Orlova, Olena

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Olena Orlova

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
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Personal preferences in networks

Olena Orlova *

Abstract

We consider a network of players endowed with individual preferences and involved in interactions of various patterns. We show that their ability to make choices according to their preferences is limited, in a specific way, by their involvement in the network. The earlier literature demonstrated the conflict between individuality and peer pressure. We show that such a conflict is also present in contexts in which players do not necessarily aim at conformity with their peers. We investigate the consequences of preference heterogeneity for different interaction patterns, characterize corresponding equilibria and outline the class of games in which following own preferences is the unique Nash equilibrium. The introduction of personal preferences changes equilibrium outcomes in a non-trivial fashion: some equilibria disappear, while other, qualitatively new, appear. These results are robust to both independent and interdependent relationship between personal and social utility components.

1 Introduction

It is quite common for economic decisions of an individual to be influenced not only by her personal preferences, but also by analogous decisions of her social environment. While the first factor is classical in economic theory, the importance of the second one was recognized more recently and modeled within one of the branches of network economics – games on networks.

In the last decade the economic networks literature has undertaken the in-depth study of how social networks influence individuals' decisions.[1] However, in order to investigate the

*Center for Mathematical Economics, Bielefeld University; Centre d'Economie de la Sorbonne, University Paris 1 Pantheon-Sorbonne; e-mail: olena.orlova@uni-bielefeld.de

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[1]The latest survey is Bramoullé and Kranton (2016). See also the seminal work of Galeotti et al. (2010) or an extensive survey of Jackson and Zenou (2014).

role of a network in shaping individual behavior, personal preferences are usually removed from the analyses. In most of the literature, the only dimension of heterogeneity between players is their structural position in the network, since such an approach allows to isolate the network effects.[2] *In this paper, heterogeneity between players stems both from their network positions and from their personal preferences over action choices in a network game.*

The introduction of personal preferences into analysis of games played on networks is important not only for the reason of increasing heterogeneity between players. More importantly, it allows to integrate into one model two utility sources that determine players' decisions in their network interactions. The first utility component – *personal utility* – originates from the concordance of choices with personal preferences. The second one – *social*, or *network utility* – owes its origins to interactions with neighbors and depends on the nature of a particular game. *The aim of this paper is to study the specific interplay between personal preferences and various types of interactions in a network.*

For this purpose, we consider a variety of games with strategic complements or substitutes and analyze network outcomes in these games when players have heterogeneous preferences over actions.[3] Then we compare these network outcomes to the corresponding ones in the absense of personal preferences. In our baseline model, the two utility components – personal and social – are interdependent, which represents a departure from the traditional assumption.[4] We then consider an alternative model specification with independent (additively separable) utility components and compare the results in the two models.

It should be justly mentioned that this paper is not the first attempt in the literature on games on networks to account for individual preferences. Hernández et al. (2013) introduce preferences over actions in the binary-action setting for two specific games, where network utility arises either from coordination with neighbors (action matching) or from anti-coordination (action mismatching). In the follow-up paper, Hernández et al. (2017), the authors extend the analysis for the case of coordination and present some existence/uniqueness conditions for different types of equilibria. Two experimental papers, Ellwardt et al. (2016) and Goyal et al. (2017), test the theoretical predictions for the pure coordination case, adding

[2]In theoretical literature, exceptions usually concern models of network formation, which introduce heterogeneity in the cost of interaction (Golub and Livne (2011)) or in benefits from socialization (Currarini et al. (2009), Cabrales et al. (2011)). In empirical applications there are more models with heterogeneity in players' characteristics: see e.g. Calvó-Armengol et al. (2009) for education, or Patacchini and Zenou (2012) for crime.

[3]In a game with strategic complements, a player is more inclined to choose a particular action as more of her network neighbors choose it. For strategic substitutes the opposite holds: the less chosen by the neighbors the more attractive an action is for a player. Typical examples of network interactions with strategic complements include peer pressure or buying compatible products, while strategic substitutes are usually used to model local public good provision, costly experimentation or competition for a certain resource.

[4]With a very few exceptions, additive separability is a typical assumption in modeling the relationship between within-group and across-group discordance (or more specifically, between the deviation from a personal prior and the deviation from social expectations or social average). This is the case in the models of opinion formation in networks (DeGroot (1974), Golub and Jackson (2010), among others), as well as in the literature on evolution of cultural traits (for example, Kuran and Sandholm (2008)) or dynamics of social norms (Olcina et al. (2018), Della Lena and Dindo (2018)).

to the game the network formation stage.

We extend the framework of Hernández et al. (2013) and analyze two different model specifications for a wide range of games on a fixed, exogenous network. We consider games that allow players to benefit from their network interactions regardless of whether their action choices match those of their neighbors. We call these games *matching (mismatching) games with tolerance*.

As an illustration let us consider the following technology adoption scenario. Let players – firms or individuals – be connected in a network and run two-party projects with their network partners. Each player has to make a choice between two available technologies and to use the chosen technology in all her projects. First, assume that an important (though not decisive) factor for project profitability is compatibility of technologies of the parties involved in a project. If the parties use the same technology, they get higher payoffs in this project, while if they use different technologies, they get lower (though still positive) payoffs. This positivity of payoffs despite difference in technologies captures what we call *tolerance* in this matching game. Next, consider the contrary assumption and let the technologies be complementary. Now if the parties in a project use the same technology, they get lower (though still positive) payoffs than if they have both technologies at their disposal. This is an example of a mismatching game with tolerance. Finally, every player might have a personal preference over available technologies and get an additional utility payoff if she adopts her preferred technology.

Further examples of matching games with tolerance and personal preferences include deviant behavior among teenagers, a decision to persue or not higher education, a school choice, or a choice of a place to live. Mismatching games with tolerance appear in contexts like attending overlapping information events, buying complementary goods or developing complementary skills, or whenever there are gains from variety or differentiation.

Thus, our setting is the following. Players connected in a network simultaneously choose between two alternative actions, over which they have individual preferences. These preferences are exogenously given and stable over time, and the players have complete information about the network and the distribution of preferences. Action choices of a player's network neighbors influence her own decision in a particular way that depends on the nature of interactions, that is, on a specific game. The game is either matching or mismatching game with a given degree of tolerance and the given strength of personal preferences. We analyze Nash equilibrium sets of such network games with heterogeneous preferences, and in particular, existence of consensus equilibria (symmetric equilibria, in terms of the action choice), disagreement (asymmetric) equilibria and so-called *fully-satisfying equilibria* (in which all players satisfy their personal preferences).

As it has been already specified, in our framework players' utility comes from two different sources that, generally speaking, can be independent or interdependent. The two model specifications correspond to different situations and, depending on the context, one or another can be more favorable. The interdependent one (multiplicative utility function) implies that the utility bonus for choosing own preferred action accompanies each connection of a player. In the above technology adoption example, if an agent adopts her preferred tech-

nology, she receives an additional payoff for every project in which she uses this technology. In the independent specification (additive utility function), the utility bonus for choosing the preferred alternative does not depend on the degree of connectivity of a player, that is, a player enjoys the fact of being able to satisfy her preference separately from matching or mismatching with her neighbors. In our technology adoption example, this specification would be appropriate if the technologies were intended for individual use as well as for the two-party projects.

Our results show that the whole range of games that we consider splits into three qualitatively different classes, which differ in best response strategies and thus in equilibria outcomes. It appears that there exists a nonempty class, including both games of strategic complements and those of strategic substitutes, for which the only individually rational strategy is to follow own preference. This result is robust to both additive and multiplicative relationship between the two utility components.

Furthermore, we derive necessary and sufficient conditions for existence of a fully satisfying equilibrium. These conditions are completely determined by the game, network structure and the distribution of preferences over the network.

The rest of the paper is organized as follows. Section 2 presents the model. In Section 3 the best response functions of players are derived for different classes of games. Equilibrium analysis and some illustrations of the results for standard network structures are contained in Section 4. Section 5 investigates the impact of preference heterogeneity by comparing our framework with the no-preference framework, and then discusses an alternative, additive specification of the utility function. Section 6 briefly concludes and appendix contains proofs of the results.

2 The model

2.1 Matching and mismatching games with tolerance

Let G be a network (graph) with the set of nodes $N = \{1, 2, ...n\}$ and links represented by an adjacency matrix. We consider undirected unweighted networks, i.e. the adjacency matrix is symmetric with entries $G_{ij} \in \{0, 1\}$ for all $i, j \in N$ (with 1 implying a link between i and j, and 0 implying no link). We assume that $G_{ii} = 0$ for all $i \in N$.

For a node $i \in N$, we denote the set of i's neighbors in G by $N_i(G) = \{j \in N \mid G_{ij} = 1\}$ and the cardinality of this set, called also i's *degree*, by d_i.

Given a network G, we let the set of nodes $N = \{1, 2, ...n\}$ be the set of players and $X = \{0, 1\}$ be the action set (the same for all players). For a player $i \in N$, $x_i \in X$ denotes i's action in an action profile $\bar{x} = (x_1, ..., x_n)$ and $\bar{x}_{N_i(G)} \in X^{d_i}$ denotes the vector of actions of i's neighbors in G.

The payoff for a player i with the set of neighbors $N_i(G)$ is defined as follows:

$$u_i(x_i, \bar{x}_{N_i(G)}) = \delta \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i = x_j\}} + (1 - \delta) \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i \neq x_j\}},$$

where $\delta \in [0;1]$. That is, the payoff of a player depends on her own action and the actions of her neighbors, and the game is semi-anonymous in the sense that only the number of neighbors choosing one or another action matters but not their identity.[5]

The parameter δ reflects relative advantage of matching versus mismatching of own action with the actions of neighbors. If $\delta = 1$ it is a pure coordination game, where players benefit exclusively from action matches with their neighbors, while if $\delta = 0$ it is a pure anti-coordination game. If $\delta \in (0;1)$ players benefit (to a different extent) both from matches and mismatches with their neighbors. We say that *tolerance* is higher when the difference $|\delta - \frac{1}{2}|$ is smaller.

For an isolated pair of connected players the game can be represented in normal form (Table 1). Here if $\delta > \frac{1}{2}$ the game is a *matching game with tolerance*, and if $\delta < \frac{1}{2}$ it is a *mismatching game with tolerance*. Obviously, the first case belongs to the class of games on networks with strategic complements, and the second – to games with strategic substitutes.

	0	**1**
0	$\delta,\ \delta$	$1-\delta,\ 1-\delta$
1	$1-\delta,\ 1-\delta$	$\delta,\ \delta$

Table 1: The game between two players: matching with tolerance ($\delta > \frac{1}{2}$) or mismatching with tolerance ($\delta < \frac{1}{2}$)

2.2 Personal preferences of the players

We assume that all players have strict personal preferences over the action set $X = \{0,1\}$, which are given exogenously and do not change throughout the game. Obviously, the set of possible personal preferences coincides with the set of available actions: $\Theta = \{0,1\}$.

Similar to an action profile, a *preference profile* $\bar{\theta} = (\theta_1, ..., \theta_n)$ is a vector of preferences of all players in the network. We call a preference profile *homogeneous* if $\theta_i = \theta_j$ for all $i, j \in N$, otherwise we call it *heterogeneous*. Whenever it does not create confusion with the common terminology, we use the term *network* to refer to the pair $(G, \bar{\theta})$, combining network structure and the distribution of preferences in this network, which is assumed to be common knowledge prior to the game.

Now a player's payoff depends also on her preference (type). The payoff for a player i with preference θ_i and the set of neighbors $N_i(G)$ is defined as follows:

$$u_i(\theta_i, x_i, \bar{x}_{N_i(G)}) = \left(\delta \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i = x_j\}} + (1-\delta) \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i \neq x_j\}} \right) \left(1 + \lambda \cdot \mathbb{1}_{\{x_i = \theta_i\}}\right),$$

where $\delta \in [0;1]$ and $\lambda \in (0;\infty)$.

The second parameter λ determines the (relative) utility bonus a player gets if she chooses her preferred action. Thus λ reflects the strength of personal preference: the higher λ, the

[5]For more on semi-anonymous graphical games, see chapter 9.3 in Jackson (2008).

stronger personal preference and the larger utility loss if the player cannot choose her preferred action. Since we are interested in the impact of introducing heterogeneous preferences, we consider $\lambda \in (0; \infty)$. The case $\lambda = 0$ corresponds to the no-preference framework and is discussed separately in Section 5.

To better illustrate the nature of the game, let us consider again a pair of connected players. Tables 2 and 3 represent their interactions in normal form in the case when two players have the same personal preference and in the case when their preferences differ.

	0	**1**
0	$\delta(1+\lambda),\ \delta(1+\lambda)$	$(1-\delta)(1+\lambda),\ 1-\delta$
1	$1-\delta,\ (1-\delta)(1+\lambda)$	$\delta,\ \delta$

Table 2: The game between two players with the same preference: $\theta_i = \theta_j = 0$

	0	**1**
0	$\delta(1+\lambda),\ \delta$	$(1-\delta)(1+\lambda),\ (1-\delta)(1+\lambda)$
1	$1-\delta,\ 1-\delta$	$\delta,\ \delta(1+\lambda)$

Table 3: The game between two players with different preferences: $\theta_i = 0$ for the row player and $\theta_j = 1$ for the column player

Note that the utility function defined above has multiplicative form with regard to δ and λ. It implies that once a player chooses her preferred action, she enjoys a utility supplement from each of her connections. In Section 5 we consider an alternative specification of the utility function, which takes additive form so that the bonus from choosing the preferred action does not depend on a player's degree.

Note also that if a player is isolated (has no neighbors), her utility in the network is zero regardless of her preference and the action she chooses. As long as network formation is not explicitly modelled, such nodes do not matter for equilibrium analysis. In the rest of the paper we assume that no player is isolated, i.e. $d_i > 0$ for every $i \in N$.

Moreover, we assume that G is a connected network, that is, every two nodes are connected by some path in the network. If the network is disconnected, each of its components (maximal connected subnetworks) can be analyzed separately. In this case all the results of this paper also hold, componentwise.

2.3 Equilibrium concept

We consider the complete information setting with rational players. Each player, given her personal preference and the network $(G, \bar{\theta})$, chooses an action from X so that to maximize her utility payoff. All the players decide simultaneously. The equilibrium concept used in this paper is the n-player Nash for a fixed network.

In our further analysis we differentiate between two types of equilibria – symmetric and asymmetric. In symmetric equilibria all players choose the same action, hence the second

name – *consensus equilibria*. In our binary setting there are two possibilities: either consensus on 0 or on 1. Asymmetric equilibria are disagreement equilibria, since different actions are chosen.

What also matters for comparison of equilibria in our framework is whether players are able to follow their preferences or, due to social pressure, have to surrender them and make unfavored choices. Based on the idea of frustration accompanying the choice of an unfavored action under social pressure, we define satisfying and frustrating actions.[6] We call an action x_i chosen by player i with preference θ_i *satisfying* if $x_i = \theta_i$, otherwise we call it *frustrating*. A player who chooses a satisfying (frustrating) action in $\bar{x}$ is called a *satisfied* (*frustrated*) player.

In some contexts it might be of interest to single out this personal aspect of utility and analyze the level of individuals' satisfaction in a network. In this paper we focus on a particular type of equilibria – those characterized by the maximum level of preference satisfaction.

Definition 1. In a network $(G, \bar{\theta})$ an action profile $\bar{x}$ is called *fully satisfying* if $x_i = \theta_i \ \forall i \in N$. If a fully satisfying action profile constitutes an equilibrium, it is called a *fully satisfying equilibrium*.

Obviously, in any given network there is only one fully satisfying action profile, and thus at most one fully satisfying equilibrium.

The last two definitions, that will prove useful in this paper, characterize player's neighbors with respect to whether they choose i's preferred action. For a player i with preference θ_i, her neighbor j is called i's *companion* if she chooses i's preferred action, i.e. $x_j = \theta_i$, otherwise j is called i's *opponent*.

3 Individual best responses

In this section we analyze individual best response strategies that characterize the set of all behaviors that can potentially be present in equilibrium.

