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## Investment Rules and Time Invariance under Population Growth

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# Investment Rules and Time Invariance under Population Growth

## Abstract

We propose an adaptation of Hartwick's investment rule to models with population growth and show that following Hartwick's rule is equivalent to a time-invariant real per capita net national product. In the so-called DHSS model of capital accumulation and resource depletion the proposed Hartwick's rule equates the accumulation of per capita capital, net of the capital dilution effect of population growth, to the value of the depletion of the resource, gross of the capital dilution effect. We investigate why this asymmetry arises by analyzing a general model with multiple capital goods, in which we obtain a formulation of Hartwick's investment rule where capital gains play a role if population growth is positive. Since capital gains accrue only to the resource but not to capital, we get the apparent asymmetry in the DHSS model. In both models we obtain as a corollary that keeping the value of net investments equal to zero leads to constant consumption if population is constant.

JEL-Codes: D630, O410, Q010.

Keywords: sustainable development, population growth, intergenerational equity.

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# 1 Introduction

Hartwick’s investment rule prescribes reinvesting resource rents in reproducible capital, thus keeping the value of net investments equal to zero, in order to keep consumption constant. Starting with Hartwick’s original article (Hartwick, 1977), a series of papers (among others, Hartwick, 1978a,b; Dixit, Hoel and Hammond, 1980; Withagen and Asheim, 1998; Mitra, 2002; Asheim, Buchholz and Withagen, 2003; Buchholz, Dasgupta and Mitra, 2005; Mitra, Asheim, Buchholz and Withagen, 2013) have contributed to our understanding of the connection between Hartwick’s investment rule and a sustainable development with constant wellbeing. This literature shows that Hartwick’s result—keeping the value of net investments equal to zero leads to constant wellbeing—is robust, as it holds in a variety of different models and technologies.

Still, these findings are established under assumptions that are rather strict: the economy is assumed to have constant technology and constant population. One example is the model of capital accumulation and resource depletion, the so-called Dasgupta-Heal-Solow-Stiglitz (DHSS) model, that Solow (1974) and Hartwick (1977) used to analyze constant consumption paths in the presence of resource constraints. Furthermore, the economy is assumed to implement an efficient path in continuous time. There are ways to relax each of these assumptions and still obtain some variant of Hartwick’s result (see Asheim, 2013, Generalizations, for an overview). For example, if there is exogenous technological progress in the sense of a time-dependent technology, Hartwick’s result is restored by including time as an additional stock. Also, it is possible to relax the assumption that time is continuous, see Asheim and Mitra (2020).

In this paper we focus on how to relax the assumption that population is constant, which is an exercise that has not yet been made subject to thorough investigation. The case where population is exponentially increasing instead of constant is simple to handle in the one-sector Ramsey model without resource constraints:

By assuming that there is an underlying constant-returns-to-scale production function of capital and labor and appropriately redefining the rate of depreciation  $\delta$ , the model can be interpreted in per capita terms. Thus, maintaining a constant per capita consumption along an efficient path can be associated with keeping per capita capital constant. However, in other models, such as the Cobb–Douglas version of the DHSS technology, exponential population growth is incompatible with the existence of an

efficient and egalitarian path. (Asheim, 2013, p. 319)

Indeed, in more general models and with less regular forms for population growth it seems futile to look for a simple investment rule that leads to constant per capita consumption. In particular, the capital dilution effect of population growth varies widely if the rate of population growth changes abruptly, influencing the portion that remains available for consumption for given investments net of the dilution of per capita capital stocks that population growth causes.

In this paper we formulate an easily interpretable definition of Hartwick’s investment rule to the population growth setting that can be applied for any exogenously given population growth function and any technology. We obtain this result by focusing, not on constant per capita consumption, but on constant per capita net national product, defined as the maximized per capita value of the flows of goods and services that are produced by the productive assets. Without population growth, net national product equals consumption plus the value of net investments. This leads to the observation that following Hartwick’s original investment rule leads to *both* constant consumption *and* constant net national product, as the rule prescribes zero value of net investments. With population growth, net national product still equals total consumption plus the value of total net investments. However, a given investment rule cannot yield both constant per capita consumption and constant per capita net national product due to the capital dilution effect of population growth, except for very special combinations of technologies and population growth functions. There is a choice to be made, and we argue that a reorientation towards the constancy of net national product cuts the Gordian Knot.

Thus, we consider an adaptation of Hartwick’s rule to the population growth setting that has following property: *Investment behavior keeps real per capita net national product constant if and only if it follows our proposed rule.* We suggest such a reinterpretation of the time invariance that follows from observing Hartwick’s rule—towards the constancy of per capita net national product, rather than the constancy of per capita consumption—for three reasons:

- Even without population growth the relationship between Hartwick’s rule and the constancy of net national product is more basic as it holds as an equivalence result: Not only does the rule imply a constant net national product, but a constant net national product implies the rule. In contrast, while the Hartwick rule implies constant consumption, as pointed out already by Hartwick (1977, 1978a,b), the converse is a harder question that remained open for many decades. In particular, without further assumptions also a generalization of

Hartwick's rule as proposed by Dixit, Hoel and Hammond (1980) in a constant population setting (see also Sato and Kim, 2002; Hamilton and Hartwick, 2005) leads to constant consumption.

- Also with population growth the relationship between an appropriately defined Hartwick's rule and the constancy of per capita net national product can be established as an easily interpretable equivalence result without restrictive assumptions on population growth and technology. The lack of appealing results on an investment rule designed to correspond to constant per capita consumption with population growth might reflect, not a lack of interest, but its unavailability.<sup>1</sup>
- From a normative perspective one might argue that the current generation should not bear the burden of protecting future generations from an accelerating future population growth. Rather, it seems more reasonable to ask current people to compensate for the dilution of the per capita productive capacity that the current population growth leads to, by keeping per capita net national product constant.

We start in Section 2 by illustrating in the DHSS model our proposed adaptation of Hartwick's rule to the population growth setting. We assume that the production function has the usual neoclassical properties, and we impose Hotelling's no-arbitrage rule as a condition for short-run efficiency. We consider the investment rule of reinvesting resource rents in per capita capital accumulation, net of the capital dilution effect of population growth. We show that following this rule is equivalent to constant real per capita net national product, where net national product is equal to production minus resource rents in this model.

