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INVESTMENT AND THE VALUATION OF FIRMS WHEN THERE IS AN OPTION TO SHUT DOWN

by

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ABSTRACT

This paper examines the investment decision of firms undertaking risky investment projects, assuming that firms can shut-down production if variable costs exceed revenues, and that claims on the firms are owned by risk-averse investors. A formula is obtained for the value of a claim on uncertain future profits. We find that increases in the variance of the output price can either raise or lower the value of a project; that claims on uncertain profits in the medium-term future can have a greater current value than claims on uncertain profits in the near future; and that changes in the risk-free rate will change the composition of the capital stock.

I. INTRODUCTION

This paper examines the investment decisions of firms undertaking risky investment projects and departs from previous economic studies of firm behavior under uncertainty in two important ways: firms have a shut-down option (i.e., they will not produce if variable costs exceed revenues); and claims on the firm are owned by risk-averse investors.[1] We do not model the investment decision directly (as do Abel [1981b] and Hartman [1972]), but concentrate on the valuation of a risky project as an all-or-nothing decision (as in Dietrich and Heckerman [1980]). The structure of the problem is similar to that of valuing a call option on a stock, and this allows us to exploit existing techniques for valuing options, as developed by Black and Scholes [1973] and Smith [1976].

There have been previous attempts to introduce risk aversion into models of firm behavior. Sandmo [1971] analyzed the production decisions of an explicitly risk-averse firm (i.e., one which maximizes the expected utility of profits), but our model differs in that the firm is a risk-neutral, value-maximizing price-taker, which is owned by risk-averse investors. The two models will generally give different comparative static results.[2] Our model is widely used in the finance literature, but has not been much exploited by economists.[3]

These features of our model lead to results different from those obtained by others. Some of our principal results are:

1) Increases in the variance of the output price can either raise or lower the value of a project. An increase in variability raises expected profits for a given capital stock at a given point in time (a result obtained by Hartman and others), but may lower the present value of a claim on future

profits. The latter effect depends upon the extent to which cash flows from the project covary with other uncertain income streams in the economy.

2) For a given project with a fixed capital stock, it is possible for claims on uncertain profits several years in the future to have greater current value than claims on uncertain profits in the immediate future. However, claims on profits sufficiently far in the future will always have a current value below that of claims on profits in the near future.

3) Changes in the risk-free rate will change the composition of the aggregate capital stock in a way which is, in principle, predictable.

The paper is organized as follows. Section II values an investment project with a simple production technology and a shut-down option, assuming that investors are risk-neutral. Section III demonstrates how the model can be applied directly to value profit streams from Leontief and Cobb-Douglas production technologies. Section IV allows investors to display risk-aversion, and it is shown that the solution is essentially unchanged from the risk neutral case. The only difference is in a term which has the interpretation of being a futures price. Option pricing techniques are used together with the Capital Asset Pricing Model to introduce risk-averse investors in a general equilibrium context, in a tractable way. Section V interprets the solution of the model as the Black-Scholes call option formula written in terms of futures prices. Section VI presents comparative static results for a Leontief technology. Section VII introduces a stochastic wage into the model. Section VIII concludes the paper.

II. THE VALUATION PROBLEM

Consider capital, which, at each time t, can produce one unit of output selling for P_t, while incurring a variable unit production cost C_t. (In a technology with only capital and labor, C_t can be thought of as the wage bill per unit of output.) The profit at time t, exclusive of capital costs, is $P_t - C_t$ if the firm decides to produce. If the firm decides not to produce, profit is zero. This decision need not be made until time t. The profit flow at time t is therefore

$$\pi_t = \max[0, P_t - C_t] \tag{1}$$

Because the firm has a shut-down option, profits are a convex function of price, as is clear from Figure I. Consequently, if output price is uncertain, then from Jensen's inequality we have

$$E_0(\max[0, P_t - C_t]) \geq \max[0, E_0(P_t) - C_t]$$

from which it follows the uncertain profits will have at least as great an expected value as certain profits.

Convexity of the profit function is responsible for the results of Abel [1981b], Hartman [1972], and Dietrich and Heckerman [1980] that increases in output price uncertainty raise the value of a firm. Those papers have profit functions which are convex due to the nature of the technology; this paper emphasizes profit functions which are convex due to the shut-down option, although we include the Cobb-Douglas model of Dietrich and Heckerman as a special case. However, as we show later, when investors are risk-averse, it will not necessarily follow that an increase in expected future profits leads