Recall that the payoff u_i of a player i depends on the actions of i's neighbors in an anonymous way. What matters for i's decision is just the total number of her neighbors choosing each of the actions. Without loss of generality, we denote the number of i's neighbors who choose action 1 by τ_i. The number of i's neighbors who choose action 0 then equals $d_i - \tau_i$.

Due to linearity of payoffs with respect to τ_i, best responses take the form of threshold functions. However, the optimal actions below and above the thresholds are not always the same, they depend on the relationship between parameters δ and λ. Below we show that the whole range of possible parameter values naturally splits into three regions (see Figure 1) that correspond to qualitatively different behaviors, and eventually different equilibria. These are the region of strong advantage of matching, $R_M = \{(\delta, \lambda) : \frac{1+\lambda}{2+\lambda} < \delta \leq 1, \lambda > 0\}$,

[6] The terminology is partly borrowed from Hernández et al. (2013).

the region of strong advantage of mismatching, $R_{MM} = \{(\delta, \lambda) : 0 \leq \delta < \frac{1}{2+\lambda}, \lambda > 0\}$, and the in-between region of no well-pronounced advantage, $R_{NP} = \{(\delta, \lambda) : \frac{1}{2+\lambda} \leq \delta \leq \frac{1+\lambda}{2+\lambda}, \lambda > 0\}$.[7]

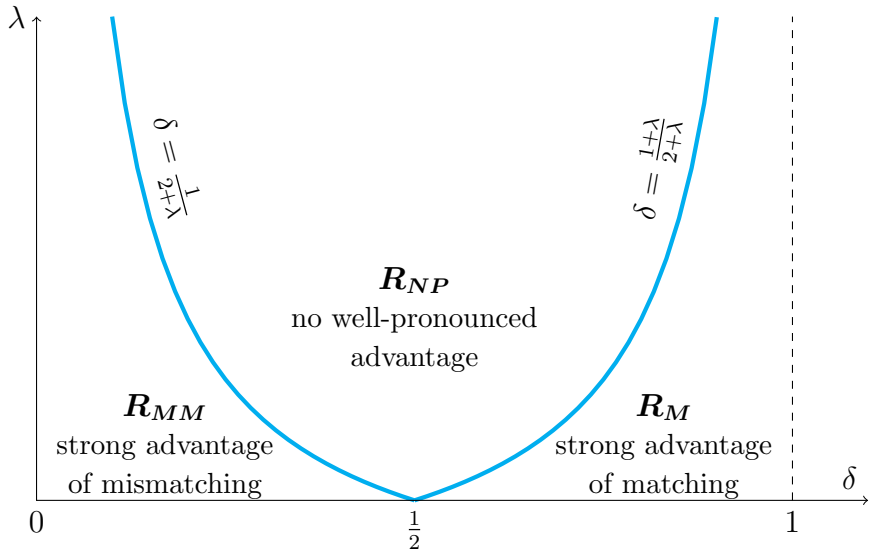


Figure 1: Three regions of parameter values

As long as both δ and λ are the same for all players, each particular pair of parameter values – a point in the (δ, λ)-space – represents a particular game.[8] We analyze these three qualitatively different classes of games in turn.

3.1 Games with strong advantage of matching

The region R_M corresponds to the class of games with strong advantage of matching own action with the actions of the neighbors. The games that fall into this region are games with strategic complements.

Before we characterize the players' best responses, let us define special partitions of R_M consisting of L parts, where $L = \lceil \frac{d_i}{2} \rceil$.[9,10] As we will see later, within each of these parts (subregions) the best responses of the players of degree d_i are the same.

For a player i of degree d_i the partition $\left\{R_M^1(d_i), ..., R_M^L(d_i)\right\}$ is defined as follows (for a

[7]What we call *matching (mismatching)* here is also referred to as *coordination (anti-coordination)* in the networks literature: e.g. Bramoullé (2007), Bramoullé et al. (2004), Jackson and Watts (2002).

[8]In principle, the relative advantage of matching versus mismatching, as well as the strength of personal preference, might not be the same for all players. However, this possibility lies outside the scope of our analysis.

[9]Although L is a function of degree d_i, we omit the argument whenever it does not create confusion, in order to avoid cumbersome notation.

[10]Here $\lceil x \rceil$ denotes the ceiling of x.

graphical illustration see Figure 2):

$$R_M^l(d_i) = \{(\delta, \lambda) \in R_M : \frac{(d_i - (l-1))(1+\lambda) - (l-1)}{(d_i - 2(l-1))(2+\lambda)} < \delta \leq \frac{(d_i - l)(1+\lambda) - l}{(d_i - 2l)(2+\lambda)}\}$$

for $l = 1, ..., L-1$, and

$$R_M^L(d_i) = \{(\delta, \lambda) \in R_M : \frac{(d_i - (L-1))(1+\lambda) - (L-1)}{(d_i - 2(L-1))(2+\lambda)} < \delta \leq 1\}.$$

The following proposition characterizes the best responses of the players, which take the form of threshold functions and where particular thresholds depend on the players' preferences, their degrees and the (subregion of) parameter values δ and λ.

Proposition 1 [Best responses. Strong advantage of matching]
In a game $(\delta, \lambda) \in R_M$ the best response function of a player i with preference θ_i and d_i neighbors, τ_i of whom play 1, is

$$BR_i(\theta_i, d_i, \tau_i) = \begin{cases} 1, \textit{ if } \tau_i > \tau_{\delta,\lambda}^{\theta_i}(d_i) \\ 0, \textit{ if } \tau_i < \tau_{\delta,\lambda}^{\theta_i}(d_i) \\ \theta_i, \textit{ if } \tau_i = \tau_{\delta,\lambda}^{\theta_i}(d_i) \end{cases}$$

where $\tau_{\delta,\lambda}^{\theta_i}(d_i) = \theta_i l + (1-\theta_i)(d_i - l)$ for $(\delta, \lambda) \in R_M^l(d_i)$, $l = 1, ..., L$.[11]

We assume hereafter the following, quite natural tie-breaking rule: whenever on the threshold, a player chooses her preferred action.[12] If the number of neighbors choosing action 1 is above the threshold, a player also chooses 1 (because of the matching advantage); if this number is below the threshold, she chooses action 0.

To visualize the results, consider Figure 2 depicting subregions of parameter values corresponding to different thresholds. Games with strong advantage of matching are contained in the area between two blue curves.

[11] As mentioned above, $BR_i(\theta_i, \bar{x}_{N_i(G)}) = BR_i(\theta_i, d_i, \tau_i)$ due to semi-anonymity of the games considered in this paper.

[12] Introducing a tie-breaking rule allows to avoid set-valued best responses and focus on pure equilibrium characterizations.

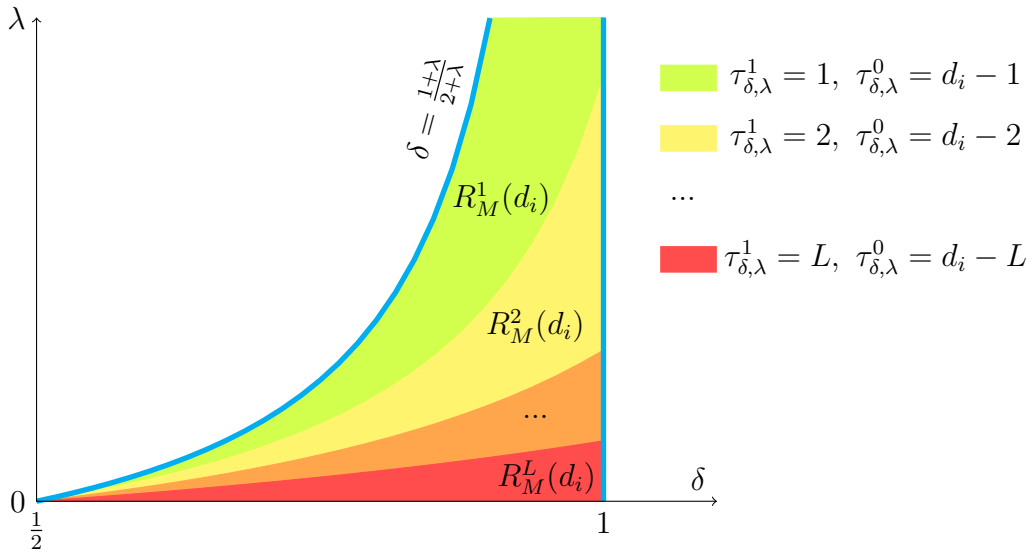


Figure 2: Decision thresholds for a player of degree d_i for different games with strong advantage of matching

The subregion $R_M^1(d_i)$ covers the cases with high strength of personal preference relative to the matching advantage. Here the decision threshold for a player i with preference 1 equals 1, while for a player with preference 0 it equals $d_i - 1$. In other words, for choosing her preferred action it is sufficient for a player i to have at least one companion, and only if she has no companions at all would she follow her neighbors' choice.

Intuitively, the less a player is concerned about matching her action with neighbors (the lower δ) the fewer companions she needs in order to switch to her preferred action. The same is true with respect to the strength of preference: the stronger personal preference is (the higher λ) the fewer companions are needed in order to follow it. Thus, for a given degree d_i, the need for companions monotonically increases with δ and decreases with λ, and the maximum possible companion requirement is $\lceil \frac{d_i}{2} \rceil$ – the majority of neighbors.[13]

Note that if none of i's neighbors chooses her preferred action, i will not choose it either, regardless of the strength of her preference. This important observation, which we use intensively later in the paper, is summarized in the following corollary.

Corollary 1 (Minimum companion requirement)**.** *For* $(\delta, \lambda) \in R_M$

$$\forall i \in N: \ BR_i(\theta_i, d_i, \tau_i) = \theta_i \ \Rightarrow \ \exists j \in N_i(G) \ s.t. \ x_j = \theta_i.$$

In other words, in games with strong advantage of matching every player needs at least one companion in order to follow personal preference.

[13]When the number of neighbors choosing i's preferred action exceeds $\lceil \frac{d_i}{2} \rceil$, the player i always chooses this action, since there is no longer conflict between following personal preference and matching the neighbors' choices.

We now present subsidiary results concerning the number of required companions for a player who adds or loses one link. For notational convenience, we set $R_M^l(d_i) = \emptyset$ whenever $l \notin \{1, ..., L\}$.

Corollary 2 (Adding/deleting a link)**.** *For all degrees $d_i \geq 2$ and $l = 1, ..., L$ the following holds:*

- $R_M^l(d_i) \subseteq R_M^l(d_i + 1) \cup R_M^{l+1}(d_i + 1)$, *and*
- $R_M^l(d_i) \subseteq R_M^l(d_i - 1) \cup R_M^{l-1}(d_i - 1)$.

That is, a player with d_i links who needs l companions in order to follow her personal preference, would increase this companion requirement by at most one if she gets an additional link, and would decrease this requirement by at most one if she loses one existing link.

3.2 Games with strong advantage of mismatching

Next, we will study optimal behavior in the region R_{MM}, corresponding to the class of games with strong advantage of action mismatching. The games that fall into this region are games with strategic substitutes.

Similarly to the previous case, let us define the following partition of R_{MM} for a given degree d_i (see Figure 3): $\left\{R_{MM}^1(d_i), ..., R_{MM}^L(d_i)\right\}$, where

$$R_{MM}^l(d_i) = \{(\delta, \lambda) \in R_{MM} : \frac{d_i - (2+\lambda)l}{(d_i - 2l)(2+\lambda)} \leq \delta < \frac{d_i - (2+\lambda)(l-1)}{(d_i - 2(l-1))(2+\lambda)}\}$$

for $l = 1, ..., L-1$, and

$$R_{MM}^L(d_i) = \{(\delta, \lambda) \in R_{MM} : 0 \leq \delta < \frac{d_i - (2+\lambda)(L-1)}{(d_i - 2(L-1))(2+\lambda)}\}.$$

Note that for each degree d_i this partition is symmetric to the corresponding partition of R_M. The following proposition characterizes players' best responses in the region R_{MM}.

Proposition 2 [Best responses. Strong advantage of mismatching]
In a game $(\delta, \lambda) \in R_{MM}$ the best response function of a player i with preference θ_i and d_i neighbors, τ_i of whom play 1, is

$$BR_i(\theta_i, d_i, \tau_i) = \begin{cases} 1, \text{if } \tau_i < \tau_{\delta,\lambda}^{\theta_i}(d_i) \\ 0, \text{if } \tau_i > \tau_{\delta,\lambda}^{\theta_i}(d_i) \\ \theta_i, \text{if } \tau_i = \tau_{\delta,\lambda}^{\theta_i}(d_i) \end{cases}$$

where $\tau_{\delta,\lambda}^{\theta_i}(d_i) = \theta_i(d_i - l) + (1 - \theta_i)l$ for $(\delta, \lambda) \in R_{MM}^l(d_i)$, $l = 1, ..., L$.

The optimal choices below and above the threshold here are different than in the case of matching advantage. Now action 1 is the best response for player i if the number of i's neighbors choosing 1 is sufficiently low, while the more neighbors choose 1 the less attractive this action becomes for i.

For visualization purposes consider Figure 3, depicting the subregions of parameter values (δ, λ) corresponding to different thresholds for the case of mismatching advantage (symmetric to the corresponding subregions from Figure 2).

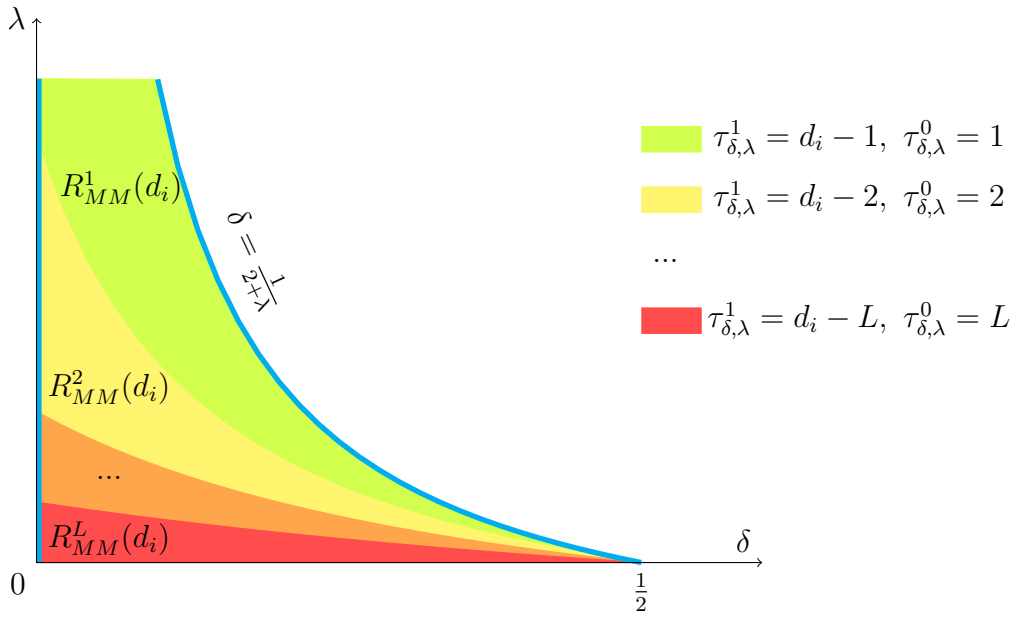


Figure 3: Decision thresholds for a player of degree d_i for different games with strong advantage of mismatching

Again, individual best responses for players with preference 0 and those with preference 1 are symmetric. If either personal preference is very strong or advantage of mismatching is small (subregion $R^1_{MM}(d_i)$), having one opponent is sufficient for following i's preference. The minimal number of such opponents that i would require in order to follow her preference increases with advantage of mismatching and with weaker personal preference until it reaches its maximum possible value – $\lceil \frac{d_i}{2} \rceil$.[14]

Similar to the matching case, the following corollary provides a necessary condition for choosing the preferred action in games with strong advantage of mismatching.