If population is constant, in which case the proposed investment rule corresponds to Hartwick's original rule, we immediately get Hartwick's result, namely that reinvesting resource rents leads to constant consumption, as a corollary.

Under positive population growth we show that per capita constant consumption obtains if the dilution of per capita capital accumulation caused by population growth is constant. This amount to restrictions on the exogenously determined population growth function. These restrictions can be made explicit if the production function is of the Cobb-Douglas form. In particular, building

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<sup>1</sup>As pointed out by a referee, the question of how to invest in order to implement constant per capita consumption can be raised. However, it seems impossible to obtain an investment rule that depends only on the current value of investments, the current capital gains, and the current population growth rate, as we do in this paper.

on calculation in Asheim, Buchholz, Hartwick, Mitra and Withagen (2007) we show that the population growth function must have a quasi-arithmetic form.

Our definition of Hartwick’s investment rule in the DHSS model with population growth equates the accumulation of per capita capital, net of the capital dilution effect of population growth, to the value of the depletion of the resource, gross of the capital dilution effect of population growth. To shed light on this asymmetry we consider in Section 3 a general model with multiple capital goods. By studying competitive paths in this more general model we obtain a formulation of Hartwick’s investment rule where capital gains play a role if population growth is positive. To be specific, to calculate the investments that must equal zero, in addition to the value of per capita capital accumulation, one can also add the capital gains multiplied by the ratio of the population growth rate and the real interest rate. Since capital gains accrue only to the resource but not to capital, we get the apparent asymmetry in the DHSS model. Also in the general model with multiple capital goods, we obtain as a corollary that keeping the value of net investments equal to zero leads to constant consumption if population is constant.

We discuss in the final Section 4 the observation that the propositions on time invariance obtained from our specification of Hartwick’s rule with population growth are equivalence results: not only does Hartwick’s rule lead a constant real per capita net national product, but it follows also that a time invariant real per capita net national product requires that Hartwick’s rule is obeyed. Also, we discuss the relationship between our findings and results on time invariant dynamic welfare in the literature on comprehensive (or green) national accounting. Finally, we show how the suggested adaptation of Hartwick’s rule to the population growth setting can easily be interpreted, including the role that capital gains play in our proposed rule.

## 2 Hartwick’s rule in the Dasgupta-Heal-Solow-Stiglitz model

The Dasgupta-Heal-Solow-Stiglitz (DHSS) model has one produced good, which serves as both capital and consumption good (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974). This good is produced with a stock of reproducible capital ( $K$ ), an extraction flow ( $R$ ) of input from a non-renewable and exhaustible resource, and labor ( $N$ ). The production function  $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  satisfies the following two assumptions:

(F1)  $F$  is continuous, non-decreasing, concave, and homogenous of degree 1 in  $(K, R, N)$  on  $\mathbb{R}_+^3$ .



(F2)  $F$  is twice continuously differentiable in  $(K, R, N)$  on  $\mathbb{R}_{++}^3$ , with  $F_K(K, R, N) > 0$ ,  $F_R(K, R, N) > 0$ , and  $F_N(K, R, N) > 0$  for all  $(K, R, N) \in \mathbb{R}_{++}^3$ .

In the case where  $F$  is of the Cobb-Douglas form:

$$F(K, R, N) = K^\alpha R^\beta N^{1-\alpha-\beta} \quad \text{where } \alpha > 0, \beta > 0, \text{ and } 1 - \alpha - \beta > 0, \quad (1)$$

the function  $F$  also satisfies that  $F(0, R, N) = F(K, 0, N) = F(K, R, 0) = 0$  for all  $(K, R, N) \in \mathbb{R}_+^3$ . In particular, the resource is essential in the sense that there is no production without a positive flow of resource input. However, we need not make this assumption in our general analysis of the DHSS model.

Labor is throughout taken to be equal to the population ( $N$ ) and assumed to be exogenously given. We assume that  $N(t)$  is a continuously differentiable and non-decreasing function of  $t$ :

$$\dot{N}(t) = g(t)N(t) \quad \text{for } t \geq 0, \quad N(0) > 0, \quad (2)$$

where  $g(t) \geq 0$  for  $t \geq 0$  represents the growth rate of population.

A *path* from initial stocks  $(K, S) \in \mathbb{R}_+^2$  of capital and resource is described by the functions  $(C(t), I(t), R(t), K(t), S(t), N(t))$ , where  $C : [0, \infty) \rightarrow \mathbb{R}_+$ ,  $I : [0, \infty) \rightarrow \mathbb{R}$ ,  $R : [0, \infty) \rightarrow \mathbb{R}_+$ ,  $K : [0, \infty) \rightarrow \mathbb{R}_+$ , and  $S : [0, \infty) \rightarrow \mathbb{R}_+$  are continuously differentiable functions of  $t$  satisfying

$$I(t) = \dot{K}(t) = F(K(t), R(t), N(t)) - C(t),$$

$$R(t) = -\dot{S}(t),$$

$$K(0) = K,$$

$$S(0) = S.$$

Production is split into consumption ( $C$ ) and net investment in reproducible capital ( $I = \dot{K}$ ), while resource input is drawn from the stock ( $S$ ) of the non-renewable and exhaustible resource.

A path  $(C(t), I(t), R(t), K(t), S(t), N(t))$  from initial stocks  $(K, S) \in \mathbb{R}_+^2$  is called *interior* if  $K(t) > 0$  and  $R(t) > 0$  for all  $t \geq 0$ . An interior path  $(C(t), I(t), R(t), K(t), S(t), N(t))$  from  $(K, S) \in \mathbb{R}_+^2$  is called *competitive* if, for all  $t \geq 0$ , it satisfies *Hotelling's no-arbitrage rule* equating the returns on the capital good and the exhaustible resource:

$$F_K(K(t), R(t), N(t)) = \frac{\dot{F}_R(K(t), R(t), N(t))}{F_R(K(t), R(t), N(t))}. \quad (\text{HotR})$$

A path that satisfies (HotR) for all  $t \geq 0$  is *short-run efficient*, implying that consumption in an interval  $[t', t'']$  with  $t' < t''$  cannot be increased compared to  $C(t)$  in some non-trivial subinterval without being decreased in some other subinterval, given  $(K(t'), S(t'))$  as initial stocks and  $(K(t''), S(t''))$  as eventual stocks. For an interior and competitive path, real *Net National Product* (NNP),  $Y(t)$ , at time  $t$  equals production net of the value of resource input:

$$\begin{aligned} Y(t) &= C(t) + \dot{K}(t) + F_R(K(t), R(t), N(t)) \cdot \dot{S}(t) \\ &= F(K(t), R(t), N(t)) - F_R(K(t), R(t), N(t)) \cdot R(t). \end{aligned}$$

