infty} e^{-rs} f(H(s,t,Q(t))) ds dt, \quad (10)$$

$$\text{s.t.} \quad Q(T) = X$$

$$\text{for } 0 \leq T; 0 \leq X \leq \bar{Q}.$$

(The requirement that  $H(s,t,X)$  be continuous in  $s$  follows from the assumption that the seller may not sell a discrete quantity at any date.)

Since the inverse demand function for the services of the good is stationary over time, it is reasonable to suppose that if any REE exists, the function  $H(\cdot, \cdot, \cdot)$  is stationary, i.e., is of the form:

$$H(s,t,X) = h(s-t,X) \quad , \quad 0 \leq t \leq s, 0 \leq X \leq \bar{Q}.$$

for some function  $h(\cdot, \cdot)$ . That is, at any date  $t$  buyers' expectations about the stock and the seller's strategy for the stock at any date  $s$  in the future depend only on the current stock  $X$  and on the length of time  $s-t$ ; expectations and the sales strategy do not depend on the calendar dates  $t$  and  $s$  independently. It is shown in the following theorem that no stationary REE exists.

Theorem 2: For no function  $f: [0, \bar{Q}] \rightarrow [0, P]$  satisfying (5) does there exist a function  $[h(s,X), 0 \leq s; 0 \leq X \leq \bar{Q}]$ , continuous in  $s$  and satisfying:

$$h(0,X) = X \quad , \quad 0 \leq X \leq \bar{Q}; \quad (11a)$$

$$h(s,X) = h(s-t, h(t,X)) \quad , \quad 0 \leq t \leq s; 0 \leq X \leq \bar{Q}; \quad (11b)$$

$$[h(s,X), 0 \leq s] \in \arg \max_{[Q(s)]} \int_0^{\infty} Q'(t) \int_t^{\infty} e^{-rs} f(h(s-t, Q(t))) ds dt \quad , \quad (11c)$$

$$\text{s.t.} \quad Q(0) = X \quad ; \quad 0 \leq X \leq \bar{Q} \quad .$$



Proof: Suppose that  $h(\cdot, \cdot)$  satisfies the required conditions. The continuity of  $h(s, X)$  in  $s$  together with condition (11a) implies that  $h(s, X)$  is also continuous in  $X$ . Moreover, the range of  $h(\cdot, \cdot)$  is the interval  $[0, \bar{Q}]$ . Since  $h(s, X)$  is increasing in  $s$  and  $\quad$  in  $X$ , it is differentiable almost everywhere. Therefore, applying the standard variational method to (11c), a necessary condition for profit to be at a maximum is:

$$L(t, X) \equiv e^{-rt} f(h(t, X)) + h_1(t, X) \int_t^{\infty} e^{-rs} f'(h(s, X)) h_2(s-t, h(t, X)) ds = 0, \quad (12)$$

or, equivalently, that:

$$L(0, X) = 0 \quad \text{and} \quad L_1(t, X) = 0 \quad .$$

Using (11b):

$$h_1(s, X) = h_1(0, h(s, X)) \quad (13a)$$

$$0 = -\dot{h}_1(s-t, h(t, X)) + h_2(s-t, h(t, X)) h_1(t, X) \quad (13b)$$

$$0 = -h_{11}(s-t, h(t, X)) + h_{12}(s-t, h(t, X)) h_1(t, X) \quad (13c)$$

Substituting (13b) into (12), the required condition is:

$$L(t, X) = e^{-rt} f(h(t, X)) + \int_t^{\infty} e^{-rs} f'(h(s, X)) h_1(s-t, h(t, X)) ds = 0 \quad .$$

Using (13a) and (13c), this condition is satisfied only if:

$$\begin{aligned} 0 &= L_1(t, X) = -re^{-rt} f(h(t, X)) + e^{-rt} f'(h(t, X)) h_1(t, X) - e^{-rt} f'(h(t, X)) h_1(0, h(t, X)) \\ &+ \int_t^{\infty} e^{-rs} f'(h(s, X)) [-h_{11}(s-t, h(t, X)) + h_{12}(s-t, h(t, X)) h_1(t, X)] ds \\ &= -re^{-rt} f(h(t, X)) \quad . \end{aligned} \quad (14)$$

Condition (14) is satisfied only if:

$$h(t, X) = \bar{Q} \quad , \quad 0 \leq t; 0 \leq X \leq \bar{Q}. \quad (15)$$

However, the function  $h(\cdot, \cdot)$  defined in (15) violates (11a).