Corollary 3 (Minimum opponent requirement)**.** *For* $(\delta, \lambda) \in R_{MM}$

$$\forall i \in N: \ BR_i(\theta_i, d_i, \tau_i) = \theta_i \ \Rightarrow \ \exists j \in N_i(G) \ s.t. \ x_j = 1 - \theta_i.$$

[14]Similar to the previous case, when the number of neighbors choosing i's unfavored action exceeds $\lceil \frac{d_i}{2} \rceil$, she always chooses her preferred action, since there is no longer conflict between following personal preference and mismatching neighbors' actions.

Thus, in games with strong advantage of mismatching every player needs at least one opponent in order to follow personal preference.

The next corollary describes the change in the number of required opponents as a player adds or loses one link. Since the partitions of R_M and R_{MM} are symmetric for any given degree d_i, the following result is similar to the one of the previous subsection (Corollary 2).

Corollary 4 (Adding/deleting a link)**.** *For all degrees $d_i \geq 2$ and $l = 1, ..., L$ the following holds:*

- $R^l_{MM}(d_i) \subseteq R^l_{MM}(d_i + 1) \cup R^{l+1}_{MM}(d_i + 1)$*, and*
- $R^l_{MM}(d_i) \subseteq R^l_{MM}(d_i - 1) \cup R^{l-1}_{MM}(d_i - 1)$*.*

That is, similar to the matching case, a player with d_i links who needs l opponents in order to follow her personal preference would increase her opponents requirement by at most one if she gets an additional link, and would decrease this requirement by at most one if she loses one existing link.

3.3 Games with no well-pronounced interactional advantage

A particularly interesting result is the one for the in-between region R_{NP}, in which advantage of neither action matching nor mismatching is well-pronounced. Notably, this region contains both games of strategic complements and those of strategic substitutes. It appears that for this class of games the unique individually rational strategy is to follow personal preference.

The intuition behind this result is simple: when personal preference becomes relatively more important than interactional advantage (the personal utility bonus outweights the utility difference between matching and mismatching a neighbor's action), players no longer take the interactional advantage into account and choose exclusively according to their personal preference. Acting according to own preference becomes the strictly dominant strategy for every player. The following proposition formalizes this result.

Proposition 3 [Best responses. No well-pronounced advantage]
In a game $(\delta, \lambda) \in R_{NP}$ the best response for every player is her preferred action:

$$BR_i(\theta_i, d_i, \tau_i) = \theta_i \quad \forall i \in N.$$

Proof. A player i with preference $\theta_i \in \{0, 1\}$ has two possible actions: θ_i or $1 - \theta_i$. The utility gain she gets from each her connection depends on her own action (rows) and that of the corresponding neighbor (columns):

	$\boldsymbol{\theta_i}$	$\boldsymbol{1 - \theta_i}$
$\boldsymbol{\theta_i}$	$\delta(1 + \lambda)$	$(1 - \delta)(1 + \lambda)$
$\boldsymbol{1 - \theta_i}$	$1 - \delta$	δ

If $\delta \in \left(\frac{1}{2}; \frac{1+\lambda}{2+\lambda}\right]$ then $1-\delta < \delta \leq (1-\delta)(1+\lambda) < \delta(1+\lambda)$. This means that the strategy θ_i is strictly dominant for player i (in the case of indifference, the tie-breaking rule in favor of personal preference is applied.)

If $\delta \in \left[\frac{1}{2+\lambda}; \frac{1}{2}\right)$ then $\delta < 1-\delta \leq \delta(1+\lambda) < (1-\delta)(1+\lambda)$. Hence, the strategy θ_i is strictly dominant for i also in this case.

Finally, if $\delta = \frac{1}{2}$ then strict dominance of the strategy θ_i follows from the fact that $\lambda > 0$.

As the above is true for every i's neighbor in G, we can conclude that the only best response of player i is θ_i. □

4 Equilibrium analysis

Using the best response functions derived in the previous section, we seek to characterize the sets of n-player Nash equilibria for different classes of games on a given network $(G, \bar{\theta})$. The first, preparatory subsection containing some useful notation and definitions. The next three subsections deal with existence and uniqueness of different types of equilibria for a general network structure and an arbitrary preference profile for three classes of games – R_M, R_{MM} and R_{NP} – respectively. Finally, the last subsection further specifies and illustrates the results for several standard network structures.

4.1 Preliminaries

Before we proceed, let us define a function $l : [0,1] \times (0; +\infty) \times \mathbb{N} \to \mathbb{N} \cup \{0\}$ that maps every game (δ, λ) and every possible degree $d_i \in \mathbb{N}$ of a player to the minimum number of companions (for $\delta \geq \frac{1}{2}$) or opponents (for $\delta \leq \frac{1}{2}$) the player of this degree needs in order to follow her preference. That is, the value of the function l corresponds to the natural number labelling the corresponding subregion $R^l_M(d_i)$ or $R^l_{MM}(d_i)$ (or to zero for R_{NP}).

It is straightforward to check that l is weakly decreasing in λ (the stronger personal preference is, the fewer companions/opponents a player needs in order to follow it), increasing in $|\delta - \frac{1}{2}|$ (the stronger advantage of considering neighbors' actions, the more companions/opponents are needed in order to act according to personal preference) and increasing in d_i (the more neighbors a player has, the more restricted she is in her choices).

For analysis of equilibria in different games, it is convenient to consider δ and λ as parameters and use the function l as a function of a single argument – a player's degree: $l(\delta, \lambda, d_i) = l_{\delta,\lambda}(d_i)$. The following technical lemma allows to explicitly calculate the value of l for given δ, λ and d_i.

Lemma 1 (Companion/opponent requirement)**.**

(i) In games with strong advantage of matching (mismatching) the minimum number of companions (opponents) that a player of degree d_i needs in order to follow her preference equals

$$l_{\delta,\lambda}(d_i) = l^* + \mathbb{1}_{\lambda < \tilde{\lambda}(l^*)},$$

where $l^* = \underset{m=1...L}{\operatorname{argmin}} |\lambda - \tilde{\lambda}(m)|$ *and* $\tilde{\lambda}(m) = \frac{2 \cdot |2\delta - 1| \cdot (d_i - 2m)}{(1 - |2\delta - 1|)(d_i - 2m) + 2m}$.[15,16]

(ii) In games with no well-pronounced advantage a player does not need any companions or opponents to follow her preference: $l_{\delta,\lambda}(d_i) = 0$.

To simplify the further exposition, let us denote by $N^\theta (\subseteq N)$ the subset of players with preference $\theta \in \{0, 1\}$. Obviously, for a heterogeneous preference profile, $\{N^0, N^1\}$ forms a partition of the set of players N. We will refer to N^0 and N^1 as corresponding *preference groups* of players.

In equilibrium characterizations we will also utilize the definition of $(r_1, ..., r_K)$-cohesive partitions of a subset, which is based on the definition of r-cohesive subsets.[17]

Definition 2 (Morris, 2000)**.** A subset of nodes $S \subset N$ in a network G is *r-cohesive* if

$$\min_{i \in S} \frac{|N_i(G) \cap S|}{|N_i(G)|} \geq r.$$

That is, a subset of nodes is r-cohesive if for its every node the share of inward-looking links (the links with nodes from the same subset) is at least r. Now let us partition the subset of nodes and look at the same measure for each of the parts specifically.

Definition 3. A partition $\{S_1, ..., S_K\}$ of a subset of nodes $S \subset N$ in a network G is *$(r_1, ..., r_K)$-cohesive* if for $k = 1, ..., K$:

$$\min_{i \in S_k} \frac{|N_i(G) \cap S|}{|N_i(G)|} \geq r_k.$$

Obviously, if $\{S_1, ..., S_K\}$ is a $(r_1, ..., r_K)$-cohesive partition of S, then S is $\min\{r_1, ..., r_K\}$-cohesive. On the other hand, if S is r-cohesive, then any partition $\{S_1, ..., S_K\}$ of S is $(r, ..., r)$-cohesive.

Furthermore, let us introduce related definitions of r-outward subsets and $(r_1, ..., r_K)$-outward partitions of a subset.

Definition 4. A subset of nodes $S \subset N$ in a network G is *r-outward* if

$$\min_{i \in S} \frac{|N_i(G) \cap (N \setminus S)|}{|N_i(G)|} \geq r.$$

That is, for every node the share of outward-looking links must be at least r (or the share of inward-looking links must be at most $1 - r$).

[15] Here $|x|$ denotes the absolute value of x.

[16] Note that, for a given degree d_i, the curve $\tilde{\lambda}(m)$ separates subregion R_M^m from R_M^{m+1} and subregion R_{MM}^m from R_{MM}^{m+1}.

[17] See Morris (2000), as well as chapter 9.6 in Jackson (2008).

Definition 5. A partition $\{S_1, ..., S_K\}$ of a subset of nodes $S \subset N$ in a network G is $(r_1, ..., r_K)$-*outward* if for $k = 1, ..., K$:

$$\min_{i \in S_k} \frac{|N_i(G) \cap (N \setminus S)|}{|N_i(G)|} \geq r_k.$$

Similar to the cohesion case, if $\{S_1, ..., S_K\}$ is a $(r_1, ..., r_K)$-outward partition of S, then S is $\min\{r_1, ..., r_K\}$-outward. On the other hand, if S is r-outward, then any partition $\{S_1, ..., S_K\}$ of S is $(r, ..., r)$-outward.

Finally, we will need the definition of a degree partition of a network.

Definition 6 (Mahadev and Peled, 1995)**.** Let G be a network with distinct positive degrees $d_{(1)} < ... < d_{(M)}$. Define $D_m = \{i \in N \mid d_i = d_{(m)}\}$ for $m = 1, ..., M$. Then the set-valued vector $D(G) = (D_1, ..., D_M)$ is called the *degree partition* of G.

4.2 Games with strong advantage of matching

We can now fully characterize equilibrium sets for games with strong advantage of matching, given an arbitrary network $(G, \bar{\theta})$. The following theorem provides existence conditions separately for symmetric and asymmetric equilibria. As we will see, there always exist at least two equilibria – two consensus equilibria – for this class of games, thus equilibrium multiplicity is unavoidable. At the same time, existence of disagreement equilibria is not always guaranteed (counterexample – a star network).

Theorem 1 [Equilibria. Strong advantage of matching]
For a network $(G, \bar{\theta})$ and game $(\delta, \lambda) \in R_M$:

(i) two symmetric equilibria exist,

(ii) an asymmetric equilibrium exists if and only if there exists such a partition $\{S^0, S^1\}$ of N that the following conditions are satisfied for $\theta = 0, 1$:

- $\forall i \in S^\theta \cap N^\theta : \ |N_i(G) \cap S^\theta| \geq l_{\delta,\lambda}(d_i)$ *and*
- $\forall i \in S^\theta \cap N^{1-\theta} : \ |N_i(G) \cap S^\theta| > d_i - l_{\delta,\lambda}(d_i)$.[18]

The first conclusion of the theorem, namely that both symmetric action profiles are equilibrium profiles (part (i) of the theorem), is not very surprising. The same result holds for the games of strategic complements without preference heterogeneity.[19] Theorem 1 confirms this result in a more general setting. It holds true even if the strength of personal preferences is very high: as long as advantage of matching is big enough, none would deviate from the symmetric equilibrium action.

[18]Here $|S|$ denotes the cardinality of a set S.
[19]See, for instance, Galeotti et al. (2010).

Part (ii) of the theorem, concerning asymmetric equilibria, states that existence of an asymmetric equilibrium is equivalent to existence of a partition $\{S^0, S^1\}$ of players satisfying several interconnectivity conditions. If we interpret this partition as the partition of players by chosen action and consider its refinement by the partition of N into preference groups – $\{S^0 \cap N^0, S^0 \cap N^1, S^1 \cap N^0, S^1 \cap N^1\}$ (some of the four subsets can be empty) – then the conditions of the theorem guarantee that the chosen actions are the best responses for all satisfied ($i \in S^\theta \cap N^\theta$) as well as for all frustrated ($i \in S^\theta \cap N^{1-\theta}$) players.

Hence, Theorem 1 provides an algorithm for practical derivation of the set of asymmetric equilibria for any given network by enumerating all possible partitions of the set of players into two subsets and checking whether they satisfy well-defined connectivity conditions. Since (ii) provides necessary and sufficient conditions for existence of an asymmetric equilibrium, the number of asymmetric equilibria is given by the number of such partitions.

We can use the definition of $(r_1, ..., r_K)$-cohesive partitions of a subset of nodes to formulate necessary and sufficient conditions for existence of asymmetric equilibria in cohesion terminology. For this purpose, we refine a partition $\{S^0, S^1\}$ of N using the degree partition of the network and the partition of N into preference groups.

Corollary 5 (Asymmetric equilibria. Strong advantage of matching)**.** *For a network $(G, \bar{\theta})$ with degree partition $(D_1, ..., D_M)$ and a game $(\delta, \lambda) \in R_M$, an asymmetric equilibrium exists if and only if there exists such a partition $\{S^0, S^1\}$ of N that for $\theta \in \{0, 1\}$ the (possibly trivial) partition $\{S^\theta \cap N^\theta \cap D_1, ..., S^\theta \cap N^\theta \cap D_M, S^\theta \cap N^{1-\theta} \cap D_1, ..., S^\theta \cap N^{1-\theta} \cap D_M\}$ of S^θ is $\left(\frac{l_{\delta,\lambda}(d_{(1)})}{d_{(1)}}, ..., \frac{l_{\delta,\lambda}(d_{(M)})}{d_{(M)}}, 1 - \frac{l_{\delta,\lambda}(d_{(1)})-1}{d_{(1)}}, ..., 1 - \frac{l_{\delta,\lambda}(d_{(M)})-1}{d_{(M)}}\right)$-cohesive.*[20]

This formulations provides additional intuition for Theorem 1: *for maintaining variation in behavior in a network, sufficient interconnection within two groups of players is important.* And it does not matter for existence of an asymmetric equilibrium *whether these groups coincide with preference groups or not*: sufficient connectivity within groups guarantees that the action choices are the best responses for both satisfied and frustrated players.

Next, we examine the existence of fully satisfying equilibria as such that guarantee the highest degree of satisfaction of personal preferences in a network. The following theorem provides necessary and sufficient conditions for existence of the fully satisfying equilibrium in an arbitrary network.

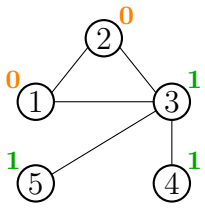
Theorem 2 [Existence of the fully satisfying equilibrium]
For a network $(G, \bar{\theta})$ and game $(\delta, \lambda) \in R_M$ the fully satisfying equilibrium exists if and only if the following condition is satisfied:

$$\forall i \in N: \ |N_i(G) \cap N^{\theta_i}| \geq l_{\delta,\lambda}(d_i).$$

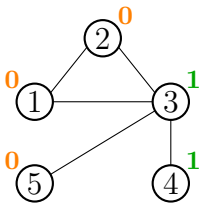
That is, such an equilibrium exists if and only if every player has at least $l_{\delta,\lambda}(d_i)$ distinct neighbors whose preferences coincide with her own. Given a network and a preference profile, this condition is very easy to check.

[20]Here a *trivial* partition is such that contains empty subsets.

Let us consider an example. Figure 4 depicts a network with two different preference profiles. The players are numbered (numbers inside circles) and their personal preferences over actions are depicted as coloured numbers outside circles. Degrees of the players are 1, 2 or 4. In the network (a) the fully satisfying action profile constitutes an equilibrium, since every player has sufficiently many neighbors with the same preference. The required number of such neighbors is one for players 1, 2, 4 and 5, and one or two – depending on the game (δ, λ) – for player 3. In the network (b) the fully satisfying action profile is not an equilibrium, since the condition of Theorem 2 for player 5 (and, for some (δ, λ), also for player 3) is not satisfied.



(a)



(b)

$(\delta, \lambda) \in R_M$

$l_{\delta,\lambda}(1) = 1$
$l_{\delta,\lambda}(2) = 1$
$l_{\delta,\lambda}(4) = 1$ or 2

Figure 4: Fully satisfying equilibrium exists in (a); does not exist in (b)

Thus, distribution of personal preferences on a network is crucial for existence of the fully satisfying equilibrium. But even prior to the preference distribution, some minimal preconditions regarding the size of preference groups must be satisfied.