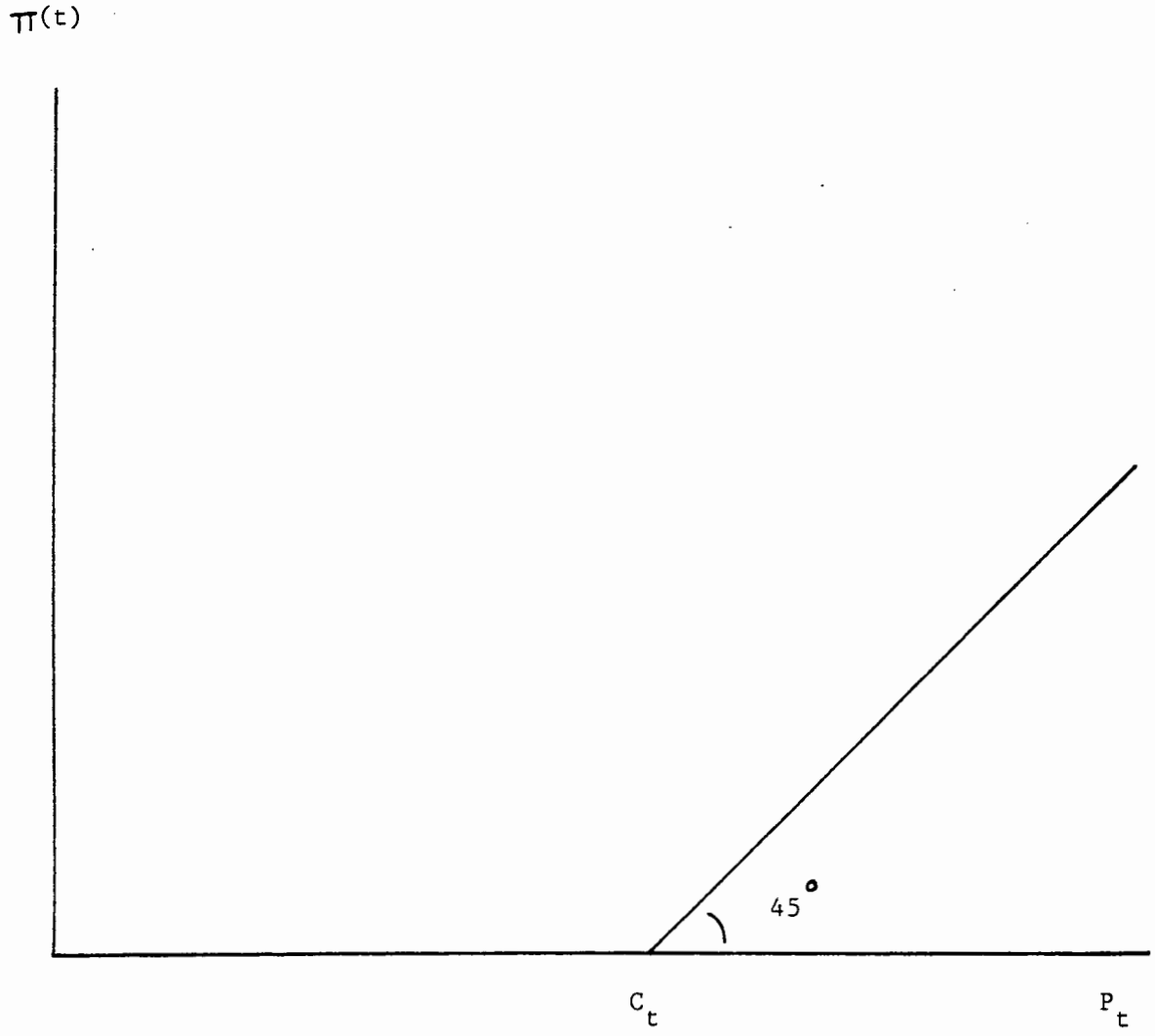

FIGURE I

to an increase in the current value of a claim on those uncertain future profits. The market value of the firm can fall when there is an increase in output price variability.

Valuation Under Risk Neutrality

We now derive a formula for $V_0(t)$, the value of a claim on time t profits, conditional information at time 0. For simplicity, we assume that all investors are risk-neutral. We relax this assumption in Section V. Once we have found the current value of a claim on future profits, $V_0(t)$, the value of a machine which generates the stream of future profits is found by summing the values of the claims:

$$J = \int_0^T V_0(t)dt$$

where it is assumed that the machine lives for T periods.

A) Assumptions

To derive the explicit valuation formula, we make several assumptions.

1) The firm is a price-taker in the market for output, and the output price follows the continuous time stochastic process[4]

$$\frac{dP}{P} = \alpha_p dt + \sigma_p dz_p \tag{2}$$

where α_p is the expected growth rate of the output price, σ_p^2 is the per unit time variance of that growth rate and dz_p is the random increment to the Wiener process z_p. This process assumes that the output price is known with certainty at time zero, but becomes increasingly uncertain over time.[5] Appendix I shows that this process implies that P_t is log-normally distributed

and that

$$E_0[P_t] = P_0 e^{\alpha_p t} \tag{3}$$

As we will discuss in Section V, the ability to observe futures prices for the commodity (or to construct a "shadow" futures price) will allow us to assume an arbitrary process generating P and the solution will be obtained in terms of the futures price, rather than the commodity spot price.

2) The variable unit production cost, C_t, is known at time zero with certainty. We will relax this assumption in a later section.

3) The risk free rate of interest, r, is constant and known with certainty.

B) Valuing the Cash Flow

In a world with risk-neutral investors, the present value of an uncertain cash flow is equal to the epectation of the cash flow discounted by the risk-free interest rate. Therefore

$$V(P,C,t) = e^{-rt} E_0[\pi_t] \tag{4}$$

$$= e^{-rt} E_0[\max(P_t - C_t, 0)]$$

if $g(P_t; \alpha_p, \sigma_p, P_0)$ represents the probability density function of P_t, conditional on P_0, then (4) becomes

$$V(P,C,t) = e^{-rt} \int_0^\infty \pi_t g(P_t; \alpha_p, \sigma_p, P_0) dP_t \tag{4'}$$

$$= e^{-rt} \int_{C_t}^\infty (P_t - C_t) g(P_t; \alpha_p, \sigma_p, P_0) dP_t$$

In Appendix I, we show that the solution to (4') is

$$(5) \qquad V(P_0,C_t,t) = P_0 e^{-\delta t} N(d_1) - C_t e^{-rt} N(d_2)$$

$$d_1 = [\ln(P_0/C_t) + (r - \delta + \sigma_p^2/2)t]/\sigma_p\sqrt{t}$$

$$d_2 = d_1 - \sigma_p\sqrt{t}$$

and $\delta = r - \alpha_p$.[6]

While it may appear that this solution has already made a strong assumption about the form of the production technology, we will show that the problem of valuing a claim on output produced by a Cobb-Douglas technology also has a solution of the form (5), although with an appropriate redefinition of variables.

In Section IV, we show that the introduction of risk-averse investors leaves (5) unchanged except that δ will involve terms which reflect risk aversion.

III. EXAMPLES

In this section we apply the above valuation technique to machines with Leontief and Cobb-Douglas technologies. In both examples, we retain the above assumptions about output price, factor costs and the risk-free interest rate.

Leontief Production Function

Assume that the machine's technology is Leontief, so that the capital-labor ratio is fixed. Then for a given level of capital K, the firm will, at

each time t, produce either

$$\bar{Q} = f(\bar{K}, L[\bar{K}]) = f(\bar{K}, \bar{L}) \tag{6}$$

or zero. Then the profit flow at time t is

$$\max[P_t\bar{Q} - W_t\bar{L},\ 0]$$

where W_t is the time t wage rate. By redefinition of units this expression can be rewritten as

$$\max[P_t\bar{Q} - C_t\bar{Q},\ 0] = \bar{Q}\max[P_t - C_t,\ 0] \tag{7}$$

where C_t is the wage bill per unit of output. Using (4), the present value of the expected cash flow in (7) is then

$$\bar{Q}V(P_0, C_t, t)$$

where $V(P_0, C_t, t)$ is calculated from (5).

Cobb-Douglas Production Function

Consider now a Cobb-Douglas technology with putty-putty capital (i.e., the capital-labor ratio can be changed at any time):

$$Q(K,L) = K^a L^b \qquad a + b < 1 \tag{8}$$

Assume that the quantity of capital is fixed at $\bar{K}$. At time t, the firm chooses L_t to satisfy

$$\max_{L_t} [P_t \bar{K}^a L^b - WL_t] \tag{9}$$

where the wage W is constant and known with certainty. It is straightforward to show that labor usage is then given by

$$L_t = (P_t/W)^{\frac{1}{1-b}} K^{\frac{a}{1-b}} b^{\frac{1}{1-b}} \tag{10}$$

Inserting (10) into (9) yields profits at time t of

$$\phi(P_t) = BP_t^{\gamma} \tag{11}$$

where

$$\gamma = \frac{1}{1-b}$$

$$B = \bar{K}^{\frac{a}{1-b}} \bar{W}^{\frac{-b}{1-b}} (b^{\frac{b}{1-b}} - b^{\frac{1}{1-b}})$$

Assume now that there is a quasi-fixed cost, C_t, at each time t, which is known with certainty and which is incurred by deciding to produce a nonzero quantity of output. This cost is independent of the level of production. For example, suppose that a plant must be heated and lighted no matter what the level of production. With no production, however, the heat and light can be shut off. Profits are therefore given by

$$\pi_t = \max[\phi(P_t) - C_t, 0]$$

The firm with a Cobb-Douglas technology will not produce unless profits exceed the quasi-fixed cost. The Cobb-Douglas profit function $\phi(P_t)$ plays a role analagous to that of P_t in the Leontief case.