Q.E.D.

The non-existence of an REE can best be understood by examining a discrete-time version of the model above. In the discrete-time model an REE does exist, and examining the behavior of these equilibria as the period shrinks provides an explanation for the non-existence of an REE in the continuous-time model.

#### 4. A Discrete-time Model: the Existence of an REE

In this section a discrete-time version of the model above is examined. The length of the period is explicitly included as a variable, so that the effect of shortening the period can be studied.

A unit of time can be chosen arbitrarily. This unit of time is held constant throughout the analysis. As before  $r$  is the (constant) rate of interest per unit time. It is assumed throughout that interest is compounded continuously. The choice of the time unit does not affect the equilibrium in any way.

A "period" in this model is defined by the length of time between selling dates. Hence it is the length of time over which the stock of the durable is constant. Let  $z$  denote the length of a period, measured in terms of the chosen units of time. (E.g., if the unit of time is chosen to be one year and a period is one month, then  $z = 1/12$ .) A period may be any multiple or any fraction of the unit of time. The length of a period, (measured in fixed units of time), as will be shown below, is a critical determinant of the REE path of production over time.

As before, let  $f(X)$  denote the stationary demand curve for the flow of services of the durable. Let  $F(X,z)$  denote the (implicit) market-clearing rental, when the supply is  $X$ , for the use of the durable during one period of length  $z$ . The rent is expressed in beginning-of-period currency, so that:

$$F(X,z) = f(X) \int_0^z e^{-rs} ds = f(X)(1-e^{-rz})/r$$

As before, it is assumed that the seller and all buyers know the demand curve.

Let  $\{G(s,t,X,z), s = t, t+1, \dots; t = 0, 1, \dots; 0 \leq X \leq \bar{Q}; 0 < z\}$  denote the stock of the good buyers expect in period  $t$  to be available in period  $s$  if the current stock in period  $t$  is  $X$  and the length of the period is  $z$ . Buyers are assumed to observe the current stock (including current sales), so that:

$$G(t,t,X,z) = X, \quad \forall t, X, z; \quad (1')$$

and their expectations are assumed to be consistent, so that:

$$G(s',t,X,z) = G(s',s,G(s,t,X,z),z), \quad \forall s, s', t, X, z. \quad (2')$$

Assume buyers know the interest rate and the length of the period, as well as the inverse demand function for the flow of services of the good, so that  $R^G(s,t,X,z)$ , the implicit rental rate buyers expect in period  $t$  to prevail in period  $s$  if the stock in period  $t$  is  $X$ , is given by:

$$R^G(s,t,X,z) = F(G(s,t,X,z),z), \quad \forall s, t, X, z.$$

Consequently  $p^G(t,X,z)$ , the price in period  $t$  when the cumulative stock (including current sales) is  $X$ , is given by:

$$p^G(t,X,z) = \sum_{n=0}^{\infty} e^{-rzn} R^G(t+n,t,X,z), \quad \forall t, X, z.$$

As before, an REE is a function  $G(s,t,X,z)$  that describes both buyers' expectations and the seller's profit-maximizing sales strategy. I.e., an REE is described by a function  $G(s,t,X,z)$  satisfying (1') and (2'), and having the property that :

$[G(s,t,X,z), s=t,t+1,\dots] \in$

$$\begin{aligned} \arg \max_{[Q_t, Q_{t+1}, \dots]} \sum_{s=t+1}^{\infty} e^{-rz(s-t)} (Q_s - Q_{s-1}) \sum_{n=0}^{\infty} e^{-rzn} F(G(s+n, s, Q_s, z), z) , \\ \text{s.t. } Q_t = X ; \end{aligned} \quad (10')$$

$\forall t, X.$

Given any demand curve  $f(\cdot)$ , interest rate  $r > 0$ , and period length  $z > 0$ , an REE can be found using the standard method of dynamic programming. If the demand curve satisfies standard conditions (e.g., if marginal revenue is strictly decreasing--  $2f'(X) + Xf''(X) < 0, 0 \leq X \leq \bar{Q}$ ), then there is a unique REE. Moreover, the function  $G(\cdot, \cdot, \cdot, \cdot)$  describing the REE will clearly be stationary. (If the uniqueness conditions fail and there are multiple REE's, some of them will be stationary.)