Corollary 6 (Necessary condition for the fully satisfying equilibrium)**.** *For a network $(G, \bar{\theta})$ with a heterogeneous preference profile and a game $(\delta, \lambda) \in R_M$ the fully satisfying equilibrium exists only if $|N^\theta| \geq 2 \ \forall \theta \in \{0, 1\}$.*

That is, for a heterogeneous preference profile the fully satisfying equilibrium exists only if each preference group contains at least two players. If this condition is not satisfied, there is a player who has no companions in the fully satisfying action profile, which contradicts the minimum companion requirement (Corollary 1).

On the other hand, network structure itself might already be decisive. There are such network structures in which fully satisfying equilibria never exist, regardless of the preference distribution and even of the strength of personal preferences. A star is a good example of such network structure: it never allows for the fully satisfying equilibrium if players have heterogeneous preferences.

4.3 Games with strong advantage of mismatching

The following theorem characterizes equilibrium sets for games with strong advantage of mismatching. Similar to the matching case, it considers symmetric and asymmetric equilibria separately. Since no symmetric equilibria exist for this class of games, equilibrium existence in general is no longer guaranteed.

Theorem 3 [Equilibria. Strong advantage of mismatching]
For a network $(G, \bar{\theta})$ *and game* $(\delta, \lambda) \in R_{MM}$*:*

(i) no symmetric equilibria exist,

(ii) an asymmetric equilibrium exists if and only if there exists such a partition $\{S^0, S^1\}$ *of* N *that the following conditions are satisfied for* $\theta = 0, 1$*:*

- $\forall i \in S^\theta \cap N^\theta : \ |N_i(G) \cap S^{1-\theta}| \geq l_{\delta,\lambda}(d_i)$ *and*
- $\forall i \in S^\theta \cap N^{1-\theta} : \ |N_i(G) \cap S^{1-\theta}| > d_i - l_{\delta,\lambda}(d_i)$.

Again, the first result concerning non-existence of (pure-strategy) symmetric equilibria goes along with typical conclusions for anti-coordination games.[21] Consequently, existence of equilibria in the general case is no longer guaranteed, while in some networks multiplicity of equilibria is still an issue. The second result of the theorem is similar to the corresponding result of Theorem 1: existence of an asymmetric equilibrium is equivalent to existence of a partition $\{S^0, S^1\}$ of network nodes that satisfies specific interconnectivity conditions. Theorem 3 provides an algorithm for practical derivation of all Nash equilibria in a given network for games with strong advantage of mismatching.

For this class of games, it is impossible to reformulate the necessary and sufficient conditions for existence of asymmetric equilibria in cohesion terminology. However, here we can employ the notion of outwardness of a subset (partition of a subset).

Corollary 7 (Asymmetric equilibria. Strong advantage of mismatching)**.** *For a network* $(G, \bar{\theta})$ *with degree partition* $(D_1, ..., D_M)$ *and a game* $(\delta, \lambda) \in R_{MM}$*, an asymmetric equilibrium exists if and only if there exists such a partition* $\{S^0, S^1\}$ *of* N *that for* $\theta \in \{0, 1\}$ *the (possibly trivial) partition* $\{S^\theta \cap N^\theta \cap D_1, ..., S^\theta \cap N^\theta \cap D_M, S^\theta \cap N^{1-\theta} \cap D_1, ..., S^\theta \cap N^{1-\theta} \cap D_M\}$ *of* S^θ *is* $\left(\frac{l_{\delta,\lambda}(d_{(1)})}{d_{(1)}}, ..., \frac{l_{\delta,\lambda}(d_{(M)})}{d_{(M)}}, 1 - \frac{l_{\delta,\lambda}(d_{(1)})-1}{d_{(1)}}, ..., 1 - \frac{l_{\delta,\lambda}(d_{(M)})-1}{d_{(M)}}\right)$*-outward.*

Now, we will state necessary and sufficient conditions for existence of the equilibrium that satisfies personal preferences of all players in a network.

Theorem 4 [Existence of the fully satisfying equilibrium]
For a network $(G, \bar{\theta})$ *and game* $(\delta, \lambda) \in R_{MM}$ *the fully satisfying equilibrium exists if and only if the following condition is satisfied:*

$$\forall i \in N : \ |N_i(G) \cap N^{1-\theta_i}| \geq l_{\delta,\lambda}(d_i).$$

In other words, the fully satisfying equilibrium exists if and only if every player has at least $l_{\delta,\lambda}(d_i)$ distinct neighbors whose preferences do not coincide with i's preference. Thus, what matters now is the number of neighbors with a different preference. As Corollary 3 suggests, for each player this number is greater or equal to one, hence fully satisfying equilibria are possible only for heterogeneous preference profiles.

[21] See, in particular, Bramoullé (2007) and Galeotti et al. (2010).

Figure 5 provides an example. Again, numbers inside circles identify players and coloured numbers outside circles correspond to players' preferences. For a preference profile in (a) the fully satisfying equilibrium exists, since every player has the required number of neighbors with different preference (the requirements are listed in the right part of the figure). In (b) the fully satisfying action profile is not an equilibrium, since for player 4 the requirement is not satisfied.

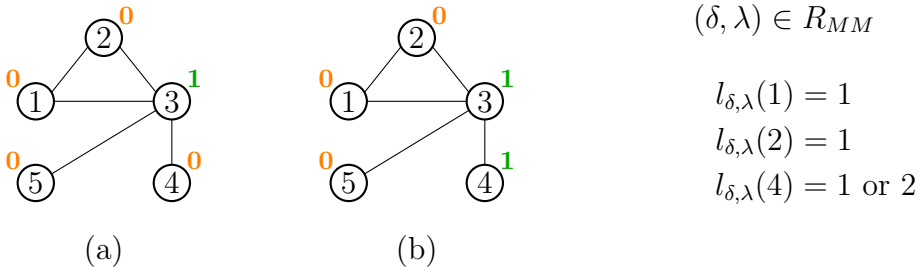


Figure 5: Fully satisfying equilibrium exists in (a); does not exist in (b)

4.4 Games with no well-pronounced interactional advantage

As it follows from Proposition 3, whenever the advantage of matching or mismatching is not well-pronounced, every player chooses her preferred action. The following theorem fully characterizes the set of equilibria for this class of games.

Theorem 5 [Equilibria. No well-pronounced advantage]
For a network $(G, \bar{\theta})$ and game $(\delta, \lambda) \in R_{NP}$ there always exists a unique equilibrium – the fully satisfying equilibrium.

The proof is straightforward. Since following personal preference is the unique best response for each player, in equilibrium every player chooses her preferred action: $x_i = \theta_i$ $\forall i \in N$. Uniqueness follows from the assumption that players' preferences over actions are strict. Thus, the equilibrium action profile coincides with the preference profile of the network.

Let us note immediately one interesting feature of such equilibria: they are characterized by the minimal possible level of network frustration, as they allow for complete concordance of chosen actions with the preferred ones.

Thus, there exists a nonempty class of games, including both games of strategic complements and those of strategic substitutes, such that the only possible equilibrium is the one in which every player follows her own preference. Note that since every player chooses to follow her preference regardless of the choices of her neighbors, this equilibrium is strong Nash, and thus it is Pareto optimal.

4.5 Existence of asymmetric equilibria in standard networks

In this subsection we provide the complete characterization of the Nash equilibrium set for several standard network structures, namely a star, a circle and a complete network. The focus here is existence and uniqueness of asymmetric equilibria, since for symmetric equilibria the general results of Theorems 1, 3 and 5 apply. At the end of the subsection, we also present sufficient conditions for existence of asymmetric equilibria in regular networks of an arbitrary degree.

The following propositions provide necessary and sufficient conditions for existence of asymmetric equilibria in games with strong advantage of matching or mismatching. For games with no well-pronounced interactional advantage the general result applies (Theorem 5): an asymmetric equilibrium exists if and only if the preference profile is heterogeneous, and the equilibrium is always unique.

Proposition 4 [Star networks]
Let $(G, \bar{\theta})$ be a star network with a heterogeneous preference profile.

(i) In a game $(\delta, \lambda) \in R_M$ no asymmetric equilibria exist.

(ii) In a game $(\delta, \lambda) \in R_{MM}$ two asymmetric equilibria exist.

Thus, any game with strong advantage of matching or mismatching has two equilibria in a star network. In the case of matching these are two symmetric equilibria, in the case of mismatching – two asymmetric. Figure 6 illustrates the latter case (here colours correspond to the chosen actions: yellow – to action 0, green – to action 1). Note that the equilibrium set is completely independent of a preference profile for star networks.

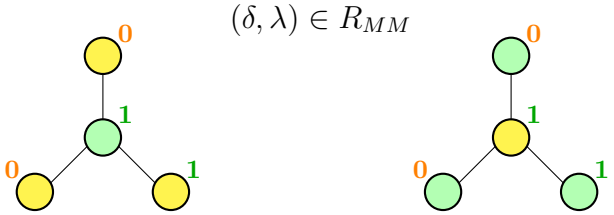


Figure 6: Star networks. Asymmetric equilibria in games with strong advantage of mismatching

Proposition 5 [Circle networks]
Let $(G, \bar{\theta})$ be a circle network with n players and a heterogeneous preference profile, and let the players be numbered consecutively.

(i) In a game $(\delta, \lambda) \in R_M$ an asymmetric equilibrium exists if and only if there exist four distinct players such that $\theta_i \neq \theta_{i+1} = \theta_j \neq \theta_{j+1}$. It is a unique asymmetric equilibrium if and only if there are no other players satisfying the above condition.

(ii) In a game $(\delta, \lambda) \in R_{MM}$ an asymmetric equilibrium always exists. It is unique if and only if n is odd and $\exists! i \in N$ s.t. $\theta_i = \theta_{i+1}$.

Hence, in a circle network existence of an asymmetric equilibrium is only guaranteed for games with strong advantage of mismatching, while for the matching games it might be that only consensus is possible in equilibrium. At the same time, multiplicity of (asymmetric) equilibria is not excluded for both classes of games.

Figure 7 provides an example of equilibria multiplicity for a game with strong advantage of matching. It depicts a circle network with heterogeneous preference profile $\bar{\theta} = (0, 1, 1, 0, 0, 1, 0)$ (the players numbered clockwise starting with the top one). Obviously, this network satisfies the condition in part (i) of the proposition: such 4-tuples of players are $\{1, 2, 3, 4\}$, $\{1, 2, 6, 7\}$ and $\{3, 4, 5, 6\}$. The figure shows three respective asymmetric equlibria for this network.

$$(\delta, \lambda) \in R_M$$

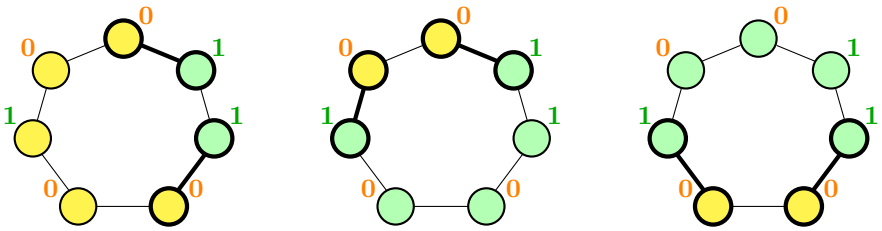


Figure 7: Circle networks. Asymmetric equilibria in games with strong advantage of matching

For games with strong advantage of mismatching, as expected, existence of asymmetric equilibria does not impose any conditions. For even number of players, there are at least two asymmetric equilibria (those with alternating actions) regardless of a preference profile. For odd number of players, an asymmetric equilibrium might be unique (and then, fully-satisfying). The necessary and sufficient condition for this is that in a network there are only two neighboring players with the same preference. If not, each pair of such neighbors corresponds to a different asymmetric equilibrium (see Figure 8).

$$(\delta, \lambda) \in R_{MM}$$

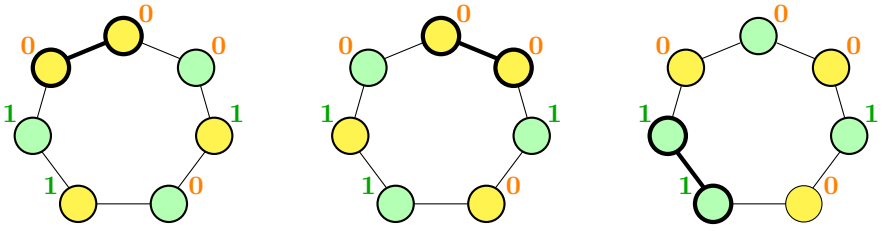


Figure 8: Circle networks. Asymmetric equilibria in games with strong advantage of mismatching

Proposition 6 [Complete networks]
Let $(G, \bar{\theta})$ be a complete network with n players and a heterogeneous preference profile.

(i) In a game $(\delta, \lambda) \in R_M$ *an asymmetric equilibrium exists if and only if* $|N^\theta| \geq l_{\delta,\lambda}(n-1)+1$ *for* $\theta = 0, 1$*. If it exists, it is a unique asymmetric equilibrium.*

(ii) In a game $(\delta, \lambda) \in R_{MM}$ *an asymmetric equilibrium always exists. It is unique if and only if* $|N^\theta| \geq l_{\delta,\lambda}(n-1)$ *for* $\theta = 0, 1$ *(strict for at least some* θ*).*

Again, for the matching case existence of an asymmetric equilibrium is not guaranteed (e.g. it does not exist for games with very strong advantage of matching, $(\delta, \lambda) \in R_M^L(n-1)$). If the advantage of matching is smaller and the preference minority (the smaller of the two preference groups) is sufficiently large, then an asymmetric equilibrium is possible, but only the fully satisfying one. That is, in games from R_M all asymmetric equilibria are fully satisfying equilibria. Intuition for this is quite straightforward: in a maximally connected network with strong advantage of action matching, departure from consensus is only possible if it allows players to satisfy their personal preferences.

It is also quite intuitive that if action mismatching brings strong advantage, then asymmetric equilibria always exist and equilibrium multiplicity is a common issue. While in the matching games a large minority is necessary for asymmetric equilibrium existence, in the mismatching games it is a necessary and sufficient condition for equilibrium uniqueness. The intuition is the following: if two preference groups are too unequal in size, some players from the majority group have to surrender their personal preferences to make the action profile more balanced, and the larger majority the more potential players who could take these roles. If, on the contrary, the minority is large enough and the equilibrium is unique, it is always fully satisfying.

Before we state sufficient conditions for existence of asymmetric equilibria in regular networks, let us recall several definitions from the graph theory.[22]

For a graph G with the set of nodes N, an *induced subgraph* $G[S]$ is another graph with the set of nodes $S \subset N$ and all the links in G connecting pairs of nodes in S (that is, $G[S]_{ij} = 1 \Leftrightarrow i, j \in S \wedge G_{ij} = 1$).

A graph G is *bipartite* if its set of nodes N can be partitioned into two subsets $\{S_1, S_2\}$ so that every link has its ends in different subsets: formally, $G_{ij} = 1 \wedge i \in S_1 \Rightarrow j \in S_2$, and $G_{ij} = 1 \wedge i \in S_2 \Rightarrow j \in S_1$. If any two nodes from different subsets are linked $(i \in S_1, j \in S_2 \Rightarrow G_{ij} = 1)$, then G is a *complete bipartite* graph.

The next proposition provides sufficient conditions for existence of asymmetric equilibria in regular networks.

Proposition 7 [Regular networks. Sufficient conditions]
Let $(G, \bar{\theta})$ *be a regular network of degree* d*.*

(i) In a game $(\delta, \lambda) \in R_M$ *an asymmetric equilibrium exists if* $\forall \theta \in \{0, 1\}\ \exists X^\theta \subseteq N^\theta$ *such that* $|X^\theta| \geq l_{(\delta,\lambda)}(d)+1$ *and the induced subgraph* $G[X^\theta]$ *is complete.*

[22] See, for instance, Diestel (2017), Bondy and Murty (2008) or Harary (1969).

(ii) In a game $(\delta, \lambda) \in R_{MM}$ an asymmetric equilibrium exists if $\forall \theta \in \{0,1\}$ $\exists X^\theta \subseteq N^\theta$ such that $|X^\theta| \geq l_{(\delta,\lambda)}(d)$ and $K[X^0, X^1] \subseteq G$, where $K[X^0, X^1]$ is a complete bipartite graph with partition $\{X^0, X^1\}$.