To place this problem within the framework developed above, it is necessary to derive the stochastic process governing $\phi(P_t)$. An application if Ito's lemma[7] yields

$$(12) \qquad d\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial P} dP + \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 \phi}{\partial P^2} dt$$

We assume that $\phi_t = 0$.[8] Substituting the price dynamics for P_t from (2) into (12), we have

$$(13) \qquad d\phi = (\frac{\partial \phi}{\partial P}\alpha_p P + \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 \phi}{\partial P^2})dt + \frac{\partial \phi}{\partial P}\sigma_p P dz_p$$

Finally, using (11) yields

$$(14) \qquad \frac{d\phi}{\phi} = (\alpha_p + \frac{1}{2}\sigma_p^2 \gamma[\gamma - i])dt + \sigma_p dz_p$$

$$= \alpha_\phi dt + \sigma_\phi dz_p$$

It is now possible to use our formula (5), simply be replacing P with ϕ and the parameters describing the prices for P with those describing the process for ϕ. That is, replace α_p with α_ϕ and σ_p with σ_ϕ.

An interesting feature of this solution can be seen by setting $C_t = 0$. Then (5) reduces to

$$(15) \qquad \phi(P_0)e^{-(r-\alpha_\phi)t}$$

because $N(d_1) = N(d_2) = 1$. Using (11) and (14), (15) becomes

$$V(P_0,W,\bar{K},T) = BP_0^{\gamma} e^{-(r-\alpha_p - \frac{1}{2}\sigma_p^2\gamma[\gamma-1])t} \tag{16}$$

The value of the stream of profits from the payouts of the project is then

$$J = \int_0^T BP_0^{\gamma} \, e^{-(r-\alpha_p - \frac{1}{2}\sigma_p^2\gamma[\gamma-1])t} \, dt \tag{17}$$

If we set $T = \infty$, the solution to (17) is

$$\frac{BP_0^{\gamma}}{r - \alpha_p - \frac{1}{2}\sigma_p^2 \gamma(\gamma - 1)} \tag{18}$$

Dietrich and Heckerman (1980) show that this is also the solution to

$$\max_{L_t} E_0 \int_0^{\infty} \pi_t e^{-rt} dt \tag{19}$$

This should be no surprise, since equation (5) was derived as the present value of expected profits. $C_t = 0$ is just a special case.

IV. VALUATION UNDER RISK AVERSION

We have derived the formula for the current value of a claim on the risky cash flow $V(P_0,C_t,t)$ in a risk neutral world. In this section we show that essentially the same formula can be derived when claims on the firm are valued by risk-averse investors, using an equilibrium asset pricing model developed by Merton [1973]. This model, the Intertemporal Capital Asset Pricing Model (ICAPM) has the implication that for any asset the expected rate of return in

excess of the risk-free rate will be proportional to σ_{im}, the covariance of the returns of the asset with the returns of all other assets in the economy. The equilibrium expected rate of return for asset i is given by

$$(20) \qquad \alpha_i - r = \beta_i(\alpha_m - r)$$

where α_m is the expected rate of return on a portfolio of all assets in the economy (the market portfolio) and

$$(21) \qquad \beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\rho_{im}\sigma_i}{\sigma_m}$$

where σ_m^2 is the variance of the rate of return on the market portfolio; ρ_{im} is the correlation coefficient between the rate of return on the asset and that on the market; and σ_i is the standard deviation of the rate of return on asset i. Suppose that the claim on future profits, $V(P_0,C_t,t)$ is traded. For an investor to willingly hold this claim in a portfolio, its expected rate of return must be given by equation (20). We can calculate the rate of return on $V(P_u,C_t,u)$ using Ito's lemma, and we then obtain

$$(22) \qquad \frac{dV}{V} = \frac{1}{V}\left[\frac{\partial V}{\partial u} + \alpha_p P\frac{\partial V}{\partial P} + \frac{1}{2}\sigma_p^2 P^2\frac{\partial^2 V}{\partial P^2}\right]du + \frac{P}{V}\sigma_p\frac{\partial V}{\partial P}dz_p$$

where u is calendar time, and t is the date at which the production decision will be made. Note that the bracketed term contains all the nonstochastic components of dV, and only the last term is stochastic. If the unexpected component of the rate of return on the market is given by $\sigma_m dz_m$, then the covariance between the rate of return on the claim and that on the market is[9]

$$(23) \qquad \frac{1}{dt}\left[\frac{dV}{V} - \frac{E(dV)}{V}\right] \sigma_m dz_m = \frac{P}{V}\frac{\partial V}{\partial P}\sigma_m\sigma_p\rho_{pm} = \frac{P}{V}\frac{\partial V}{\partial P}\sigma_{pm}$$

The beta of the claim on future profits (using (21)) is therefore this covariance divided by the variance of the rate of return on the market:

$$(24) \qquad \beta_v = \frac{P}{V}\frac{\partial V}{\partial P}\frac{\sigma_{pm}}{\sigma_m^2} = \frac{P}{V}\frac{\partial V}{\partial P}\beta_p$$

where β_p is the beta of the commodity. We can now impose the constraint that the expected rate of return on the claim on profits be given by the ICAPM:

$$(25) \qquad \frac{E(dV)}{V} = r + \beta_v(r_m - r)$$

Using (22) and (24), (25) becomes a partial differential equation:

$$(26) \qquad \frac{\partial V}{\partial u} = rV - (r - \delta)P\frac{\partial V}{\partial P} - \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 V}{\partial P^2}$$

where

$$\delta = \alpha_s - \alpha_p$$

$$\alpha_s = r + \beta_p(r_m - r)$$

α_s is the equilibrium rate of return on a stock which has the same beta as the commodity price. (26) can be solved, subject to the boundary condition that, at the time at which the production decision must be made, the value is

$$(27) \qquad V(P_t, C_t, t) = \max\,(0,\ P_t - C_t).$$

Smith ([1976], p. 26)] shows that the solution to (26) subject to (27) is identical to the formula in the risk-neutral case (5) except that $\delta = \alpha_s - \alpha_p$ instead of $r - \alpha_p$.[10] In the absence of risk aversion, the expected return on the market, r_m, would equal the risk-free rate r. Thus, it is only due to the presence of risk aversion that α_s is different from the risk-free rate.

Note that, in general, the value of a project will depend upon both the beta of the commodity price and upon the total variance of the commodity price.