Since the purpose here is merely to provide insight into the non-existence of an REE in the continuous-time model, for simplicity a linear demand curve will be used in the rest of this section. Let,

$$f(X) = \alpha - \beta X \quad , \quad 0 \leq X \leq \alpha/\beta. \quad (16)$$

Since marginal revenue is strictly decreasing in this case, there is a unique REE. It can be solved for explicitly by hypothesizing that  $G(\cdot, \cdot, \cdot, \cdot)$  has the form:

$$G(t+1, t, X, z) = a_0 + a_1 X ;$$

and solving for the coefficients  $a_0$  and  $a_1$ . The solution is presented in Theorem 3.

Theorem 3: For any  $r > 0$  and  $z > 0$ , the solution of (1'), (2'), (10') when  $f(\cdot)$  is given by (16) is the function:

$$G(s, t, X, z) = \alpha/2 + \mu(z)^{s-t} (X - \alpha/3), \quad \forall s, t, X, z; \quad (17)$$

where  $\mu(z) \equiv (1 - \sqrt{1 - e^{-rz}}) e^{rz} < 1, \quad \forall z$ .

Proof: Obviously the solution in (17) satisfies (1') and (2'). To see that it also satisfies (10') define:

$$g(s, t, X, z) \equiv G(s, t, X, z) - G(s-1, t, X, z), \quad \forall s, t, X, z.$$

Differentiating (10') and using (2') and (16), profit-maximizing requires:

$$e^{-rz(T-t)} [\alpha - \beta G(T, t, X, z) - \beta g(T, t, X, z) \sum_{n=0}^{\infty} e^{-rzn} G_3(T+n, T, G(T, t, X, z), z)] = 0.$$

Substituting from (17), the required condition is:

$$\alpha - [\alpha + \mu(z)^{T-t} (\beta X - \alpha)] - \mu(z)^{T-t-1} (\mu(z) - 1) (\beta X - \alpha) \sum_{n=0}^{\infty} e^{-rzn} \mu(z)^n = 0,$$

or,

$$\mu(z) + (\mu(z) - 1)(1 - e^{-rz} \mu(z))^{-1} = 0,$$

or

$$e^{-rz} \mu(z)^2 - 2\mu(z) + 1 = 0;$$

which is satisfied.

Finally,  $\mu(z) < 1$ , since  $1 > 1 - e^{-rz}$ , so that:

$$1 - e^{-rz} < \sqrt{1 - e^{-rz}},$$

and

$$1 - \sqrt{1 - e^{-rz}} < e^{-rz}.$$

Q.E.D.

The effect of varying the length of the period can be studied using (17). Let  $\tau$  and  $\tau+\Delta$  be arbitrary dates, and suppose that the stock at time  $\tau$  is  $X$ . The stock at time  $\tau+\Delta$ , when the period is of length  $z$ , is given by:

$$G\left(\frac{\tau+\Delta}{z}, \frac{\tau}{z}, X, z\right) = \alpha/\beta + \left[ (1 - \sqrt{1 - e^{-rz}}) e^{rz} \right]^{\Delta/z} (X - \alpha/\beta)$$

As the period shrinks,

$$\lim_{z \rightarrow 0} (1 - \sqrt{1 - e^{-rz}})^{\Delta/z} e^{rz} = 0$$

and  $\lim_{z \rightarrow 0} G\left(\frac{\tau+\Delta}{z}, \frac{\tau}{z}, X, z\right) = \alpha/\beta$ . Hence for any fixed length of time  $\Delta$ ,

as  $z \rightarrow 0$  the stock of the good approaches its limiting value  $\alpha/\beta$  within the length of time  $\Delta$ .<sup>2</sup>

This result is illustrated in Figure 3. As the period shrinks, sales per period are approximately unaffected. (A minor adjustment occurs through the effect of discounting, but this effect becomes negligible as the period gets very short.) Since as the period shrinks there are more periods per unit of time, eventually virtually all sales occur within an arbitrarily short (but non-zero) length of time. As  $z \rightarrow 0$ , the function  $G(\cdot, \cdot, \cdot, z)$  approaches the limit  $[G(s, t, X, 0) = \bar{Q}, \forall s, t, X]$ . However, this potential solution for the continuous-time model violates the requirement that the path of the stock over time be continuous, as well as the requirement that  $G(t, t, X, z) = X$ .