In other words, in games with strong advantage of matching, if each preference group contains at least $l_{(\delta,\lambda)}(d) + 1$ fully interconnected players, then an asymmetric equilibrium exists. In games with strong advantage of mismatching, an asymmetric equilibrium exists if there are at least $l_{(\delta,\lambda)}(d)$ players from each preference group and they are fully intraconnected (connected to each player of the other subgroup). Thus, sufficiently large minority and specific connectivity conditions for the preference groups guarantee existence of asymmetric equilibria in regular networks.

Figure 9 shows some asymmetric equilibria in a cubic (3-regular) network with different heterogeneous preference profiles. The first two examples correspond to games with strong advantage of matching, where the companion requirement is either 1 or 2. The last example corresponds to games with strong advantage of mismatching and the opponent requirement of 2. In all networks the respective sufficient conditions of Proposition 7 are satisfied.

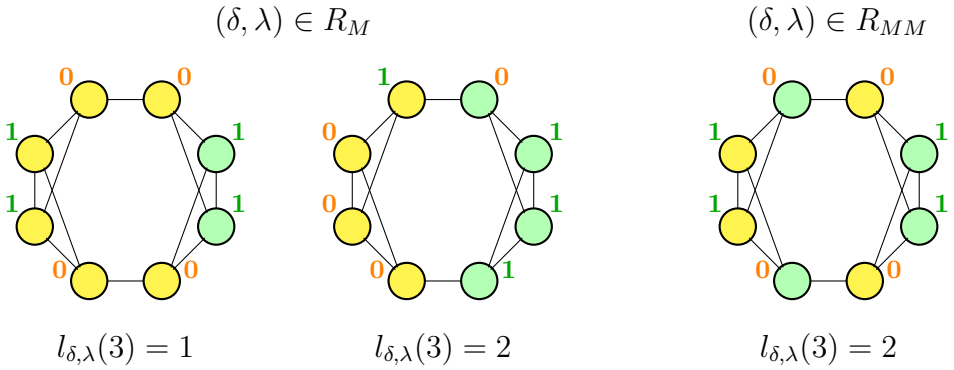


Figure 9: Asymmetric equilibria in regular networks

5 Discussion

5.1 No-preference framework

Let us briefly outline the results for the case when players do not have any preferences over action choices (and hence $\lambda = 0$). It would allow us to compare the two frameworks – with and without personal preferences.

The payoff function of player i with the set of neighbors $N_i(G)$ takes the form:

$$u_i(x_i, \bar{x}_{N_i(G)}) = \delta \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i = x_j\}} + (1 - \delta) \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i \neq x_j\}},$$

where $\delta \in [0; 1]$.

The best responses of a player are given by the following lemma.

Lemma 2 (Best responses. No-preference framework). *In the no-preference framework, the best response of a player i of degree d_i is*

$$BR_i(d_i, \tau_i) = \begin{cases} 1, & \text{if } \tau_i \geq \frac{d_i}{2} \\ 0, & \text{if } \tau_i \leq \frac{d_i}{2} \end{cases} \quad for \quad \delta > \frac{1}{2}$$

and

$$BR_i(d_i, \tau_i) = \begin{cases} 0, & \text{if } \tau_i \geq \frac{d_i}{2} \\ 1, & \text{if } \tau_i \leq \frac{d_i}{2} \end{cases} \quad for \quad \delta < \frac{1}{2},$$

where τ_i is the number of i's neighbors who choose action 1.

It is easy to see that whenever $\delta > \frac{1}{2}$ the best response for every player is to choose the same action as the majority of her neighbors, while whenever $\delta < \frac{1}{2}$ the best response is to choose the opposite action. The two cases correspond to games with strategic complements and those with strategic substitutes respectively. The equilibrium characterizations for the two cases are given by the following propositions.

Proposition 8 [Equilibria. Strategic complements]
In the no-preference framework, for a network G and $\delta > \frac{1}{2}$:

(i) two symmetric equilibria exist;

(ii) an asymmetric equilibrium exists if and only if there exists such a partition $\{S^0, S^1\}$ of N that the following condition is satisfied for $\theta = 0, 1$:

$$\forall i \in S^\theta : \ |N_i(G) \cap S^\theta| \geq \frac{d_i}{2}.$$

Proposition 9 [Equilibria. Strategic substitutes]
In the no-preference framework, for a network G and $\delta < \frac{1}{2}$:

(i) no symmetric equilibria exist;

(ii) an asymmetric equilibrium exists if and only if there exists such a partition $\{S^0, S^1\}$ of N that the following condition is satisfied for $\theta = 0, 1$:

$$\forall i \in S^\theta : \ |N_i(G) \cap S^{1-\theta}| \geq \frac{d_i}{2}.$$

The necessary and sufficient conditions for existence of asymmetric equilibria in the case of strategic complements (Propositions 8) can be easily reformulated using the definition of r-cohesive subsets of nodes. Specifically, for a network G and $\delta > \frac{1}{2}$ asymmetric equilibria exist if and only if there exists such a partition $\{S^0, S^1\}$ of N that both S^0 and S^1 are $\frac{1}{2}$-cohesive. In the case of strategic substitutes ($\delta < \frac{1}{2}$), Proposition 9 can be similarly reformulated using the notion of r-outwardness of a subset: asymmetric equilibria exist if and only if there exists such a partition $\{S^0, S^1\}$ of N that both S^0 and S^1 are $\frac{1}{2}$-outward.

It is notable that if we compare, for a given network and game, the set of equilibria in the no-preference framework to the corresponding set in our framework with preferences, there is generally no inclusion in *either* direction. On the one hand, equilibria in the framework without preferences do not necessarily remain equilibria if we allow players to have preferences. On the other hand, some equilibria in the framework with preferences are never possible in the framework without.

Figure 10 illustrates this fact. Five players are connected in a network and play a game with strategic complements ($\delta > \frac{1}{2}$). For the framework with preferences, we additionally assume a heterogeneous preference profile $\bar{\theta} = (0, 0, 1, 1, 0)$ and the strength of personal preferences such that $(\delta, \lambda) \in R_M^1(3)$. In the no-preference framework, $\bar{x} = (0, 0, 0, 1, 1)$ is an equilibrium (Figure 10.a), while $\bar{x} = (0, 0, 1, 1, 1)$ is not, as player 3 has an incentive to deviate and follow the majority of her neighbors (Figure 10.b). However, if we allow players to have personal preferences, the situation is reverse: $\bar{x} = (0, 0, 0, 1, 1)$ is not an equilibrium anymore, as player 3 has enough companions to switch to her preferred action 1 (Figure 10.a), while $\bar{x} = (0, 0, 1, 1, 1)$ is an equilibrium for this very reason (Figure 10.b).

Therefore, we can neither claim that the introduction of personal preferences in general extends the equilibrium set, nor that it shrinks it. What happens depends on a particular network, preference profile and the strength of personal preferences.

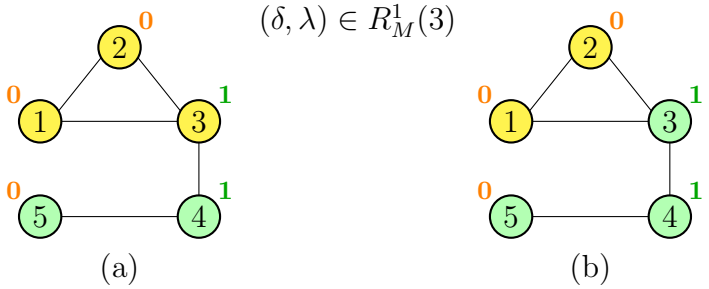


Figure 10: (a) is an equilibrium in the no-preference framework, but not an equilibrium in the framework with preferences; (b) – vice versa.

5.2 Additively separable utility

In this subsection we consider an alternative – additive – specification of the utility function, that assumes additive separability of personal and social utility components. The payoff for player i with preference θ_i and the set of neighbors $N_i(G)$ is now defined as

$$u_i(\theta_i, x_i, \bar{x}_{N_i(G)}) = \left(\delta \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i = x_j\}} + (1 - \delta) \sum_{j \in N_i(G)} \mathbb{1}_{\{x_i \neq x_j\}} \right) + \lambda \cdot \mathbb{1}_{\{x_i = \theta_i\}}.$$

As the following figure shows, the borders between the three regions – R_M, R_{MM} and R_{NP} – now depend on d_i. It means that the interactional advantage might not be of the same kind for all players. More precisely, there are games in which more connected players have

strong advantage of matching (or mismatching), whereas less connected ones have no well-pronounced interactional advantage (as compared to benefits from following their personal preference).

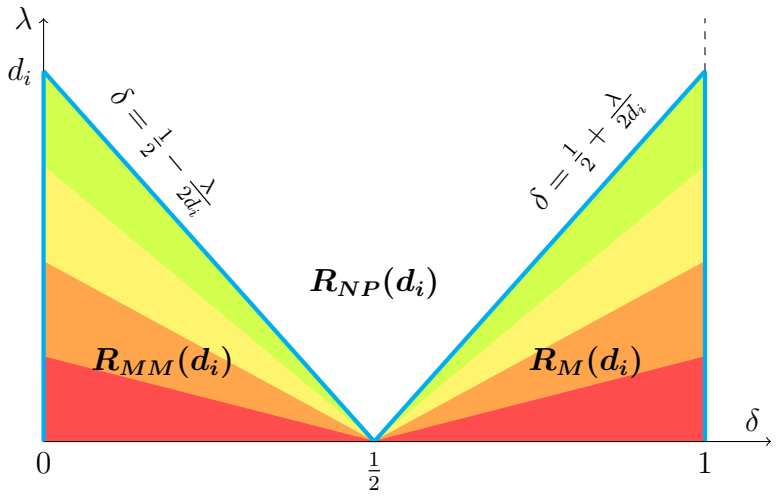


Figure 11: Additive utility. Regions of different best response strategies for a player of degree d_i

Let us fix a game (δ, λ) and partition the players according to the type of their interactional advantage in this game (and thus their best response strategy): $N = N^M_{\delta,\lambda} \cup N^{MM}_{\delta,\lambda} \cup N^{NP}_{\delta,\lambda}$, where $N^t_{\delta,\lambda} = \{i \in N \mid (\delta, \lambda) \in R_t(d_i)\}$ with $t \in \{M, MM, NP\}$. Obviously, $N^{MM}_{\delta,\lambda} = \emptyset$ for $\delta \geq \frac{1}{2}$ and $N^M_{\delta,\lambda} = \emptyset$ for $\delta \leq \frac{1}{2}$. If a player belongs to $N^M_{\delta,\lambda}$, $N^{MM}_{\delta,\lambda}$ or $N^{NP}_{\delta,\lambda}$, her best response function is given by Proposition 1, 2 or 3 respectively, with the only difference that subregions $R^l_M(d_i)$ and $R^l_{MM}(d_i)$ are accordingly redefined.

Given a network G, we can now specify such partition of the set of players for each admissible game (δ, λ). Let $(D_1, ..., D_K)$ be the degree partition of G with corresponding (positive) degrees $d_{(1)} < ... < d_{(K)}$. Obviously, the more neighbors a player has the more restricted she is in following her personal preference: $R_{NP}(d_{(k)}) \subset R_{NP}(d_{(k-1)})$ for any $k = 2, .., K$ (see Figure 11). This nestedness implies that the right half of the parameter space $(\delta \geq \frac{1}{2})$ can be partitioned into following subregions (Figure 12): $R_{NP}(d_{(K)})$, $R_M(d_{(K)}) \cap R_{NP}(d_{(K-1)})$, ..., $R_M(d_{(2)}) \cap R_{NP}(d_{(1)})$, and $R_M(d_{(1)})$. The union of all inner subregions is denoted by $\tilde{R}_M(d_{(1)}, d_{(K)})$. For the left half of the parameter space $(\delta \leq \frac{1}{2})$ the partition is symmetric and the corresponding union of all inner subregions is denoted by $\tilde{R}_{MM}(d_{(1)}, d_{(K)})$.

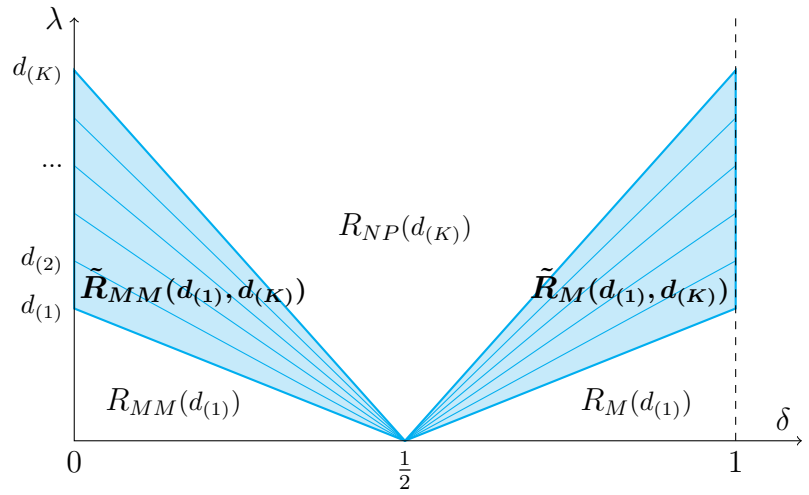


Figure 12: Additive utility. Regions of different player partitions for a network with degree partition $(D_1, ..., D_K)$

Note that for regular networks both $\tilde{R}_M(d, d)$ and $\tilde{R}_{MM}(d, d)$ are empty, hence the parameter space splits, as before, into three regions – $R_M(d)$, $R_{MM}(d)$ and $R_{NP}(d)$. However, let us specify the player partition in the general case.

Lemma 3 (Additive utility. Player partition)**.** *For a network G with degree partition $(D_1, ..., D_K)$ and a game (δ, λ), the (possibly trivial) partition of players according to their interactional advantage is*

$$N^{NP}_{\delta,\lambda} = D_1 \cup ... \cup D_k, \;\; N^{t}_{\delta,\lambda} = D_{k+1} \cup ... \cup D_K$$

with $t = M$ for $\delta > \frac{1}{2}$, $t = MM$ for $\delta < \frac{1}{2}$ and $k = k^* - \mathbb{1}_{\lambda < \tilde{\lambda}(k^*)}$, where $k^* = \underset{m=1...K}{\operatorname{argmin}} |\lambda - \tilde{\lambda}(m)|$ and $\tilde{\lambda}(m) = |(2\delta - 1) \cdot d_{(m)}|$.[23,24]

Note that $(\delta, \lambda) \in R_{NP}(d_{(K)})$ implies $N = N^{NP}_{\delta,\lambda}$, and thus for all players the best responses are their preferred actions. According to Theorem 5, a unique, fully satisfying equilibrium exists in this case. Similarly, if $(\delta, \lambda) \in R_M(d_{(1)})$ (or $R_{MM}(d_{(1)})$) then $N = N^{M}_{\delta,\lambda}$ (or $N^{MM}_{\delta,\lambda}$ respectively), and thus Theorem 1 (Theorem 3) applies for the equilibrium characterization.

In the remaining regions of games, $\tilde{R}_M(d_{(1)}, d_{(K)})$ and $\tilde{R}_{MM}(d_{(1)}, d_{(K)})$, the best response strategies partition the set of players into two proper subsets: $N^{NP}_{\delta,\lambda} = D_1 \cup ... \cup D_k$ and $N^{M}_{\delta,\lambda}$ (or $N^{MM}_{\delta,\lambda}$)$= D_{k+1} \cup ... \cup D_K$ with some $k \in \{1, ..., K-1\}$. That is, only for players with sufficiently high degrees interactional advantage (of matching or mismatching) is well-pronounced, while for players with lower degrees it is never reasonable to deviate from their preferred actions. Theorems 6 and 7 provide equilibrium characterizations for games from regions $\tilde{R}_M(d_{(1)}, d_{(K)})$ and $\tilde{R}_{MM}(d_{(1)}, d_{(K)})$ respectively.