V. INTERPRETATION

Consider the expression

$$F_0^t = e^{-\alpha_s t} E(P_t) = e^{-\delta t} P_0 \tag{28a}$$

where we have used (3) to evaluate $E(P_t)$. F_0^t is the price an investor would pay today in order to receive future delivery of the commodity--that is, it is the price of a futures contract which must be purchased in full at the time the contract is arranged.[11]

Consider also the expression

$$G_0^t = e^{-rt} C_t \tag{28b}$$

This is the futures price for the certain cost of production, C_t. If we substitute (28) into (5), we can write the value of a claim on future profits in terms of the current futures prices for the commodity being purchased and for the production cost:

$$(29) \qquad V(F_0^t, G_0^t, t) = F_0^t N(d_1^*) - G_0^t N(d_2^*)$$

$$d_1^* = [\ln(F_0^t/G_0^t) + (\tfrac{1}{2}\sigma_p^2)t]/\sigma_p\sqrt{t}$$

$$d_2^* = d_1^* - \sigma_p\sqrt{t}$$

Equation (29) is a slightly rewritten version of the Black-Scholes formula for the value of a call option, where the stock price has been replaced by the futures price of the commodity and the exercise price has been replaced by a futures price for the production cost. Owning a machine provides the owner with the right to produce a commodity at the variable production cost C_t (analogous to the exercise price of a call option), and to receive the commodity price P_t (analagous to the stock price). A call option on a stock need not be exercised if, at maturity, the exercise price is greater than the price of the stock. The owner of physical capital can similarly avoid a loss by shutting down the plant if the variable cost of production exceeds sales revenue.

If there is a futures market for the commodity being produced and for the production cost, where the futures contracts require full payment at the time the contract is made, then (29) can be used directly. If there is no futures market, shadow futures prices can be constructed using (28).

Because the firm produces a commodity, it should be no surprise that (29) is the formula for a commodity call option. (29) is identical to Black's [1976] formula for the value of a commodity call option, except that Black's formula uses a futures price where payment is made when the commodity is delivered.

We are now in a position to understand why the assumption that the price follows a geometric Wiener process as in (2) is not critical. It is possible to derive (29) directly by repeating the derivation in Section 3, and writing the value of the claim, V, directly as a function of the futures prices: $V = V(F,G,u)$. While it may not be reasonable to expect the output price in a competitive industry to follow equation (2), it is reasonable to expect the price of a futures contract with a given expiration date to follow a process like (2):

$$\frac{dF}{F} = \alpha_F dt + \sigma_F dz_F \tag{2'}$$

Furthermore, a futures contract of the type we have described is a financial asset which must have an expected rate of return sufficient to induce investors to hold it.[12] Thus, the expected rate of price increase of a futures contract will be determined by the Capital Asset Pricing Model:[13]

$$\alpha_F = r + \beta_F(r_m - r) \tag{20'}$$

Using (2') and (20'), and repeating the section IV derivation, will yield (29). Having noted that the model is really more general than it appears to be, for simplicity we will continue to assume that the price process is given by equation (2).

To completely understand (29), it is useful to distinguish between two classes of commodities, stored and non-stored commodities.

Stored vs. Non-Stored Commodities

If a commodity is actually stored, then it must be that its price (net of physical storage costs) is rising at a rate which compensates the owner for the risk involved in storing the commodity. In effect, the commodity is like any financial asset. The futures price for such a commodity will simply be the current spot price, if payment for delivery must be made when the contract is struck.[14]

If the commodity price is expected to rise at a rate which is too low to compensate for the risk involved in storing the commodity, then no one will store the commodity. The futures price will be less than the current spot price, and is given by (28).

Many commodities are non-stored because they may easily be reproduced at constant marginal cost, and potential production prevents a rise in their price. Pencils and shoes are two examples. In equilibrium, we would not expect the price of pencils or shoes to rise at the rate of interest.[15]

The distinction between stored and non-stored commodities is important. We will show in the next section that an infinitely-lived, non-depreciating machine which produced a stored commodity ($\delta = 0$) would have an infinite present value, while the present value of a machine producing non-stored commodity ($\delta > 0$) will always be infinite.

VI. COMPARATIVE STATICS

We showed in the previous section that our expression for the valuation of a claim on uncertain future profits is equivalent to the formula for a call option on a stock with a futures price in place of the more usual stock price. The comparative static properties of call options are well-known [Smith (1976)]. However, our formula exhibits some unusual properties because changes in parameters affect both the value of the claim given the futures prices, and the value of the futures prices.

To interpret the results in this section, one should recall that the purpose of the exercise is to compute the present value of a risky cash flow. This present value can be thought of as having an option component (when there is a right to limit losses by not producing) and a futures value component (reflecting the current price of a contract for future delivery of a commodity). When there is no option component (as, e.g., when the variable production cost C_t is zero), then the formula (5) reduces to $V(P_0,t) = P_0 e^{-\delta t} = F_0^t$ which is just the futures price. In this case, futures prices by themselves are a guide to production decisions. When there is no meaningful futures price component (when $\delta = 0$)[16] the formula reduced to a standard call option. It is possible to decompose all of the comparative static results in the following way:

$$\frac{dV}{d\theta} = \frac{\partial V}{\partial F}\frac{\partial F}{\partial \theta} + \frac{\partial V}{\partial \theta}$$

That is, when the parameter θ changes, it will affect the value of the claim on future profits both by affecting the futures price (the first term) and by affecting the value of the claim given the futures price (the second term). When the futures price is unaffected, our comparative static results are identical to those for a call option on a stock. All of the calculations are

made for V(t), and may be extended to the value of a project by calculating

$$\int_0^T \frac{dV(t)}{d\theta}\, dt$$

For future reference, note that differentiating (29) yields

$$\frac{\partial V}{\partial F} = N(d_1) + FN'(d_1)\frac{\partial d_1}{\partial F} - Ce^{-rt}N'(d_2)\frac{\partial d_2}{\partial F}$$

It is straightforward to verify, however, that

$$FN'(d_1) = Ce^{-rt}N'(d_2)$$

and that

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F}$$

so that

$$\frac{\partial V}{\partial F} = N(d_1) \tag{30}$$

Equation (30) is frequently useful in the calculations in this section. The implication of (30) is that a rise in the futures price raises the value of a claim on future profits, but in general by less than the value of the futures price increase.

In what follows, we show the effect on our valuation formula of a change in the: time to production, variance of the rate of change of the commodity price, risk-free rate, variable production cost and δ. We treat only the case

of a Leontief technology; the results are easily extended to the Cobb-Douglas case, provided that P_0 is replaced with $\phi(P_0)$ along with the other changes mentioned in Section 2 below.[17]

i) Time to Production

Differentiating equation (29) with respect to t yields

$$\frac{dV}{dt} = \frac{\partial V}{\partial F}\frac{\partial F}{\partial t} + \frac{\partial V}{\partial t} = -\delta e^{-\delta t}N(d_1) + re^{-rt}CN(d_2) + e^{-\delta t}\frac{P\sigma_p}{2\sqrt{t}}N'(d_1)$$

where $\delta = \alpha_s - \alpha_\phi$ and d_1, d_2 are as defined in (29). When $\delta = 0$, we have the standard result that the value of the claim (due to the option component) increases with time to maturity. When $\delta > 0$, the sign of the derivative is ambiguous. We will first show analytically that for large enough t, V(t) is approximately zero and then we will present several simulations to show under what conditions $dV/dt > 0$.