Comparing REE's as the length of the period,  $z$ , varies, the seller's profit is a monotonically increasing function of  $z$  and consumers' surplus a monotonically decreasing function of  $z$ . This follows directly from the

fact that the stock at any date is an increasing function of  $z$ , so that implicit rents and prices are decreasing functions of  $z$  (see Figure 3).

The effect of having sales occur only at fixed intervals is to enable the seller, after making sales at the beginning of one period, to guarantee to buyers that the stock will not increase any more over the course of the period. This <sup>is</sup> valuable to the seller, since it is his only effective means of convincing buyers that the total supply available to them will remain limited over the near future.

It has been shown (Stokey [1979]) that if the monopolist can make binding commitments to limit his future production, his optimal strategy is to sell the quantity  $Q^M \in \arg \max_X Xf(X)$  at date  $t = 0$ , and to commit himself to make no sales at later dates. This strategy maximizes the implicit rents  $Q(t)f(Q(t))$  at every date, and the seller collects the present discounted value of these rents,  $Q^M f(Q^M)/r$ , as sales revenues. (As Bulow [1979] has pointed out, a monopolist renter would use an equivalent strategy, renting the quantity  $Q^M$  at every date, and would reap the same profits.)

In the model examined here, the seller cannot make explicit commitments about his future sales. However, a period model--where sales occur only at fixed time intervals of length  $z$ --is similar to a continuous-time model where the seller can make commitments about his sales over a length of time  $z$  into the future.

In the discrete-time model, as the period gets very long it is as if the seller can make commitments about his entire future sales strategy. As  $z \rightarrow \infty$  the monopolist's optimal strategy is to sell  $Q(0) = Q^M$  in period 0, at a price of  $p(0, Q(0)) = f(Q^M)/r$ . I.e., as the period gets infinitely long the seller's profit approaches the maximum that it would attain even if any type of commitments about future sales were allowed.



## 5. Conclusion

The sensitivity of some results to the length of the time period is a problem that has long plagued macroeconomists. Their response has taken the form of a rule, concisely stated by Foley [1975] as follows: "No substantive prediction or explanation in a well-defined macroeconomic period model should depend on the real time length of the period." Foley goes on to say:

Period models are attractive because they are easy to conceptualize. This advantage is bought at a price, however, which is the danger that hidden assumptions may enter the model under the cover of the fixed period length. If the results of a period model do not depend in any important way on the period, the model can be formulated as a continuous model. (p. 311)

The REE's in the discrete-time model examined above obviously violate this rule. However, it seems likely that introducing capacity constraints and/or increasing marginal costs of production would make the equilibria robust against changes in the length of the period. Moreover, it seems likely that the limit of these equilibria as the length of the period shrinks to zero would be an REE of the continuous-time model.

Thus, the analysis above suggests that decisions about investment in fixed capital are particularly important to the seller of a durable. Not only do these decisions affect his production costs (as they do for all firms), but they may also play an important role in shaping buyers' expectations about future rates of production. Limiting his production capacity may be the only means by which the seller can convince rational buyers that he will restrict production. Buyers' expectations, in turn, are a crucial determinant of the price. Consequently, because of the effect on expectations, building a small capacity plant may be an effective means of raising the price.

The seller's strategic decisions then are not decisions about rates of production, but rather are decisions about plant capacity (e.g., the capacity at which minimum average cost on a U-shaped average cost curve is attained) and plant flexibility (the size of the range over which average cost is approximately constant). Some of these questions have been analyzed in Bulow [1979]. However, the assumption about how expectations are formed used there is quite different from the assumptions used here.