[23]The curve $\tilde{\lambda}(m)$ defines the boundary of $R_{NP}(d_{(m)})$ for $m = 1, ..., K$ (blue lines in Figure 12).

[24]Note that $k \in \{0, ..., K\}$. If $k = 0$ then $N = N^{M}_{\delta,\lambda}$ (or $N^{MM}_{\delta,\lambda}$). If $k = K$ then $N = N^{NP}_{\delta,\lambda}$.

Theorem 6 [Additive utility. Equilibria for games with heterogeneous interactional advantage. Matching]
For a network $(G, \bar{\theta})$ *with degree partition* $(D_1, ..., D_K)$ *and a game* $(\delta, \lambda) \in \tilde{R}_M(d_{(1)}, d_{(K)})$*:*

(i) a (unique) symmetric equilibrium exists if and only if

$$\forall i, j \in N^{NP}_{\delta,\lambda} : \ \theta_i = \theta_j,$$

(ii) an asymmetric equilibrium exists if and only if there exists such a partition $\{S^0, S^1\}$ *of* N *that the following conditions are satisfied for* $\theta = 0, 1$*:*

- $S^\theta \cap N^{1-\theta} \cap N^{NP}_{\delta,\lambda} = \emptyset$,
- $\forall i \in S^\theta \cap N^\theta \cap N^M_{\delta,\lambda} : \ |N_i(G) \cap S^\theta| \geq l_{\delta,\lambda}(d_i)$ *and*
- $\forall i \in S^\theta \cap N^{1-\theta} \cap N^M_{\delta,\lambda} : \ |N_i(G) \cap S^\theta| > d_i - l_{\delta,\lambda}(d_i)$.[25]

It is noteworthy that existence of symmetric equilibria is no longer guaranteed for $\tilde{R}_M(d_{(1)}, d_{(K)})$ as compared to $R_M(d_{(1)})$. The necessary and sufficient condition for existence of (one) symmetric equilibrium is that all the players that fall into $N^{NP}_{\delta,\lambda}$ (less connected players) prefer the same action; in such a case, this action is the symmetric equilibrium action.

Thus, some consensus equilibria are ruined if additive separability of the utility function is assumed. Among those can also be efficient equilibria (such that maximize aggregate utility in the network), like in Figure 13. Here $(\delta, \lambda) \in R_{NP}(1) \cap R_M(2)$, implying that player 4 always follows her personal preference in equilibrium, while players 1, 2 and 3 have strong advantage of matching neighbors' actions. This nonalignment of the consensus equilibrium with efficiency is not surprising, since the equilibrium action is "dictated" by the players with lower degrees and absolutely regardless of their relative number.

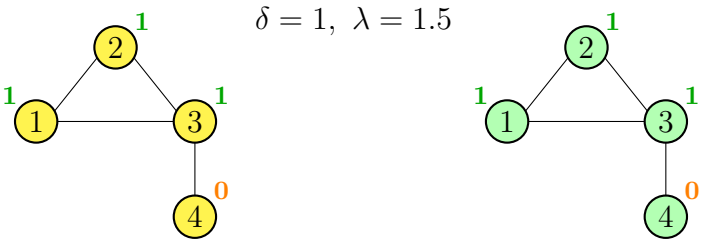


Figure 13: Multiplicative utility: both (a) and (b) are consensus equilibria. Additive utility: (a) is the unique consensus equilibrium, while (b) is the (unique) efficient action profile.

Theorem 7 [Additive utility. Equilibria for games with heterogeneous interactional advantage. Mismatching]
For a network $(G, \bar{\theta})$ *with degree partition* $(D_1, ..., D_K)$ *and a game* $(\delta, \lambda) \in \tilde{R}_{MM}(d_{(1)}, d_{(K)})$*:*

[25] Note that for derivation of $l_{\delta,\lambda}(d_i)$ in this and the following theorem, Lemma 1 must be slightly modified: $\tilde{\lambda}(m) = |2\delta - 1| \cdot (d_i - 2m)$.

(i) no symmetric equilibria exist,

(ii) an asymmetric equilibrium exists if and only if there exists such a partition $\{S^0, S^1\}$ *of* N *that the following conditions are satisfied for* $\theta = 0, 1$*:*

- $S^\theta \cap N^{1-\theta} \cap N^{NP}_{\delta,\lambda} = \emptyset$,
- $\forall i \in S^\theta \cap N^\theta \cap N^{MM}_{\delta,\lambda} : \ |N_i(G) \cap S^{1-\theta}| \geq l_{\delta,\lambda}(d_i)$ *and*
- $\forall i \in S^\theta \cap N^{1-\theta} \cap N^{MM}_{\delta,\lambda} : \ |N_i(G) \cap S^{1-\theta}| > d_i - l_{\delta,\lambda}(d_i)$.

Thus, for a given network G with degree partition $(D_1, ..., D_K)$, the qualitative results of sections 3 and 4 hold for all games from $R_M(d_{(1)}) \cup R_{MM}(d_{(1)}) \cup R_{NP}(d_{(K)})$ and with slight modifications also hold for the remaining games.

A notable difference from the multiplicative utility case is that *social influence is limited if the utility function is additively separable*. Whenever the strength of personal preferences is high enough (to be precise, higher than $d_{(K)}$ – the maximal possible network utility gain), the only thing that matters for a player is her own preference. In this case, there is a unique, fully satisfying equilibrium. Clearly, this difference follows directly from the specification of the utility function.

Another notable difference is that the in-between region R_{NP}, with the unique and fully satisfying Nash equilibrium, shrinks as long as connectivity of the network increases. The intuition behind this phenomenon is the following: as a player becomes more connected, the relative weight of the personal utility component gets lower for her, which reduces the possibility of the fully satisfying equilibrium.

6 Conclusions

Our results confirm that the introduction of individual preferences over actions makes an important difference to equilibrium outcomes. We show that, in most of the cases, the set of equilibria in a no-preference network is different from the analogous set in a network with personal preferences, and that there is no inclusion in either direction.

Extending the framework of Hernández et al. (2013), we characterize individual behavior and equilibria outcomes in a much wider range of games, for both multiplicative and additive relationship between what we call social (originating from the network) and personal (originating from individual preferences) utility components.

We demonstrate that under either specification of the utility function there exists a class of games (including both games of strategic complements and of strategic substitutes) in which everyone choosing their preferred action is a unique equilibrium, and it is a strong Nash. At the same time, there are classes of games in which such a fully satisfying action profile is never an equilibrium, no matter how strong players' preferences are.

We provide necessary and sufficient conditions for existence of different types of equilibria for different classes of games on an arbitrary network with an arbitrary preference profile of the players. If the game is such that matching neighbors' actions has strong advantage, two

consensus equilibria always exist. For maintaining different behavior in a network, sufficient interconnection between two groups of players is important – no matter whether these groups coincide with preference groups or not. If the game is such that action mismatching has strong advantage, only different behavior is possible in equilibrium.

A Appendix

Proof of Proposition 1

We will first prove an auxiliary lemma characterizing the best responses of players in a general form.

Lemma 4. *In a game $(\delta, \lambda) \in R_M$ the best response function of a player i with preference θ_i and d_i neighbors, τ_i of whom play 1, is*

$$BR_i(\theta_i, d_i, \tau_i) = \begin{cases} 1, \text{if } \tau_i > \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) \\ 0, \text{if } \tau_i < \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) \\ \theta_i, \text{if } \tau_i = \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) \end{cases}$$

where $\widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) = \frac{\delta(2+\lambda)-1-\lambda\theta_i}{(2\delta-1)(2+\lambda)} d_i$.

Proof. At the decision threshold the utility the player gets if she chooses action 1 should equal her utility from choosing action 0. It means that $(\delta\tau_i + (1-\delta)(d_i - \tau_i))(1 + \lambda\theta_i) = (\delta(d_i - \tau_i) + (1-\delta)\tau_i)(1 + \lambda(1-\theta_i))$, which gives the threshold $\widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) = \frac{\delta(2+\lambda)-1-\lambda\theta_i}{(2\delta-1)(2+\lambda)} d_i$.

It is straightforward to verify that in the region R_M action 0 gives higher utility whenever $\tau_i < \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i)$, while action 1 is preferred whenever $\tau_i > \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i)$. If the player is indifferent (which happens when $\tau_i = \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i)$), then according to the tie-breaking rule she chooses her preferred action. □

Note that for a given game $(\delta, \lambda) \in R_M$ and degree d_i we always have $\widetilde{\tau}^0_{\delta,\lambda}(d_i) > \widetilde{\tau}^1_{\delta,\lambda}(d_i)$, that is, a player with preference 0 needs more neighbors choosing action 1 in order to switch her choice from 0 to 1 than a player with preference 1 does. It goes along with the intuition: a player is more reluctant to switch from her preferred action to an unfavoured one than vice versa.

Let us now make an important observation that allows to specify further the threshold values. Since τ_i can only be a non-negative integer, if we substitute $\widetilde{\tau}^1_{\delta,\lambda}(d_i)$ by $\lceil \widetilde{\tau}^1_{\delta,\lambda}(d_i) \rceil$ and $\widetilde{\tau}^0_{\delta,\lambda}(d_i)$ by $\lfloor \widetilde{\tau}^0_{\delta,\lambda}(d_i) \rfloor$ in the above best response function, Lemma 4 still holds.[26] It implies that we could focus solely on integer thresholds. Let us denote $\lceil \widetilde{\tau}^1_{\delta,\lambda}(d_i) \rceil$ by $\tau^1_{\delta,\lambda}(d_i)$ and $\lfloor \widetilde{\tau}^0_{\delta,\lambda}(d_i) \rfloor$ by $\tau^0_{\delta,\lambda}(d_i)$. To complete the proof of the proposition we are left to verify the following:

(i) $\tau^1_{\delta,\lambda}(d_i) = l$ if and only if $(\delta, \lambda) \in R^l_M(d_i)$ for $l = 1, ..., L$, and

[26] Here $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor and ceiling of x respectively.

(ii) $\tau^0_{\delta,\lambda}(d_i) = d_i - \tau^1_{\delta,\lambda}(d_i)$.

Let us prove the first equivalence.

Note that for every $l = 1, ..., L$ the condition $\tau^1_{\delta,\lambda}(d_i) = l$ is equivalent to $l-1 < \widetilde{\tau}^1_{\delta,\lambda}(d_i) \leq l$, which in its turn can be rewritten as a conjunction of two inequalities:

$$\begin{cases} (\delta(2+\lambda) - 1 - \lambda)\, d_i > (l-1)(2\delta - 1)(2+\lambda) \\ (\delta(2+\lambda) - 1 - \lambda)\, d_i \leq l(2\delta - 1)(2+\lambda), \end{cases}$$

or equivalently,

$$\begin{cases} \delta(d_i - 2l + 2)(2+\lambda) > (d_i - l + 1)(1+\lambda) - l + 1 & (1) \\ \delta(d_i - 2l)(2+\lambda) \leq (d_i - l)(1+\lambda) - l. & (2) \end{cases}$$

Consider any $l = 1, ..., L-1$. Since $l \leq L - 1 = \lceil \frac{d_i}{2} \rceil - 1 < \frac{d_i}{2}$, the above system of inequalities can be rewritten as

$$\begin{cases} \delta > \dfrac{(d_i - (l-1))(1+\lambda) - (l-1)}{(d_i - 2(l-1))(2+\lambda)} & (3) \\ \delta \leq \dfrac{(d_i - l)(1+\lambda) - l}{(d_i - 2l)(2+\lambda)}, & (4) \end{cases}$$

which, provided that $(\delta, \lambda) \in R_M$, is precisely the condition $(\delta, \lambda) \in R^l_M(d_i)$. Thus, we have proved (i) for $l = 1, ..., L-1$.

Now consider $l = L$. Since $\frac{d_i}{2} \leq L < \frac{d_i}{2} + 1$, inequality (1) can be rewritten as (3) with $l = L$. Further, let us consider two separate cases: when d_i is even and when it is odd. If it is even, then $L = \frac{d_i}{2}$ and (2) holds trivially. Therefore, $\tau^1_{\delta,\lambda}(d_i) = L$ is equivalent to condition (3) with $l = L$ and, provided that $(\delta, \lambda) \in R_M$, to $(\delta, \lambda) \in R^L_M(d_i)$. If d_i is odd, then $\frac{d_i}{2} < L < \frac{d_i}{2} + 1$ and inequality (2) can be rewritten as

$$\delta \geq \frac{(d_i - L)(1+\lambda) - L}{(d_i - 2L)(2+\lambda)}. \quad (5)$$

Now $\tau^1_{\delta,\lambda}(d_i) = L$ is equivalent to the conjunction of (5) and (3) with $l = L$. It is straightforward to show that the right-hand-side of the former is strictly less than the right-hand-side of the latter, and thus (5) is redundant. Again, provided that $(\delta, \lambda) \in R_M$, we get that $\tau^1_{\delta,\lambda}(d_i) = L$ is equivalent to $(\delta, \lambda) \in R^L_M(d_i)$, which completes the proof of (i).

Finally, (ii) follows trivially: $\tau^0_{\delta,\lambda}(d_i) = d_i - l = d_i - \tau^1_{\delta,\lambda}(d_i)$ for every $l = 1, ..., L$.

Proof of Corollary 1

Fix $(\delta, \lambda) \in R_M$ and an arbitrary $i \in N$. We will prove the corollary by contraposition. That is, we will prove the following: $\nexists j \in N_i(G)$ s.t. $x_j = \theta_i \Rightarrow\ BR_i(\theta_i, d_i, \tau_i) = 1 - \theta_i$.

If $\nexists j \in N_i(G)$ s.t. $x_j = \theta_i$, then $x_j = 1 - \theta_i\ \forall j \in N_i(G)$. Let $\theta_i = 0$. Then, according to Proposition 1, $\tau^0_{\delta,\lambda}(d_i) = d_i - l$ where $l \in \{1, ..., \lceil \frac{d_i}{2} \rceil\}$, and thus $BR_i(0, d_i, d_i) = 1$. Let $\theta_i = 1$. In this case $\tau^1_{\delta,\lambda}(d_i) = l$ where $l \in \{1, ..., \lceil \frac{d_i}{2} \rceil\}$, and thus $BR_i(1, d_i, 0) = 0$. In either case, $BR_i(\theta_i, d_i, \tau_i) = 1 - \theta_i$, what was to be shown.

Proof of Corollary 2

Fix a degree d_i of a player and let $(\delta, \lambda) \in R_M^l(d_i)$. For the sake of conciseness, let us denote $\frac{(d_i-l)(1+\lambda)-l}{(d_i-2l)(2+\lambda)}$ by $\delta^l(d_i)$ for $l = 0, ..., L-1$ and set $\delta^L(d_i) = 1$.

We first consider $l = 1, ..., L-1$. In this case, by the definition of the partition of R_M, $(\delta, \lambda) \in R_M^l(d_i)$ is equivalent to $\delta^{l-1}(d_i) < \delta \leq \delta^l(d_i)$.

First, we need to prove that it implies $\delta^{l-1}(d_i+1) < \delta \leq \delta^{l+1}(d_i+1)$. Since $\delta^l(d_i)$ is decreasing in d_i for $l = 1, ..., L-1$, the first inequality is straightforward: $\delta^{l-1}(d_i+1) < \delta^{l-1}(d_i) < \delta$. The second inequality follows from the fact that the difference $\delta^{l+1}(d_i+1) - \delta^l(d_i)$ is positive for $l = 1, ..., L(d_i) - 1$.

Next, we need to prove that $\delta^{l-1}(d_i) < \delta \leq \delta^l(d_i)$ implies $\delta^{l-2}(d_i-1) < \delta \leq \delta^l(d_i-1)$. Again, since $\delta^l(d_i)$ is decreasing in d_i, it follows that $\delta^l(d_i) < \delta^l(d_i-1)$, and thus $\delta \leq \delta^l(d_i-1)$ holds true. The remaining inequality follows from the fact that the difference $\delta^{l-1}(d_i) - \delta^{l-2}(d_i-1)$ is positive.