Use lower case for per capita variables:

$$c = C/N, i = I/N, r = R/N, k = K/N, \text{ and } s = S/N.$$

Population growth causes a dilution both of per capita capital accumulation (the term  $g(t)k(t)$ ) and per capita resource conservation (the term  $g(t)s(t)$ ), reflecting that an augmented population does not only contribute additional labor to production, but also increases the number of recipients of output (Yamaguchi, 2014, p. 22):

$$\begin{aligned} \dot{k}(t) &= \frac{d\left(\frac{K(t)}{N(t)}\right)}{dt} = \frac{\dot{K}(t)}{N(t)} - \frac{\dot{N}(t)}{N(t)} \cdot \frac{K(t)}{N(t)} = i(t) - g(t)k(t), \\ \dot{s}(t) &= \frac{d\left(\frac{S(t)}{N(t)}\right)}{dt} = \frac{\dot{S}(t)}{N(t)} - \frac{\dot{N}(t)}{N(t)} \cdot \frac{S(t)}{N(t)} = -r(t) - g(t)s(t). \end{aligned}$$

Define the per capita production function  $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  by  $f(k, r) = F(k, r, 1)$  for all  $(k, r) \in \mathbb{R}_+^2$ . Then it follows from (F1) and (F2) that  $f$  is continuous, non-decreasing, and concave in  $(k, r)$  on  $\mathbb{R}_+^2$ , and that  $f$  is twice continuously differentiable in  $(k, r)$  on  $\mathbb{R}_{++}^2$ , with  $f_k(k, r) = F_K(kN, rN, N) > 0$  and  $f_r(k, r) = F_R(kN, rN, N) > 0$  for all  $(k, r) \in \mathbb{R}_{++}^2$  and all  $N \in \mathbb{R}_{++}$ . Furthermore, a *per capita path* from initial per capita stocks  $(k, s) \in \mathbb{R}_+^2$  of capital and resource is described by the functions  $(c(t), i(t), r(t), k(t), s(t))$ , where  $c : [0, \infty) \rightarrow \mathbb{R}_+$ ,  $i : [0, \infty) \rightarrow \mathbb{R}$ ,  $r : [0, \infty) \rightarrow \mathbb{R}_+$ ,  $k : [0, \infty) \rightarrow \mathbb{R}_+$ , and  $s : [0, \infty) \rightarrow \mathbb{R}_+$  are continuously differentiable functions of  $t$  satisfying

$$\begin{aligned}
c(t) &= C(t)/N(t), \\
i(t) &= \dot{k}(t) + g(t)k(t) = f(k(t), r(t)) - c(t), \\
r(t) &= -\dot{s}(t) - g(t)s(t), \\
k(t) &= K(t)/N(t), \quad \text{with } k(0) = k = K/N(0), \\
s(t) &= S(t)/N(t), \quad \text{with } s(0) = s = S/N(0),
\end{aligned} \tag{3}$$

where  $(C(t), I(t), R(t), K(t), S(t), N(t))$  is a path from initial stocks  $(K, S) \in \mathbb{R}_+^2$ . Per capita real NNP,  $y(t)$ , at time  $t$  equals:

$$y(t) = f(k(t), r(t)) - f_r(k(t), r(t)) \cdot r(t). \tag{NNPs}$$

Our analysis shows that the rule of, at each time, reinvesting resource rents  $f_r(k, r) \cdot r$  in per capita capital accumulation  $\dot{k}$ , net of the dilution  $gk$  caused by population growth, has interesting time invariance properties. Therefore, we propose the following adaptation of Hartwick's investment rule to the DHSS model with population growth. Clearly, by (3), this specification specializes to the ordinary Hartwick's rule ( $\dot{k} = i = f_r(k, r) \cdot r$ ) if there is no population growth.

**Definition 1 (Hartwick's investment rule in the DHSS model with population growth)**

*Hartwick's investment rule* is followed at time  $t$  along an interior and competitive path  $(c(t), i(t), r(i), k(t), s(t))$  with rates of population growth given by  $g(t)$  if

$$\dot{k}(t) = i(t) - g(t)k(t) = f_r(k(t), r(t)) \cdot r(t). \tag{HarRs}$$

We can now state our main result in the context of the DHSS model.

**Proposition 1** *Consider an interior and competitive path  $(c(t), i(t), r(i), k(t), s(t))$  in the DHSS model with population growth. Real per capita NNP is constant if and only Hartwick's investment rule (HarRs) is followed.*

**Proof.** Assume that  $(c(t), i(t), r(i), k(t), s(t))$  is an interior and competitive path, implying that (HotR) holds for all  $t \geq 0$  and  $f_k(k(t), r(t)) = F_K(k(t)N(t), r(t)N(t), N(t)) > 0$  for all  $t \geq 0$ .

By differentiating equation (NNPs) with respect to time, and suppressing the time variable:

$$\dot{y} = f_k \dot{k} + f_r \dot{r} - \dot{f}_r r - f_r \dot{r} = f_k \dot{k} - \dot{f}_r r = f_k \dot{k} - \frac{\dot{f}_r}{f_r} f_r r = f_k (\dot{k} - f_r r), \quad (4)$$

where the last equality of (4) follows from (HotR), recalling that  $f_k = F_K$  and  $f_r = F_R$ . Hence, for all  $t \geq 0$ ,  $\dot{y}(t) = 0$  is equivalent to  $\dot{k}(t) = f_r(k(t), r(t)) \cdot r(t)$  since  $f_k = F_K > 0$ . ■

From this result we obtain in a straightforward manner Hartwick's (1977) original result, showing that reinvesting resource rents leads to constant consumption, in the case where there is no population growth (i.e.,  $N(t) = N(0) > 0$  for all  $t \geq 0$  so that  $g(t) = 0$  for all  $t \geq 0$ ).