Another interesting area for further study is the effect potential entrants might have in changing the behavior of the incumbent monopolist. Spence [1977] has argued that an incumbent firm may find it advantageous to build excess capacity in order to deter entry, even if this capacity is not utilized (or not fully utilized) before entry. The analysis above suggests that in the market for a durable, the advantages of increasing capacity in order to deter entry must be weighed against the advantages of restricting capacity in order to affect buyers' expectations.

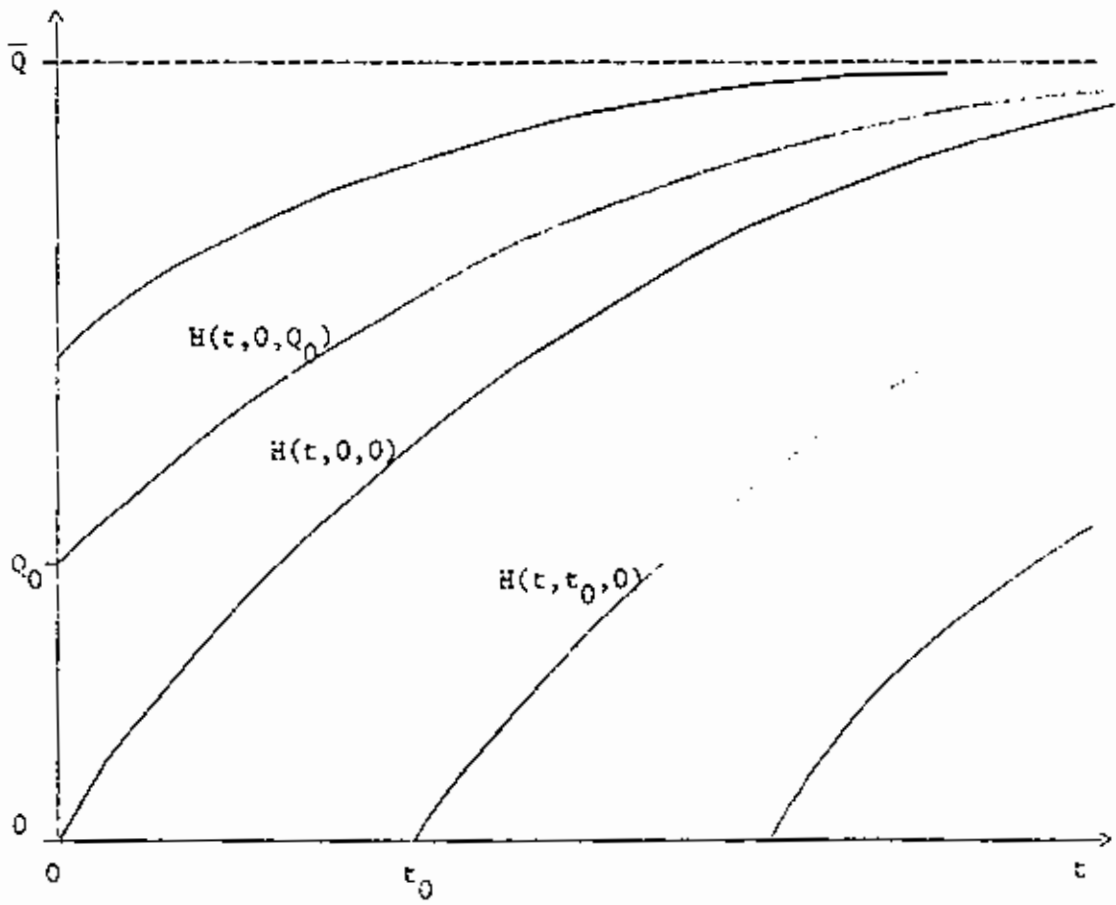


Figure 1

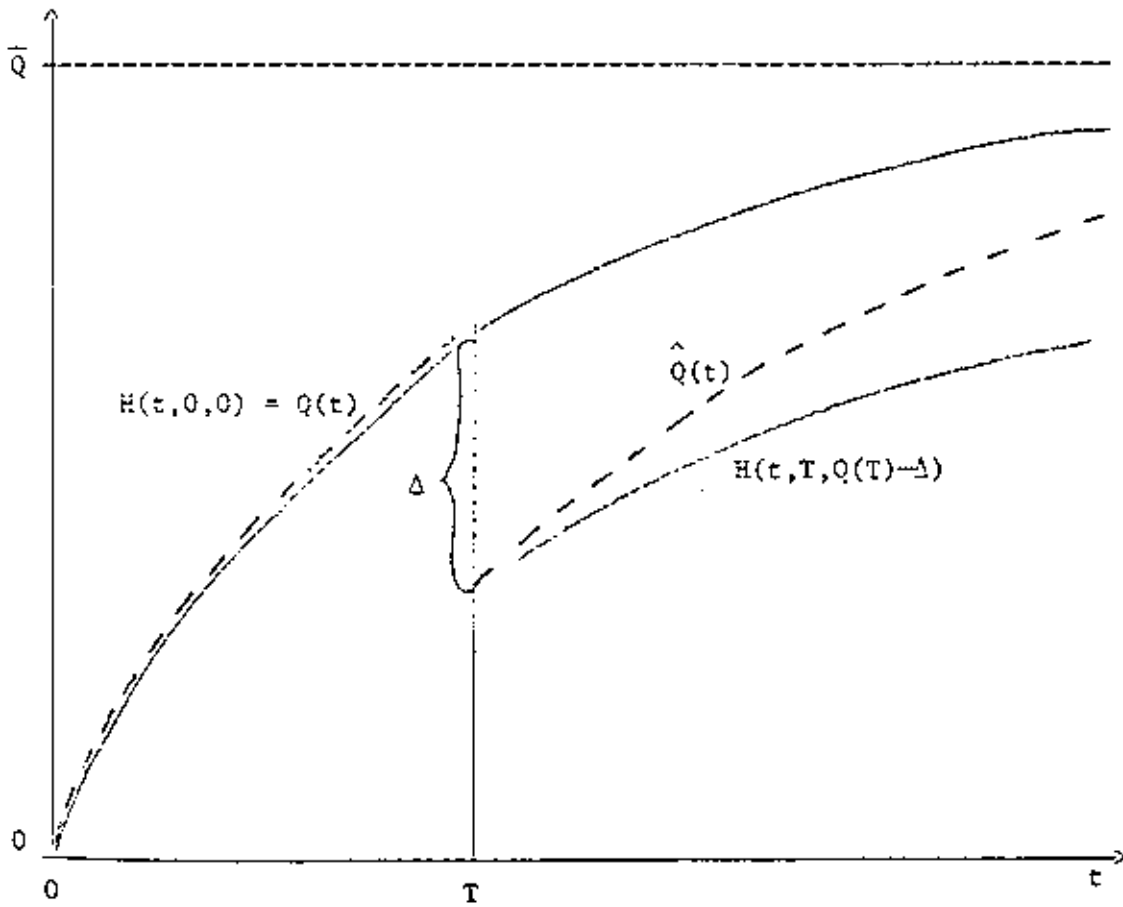