Assuming that all limits are finite,

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} Pe^{-\delta t} \lim_{t\to\infty} N(d_1) - \lim_{t\to\infty} Ce^{-rt} \lim_{t\to\infty} N(d_2)$$

since the exponential terms have limits of zero, and the limiting values of $N(d_1)$ and $N(d_2)$ are either zero or one,[18]

$$\lim_{t\to\infty} V(t) = 0$$

To examine the behavior of V(t) for intermediate t, we present the results of several simulations in Figure 2. Of particular interest is the hump-shaped time profile of the value of the claim, which has the interpretation that a claim on the cash flows from a project several years in

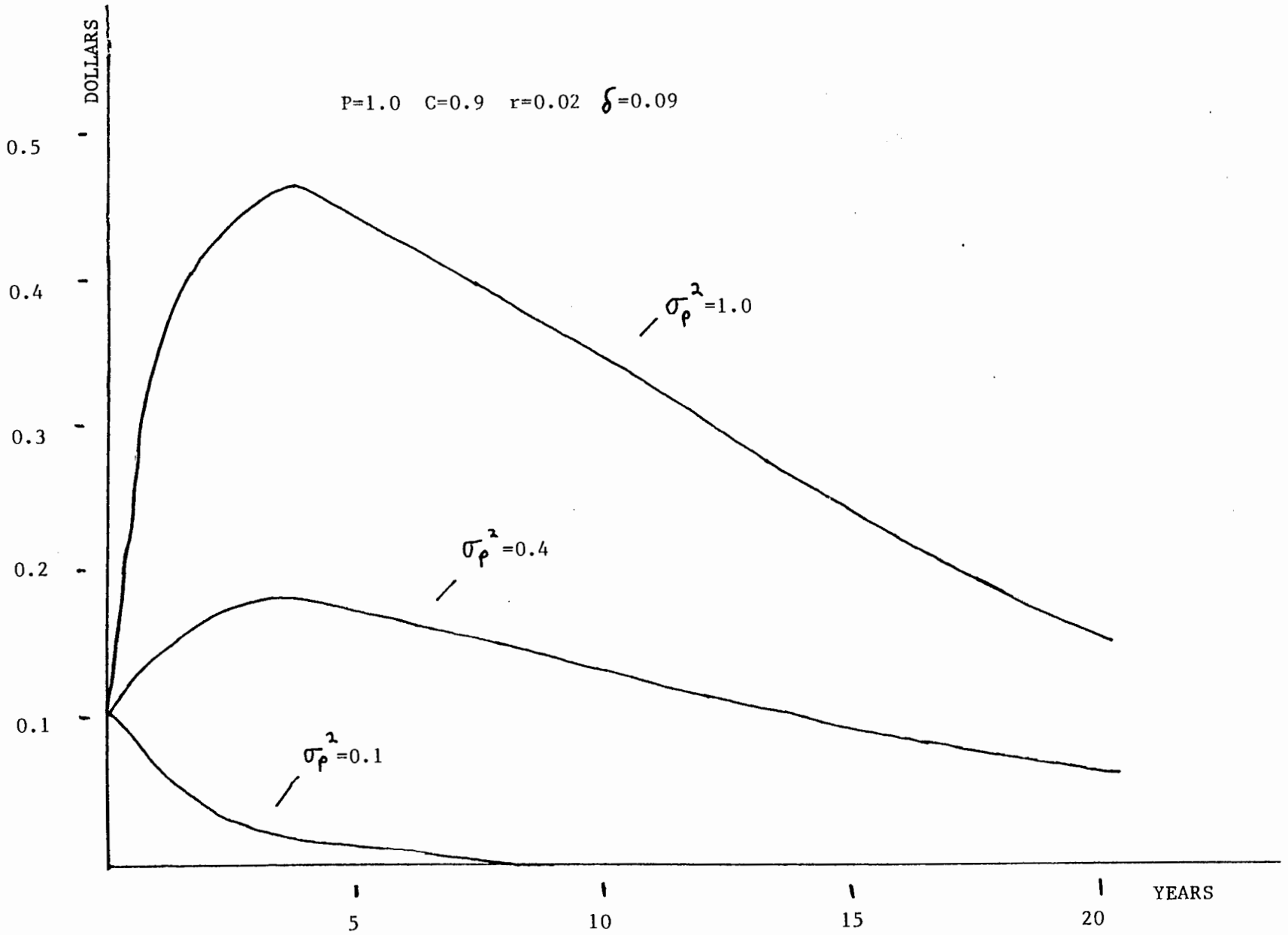
DOLLARS
P=1.0 C=0.9 r=0.02 δ=0.09
0.5
0.4
0.3
0.2
0.1
σ_ρ^2=1.0
σ_ρ^2=0.4
σ_ρ^2=0.1
5
10
15
20
YEARS

FIGURE II

the future may be worth more than a claim on cash flows from the same project tomorrow or in the far future.

It is the option component that causes the rising portion of the time profile. The hump-shape can always be made into a monotonic decline by changing other parameters so as to lower the value of the option component or increase the futures price component, so that the latter dominates. In particular, the hump-shape can be made to vanish if we (a) lower the variance of the rate of price increase of the commodity, (b) lower the exercise price, or (c) lower the futures price. Conversely, the hump-shape can be made to appear by doing the opposite. The first two parameters affect the option component, while the third affects the futures price component.

If $\delta = 0$ and a machine is infinitely-lived and non-depreciating, we have

$$\int_0^\infty V(t)\, dt = \infty$$

so such a machine would have an infinite present value.

ii) Variance

Differentiating (29) with respect to the variance of the rate of change of the price of commodity yields

$$(31) \qquad \frac{dV}{d\sigma_p^2} = \frac{\partial V}{\partial F}\frac{\partial F}{\partial \sigma_p^2} + \frac{\partial V}{\partial \sigma_p^2} = -t\,\frac{\partial \delta}{\partial \sigma_p^2}\, e^{-\delta t} PN(d_1) + e^{-rt} CN'(d_2)\sigma_p \frac{\sqrt{t}}{2}$$

If $\delta = 0$, then $\partial F/\partial \sigma_p^2 = 0$ and (31) gives the standard result that increases in variance raise the value of the option. If $\delta \neq 0$, then we must examine $\partial \delta/\partial \sigma_p^2$.

If we assume that the secular rate of price increase of the commodity is α_p, then $\delta = \alpha_s - \alpha_p = r + \beta_p(r_m - r) - \alpha_p$, hence

$$\frac{\partial \delta}{\partial \sigma_p^2} = (r_m - r) \frac{\partial \beta_p}{\partial \sigma_p^2}$$

$$= (r_m - r) \frac{1}{2} \frac{\sigma_{mp}}{\sigma_m \sigma_p}$$

$$= \frac{1}{2\sigma_p^2} (r_m - r) \beta_p$$

This calculation assumes the the correlation coefficient between the return on the market and the rate of change of the commodity price, ρ_{mp}, is unchanged by changes in the variance of the rate of change of the commodity price. If the commodity is positively correlated with the market, then an increase in variance raises the discount rate for the project and thus tends to lower the value of the claim on future profits.