**Corollary 1 (Hartwick's result without population growth; Hartwick, 1977)** *Consider an interior and competitive path  $(c(t), i(t), r(t), k(t), s(t))$  in the DHSS model without population growth. Per capita consumption is constant if Hartwick's investment rule (HarRs) is followed.*

**Proof.** Without population growth, by (3), per capita NNP is given by:  $y = f - f_r r = c + \dot{k} - f_r r$ . It follows from Proposition 1 that  $0 = \dot{y} = \dot{c} + \frac{d}{dt} (\dot{k} - f_r r)$  if Hartwick's investment rule,  $\dot{k} = f_r r$ , is followed. Since, with  $\dot{k} = f_r r$ ,  $\frac{d}{dt} (\dot{k} - f_r r) = \frac{d}{dt} (0) = 0$ , it follows that  $\dot{c} = 0$ . ■

Corollary 1 can be summarized as follows. As shown in (4), it holds along any short-run efficient path that  $\dot{y} = f_k (\dot{k} - f_r r)$ . Furthermore, without population growth,  $f = c + \dot{k}$  by (3), and  $f = y + f_r r$  by (NNPs), implying that  $\dot{k} - f_r r = y - c$ . By combining these results we obtain  $\dot{y} = f_k (y - c)$ . Therefore, since  $f_k > 0$ , Definition 1 leads to the following equivalence:

$$\text{Hartwick's investment rule} \quad \Leftrightarrow \quad \dot{y} = 0 \quad \Leftrightarrow \quad y = c.$$

If any one of the three conditions stated in this equivalence is satisfied, then  $\dot{c} = 0$  holds, so that consumption is constant.

It is of interest to note that the converse of Hartwick's result without population growth—that constant per capita consumption implies the reinvestment of resource rents—is not an immediate consequence of Proposition 1, even though this converse does hold in the DHSS model (Withagen and Asheim, 1998; Mitra, 2002; Buchholz, Dasgupta and Mitra, 2005). We have that:

$$\dot{c} = \frac{d}{dt} (f - \dot{k}) = f_k \dot{k} + f_r \dot{r} - \ddot{k} = \frac{\dot{f}_r}{f_r} \dot{k} - \ddot{k} + f_r \dot{r} \quad (\text{by (3) and (HotR)})$$

$$= f_r \left( \frac{\dot{f}_r}{f_r} \frac{\dot{k}}{f_r} - \frac{\ddot{k}}{f_r} + \dot{r} \right) = -f_r \cdot \frac{d}{dt} \left( \frac{\dot{k}}{f_r} - r \right).$$

Hence, constant consumption implies a generalized Hartwick's rule proposed by Dixit, Hoel and Hammond (1980) and discussed by Sato and Kim (2002, Section 4) and Hamilton and Hartwick (2005), namely that the present value of net investments,  $(\dot{k}/f_r) - r$ , is constant, but not necessarily zero. To show that the value of net investment must be zero along an efficient path with constant consumption, one must show that  $(\dot{k}/f_r) - r$  equal to a positive constant is inefficient and  $(\dot{k}/f_r) - r$  equal to a negative constant is infeasible. However, Hotelling's no-arbitrage rule, as a condition for short-run efficiency, is not sufficient to show this; rather, one has to consider the properties of the path as time goes not infinity.

Hartwick's result does not hold in the DHSS model with population growth without imposing additional assumptions.

**Proposition 2 (Hartwick's result with population growth)** *Consider an interior and competitive path  $(c(t), i(t), r(i), k(t), s(t))$  in the DHSS model with population growth. Per capita consumption is constant if Hartwick's investment rule (HarRs) is followed and the dilution,  $g(t)k(t)$ , of per capita capital accumulation caused by population growth is constant.*

**Proof.** It follows from Proposition 1 that  $\dot{y} = 0$  if (HarRs) is followed. By (NNPS), (3), and (HarRs),  $y = f - f_r r = c + \dot{k} + gk - \dot{k} = c + gk$ . Thus, if (HarRs) is followed, then  $\dot{c} = -\frac{d}{dt}(gk)$ , implying that  $\dot{c} = 0$  if  $gk$  is constant. ■

Proposition 2 can be summarized before. As we have noted before, any short-run efficient path satisfies  $\dot{y} = f_k (\dot{k} - f_r r)$ . Furthermore, with population growth,  $f = c + \dot{k} + gk$  by (3), and  $f = y + f_r r$  by (NNPS), implying that  $\dot{k} - f_r r = y - (c + gk)$ . By combining these results we obtain  $\dot{y} = f_k (y - (c + gk))$ . Therefore, since  $f_k > 0$ , Definition 1 leads to the following equivalence:

$$\text{Hartwick's investment rule} \quad \Leftrightarrow \quad \dot{y} = 0 \quad \Leftrightarrow \quad y = c + gk.$$

Constant per capita consumption ( $\dot{c} = 0$ ) follows if Hartwick's rule is followed *and*  $gk$  is constant.

When Hartwick's investment rule (HarRs) is combined with Hotelling's no-arbitrage rule (HotR) as a condition for short-run efficiency, the development of  $k(t)$  is fully determined by the initial condition  $k(0) = k = K/N(0)$  and its time derivative  $\dot{k} = f_r r$ . Hence, the constancy of the dilution,  $g(t)k(t)$ , of per capita capital accumulation caused by population growth imposes

restrictions on the population growth function  $N(t)$  that can be satisfied only by coincidence if we maintain the assumption that population growth is exogenous.

These restrictions on  $N(t)$  can be made explicit if the production function  $F$  is of the Cobb-Douglas form (1). Then per capita production is given by  $f(k, r) = k^\alpha r^\beta$ , and the net of population growth per capita capital accumulation is given by  $\dot{k} = f_r(k, r) \cdot r = \beta k^\alpha r^\beta = \beta f(k, r)$ . Hence, if (HarRs) is followed, then the constancy of  $y = f(k, r) - f_r(k, r) \cdot r = (1 - \beta)k^\alpha r^\beta$  implies that production  $f(k, r) = k^\alpha r^\beta$  and per capita capital accumulation  $\dot{k} = \beta k^\alpha r^\beta$  are also constant. Hence, the constancy of  $gk$ , as required by Proposition 2, implies that  $i = \dot{k} + gk$  is constant as well, which combined with the constancy of  $f(k, r)$  implies that even the *gross of population growth savings rate*  $a = i/f(k, r)$  is constant. The following result now follows from Theorem 9 of Asheim, Buchholz, Hartwick, Mitra and Withagen (2007). It specifies limits to population growth under exhaustible resource constraints, as first investigated by Mitra (1983), determining how much population growth can be accommodated with per capita consumption remaining unchanged and the stock of the essential resource being finite.