Figure 2

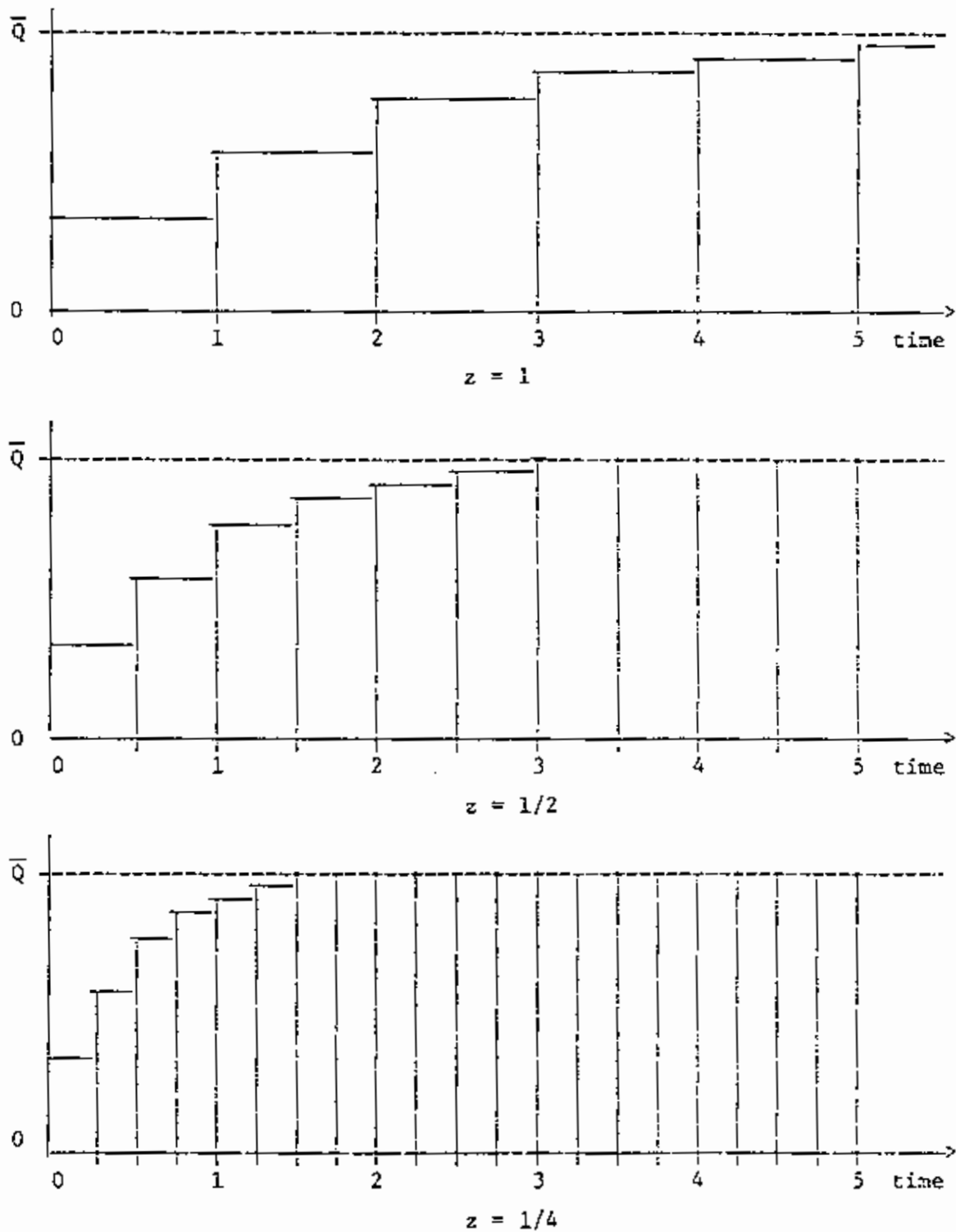


Figure 3

Graphs of  $G(t, 0, 0, z)$  for  $z = 1, 1/2, 1/4$

Footnotes

<sup>1</sup> This condition is similar to the "perfectness" condition used in game theory (see, for example, Selten [1975]).

<sup>2</sup> Note that there is never an REE in which the stock reaches  $\bar{Q}$  in finite time. For suppose that along some equilibrium path the stock  $Q(t)$  reached  $\bar{Q}$  in period  $T$ . Price, and consequently revenue, in period  $T$  would be zero, while if sales were limited to any positive quantity smaller than  $\bar{Q} - Q(T)$  the price would be positive and the seller would reap a positive profit.

References

- Bulow, Jeremy I., "Durable Goods Monopolists," unpublished manuscript Stanford University, (1979).
- Foley, Duncan, "On Two Specifications of Asset Equilibrium in Macroeconomic Models," Journal of Political Economy, Vol. 83, No. 2 (April, 1975), pp. 303-324.
- Selten, R., "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory, Vol. 4, No. 1 (1975), pp. 23-55.
- Spence, A. Michael, "Entry, Capacity, Investment, and Oligopolistic Prices," The Bell Journal of Economics, Vol. 8, No. 2 (Autumn, 1977), pp. 534-544.
- Stokey, Nancy L., "Intertemporal Price Discrimination," The Quarterly Journal of Economics, Vol. 93, No. 3 (August, 1979), pp. 355-371.