Now we consider the second case, $l = L$. We need to prove that $\delta^{L-1}(d_i) < \delta \leq 1$ implies $\delta^{L-1}(d_i+1) < \delta \leq 1$ and $\delta^{L-2}(d_i-1) < \delta \leq 1$. Similarly to the above, the first implication is a consequence of the fact that $\delta^l(d_i)$ is decreasing in d_i for $l = 1, ..., L-1$, and the second implication follows from positivity of $\delta^{L-1}(d_i) - \delta^{L-2}(d_i-1)$.

Proof of Proposition 2

Similar to the proof of Proposition 1, we will first prove an auxiliary lemma characterizing the best responses of players in a general form.

Lemma 5. *In a game $(\delta, \lambda) \in R_{MM}$ the best response function of a player i with preference θ_i and d_i neighbors, τ_i of whom play 1, is*

$$BR_i(\theta_i, d_i, \tau_i) = \begin{cases} 1, \text{ if } \tau_i < \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) \\ 0, \text{ if } \tau_i > \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) \\ \theta_i, \text{ if } \tau_i = \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) \end{cases}$$

where $\widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i) = \frac{\delta(2+\lambda)-1-\lambda\theta_i}{(2\delta-1)(2+\lambda)} d_i$.

Proof. See the proof of Lemma 4 for derivation of the decision threshold $\widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i)$. It can be verified that in R_{MM} action 1 gives higher utility whenever $\tau_i < \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i)$ and action 0 does so whenever $\tau_i > \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i)$. The tie-breaking rule manages the remaining case of $\tau_i = \widetilde{\tau}^{\theta_i}_{\delta,\lambda}(d_i)$, in which the preferred action θ_i is chosen. □

Compared to Lemma 4, the relationship between the thresholds changes: now $\widetilde{\tau}^0_{\delta,\lambda}(d_i) < \widetilde{\tau}^1_{\delta,\lambda}(d_i)$, that is a player with preference 0 needs fewer neighbors choosing action 1 in order to switch her choice from 1 to 0, since now she is switching to her preferred action. If we substitute $\widetilde{\tau}^1_{\delta,\lambda}(d_i)$ by $\lfloor \widetilde{\tau}^1_{\delta,\lambda}(d_i) \rfloor$ and $\widetilde{\tau}^0_{\delta,\lambda}(d_i)$ by $\lceil \widetilde{\tau}^0_{\delta,\lambda}(d_i) \rceil$ in the above best response function, Lemma 5 still holds, so we could focus solely on integer thresholds.

The rest of the proof of Proposition 2 is analogous to the above proof of Proposition 1.

Proof of Corollary 3

The proof uses contraposition, analogously to the proof of Corollary 1. Fix $(\delta, \lambda) \in R_{MM}$ and an arbitrary $i \in N$. If $\nexists j \in N_i(G)$ s.t. $x_j = 1 - \theta_i$, then $x_j = \theta_i \ \forall j \in N_i(G)$. According to Proposition 2, $\tau^0_{\delta,\lambda}(d_i) = l$ and $\tau^1_{\delta,\lambda}(d_i) = d_i - l$ where $l \in \{1, ..., \lceil \frac{d_i}{2} \rceil\}$. Then $BR_i(0, d_i, 0) = 1$ and $BR_i(1, d_i, d_i) = 0$, that is, in either case $BR_i(\theta_i, d_i, \tau_i) = 1 - \theta_i$.

Proof of Lemma 1

(i) Let us fix a player's degree d_i. As it follows from the definition of the partition of R_M (see subsection 3.1), the curve separating subregions $R^m_M(d_i)$ and $R^{m+1}_M(d_i)$ is $\delta = \frac{(d_i - m)(1+\lambda) - m}{(d_i - 2m)(2+\lambda)}$, or equivalently, $\lambda = \frac{(2\delta - 1)(d_i - 2m)}{(1-\delta)(d_i - 2m) + m}$ on the domain $\delta \in [\frac{1}{2}; 1]$, which is exactly the curve $\tilde{\lambda}(m)$ for $\delta \geq \frac{1}{2}$. Then $l^* = \underset{m=1...L}{\operatorname{argmin}} |\lambda - \tilde{\lambda}(m)|$ corresponds to the separating curve that lies closest to a given λ (and separates $R^{l^*}_M(d_i)$ and $R^{l^*+1}_M(d_i)$). It can be easily seen from Figure 2 that if λ lies above this curve, it belongs to subregion $R^{l^*}_M(d_i)$, and if it lies below this curve, it belongs to the next subregion $R^{l^*+1}_M(d_i)$. Finally, let us note that $\tilde{\lambda}(L) \leq 0$, that is why λ can never lie below the curve $\tilde{\lambda}(L)$, and thus $1 \leq l_{\delta,\lambda}(d_i) \leq L$.

The proof for R_{MM} is analogous. For a given degree d_i, the curve separating subregions $R^m_{MM}(d_i)$ and $R^{m+1}_{MM}(d_i)$ is $\delta = \frac{d_i - (2+\lambda)m}{(d_i - 2m)(2+\lambda)}$, or equivalently, $\lambda = \frac{(1-2\delta)(d_i - 2m)}{\delta(d_i - 2m) + m}$ on the domain $\delta \in [0; \frac{1}{2}]$, which is the curve $\tilde{\lambda}(m)$ for $\delta \leq \frac{1}{2}$. Then l^* corresponds to the separating curve that lies closest to a given λ. If λ is above this curve, it belongs to subregion $R^{l^*}_{MM}(d_i)$, and if it is below – to subregion $R^{l^*+1}_{MM}(d_i)$. And again, $\tilde{\lambda}(L) \leq 0$, thus λ can never lie below the curve $\tilde{\lambda}(L)$, implying $1 \leq l_{\delta,\lambda}(d_i) \leq L$.

(ii) The proof follows directly from Proposition 3.

Proof of Theorem 1

(i) Take an arbitrary (connected) network G with a preference profile $\bar{\theta}$ and consider an action profile $\bar{x} = (x_1, ..., x_n)$. If $\bar{x}$ is symmetric, then for every player i all her neighbors choose the same action: $\forall i \in N \ \forall j \in N_i(G) \ x_j = x^*$ with some $x^* \in \{0, 1\}$. If $x^* = 0$ then $\tau_i = 0$, and according to Proposition 1 player i's best response is also 0 (since for any $(\delta, \lambda) \in R_M$ the threshold $\tau^{\theta_i}_{\delta,\lambda}(d_i) \geq 1$). If $x^* = 1$ then $\tau_i = d_i$, and according to Proposition 1 player i's best response is 1 (since the threshold $\tau^{\theta_i}_{\delta,\lambda}(d_i) \leq d_i - 1$). As the above is true for all $i \in N$, $\bar{x}$ is an equilibrium.

(ii) Necessity. Fix $(\delta, \lambda) \in R_M$, a network $(G, \bar{\theta})$ and let $\bar{x} = (x_1, ..., x_n)$ be an asymmetric equilibrium. For $\theta = 0, 1$ set $S^\theta := \{i \in N \mid x_i = \theta\}$. Since $\bar{x}$ is asymmetric, both S^0 and S^1 are nonempty and thus form a partition of N. We are left to prove that this partition satisfies the conditions of the theorem:

$\forall i \in S^\theta \cap N^\theta : \ |N_i(G) \cap S^\theta| \geq l_{\delta,\lambda}(d_i)$ and

$\forall i \in S^\theta \cap N^{1-\theta}: \ |N_i(G) \cap S^\theta| > d_i - l_{\delta,\lambda}(d_i)$.

First, take a player $i \in S^1 \cap N^\theta$. That is, $x_i = 1$ and $\theta_i = \theta$ with some $\theta \in \{0,1\}$. According to Proposition 1, $BR_i(\theta, d_i, \tau_i) = 1$ iff $\tau_i \geq \tau^\theta_{\delta,\lambda}(d_i)$ (with strict inequality if $\theta = 0$). The same proposition implies that $\tau^1_{\delta,\lambda}(d_i) = l_{\delta,\lambda}(d_i)$ and $\tau^0_{\delta,\lambda}(d_i) = d_i - l_{\delta,\lambda}(d_i)$. Since $|N_i(G) \cap S^1| = \tau_i$, it follows that $|N_i(G) \cap S^1| \geq l_{\delta,\lambda}(d_i)$ for $\theta = 1$ and $|N_i(G) \cap S^1| > d_i - l_{\delta,\lambda}(d_i)$ for $\theta = 0$.

Second, take a player $i \in S^0 \cap N^\theta$. That is, $x_i = 0$ and $\theta_i = \theta$ with some $\theta \in \{0,1\}$. Similarly, Proposition 1 implies that $BR_i(\theta, d_i, \tau_i) = 0$ iff $\tau_i \leq \tau^\theta_{\delta,\lambda}(d_i)$ (with strict inequality if $\theta = 1$), which is equivalent to $d_i - \tau_i \geq d_i - \tau^\theta_{\delta,\lambda}(d_i)$ (strict if $\theta = 1$). Recall that $\tau^1_{\delta,\lambda}(d_i) = l_{\delta,\lambda}(d_i)$ and $\tau^0_{\delta,\lambda}(d_i) = d_i - l_{\delta,\lambda}(d_i)$. Since $|N_i(G) \cap S^0| = d_i - \tau_i$, it follows that $|N_i(G) \cap S^0| \geq l_{\delta,\lambda}(d_i)$ for $\theta = 0$ and $|N_i(G) \cap S^0| > d_i - l_{\delta,\lambda}(d_i)$ for $\theta = 1$, which completes the proof.

Sufficiency. Fix $(\delta, \lambda) \in R_M$ and a network $(G, \bar{\theta})$. Assume that there exists a partition $\{S^0, S^1\}$ of N satisfying the conditions of the theorem and let us prove that an asymmetric equilibrium exists. Consider an action profile $\bar{x} = (x_1, ..., x_n)$ such that $x_i = 0$ for $i \in S^0$ and $x_i = 1$ for $i \in S^1$. Since $\{S^0, S^1\}$ is a partition of N, both S^0 and S^1 are nonempty, and thus $\bar{x}$ is an asymmetric action profile. We are left to prove that it is an equilibrium.

Take a player $i \in S^0$. There are two possibilities: either $i \in N^0$ or $i \in N^1$. If $i \in S^0 \cap N^0$ then it must be that $|N_i(G) \cap S^0| \geq l_{\delta,\lambda}(d_i)$, which is equivalent to $d_i - \tau_i \geq d_i - \tau^0_{\delta,\lambda}(d_i)$ (since $\tau^0_{\delta,\lambda}(d_i) = d_i - l_{\delta,\lambda}(d_i)$, according to Proposition 1). Then $\tau_i \leq \tau^0_{\delta,\lambda}(d_i)$ and, again according to Proposition 1, $BR_i(0, d_i, \tau_i) = 0$. That is, the player i has no incentive to deviate from $x_i = 0$. Alternatively, if $i \in S^0 \cap N^1$ then it must be that $|N_i(G) \cap S^0| > d_i - l_{\delta,\lambda}(d_i)$, which is equivalent to $d_i - \tau_i > d_i - \tau^1_{\delta,\lambda}(d_i)$ (recall, $\tau^1_{\delta,\lambda}(d_i) = l_{\delta,\lambda}(d_i)$), and thus $\tau_i < \tau^1_{\delta,\lambda}(d_i)$. Proposition 1 implies in this case that $BR_i(1, d_i, \tau_i) = 0$. And again, the player i has no incentive to deviate from $x_i = 0$.

Now take a player $i \in S^1$. Either $i \in N^0$ or $i \in N^1$ must be true. If $i \in S^1 \cap N^1$ then it must be that $|N_i(G) \cap S^1| \geq l_{\delta,\lambda}(d_i)$, which is equivalent to $\tau_i \geq \tau^1_{\delta,\lambda}(d_i)$. Proposition 1 implies that $BR_i(1, d_i, \tau_i) = 1$, and thus i has no incentive to deviate from $x_i = 1$. If $i \in S^1 \cap N^0$ then it must be that $|N_i(G) \cap S^1| > d_i - l_{\delta,\lambda}(d_i)$, which is equivalent to $\tau_i > \tau^0_{\delta,\lambda}(d_i)$. According to Proposition 1, $BR_i(0, d_i, \tau_i) = 1$, implying that in this case as well i has no incentive to deviate from $x_i = 1$.

Since for all players their actions in $\bar{x}$ are the best responses, $\bar{x}$ is an asymmetric equilibrium.

Proof of Corollary 5

We will prove that the necessary and sufficient conditions for existence of an asymmetric equilibrium of this corollary are equivalent to the conditions of part (ii) of Theorem 1. Take a network $(G, \bar{\theta})$ with a degree partition $(D_1, ..., D_M)$, a game $(\delta, \lambda) \in R_M$ and a partition $\{S_0, S_1\}$ of N. Fix some $\theta \in \{0,1\}$. We will prove that the (possibly trivial)

partition $\{S^\theta \cap N^\theta \cap D_1, ..., S^\theta \cap N^\theta \cap D_M, S^\theta \cap N^{1-\theta} \cap D_1, ..., S^\theta \cap N^{1-\theta} \cap D_M\}$ of S^θ is $\left(\frac{l_{\delta,\lambda}(d_{(1)})}{d_{(1)}}, ..., \frac{l_{\delta,\lambda}(d_{(M)})}{d_{(M)}}, 1 - \frac{l_{\delta,\lambda}(d_{(1)})-1}{d_{(1)}}, ..., 1 - \frac{l_{\delta,\lambda}(d_{(M)})-1}{d_{(M)}}\right)$-cohesive if and only if the two conditions are satisfied:

$$\begin{array}{l} \forall i \in S^\theta \cap N^\theta : \ |N_i(G) \cap S^\theta| \geq l_{\delta,\lambda}(d_i), \\ \forall i \in S^\theta \cap N^{1-\theta} : \ |N_i(G) \cap S^\theta| > d_i - l_{\delta,\lambda}(d_i). \end{array} \tag{6}$$

First, let the above-mentioned partition be $(r_1, ..., r_{2M})$-cohesive, where $r_m = \frac{l_{\delta,\lambda}(d_{(m)})}{d_{(m)}}$ for $m = 1, ..., M$, and $r_m = 1 - \frac{l_{\delta,\lambda}(d_{(m-M)})-1}{d_{(m-M)}}$ for $m = M+1, ..., 2M$. According to Definition 3, it implies that for $m = 1, ..., M$:

$$\begin{array}{l} \forall i \in S^\theta \cap N^\theta \cap D_m \colon \frac{|N_i(G) \cap S^\theta|}{|N_i(G)|} \geq \frac{l_{\delta,\lambda}(d_{(m)})}{d_{(m)}}, \\ \forall i \in S^\theta \cap N^{1-\theta} \cap D_m \colon \frac{|N_i(G) \cap S^\theta|}{|N_i(G)|} \geq 1 - \frac{l_{\delta,\lambda}(d_{(m)})-1}{d_{(m)}}. \end{array} \tag{7}$$

Since $i \in D_m$, we can substitute $d_{(m)}$ by d_i in the above conditions. Further, we group all the conditions for $m = 1, ..., M$ together, since now they do not depend on m. Finally, noting that $|N_i(G)| = d_i$ by definition, we multiply all the conditions by d_i and get the following:

$$\begin{array}{l} \forall i \in S^\theta \cap N^\theta \colon |N_i(G) \cap S^\theta| \geq l_{\delta,\lambda}(d_i), \\ \forall i \in S^\theta \cap N^{1-\theta} \colon |N_i(G) \cap S^\theta| \geq d_i - l_{\delta,\lambda}(d_i) + 1. \end{array} \tag{8}$$

To see that the resulting conditions (8) are exactly the conditions (6), let us remark that both $|N_i(G) \cap S^\theta|$ and $d_i - l_{\delta,\lambda}(d_i)$ are integer expressions. This completes the first part of the proof.