**Corollary 2** *Consider an interior and competitive path  $(c(t), i(t), r(t), k(t), s(t))$  in the DHSS model with population growth where the production function  $F$  is of the Cobb-Douglas form (1) with  $\alpha > \beta$ . Per capita consumption is constant if Hartwick's investment rule (HarRs) is followed and the exogenous population growth function,  $N(t)$ , is of the following quasi-arithmetic form:*

$$N(t) = N(0) (1 + \mu t)^\varphi ,$$

where the gross of population growth savings rate  $a = i/f(k, r)$  is a constant in  $(\beta, \alpha)$ , and

$$\mu = \beta \left[ (\alpha - a)^\beta K^{\alpha-1} S^\beta N(0)^{1-\alpha-\beta} \right]^{\frac{1}{1-\beta}} , \quad (5)$$

$$\varphi = \frac{a - \beta}{\beta} . \quad (6)$$

**Proof.** If (HarRs) is followed and the production function  $F$  is of the Cobb-Douglas form (1), then the net of population growth savings rate  $b = \dot{k}/f(k, r)$  is constant and equal to  $\beta$ . Hence, it follows from equation (12) of Asheim, Buchholz, Hartwick, Mitra and Withagen (2007) that  $\sigma = \beta$ . Therefore, equations (10) and (11) of their Theorem 9 imply equations (5) and (6) in the statement of Proposition 2. ■

Our definition of Hartwick's investment rule (HarRs) with population growth in Definition 1

equates the accumulation of per capita capital  $\dot{k}$ , *net* of the dilution effect of population growth, to the value of the depletion of the resource  $f_r r$ , *gross* of the dilution effect of population growth. If both terms were net of the dilution effect of population growth, we would have had  $0 = \dot{k} + f_r \dot{s} = i - gk - f_r r - f_r g s$ , so that  $\dot{k} = f_r (r + g s)$ . Instead, we require that  $\dot{k} + f_r \dot{s} = \dot{k} - f_r r - f_r g s = -f_r g s$ . If both terms included the dilution effect of population growth, we would have had  $0 = \dot{k} + gk + f_r (\dot{s} + g s) = i - f_r r$ , so that  $i = f_r r$ . Instead, we require that  $i - f_r r = \dot{k} - f_r r + gk = gk$ . However, neither of these alternatives to  $\dot{k} = f_r r$  leads to interesting time invariance properties under population growth. To understand why this asymmetry arises naturally in the context of the DHSS model, we turn to the analysis of a general model with multiple capital goods in the next section.

### 3 Hartwick's rule in a general model with multiple capital goods

In this section we generalize the analysis of Section 2 to a general model with multiple capital goods, following the framework of Asheim (2004), but simplifying to one consumption good. It can in a straightforward manner be further generalized to a model with multiple consumption goods by letting real prices be found through the use of a Divisia consumer price index (Asheim and Weitzman, 2001; Asheim, 2004, Section 5; Sefton and Weale, 2006).

Denote by  $\mathbf{K} = (K_1, \dots, K_n)$  the non-negative vector of capital goods. This vector includes not only the usual kinds of man-made capital stocks, but also stocks of natural resources, environmental assets, human capital, and other durable productive assets. Corresponding to the stock of capital of type  $j$ ,  $K_j$ , there is a net investment flow:  $I_j = \dot{K}_j$ . Hence,  $\mathbf{I} = (I_1, \dots, I_n) = \dot{\mathbf{K}}$  denotes the vector of net investments.

The quadruple  $(C, \mathbf{I}, \mathbf{K}, N)$  is *feasible* if  $(C, \mathbf{I}, \mathbf{K}, N) \in \mathcal{Y}$ , where  $\mathcal{Y}$  is a convex cone, with free disposal of consumption flows. The set of feasible quadruples does not depend directly on time. Thus, current productive capacity depends solely on the vector of capital stocks and labor. As before, labor equals population, which is an exogenously given function satisfying (2). Since  $\mathcal{Y}$  is a cone, the technology exhibits constant returns to scale. The DHSS model analyzed in Section 2 is a special case of this general model by letting  $(I_1, I_2) = (I, -R)$ ,  $(K_1, K_2) = (K, S)$ , and

$$\mathcal{Y} = \{(C, I_1, I_2, K_1, K_2, N) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_- \times \mathbb{R}_+^3 : C + I_1 \leq F(K_1, -I_2, N)\},$$

where  $\mathcal{Y}$  is convex since  $F$  is concave and a cone since  $F$  is homogeneous of degree 1.

Society makes decisions according to a *resource allocation mechanism*,  $C^* : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$  and  $\mathbf{I}^* : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}^n$ , that assigns to any vector of capital stocks  $\mathbf{K}$  and any population  $N$  a pair  $(C^*(\mathbf{K}, N), \mathbf{I}^*(\mathbf{K}, N))$  satisfying that  $(C^*(\mathbf{K}, N), \mathbf{I}^*(\mathbf{K}, N), \mathbf{K}, N)$  is feasible. We assume that there is a continuously differentiable function  $\mathbf{K} : [0, \infty) \rightarrow \mathbb{R}_+^n$  being the unique solution to the differential equations  $\dot{\mathbf{K}}(t) = \mathbf{I}^*(\mathbf{K}(t), N(t))$  when  $\mathbf{K}(0)$  equal the exogenously given initial stocks  $\mathbf{K} \in \mathbb{R}_+^n$ . Hence,  $\mathbf{K}(t)$  is the capital path that the resource allocation mechanism implements.