Since $\partial \delta / \partial \sigma_p^2$ is proportional to the commodity beta, (31) gives us the result that increases in own variance will lower the value of projects which are highly correlated with the market, and will raise the value of projects with zero or negative correlation with the market. This result is sensible. Undertaking a project which is highly correlated with other projects in the economy (i.e., which has a high beta) adds to the risk of the market portfolio and such a project will, ceteris paribus, have a lower value. A somewhat surprising result is that for commodities with a small positive beta, it is possible that increases in own variance can increase the value of the project, because the first term on the right hand side of (31) might still dominate.[19]

iii) Risk-Free Rate

Differentiating the value of the claim with respect to the risk free rate yields

$$(32) \qquad \frac{dV}{dr} = \frac{\partial V}{\partial F}\frac{\partial F}{\partial r} + \frac{\partial V}{\partial r} = -tV + \left(1 - \frac{\partial \delta}{\partial r}\right) e^{-\delta t} PN(d_1)t$$

where

$$\frac{\partial \delta}{\partial r} = 1 + \beta_p \frac{\partial (r_m - r)}{\partial r}$$

Hence, (32) becomes

$$\frac{\partial V}{\partial r} = -tV - \beta_p \frac{\partial (r_m - r)}{\partial r} e^{-\delta t} PN(d_1)t$$

The effect on project value of a change in the risk free rate depends upon whether the commodity has a price change which is positively or negatively correlated with the return on the market, and upon whether the market risk premium rises or falls with the risk-free rate.

This has interesting implications for aggregate investment. If the risk premium rises with the risk-free rate, then an increase in the risk free rate will shift investment from high beta projects to low beta projects. If the risk premium falls with increases in the risk-free rate, the effect will be the opposite.

iv) Variable Production Cost

Taking the derivative with respect to the variable production cost yields

$$\frac{dV}{dC} = -e^{-rt} N(d_2) < 0$$

An increase in the variable cost of production lowers the value of the project.

v) δ

An increase in δ could come about by increasing the systematic risk of the commodity or by lowering the secular expected rate of increase of the commodity price. Differentiating (5) with respect to δ yields

$$\frac{dV}{d\delta} = -te^{-\delta t}PN(d_1) < 0$$

An increase in δ lowers the value of the project.

VII. STOCHASTIC COST OF PRODUCTION

We now allow the variable cost of production to be random. Suppose that the stochastic evolution of the variable production cost is governed by

$$\frac{dC}{C} = \alpha_c dt + \sigma_c dz_c$$

We can write the value of the claim as $W(P_u, C_u, u, t)$ where u is calendar time and t is time production, and the rate of return on the claim is given by

$$\frac{dW}{W} = W_u + W_p dP + W_c dC + \frac{1}{2}[\sigma_p^2 P^2 W_{pp} + \sigma_c^2 C^2 W_{cc} + 2CPW_{cp}\sigma_c\sigma_p\rho_{pc}]dt$$

where ρ_{cp} is the correlation coefficient between the rate of change of the output price and the variable production cost. The stochastic component of the return is $W_p P\sigma_p dz_p + W_c C\sigma_c dz_c$. Repeating the derivation in Section 3 gives the following partial differential equation:

$$(33) \qquad W_u = rW - (r - [\alpha_s - \alpha_p])PW_p - (r - [\alpha_x - \alpha_c])CW_c - \frac{1}{2}W_{pp}\sigma_p^2 P^2 - \frac{1}{2}W_{cc}\sigma_c^2 C^2 - PCW_{pc}\rho_{pc}\sigma_p\sigma_c$$

α_x is defined by

$$\alpha_x = r + \beta_c(r_m - r)$$

As with α_s, α_x is the required rate of return for anyone to willingly hold an asset which has the same beta as the random exercise price, C.

This equation, together with the boundary condition

$$W(C_t, P_t, t) = \max[0, P_t - C_t]$$

has the solution

$$W(C_0, P_0, t) = P_0 e^{-\delta t} N(d_1^*) - C_0 e^{-\lambda t} N(d_2^*) \tag{34}$$

$$d_1^* = [\ln(P_0/C_0) + (\lambda - \delta + \frac{\sigma^2}{2})t]/\sigma\sqrt{t}$$

$$d_2^* = d_1^* - \sigma\sqrt{t}$$

and

$$\sigma^2 = \sigma_p^2 + \sigma_c^2 - 2\rho_{pc}\sigma_p\sigma_c$$

$$\delta = \alpha_s - \alpha_p$$

$$\lambda = \alpha_x - \alpha_c$$

Appendix II shows by direct calculation that (34) is a solution to (33).

Notice that if we define

$$G_0^t = C_0 e^{-\lambda t} \tag{35}$$

then G_0^t has the interpretation of being a futures price for the variable production cost. If we substitute (35) and (28a) into (34), we obtain (29). When it is known for certain that C will be constant, (35) reduces to $G_0^t = Ce^{-rt}$, which when substituted into (29), yields the formula for the case of a non-stochastic cost of production.

Since (34) is equivalent to (29), we can decompose comparative static changes in the following way:

$$\frac{dW}{d\theta} = \frac{\partial W}{\partial \theta} + \frac{\partial W}{\partial F}\frac{dF}{d\theta} + \frac{\partial W}{\partial G}\frac{dG}{d\theta}$$

The first term is again the "pure" option effect, and the last two are the effects of futures price changes.

The derivatives of W with respect to time to expiration, excercise price, and δ will be like those of V when the exercise price is stochastic. The derivatives with respect to the variance of the rate of change of the output price and risk-free rate will be different, however. We will also consider the effect on project value of a change in the variance of the rate of change of the variable production cost and a change in the correlation between the rates of change of the production cost and the output price.

i) Output Price Variance

Differentiating (34) with respect to the output price variance yields

$$\frac{dW}{d\sigma_p^2} = -t \frac{\partial \delta}{\partial \sigma_p^2} N(d_1)F + Ce^{-(\alpha_x - \alpha_c)t} N'(d_2) \frac{\sqrt{t}}{2\sigma} (1 - \rho_{pc} \frac{\sigma_c}{\sigma_p})$$

In equation (5.3) we saw that $\partial\delta/\partial\sigma_p^2$ was proportional to β_p, the output price beta. Hence the first term is negative if the output price has a positive beta. The sign of the second term depends upon both the correlation of the rates of change C and P (ρ_{cp}) and their relative variances. If the commodity has a positive beta, then the entire derivative is ambiguous unless $\rho_{cp} > 0$ and σ_c^2 substantially exceeds σ_p^2. In that case an increase in σ_p^2 lowers the value of a project. If the project is uncorrelated with the market (zero beta), and if $\rho_{pc} = 0$, then an increase in σ_p^2 raises the value of the project.

iii) Wage and Output Price Correlation

Differentiating (34) with respect to the correlation between the rates of change of costs and the output price yields

$$\frac{dV}{d\rho_{pc}} = \frac{\partial W}{\partial \sigma^2} \frac{\partial \sigma^2}{\partial \rho_{pc}} = -\frac{\sqrt{t}}{\sigma} \sigma_p \sigma_c Ce^{-(\alpha_x - \alpha_c)t} N'(d_2) < 0$$

A ceteris paribus increase in this correlation lowers σ and thus lowers the value of a project. This calculation assumes that both σ_p^2 and σ_c^2 are unchanged. Note that an increase in the correlation coefficient lowers the variance of the rate of change of P-C and therefore lowers the value of the "option component"--an option which is in the money now is likelier to remain in the money--without affecting the time-value component.