A *path* from initial stocks  $\mathbf{K} \in \mathbb{R}_+^n$  is described by the functions  $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$ , where  $C : [0, \infty) \rightarrow \mathbb{R}_+$  and  $\mathbf{I} : [0, \infty) \rightarrow \mathbb{R}^n$  are determined by  $C(t) = C^*(\mathbf{K}(t), N(t))$  and  $\mathbf{I}(t) = \mathbf{I}^*(\mathbf{K}(t), N(t))$  for all  $t \geq 0$ . A path  $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$  from initial stocks  $\mathbf{K} \in \mathbb{R}_+^n$  is called *interior* if  $C^* : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$  and  $\mathbf{I}^* : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}^n$  are continuously differentiable at all  $(\mathbf{K}', N')$  such that  $(\mathbf{K}', N') = (\mathbf{K}(t), N(t))$  for some  $t \geq 0$ . This implies that also  $C : [0, \infty) \rightarrow \mathbb{R}_+$  and  $\mathbf{I} : [0, \infty) \rightarrow \mathbb{R}^n$  are continuously differentiable functions. A path  $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$  from initial stocks  $\mathbf{K} \in \mathbb{R}_+^n$  is called *competitive* if

- (C) for all  $t \geq 0$ , there exist present-value prices of the flows of consumption, labor input, and investment,  $p_0(t)$ ,  $w(t)$ , and  $\mathbf{p}(t)$ , with  $\mathbf{p}(t) \geq 0$ , such that  $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$  maximizes profits  $p_0(t)C' - w(t)N' + \mathbf{p}(t)\mathbf{I}' + \dot{\mathbf{p}}(t)\mathbf{K}'$  over all  $(C', \mathbf{I}', \mathbf{K}', N') \in \mathcal{Y}$ .

A competitive path is short-run efficient. By differentiating  $p_0(t)C' - w(t)N' + \mathbf{p}(t)\mathbf{I}' + \dot{\mathbf{p}}(t)\mathbf{K}'$  with respect to  $N'$  and the components of  $\mathbf{K}'$  and recalling that  $C^*$  and  $\mathbf{I}^*$  are continuously differentiable at all  $(\mathbf{K}(t), N(t))$  along an interior path, it follows from (C) that, for all  $t \geq 0$ ,

$$w(t) = p_0(t) \frac{\partial C^*(\mathbf{K}(t), N(t))}{\partial N} + \mathbf{p}(t) \frac{\partial \mathbf{I}^*(\mathbf{K}(t), N(t))}{\partial N}, \quad (7)$$

$$-\dot{\mathbf{p}}(t) = p_0(t) \nabla_{\mathbf{K}} C^*(\mathbf{K}(t), N(t)) + \mathbf{p}(t) \nabla_{\mathbf{K}} \mathbf{I}^*(\mathbf{K}(t), N(t)), \quad (8)$$

if a competitive path is interior. Since

$$\frac{\partial C^*}{\partial N} \cdot \dot{N} + \nabla_{\mathbf{K}} C^* \cdot \dot{\mathbf{K}} = \dot{C} \quad \text{and} \quad \frac{\partial \mathbf{I}^*}{\partial N} \cdot \dot{N} + \nabla_{\mathbf{K}} \mathbf{I}^* \cdot \dot{\mathbf{K}} = \dot{\mathbf{I}},$$

we have that (7) and (8) imply that, at each  $t$ ,

$$w(t)\dot{N}(t) - \dot{\mathbf{p}}(t)\mathbf{I}(t) = p_0(t)\dot{C}(t) + \mathbf{p}(t)\dot{\mathbf{I}}(t). \quad (9)$$



Furthermore, since  $\mathcal{Y}$  is a cone, the technology exhibits constant returns to scale, and the competitiveness condition (C) implies that, for all  $t \geq 0$ , maximized profits must be zero:

$$p_0(t)C(t) - w(t)N(t) + \mathbf{p}(t)\mathbf{I}(t) + \dot{\mathbf{p}}(t)\mathbf{K}(t) = 0, \quad (10)$$

where  $p_0(t)C(t) + \mathbf{p}(t)\mathbf{I}(t)$  is the value of outputs, and  $w(t)N(t) - \dot{\mathbf{p}}(t)\mathbf{K}(t)$  is the cost of inputs, as  $-\dot{p}_j$  can be interpreted as the cost of holding one unit of capital of type  $j$ .

The path of the present-value consumption price  $p_0(t)$  is a consumer price index that can be used, for all  $t \geq 0$ , to turn the nominal present-value wage  $w(t)$  into a *real* wage:

$$W(t) = \frac{w(t)}{p_0(t)},$$

and the nominal present-value capital prices  $\mathbf{p}(t)$  into *real* capital prices:

$$\mathbf{P}(t) = \frac{\mathbf{p}(t)}{p_0(t)}.$$

Furthermore, the real interest rate,  $\rho(t)$ , at time  $t \geq 0$  equals the rate at the present-value consumption price decreases:

$$\rho(t) = -\frac{\dot{p}_0(t)}{p_0(t)}.$$

For an interior and competitive path, real *Net National Product* (NNP),  $Y(t)$ , at time  $t$  equals consumption plus the real value of net investments:

$$Y(t) = C(t) + \mathbf{P}(t)\mathbf{I}(t). \quad (11)$$

As in Section 2, use lower case for per capita variables. And as in the DHSS model, population growth causes a dilution,  $g(t)\mathbf{k}(t)$ , of per capita capital accumulation:

$$\dot{\mathbf{k}}(t) = \frac{d\left(\frac{\mathbf{K}(t)}{N(t)}\right)}{dt} = \frac{\dot{\mathbf{K}}(t)}{N(t)} - \frac{\dot{N}(t)}{N(t)} \cdot \frac{\mathbf{K}(t)}{N(t)} = \mathbf{i}(t) - g(t)\mathbf{k}(t).$$

A *per capita path* from initial per capita stocks  $\mathbf{k} \in \mathbb{R}_+^n$  is described by the continuously differentiable functions  $(c(t), \mathbf{i}(t), \mathbf{k}(t))$ , where  $c : [0, \infty) \rightarrow \mathbb{R}_+$ ,  $\mathbf{k} : [0, \infty) \rightarrow \mathbb{R}^n$ , and  $\mathbf{i} : [0, \infty) \rightarrow \mathbb{R}^n$  are determined by

$$c(t) = C(t)/N(t)$$

$$\mathbf{i}(t) = \mathbf{I}(t)/N(t) = \dot{\mathbf{k}}(t) + g(t)\mathbf{k}(t), \quad (12)$$

$$\mathbf{k}(t) = \mathbf{K}(t)/N(t), \quad \text{with } \mathbf{k}(0) = \mathbf{k} = \mathbf{K}/N(0),$$

where  $(C(t), \mathbf{I}(t), \mathbf{K}(t), N(t))$  is a path from initial stocks  $\mathbf{K} \in \mathbb{R}_+^n$ . Per capita real NNP,  $y(t)$ , at time  $t$  in this general model equals:

$$y(t) = c(t) + \mathbf{P}(t)\mathbf{i}(t). \quad (\text{NNPg})$$

To reason about how Hartwick's investment rule under population growth generalizes to this model with multiple capital goods, it is useful to reconsider the rule we applied in the DHSS model. In terms of the DHSS model where  $\mathbf{k}(t) = (k_1(t), k_2(t)) = (k(t), s(t))$ , the competitiveness condition (C) implies Hotelling's no-arbitrage rule (HotR) so that  $\rho(t) = f_k(k(t), r(t)) > 0$ ,  $\mathbf{P}(t) = (P_1(t), P_2(t)) = (1, f_r(k(t), r(t)))$  and  $\dot{\mathbf{P}}(t) = (\dot{P}_1(t), \dot{P}_2(t)) = (0, \rho(t)f_r(k(t), r(t)))$  (as by (HotR),  $\dot{P}_2 = \dot{f}_r = f_k f_r = \rho f_r$ ). Then our suggested Hartwick's rule in the DHSS model with population growth can be rewritten as follows:

$$0 = \dot{k} - f_r r = \dot{k} + f_r \dot{s} + f_r g s = P_1 \dot{k}_1 + P_2 \dot{k}_2 + \frac{g}{\rho} (\dot{P}_1 k_1 + \dot{P}_2 k_2) = \mathbf{P}\dot{\mathbf{k}} + \frac{g}{\rho} \dot{\mathbf{P}}\mathbf{k}.$$

Essentially, in addition to the value of per capita capital accumulation, one can add the capital gains multiplied by the ratio of the population growth rate and the real interest rate. Since capital gains accrue only to the resource but not to capital, we get the apparent asymmetry discussed at the end of Section 2. This discussion motivates the following general specification of Hartwick's investment rule in the model with multiple capital goods.

**Definition 2 (Hartwick's investment rule under population growth)** *Hartwick's investment rule* is followed at time  $t$  along an interior and competitive path  $((c(t), \mathbf{i}(t), \mathbf{k}(t))$  with rates of population growth given by  $g(t)$  if

$$\rho(t)\mathbf{P}(t)\dot{\mathbf{k}}(t) + g(t)\dot{\mathbf{P}}(t)\mathbf{k}(t) = 0. \quad (\text{HarRg})$$

Clearly, also this specification specializes to the ordinary Hartwick's rule ( $\mathbf{P}\dot{\mathbf{k}} = 0$ ) if there is no population growth, provided that the real interest rate  $\rho$  is positive. We can now state our main result in the general model with multiple capital goods.

**Proposition 3** Consider an interior and competitive path  $(c(t), \mathbf{i}(t), \mathbf{k}(t))$  in the general model with multiple capital goods. Real per capita NNP is constant if and only Hartwick's investment rule (HarRg) is followed.

**Proof.** Note that

$$\dot{\mathbf{P}} = \frac{d}{dt} \left( \frac{\mathbf{P}}{p_0} \right) = \frac{\dot{\mathbf{P}}}{p_0} - \frac{\dot{p}_0}{p_0} \cdot \frac{\mathbf{P}}{p_0} = \frac{\dot{\mathbf{P}}}{p_0} + \rho \mathbf{P}, \quad (13)$$

so that (9) can be rewritten as

$$\dot{\mathbf{C}} + \mathbf{P}\dot{\mathbf{I}} + \dot{\mathbf{P}}\mathbf{I} = W\dot{N} + \rho\mathbf{P}\mathbf{I}. \quad (14)$$

It follows from (11) and (14) that

$$\dot{Y} = \dot{\mathbf{C}} + \frac{d}{dt}(\mathbf{P}\mathbf{I}) = \dot{\mathbf{C}} + \mathbf{P}\dot{\mathbf{I}} + \dot{\mathbf{P}}\mathbf{I} = W\dot{N} + \rho\mathbf{P}\mathbf{I}. \quad (15)$$

To obtain an expression for growth,  $\dot{y}$ , in real per capita NNP, note that (10) and (13) imply

$$W\dot{N} = gWN = g \cdot (\mathbf{C} + \mathbf{P}\mathbf{I} + \dot{\mathbf{P}}\mathbf{K} - \rho\mathbf{P}\mathbf{K}), \quad (16)$$

since  $\dot{N} = gN$ . By combining (15) and (16), we obtain

$$\dot{Y} - gY = \dot{Y} - g \cdot (\mathbf{C} + \mathbf{P}\mathbf{I}) = \rho\mathbf{P}\mathbf{I} + g \cdot (\dot{\mathbf{P}}\mathbf{K} - \rho\mathbf{P}\mathbf{K}).$$

Hence,

$$\dot{y} = \frac{\dot{Y}}{N} - g\frac{Y}{N} = \rho\mathbf{P}\mathbf{i} + g\dot{\mathbf{P}}\mathbf{k} - \rho\mathbf{P}\mathbf{k} = \rho\mathbf{P}\dot{\mathbf{k}} + g\dot{\mathbf{P}}\mathbf{k}, \quad (17)$$

where the last equality follows from (12). Equation (17) establishes the proposition. ■

Hence, we obtain the result that constant real per capita NNP is equivalent to the sum of the value of net investments weighted by the real interest rate and the capital gains weighted by the population growth rate being equal to zero.

From this result we directly obtain Hartwick's result (see, among others, Dixit, Hoel and Hammond, 1980) in the case where there is no population growth (i.e.,  $N(t) = N(0) > 0$  for all  $t \geq 0$  so that  $g(t) = 0$  for all  $t \geq 0$ ). This result shows that keeping the value of net investments  $\mathbf{P}\dot{\mathbf{k}}$  equal to zero in the general model with multiple capital goods leads to constant consumption.

**Corollary 3 (Hartwick's result without population growth)** Consider an interior and competitive path  $(c(t), \mathbf{i}(t), \mathbf{k}(t))$  in the general model with multiple capital goods. Per capita consump-

tion is constant if Hartwick's investment rule  $\mathbf{P}\dot{\mathbf{k}} = 0$  is followed.

**Proof.** Without population growth, by (12), per capita NNP is given by:  $y = c + \mathbf{P}\dot{\mathbf{k}}$ , since capital accumulation is not diluted when  $g = 0$ . Furthermore, without population growth,  $\mathbf{P}\dot{\mathbf{k}} = 0$  implies that  $\rho\mathbf{P}\dot{\mathbf{k}} + g\dot{\mathbf{P}}\mathbf{k} = 0$ , which in turn, by Proposition 3, implies that NNP is constant:  $0 = \dot{y} = \dot{c} + \frac{d}{dt}(\mathbf{P}\dot{\mathbf{k}})$ . Finally, as  $\mathbf{P}\dot{\mathbf{k}} = 0$  implies that  $\frac{d}{dt}(\mathbf{P}\dot{\mathbf{k}}) = \frac{d}{dt}(0) = 0$ , we obtain  $\dot{c} = 0$ . ■

However, as in the DHSS model, the converse of this result (see, among others, Mitra, 2002) does not follow as corollary from Proposition 3.

## 4 Concluding remarks

We have established that Hartwick's investment rule as adapted to models with population growth in (HarRs) and (HarRg) is equivalent to a time invariant real per capita NNP. Hence, not only does following Hartwick's rule imply that real per capita NNP is constant, but a constant real per capita NNP also implies that Hartwick's rule is followed. Moreover, this equivalence result is obtained by imposing only competitiveness as a condition of short-run efficiency.

It is interesting to contrast this result with the relationship of Hartwick's investment rule in models *without* population growth and its relationship to constant consumption. Also, in this case is short-run efficiency sufficient for showing the result (sometimes referred to as Hartwick's result) that obeying Hartwick's rule leads to constant consumption. However, to establish the converse (sometimes referred to as the converse of Hartwick's result) one must go beyond competitiveness as a condition of short-run efficiency and assume that the constant consumption path is efficient, thereby also considering the properties of the path as time goes not infinity. Indeed, the question of whether the converse of Hartwick's result holds, which was originally posed by Dixit, Hoel and Hammond (1980), was only confirmatively answered much later (Withagen and Asheim, 1998; Mitra, 2002; Buchholz, Dasgupta and Mitra, 2005).

These observations can be used to argue that the relationship between Hartwick's investment rule and constant real NNP is more basic, as it holds as an if-and-only-if result under weaker assumptions than the rule's relation to constant consumption. Moreover, the relationship between Hartwick's investment rule and constant real NNP can also be generalized as per capita results under population growth, as we have shown in the present paper.

It is also worth to mention that invariance in economic models has been studied in different

frameworks, but with similar conclusions. The study of conservation laws in economic models has shown that one of the main invariant along optimal development paths is NNP, just as in the present paper; see Sato (1999, Chapter 7) in the context of general models and Martinet (2012, Chapter 7) in a model with natural resources. Having constant consumption, however, requires additional conditions, and in particular Hartwick’s investment rule in the DHSS model (Martinet and Rotillon, 2007).

There is a link between the equivalence result relating Hartwick’s investment rule to constant NNP, as reported in the present paper, and the literature on comprehensive (or green) national accounting. Under the assumption of a stationary technology and a constant population it is well-known that the value of net investments is positive if and only if dynamic welfare is increasing. Under discounted utilitarianism, this is first proven by Weitzman (1976, eq. (14)), and reported by, among others, Hamilton and Clemens (1999), Dasgupta and Mäler (2000) and Pemberton and Ulph (2001). Moreover, Asheim and Weitzman (2001) and Asheim and Buchholz (2004) have shown how growth in real NNP can be used to indicate improvement in dynamic welfare. By combining these results we obtain a relationship between the value of net investment and real NNP growth in the case where there is no population growth. In this paper we have shown that such a relationship holds in per capita terms also under population growth. However, it does not follow from our analysis that an investment behavior that more than satisfies (HarRs) or (HarRg), in the sense that  $f - f_r r$  and  $\rho \dot{\mathbf{P}}\mathbf{k} + g \dot{\mathbf{P}}\mathbf{k}$  are positive, leads to an increase in a suitable measure of dynamic welfare. Rather, such a welfare measurement must take into account how the capital dilution effect of population growth develops over time, leading to the different kinds of welfare measures that are reported in expressions (20) and (21) of Arrow, Dasgupta and Mäler (2003) and Proposition 6 of Asheim (2004).

To interpret the suggested investment rule—keeping equal to zero the real interest rate time the real value of net investments plus population growth rate times capital gains—note that with constant population, it is a fundamental equation of comprehensive national accounting that

$$\text{change in real NNP} = \text{real interest rate} \cdot \text{the real value of net investments}.$$

As noted by Asheim (2003, p. 119), it is this equation that allows the “‘futurity’ in any welfare evaluation of any dynamic situation” (Samuelson, 1961, p. 53) to be captured by current national accounting aggregates. So without population growth it is well understood how this rule leads to

constant real NNP, The challenge in terms of interpretation is thus to explain why capital gains contribute to growth in per capita NNP with population growth.

The role of capital gains for the concept of income has been discussed extensively in the literature on economic accounting, from Hicks (1946, Chapter 14) via Hill and Hill (2003) to Cairns (2018). It is a basic insight that capital gains should not enter in the aggregate when technology is constant, as the reevaluation of capital stocks reflects future changes in the real interest rate. Still, capital gains might be a real source of income for the owners of the individual capital stocks. Indeed, in the DHSS model, capital gains are a constant source of income for resource owners, while the owners of reproducible capital must reinvest in order to compensate for the declining real interest rate (= net capital productivity) caused by the diminishing resource flow and the augmented capital stock. Hence, even though the aggregate technology is stationary, the environments for the capital owners and the resource owners are not stationary (Asheim, 1986). In particular, capital owners experience a decreasing interest rate and resource owners an increasing resource price, leading to deteriorating “terms-of-trade” for capital owners and improving “terms-of-trade” for resource owners (Asheim and Hartwick, 2011, Section 7). However, with population growth, the greater availability of labor mitigates this negative interest rate effect for capital owners and reduces their need for reinvestment in order to keep per capita income constant. This explains how capital gains play a role, also in the aggregate, with a growing population, in spite of the assumption that technology is constant.

While we have used the DHSS model for illustrative purposes, our adaptation of Hartwick’s rule to the population growth setting has been analyzed in a general model with multiple capital goods and where the population growth function might take any functional form. As our purpose has been to weaken the assumption of constant population from those that are used to establish Hartwick’s result, we have kept the remaining assumptions, including that of a constant technology. However, the general model with multiple capital goods allows us to interpret some of these capital stocks as knowledge accumulated by labor devoted to a research sector. Thus, our adaption of Hartwick’s rule can be applied to recent growth models that include such knowledge accumulation. An explicit derivation of the conditions for constant per capita NNP in specific models of this kind is beyond the scope of the present paper.

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