This is contrary to the following plausible but erroneous intuition: as P-C become less variable, given current values of P and C the cash flow increases in value because of the reduction in variability.

iv) Risk-Free Rate

From (29), we can see that the value of a claim depends on the risk-free rate only through the futures price of F and G. Hence

$$\frac{dW}{dr} = \frac{\partial W}{\partial F}\frac{dF}{dr} + \frac{\partial W}{\partial G}\frac{dG}{dr}$$

$$= -t\frac{\partial \delta}{\partial r} FN(d_1) + t\frac{\partial \lambda}{\partial r} GN(d_2)$$

$$= -t[1+\beta_p \frac{\partial (r_m - r)}{\partial r}] FN(d_1) + t[1 + \beta_c \frac{\partial (r_m - r)}{\partial r}] GN(d_2)$$

As before, the effect of a change in the risk-free rate is ambiguous, depending on the beta of the commodity being produced and the beta of the cost of the factor of production.

The extension of these results to other technologies is straightforward. In particular, it is possible to incorporate stochastic wage in the Cobb-Douglas case, as in Dietrich and Heckerman.

VIII. SUMMARY AND CONCLUSIONS

We have shown in this paper how option-pricing techniques can be applied in a straightforward way to the investment problem of the firm which has the option to shut down production if variable production costs exceed revenues. The investment model is rudimentary, however, in that no account is taken of adjustment costs, depreciation, or the possibility of other fixed factors.

Allowing for depreciation is a trivial extension when the rate of depreciation is a fixed function of time. A more interesting and certainly more difficult problem, however, is modelling the reasonable case where physical depreciation depends upon the intensity with which the factor is

used. The analytical problem in this case is the dependence of expected future profit not just on the future realized commodity price, but also on the entire sequence of commodity prices.

There is another interesting problem which is formally identical to the problem of modeling depreciation which is dependent upon usage: calculating the value of a depletable resource. The value of options to produce in the future will be affected by decisions to produce some or all of the resource now.

Our model also provides a way to deal with the investment decisions of monopolists facing a stochastic demand. While we assumed price taking behavior, this was not necessary. We could have assumed that a parameter in the demand curve followed a Wiener process, instead of having the price follow such a process.[20] Then, assuming that the firm maximizes instantaneous profits, marginal revenue would follow an Ito process and the analysis could be carried through, calculating firm value as a function of the capital stock. Determination of the optimal capital stock would then be straightforward.

Appendix I

The evolution of P is described by:

$$(1) \qquad \frac{dP}{P} = \alpha dt + \sigma dz$$

This can be integrated to give (Fischer [1975]).

$$(2) \qquad \frac{P_t}{P_0} = \exp\left[(\alpha - \frac{1}{2}\sigma^2)t + \sigma\int_0^t dz_s\right]$$

Applying Ito's Lemma to (2) will yield (1). By assumption, dz is a standard normal random variable, so that

$$(3) \qquad \ln(P_t) = (\alpha - \tfrac{1}{2}\sigma^2)t + \sigma\int_0^t dz_s + \ln(P_0)$$

is distributed normally with mean $(\alpha - \frac{1}{2}\sigma^2)t + \ln P_0 = \mu$ and variance $\sigma^2 t = s^2$. Thus P_t is lognormal, with density function

$$(4) \qquad g(P) = (Ps\sqrt{2\pi})^{-1}\exp\left[-\frac{1}{2}\left(\frac{\ln P - \mu}{s}\right)^2\right] \quad ; \quad 0 < P < \infty$$

We now proceed to calculate

$$(5) \qquad E_0[\max(0,\ P_t - C)] = \int_C^\infty (P_t - C)g(P)dP$$

Make the change variable

$$y = \frac{\ln P - \mu}{s}$$

which transforms (5) into

$$\int_{\frac{\ln C-\mu}{s}}^{\infty} (e^{ys+\mu} - C)(e^{ys+\mu}s\sqrt{2\pi})^{-1}e^{-\frac{1}{2}y^2}se^{ys+\mu}dy$$

This can be rewritten to give

$$(7) \qquad \int_{\frac{\ln C-\mu}{s}}^{\infty} - s \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} e^{\mu + \frac{1}{2}s^2} dy - C \int_{\frac{\ln C - \mu}{s}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} dy$$

where the further change of variable $z = y-s$ was made. Now define

$$(8) \qquad N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

and note that

$$(9) \qquad 1 - N(x) = N(-x)$$

Using (8) and (9), (7) becomes

$$(10) \qquad e^{\mu + \frac{1}{2}s^2} N(s - \frac{\ln C - \mu}{s}) - CN(\frac{\mu - \ln C}{s})$$

and substituting into (10) the definition of μ and s yields

$$P_o e^{\alpha t} N(d_1) - CN(d_2)$$

where

$$d_1 = [\ln(P_0/C) + (\alpha + \tfrac{1}{2}s^2)t]/s\sqrt{t}$$

$$d_2 = d_1 - s\sqrt{t}$$

Multiplying through by e^{-rt} gives equation (5) in the text.

Notice also that since $\ln(P_t)$ is normally distributed with mean μ and variance s^2, P_t is log-normally distributed with mean

$$E(P_t) = e^{\mu + \frac{1}{2}s^2} = P_0 e^{\alpha t}$$

where the second equality follows from the definitions of μ and s^2.

Appendix II: Verification of Formula (34)

In this Appendix, we verify that equation (34) is the value of a claim on future profits when the variable cost of production is stochastic. Equation (34) is verified by deriving the partial differential equation that the claim must satisfy and then demonstrating that (34) satisfies it. The partial differential equation is derived in the same way as in Section IV: We calculate the expected rate of return on a claim on future profits, and impose the condition that, in equilibrium, the claim must have an expected rate of return in accord with the ICAPM. This yields from (33) in the text.

Let $\delta = (\alpha_s - \alpha_p)$ and $\lambda = (\alpha_x - \alpha_c)$. We now evaluate the various derivatives, using (34)

$$W_u = -W_t = \delta P e^{-\delta t} N(d_1) - \lambda e^{-\lambda t} N(d_2) - C e^{-\lambda t} N'(d_2) \frac{\sigma}{2\sqrt{t}}$$

$$W_p = e^{-\delta t} N(d_1)$$

$$W_C = -e^{-\lambda t} N(d_2)$$

$$W_{pp} = e^{-\delta t} N'(d_1) \frac{1}{P\sigma\sqrt{t}}$$

$$W_{cc} = -e^{-\lambda t} N'(d_2) \frac{1}{W\sigma\sqrt{t}}$$

$$W_{pc} = -e^{-\delta t} N'(d_1) \frac{1}{C\sigma\sqrt{t}}$$

where $\sigma^2 = \sigma_p^2 + \sigma_c^2 - 2\rho_{pc}\sigma_c\sigma_p$.

A crucial simplification is obtained by noting that

$$e^{-\delta t}N'(d_1) = \frac{C}{P}e^{-\lambda t}N'(d_2)$$

which follows from the definitions

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}d_1^2}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1 - \sigma\sqrt{t})^2}$$

Note that $rW = rPW_p + rCW_c$. Making the appropriate substitutions into (33) in the text yields

$$\delta Pe^{-\delta t}N(d_1) - Ce^{-\lambda t}N(d_2) - Ce^{-\lambda t}N'(d_2)\,\frac{\sigma}{2\sqrt{t}}$$

$$= \delta Pe^{-\delta t}N(d_1) - Ce^{-\lambda t}N(d_2)$$

$$- \frac{1}{2}\, e^{-\lambda t}N'(d_2)C\frac{\sigma_p^2}{\sigma\sqrt{t}} - \frac{1}{2}\, e^{-\lambda t}N'(d_2)C\frac{\sigma_c^2}{\sigma\sqrt{t}}$$

$$+ e^{-\lambda t}N'(d_2)C\frac{\rho_{pc}\sigma_p\sigma_c}{\sigma\sqrt{t}}$$

But by the definition of σ^2, the last three terms on the right hand side of this equation can be written as $-Ce^{-\lambda t}N'(d_2)\sigma/2\sqrt{t}$. Therefore, all of the terms cancel and (34) is verified.

FOOTNOTES

1. Abel [1981a] models the choice of the energy-capital ratio for a firm with a shutdown option.

2. In Sandmo's model the utility of profit is a concave function of the output price, and hence and increase in the variance of the output price lowers the value of the firm. In risk neutral models of the firm, the profit function is convex, and hence an increase in price variance has the opposite effect.

3. It is standard in the literature on capital budgeting to calculate the present value of a cash flow stream by discounting expected cash flows using a discount rate which adjusts for the systematic risk of the cash flows. (See, for example, Brealey and Myers [1981], Chapter 9.) In effect, this is our solution procedure in Section IV.

4. See the appendix to Fischer [1975] for an excellent introduction to the use of continuous-time stochastic processes, and for further references.

5. Equation (2) is taken as exogenously given. A more satisfactory formulation would involve solving for the price process as a function of true exogenous variables, but we do not attempt that here.

A more plausible process for the output price would be the mean reverting process $dP = \alpha_p[P^* - P]dt + \sigma_p P dz_p$. Lagged entry into the industry drives price slowly back toward the long run production cost P^*. We have not been able to solve explicitly for the valuation formula using this process, but numerical methods can be used. However, see Section V below.

6. This is the value of a European call option on a stock which pays continuous dividends at the proportional rate δ. Holding the commodity whose price dynamics are given by (2), is exactly like holding a dividend-paying

stock, but receiving only capital gains and not the dividends.

7. Ito's Lemma states that if $Y=\phi(P,t)$, and P follows (2), then the stochastic differential of Y is given by

$$dY = \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial P}dP + \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 \phi}{\partial P^2}dt$$

8. ϕ_t will be non-zero if capital physically depreciates slowly over time. Thus, we are assuming that capital disintegrates at T, with no depreciation before then.

9. Here we use the formalism $dz_m dz_p = \rho_{mp}dt$.

10. Constantinides [1978] derives the same formula as the solution to valuing a call option on a "non-traded" (hence incorrectly priced) asset.

11. Standard futures contracts call for payment to be made when the commodity is delivered, rather than when the contract is struck. Because payment is to be made in the future, a standard futures contract would have the price

$$f_0^t = F_0^t e^{rt} = P_0 e^{(r-\delta)t}$$

12. See Black [1976] for a discussion of these points.

13. In this case the futures price can still be determined by (28a), but δ is not a constant, and may be a complicated function of time and the current commodity spot price.

14. It is easy to see that when the commodity is stored, $F_0^t = P_0$ is necessary to prevent the existence of arbitrage opportunities. If $F > P$, one would buy the commodity and sell it forward, making a sure profit of $F - P$. If $F < P$, one sells short the commodity and buys it forward to make a profit.

With standard futures contracts, the no-arbitrage relationship for stored comodities is $f_0^t = P_0 e^{rt}$.

15. A more complicated example is corn. Because corn is stored between harvests, the expected rate of price increase for corn must be sufficient to induce investors to store corn. In this case, $\delta = 0$, and from (28a), the futures price for corn equals the spot price. Corn will not be stored across harvest, however, since the price will typically fall at harvest time, and the investor who stores corn will not be compensated for this loss.

Consider now a production process which produced a perfect substitute for corn. If the project were to be short-lived, lasting only between harvests, it would be appropriate to set $F_0^t = P_0$. If the project were to last several years, a more complicated array of futures prices would be required to use the formula, in order to account for the expect reversion to the mean of corn prices. For example, between November and March, when corn is not produced, corn will be stored and its price will be expected to rise at the rate α_s. From January to January, however, the price of corn would be expected to remain unchanged, since (assuming that corn can be produced at constant marginal cost) an expected price increase would induce new production which would eliminate the price increase.

16. When $\delta = 0$ the current spot price is the futures price.

17. Dietrich and Heckerman perform comparative static calculations for the Cobb-Douglas case with (in our teminology) $C_t = 0$ and risk-neutrality.

18. It is impossible for $N(d_1)$ to have a limiting value of one when $N(d_2)$ has a limiting value of zero. The other three combinations are possible, however.

19. Note that this result also holds for the Cobb-Douglas case considered above where $C_t = 0$ [equation (16)]. It can be shown that if α_s is

the return on the asset which has the same systematic risk as $\phi(P)$, then the derivative of the value of the claim with respect to the variance will be positive if

$$(\alpha_s - r) < \sigma_p^2 \gamma(\gamma - 1)$$

When α_s is large ($\beta_p > 0$) this inequality will be reversed and the value of the claim will fall with an increase in variance. In the Cobb-Douglas case, an increase in variance will raise value--given the future price--because the profit function is convex even without a shutdown option.

Dietrich and Heckerman [1980], using a Cobb-Douglas production function, obtain the result that an increase in own variance raises the value of a project, but they ignore the effect on the discount rate.

20. Pindyck [1980b] studies a related problem introducing a demand curve varying according to an Ito process.

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