Second, let conditions (6) be satisfied. We will prove that the partition $\{S^\theta \cap N^\theta \cap D_1, ..., S^\theta \cap N^\theta \cap D_M, S^\theta \cap N^{1-\theta} \cap D_1, ..., S^\theta \cap N^{1-\theta} \cap D_M\}$ of S^θ is $\left(\frac{l_{\delta,\lambda}(d_{(1)})}{d_{(1)}}, ..., \frac{l_{\delta,\lambda}(d_{(M)})}{d_{(M)}}, 1 - \frac{l_{\delta,\lambda}(d_{(1)})-1}{d_{(1)}}, ..., 1 - \frac{l_{\delta,\lambda}(d_{(M)})-1}{d_{(M)}}\right)$-cohesive. It suffices to prove that conditions (7) hold for $m = 1, ..., M$.

As it is mentioned in the first part of the proof, conditions (6) are equivalent to conditions (8). Let us divide them by d_i and then group the players according to their degrees $(D_1, ..., D_M)$, which allows to substitute d_i by the corresponding $d_{(m)}$ in each degree group $m = 1, ..., M$. We get conditions (7), what was to be shown.

Proof of Theorem 2

Necessity. Fix $(\delta, \lambda) \in R_M$. Suppose that the fully satisfying action profile $\bar{x} = (\theta_1, ..., \theta_n)$ is an equilibrium, but for some player i the condition on her neighbors does not hold: $|N_i(G) \cap N^{\theta_i}| = |\{j \in N_i(G) : \theta_j = \theta_i\}| < l_{\delta,\lambda}(d_i)$. Since $x_j = \theta_j \ \forall j \in N$, it implies $|\{j \in N_i(G) : x_j = \theta_i\}| < l_{\delta,\lambda}(d_i)$.

If $\theta_i = 0$, the last inequality is equivalent to $d_i - \tau_i < l_{\delta,\lambda}(d_i)$, or $\tau_i > d_i - l_{\delta,\lambda}(d_i) = \tau^0_{\delta,\lambda}(d_i)$ (see Proposition 1 for the last equality), and thus $BR_i(0, d_i, \tau_i) = 1 \neq \theta_i$ (again, from Proposition 1). If $\theta_i = 1$ then $\tau_i < l_{\delta,\lambda}(d_i) = \tau^1_{\delta,\lambda}(d_i)$, implying $BR_i(1, d_i, \tau_i) = 0 \neq \theta_i$. In either case, the player i has an incentive to deviate from her preferred action. Hence, $\bar{x} = (\theta_1, ..., \theta_n)$ is not an equilibrium.

Sufficiency. Fix $(\delta, \lambda) \in R_M$ and suppose that the condition on neighbors' preferences holds: $|N_i(G) \cap N^{\theta_i}| = |\{j \in N_i(G) : \theta_j = \theta_i\}| \geq l_{\delta,\lambda}(d_i)\ \forall i \in N$. Let us check if the fully satisfying action profile is an equilibrium. Since $x_j = \theta_j\ \forall j \in N$, the above condition implies $|\{j \in N_i(G) : x_j = \theta_i\}| \geq l_{\delta,\lambda}(d_i)\ \forall i \in N$.

Take an arbitrary $i \in N$. If $\theta_i = 0$, the above becomes $d_i - \tau_i \geq l_{\delta,\lambda}(d_i)$, or $\tau_i \leq d_i - l_{\delta,\lambda}(d_i) = \tau^0_{\delta,\lambda}(d_i)$, and Proposition 1 implies $BR_i(0, d_i, \tau_i) = 0$. If $\theta_i = 1$ then $\tau_i \geq l_{\delta,\lambda}(d_i) = \tau^1_{\delta,\lambda}(d_i)$, and thus $BR_i(1, d_i, \tau_i) = 1$. In either case, $BR_i(\theta_i, d_i, \tau_i) = \theta_i$. Since it holds for any $i \in N$, the fully satisfying action profile is indeed an equilibrium.

Proof of Theorem 3

(i) For an arbitrary network G with a preference profile $\bar{\theta}$ consider a symmetric action profile $\bar{x}$. Fix a player i. Since the action profile is symmetric, all i's neighbors choose the same action: $\forall j \in N_i(G)\ x_j = x^*$ with some $x^* \in \{0, 1\}$. If $x^* = 0$ then $\tau_i = 0$, and according to Proposition 2 player i's best response is 1 (for any $(\delta, \lambda) \in R_{MM}$ the threshold $\tau^{\theta_i}_{\delta,\lambda}(d_i) \geq 1$). If $x^* = 1$ then $\tau_i = d_i$, and according to Proposition 2 player i's best response is 0 (the threshold $\tau^{\theta_i}_{\delta,\lambda}(d_i) \leq d_i - 1$). Since i has an incentive to deviate from x^*, $\bar{x} = (x^*, ..., x^*)$ cannot be an equilibrium action profile.

(ii) The proof builds directly on Proposition 2 and is analogous to the proof of part (ii) of Theorem 1.

Proof of Theorem 4

The proof is analogous to the proof of Theorem 2 and uses the results of Proposition 2.

Proof of Proposition 4

Without loss of generality, let player 1 be the central one in a star. Then $d_1 \geq 2$ and $d_i = 1$ for $i = 2, ..., n$. Let $\bar{x}$ be an equilibrium and consider an arbitrary $i \in \{2, ..., n\}$.

(i) In a game with strong advantage of matching, if $x_1 = \theta_i$ then $x_i = \theta_i$, and if $x_1 = 1 - \theta_i$ then $x_i = 1 - \theta_i$ (see Corollary 1). That is, i matches the action of the central player regardless of whether this action is her preferred one. Since it is true for all $i \in \{2, ..., n\}$, $\bar{x}$ is symmetric.

(ii) In a game with strong advantage of mismatching, if $x_1 = 1 - \theta_i$ then $x_i = \theta_i$, and if $x_1 = \theta_i$ then $x_i = 1 - \theta_i$ (see Corollary 3). That is, i always mismatches the action of the central player. Since it is true for all $i \in \{2, ..., n\}$, there are two possible equilibria: $\bar{x} = (0, 1, ..., 1)$ and $\bar{x} = (1, 0, ..., 0)$.

Proof of Proposition 5

(i) Fix some $(\delta, \lambda) \in R_M$.

Necessity. Let $\bar{x}$ be an asymmetric equilibrium. Then, since players are arranged in a circle, there must be four players such that $x_i \neq x_{i+1} = x_j \neq x_{j+1}$. Note that all players must be distinct (i.e. $i+1 \neq j$), otherwise $x_{j-1} = x_{j+1} \neq x_j$, which contradicts the best response behavior for player j. Moreover, it must be that $x_{k-1} \neq x_{k+1}$ $\forall k \in \{i, i+1, j, j+1\}$, otherwise it would contradict the best response behavior for player k. Due to the tie-breaking rule, $x_k = \theta_k$, and hence $\theta_i \neq \theta_{i+1} = \theta_j \neq \theta_{j+1}$.

Sufficiency. Let such players $i, i+1, j$ and $j+1$ exist: without loss of generality, $\theta_i = \theta_{j+1} = 0$ and $\theta_{i+1} = \theta_j = 1$. Consider an action profile that assigns the following actions to players: $x_1 = ... = x_i = x_{j+1} = ... = x_n = 0$ and $x_{i+1} = ... = x_j = 1$. Such an action profile is an asymmetric equilibrium.

Uniqueness. If another 4-tuple of players $\{l, l+1, m, m+1\}$ satisfies $\theta_l \neq \theta_{l+1} = \theta_m \neq \theta_{m+1}$, then the action profile $\bar{x}'$ in which $x'_1 = ... = x'_l = x'_{m+1} = ... = x'_n = \theta_l$ and $x'_{l+1} = ... = x'_m = \theta_{l+1}$ is an asymmetric equilibrium that is different from the original one (see Figure 7 for an illustration). Thus, uniqueness of such a 4-tuple of players is necessary for uniqueness of an asymmetric equilibrium.

Now let us prove the opposite: if another asymmetric equilibrium exists, then there must be another 4-tuple of players satisfying the conditions of the proposition. For a circle network, in any asymmetric equilibrium there must be four players such that $x_i \neq x_{i+1} = x_j \neq x_{j+1}$. If $\bar{x}'$ is an asymmetric equilibrium different from $\bar{x}$, there must be a different 4-tuple of players: $x'_l \neq x'_{l+1} = x'_m \neq x'_{m+1}$. As it follows from the necessity part of the proof, $x_k = \theta_k$ $\forall k \in \{i, i+1, j, j+1\}$ and $x'_k = \theta_k$ $\forall k \in \{l, l+1, m, m+1\}$. Hence, two different asymmetric equilibria imply two different 4-tuples of players satisfying the conditions of the proposition.

(ii) Fix some $(\delta, \lambda) \in R_{MM}$.

Existence. Let n be even. Then an action profile with alternating actions, $x_k = 0$ for even k and $x_k = 1$ for odd k, is always an equilibrium.

Let n be odd. Then there must exist two neighbors, denote them i and $i+1$, with the same preference. Without loss of generality, let i be odd and let $\theta_i = \theta_{i+1} = 0$. Consider the following action profile: $x_k = 0$ for odd $k \leq i$ and for even $k \geq i+1$ and $x_k = 1$ for all other players. This is an action profile with alternating actions, except for those of players i and $i+1$. It is easy to see that such a profile is a Nash equilibrium.

Uniqueness. For even n there always exist at least two asymmetric equilibria (those with alternating actions), regardless of the preference profile.

Let n be odd. Then $\exists i \in N$ s.t. $\theta_i = \theta_{i+1}$. Without loss of generality, let i be odd. The action profile $\bar{x}$ in which $x_k = \theta_i$ for odd $k \leq i$ and even $k \geq i+1$ and $x_k = 1 - \theta_i$ for all other players is an equilibrium (see the proof of the existence part). Assume

that $\exists j \in N \setminus \{i\}$ s.t. $\theta_j = \theta_{j+1}$ and, without loss of generality, let j also be odd. Then the action profile $\bar{x}'$ in which $x'_k = \theta_j$ for odd $k \leq j$ and even $k \geq j+1$ and $x'_k = 1 - \theta_j$ for all other players is also an asymmetric equilibrium, and it is different from $\bar{x}$. Thus, uniqueness of the pair of neighbors with the same preference is necessary for equilibrium uniqueness.

Finally, we need to prove that multiplicity of equilibria implies multiplicity of such pairs of same-preference neighbors. Let $\bar{x}$ be an equilibrium. Since n is odd, $\exists i \in N$ s.t. $x_i = x_{i+1}$. Note that $x_{i-1} \neq x_{i+1}$ and $x_i \neq x_{i+2}$, otherwise it would contradict the best response behavior for players i and $i+1$ respectively. Due to the tie-breaking rule, $x_i = \theta_i$ and $x_{i+1} = \theta_{i+1}$. Hence, $\theta_i = \theta_{i+1}$. If $\bar{x}'$ is a different equilibrium, it must be that $\exists j \in N \setminus \{i\}$ s.t. $x'_j = x'_{j+1}$. Thus, $\exists j \in N \setminus \{i\}$ s.t. $\theta_j = \theta_{j+1}$, what has to be shown.

Proof of Proposition 6

(i) Fix a game $(\delta, \lambda) \in R_M$. Every player has $n-1$ neighbors and needs $l_{\delta,\lambda}(n-1)$ companions to choose her preferred action (see Lemma 1). For brevity, we will drop the subscript and the argument of $l_{\delta,\lambda}(n-1)$ until the end of this proof.

Necessity. Let $\bar{x}$ be an asymmetric equilibrium and $S^\theta = \{i \in N \mid x_i = \theta\}$ for $\theta = 0, 1$. Obviously, both subsets are nonempty and $S^0 \cup S^1 = N$. We need to prove that $|N^\theta| \geq l+1$ for $\theta = 0, 1$.

Without loss of generality, suppose that $|S^0| \leq l$. Then $|\{j \in N_i(G) : x_j = 0\}| \leq l-1$ $\forall i \in S^0$. If $\theta_i = 0$ for some $i \in S^0$, it would contradict i's best response to her neighbors' actions, hence $\theta_i = 1$ $\forall i \in S^0$. Then $|S^1| \leq l-1$ (otherwise players in S^0 would switch to action 1). It implies that the total number of players $n \leq 2l-1$. However, it contradicts the fact that $l \leq \lceil \frac{n-1}{2} \rceil \leq \frac{n}{2}$. Thus, by contradiction we proved that $|S^\theta| \geq l+1$ for $\theta = 0, 1$.

Consequently, for all $i \in N$ both $|\{j \in N_i(G) : x_j = 0\}| \geq l$ and $|\{j \in N_i(G) : x_j = 1\}| \geq l$, implying $x_i = \theta_i$. Together with the previous conclusion about cardinality of S^0 and S^1, it implies in its turn that $|N^\theta| \geq l+1$ for $\theta = 0, 1$. Obviously, it also proves uniqueness of an asymmetric equilibrium.

Sufficiency. Let $|N^\theta| \geq l+1$ for $\theta = 0, 1$ and consider a fully satisfying action profile $\bar{x}$, in which $x_i = \theta_i$ $\forall i \in N$. Obviously, this action profile is asymmetric and it is an equilibrium, since the companion requirement is satisfied.

(ii) Consider a game $(\delta, \lambda) \in R_{MM}$. Every player has $n-1$ neighbors and needs $l_{\delta,\lambda}(n-1)$ opponents to choose her preferred action (Lemma 1). Again, for brevity we drop the subscript and the argument of $l_{\delta,\lambda}(n-1)$ in what follows. We also denote $S^\theta = \{i \in N \mid x_i = \theta\}$ for $\theta = 0, 1$.

Since $l \leq \lceil \frac{n-1}{2} \rceil \leq \frac{n}{2}$, it must be that either $n = 2l$ or $n > 2l$. Let us consider the first case, $n = 2l$. Let $\bar{x}$ be such an action profile that $|S^0| = |S^1| = l$. In this profile, if

$x_i = \theta_i$ for a player i then she has l opponents, otherwise she has $l-1$ opponents. Since no player has an incentive to deviate, $\bar{x}$ is an (asymmetric) equilibrium. Obviously, regardless of the preference profile, many asymmetric equilibria are possible in this case.

Let us consider now the second case, $n > 2l$, which we split into two subcases: $|N^\theta| < l$ for some $\theta \in \{0,1\}$ and $|N^\theta| \geq l \ \forall \theta \in \{0,1\}$.

Consider the first subcase. Without loss of generality, let $|N^0| < l$, and hence $|N^1| > l$. Let $\bar{x}$ be such an action profile that $N^0 \subseteq S^0$ and $|S^0| = l$. In this profile, every $i \in N^0$ has more than l opponents (since $|S^1| > l$), every $i \in N^1 \cap S^0$ has $l-1$ opponents and every $i \in N^1 \cap S^1$ has l opponents. Again, no player has an incentive to deviate, hence $\bar{x}$ is an equilibrium. Obviously, multiplicity of equilibria is also a case here, since $|N^0| < |S^0|$.

Now consider the second subcase, that is, both $|N^0| \geq l$ and $|N^1| \geq l$. Let $x_i = \theta_i$ $\forall i \in N$. Since the opponent requirement is satisfied, this fully satisfying action profile is an equilibrium. This completes the proof of existence of asymmetric equilibria in games $(\delta, \lambda) \in R_{MM}$.

Finally, let us prove that in this last subcase the fully satisfying equilibrium is a unique equilibrium. Assume the contrary: there is another equilibrium $\bar{x}$, in which $x_i \neq \theta_i$ for some $i \in N$. Without loss of generality, let $\theta_i = 0$. Then it must be that $|S^1| \leq l$ (inequality is not strict, since i is also in S^1). It follows that $|S^0| > l$, and since any $j \in N^1$ has sufficiently many opponents, $x_j = \theta_j \ \forall j \in N^1$, i.e. $N^1 \subseteq S^1$. As $|N^1| \geq l$ and $i \in S^1 \setminus N^1$, it must be that $|S^1| > l$, which contradicts $|S^1| \leq l$. Thus, $\bar{x}$ is not an equilibrium.

Proof of Proposition 7

(i) Fix $(\delta, \lambda) \in R_M$. Since every player has d neighbors, $l := l_{(\delta,\lambda)}(d)$ is the number of companions every player needs in order to follow her preference. Suppose, $\forall \theta \in \{0,1\}$ $\exists X^\theta \subseteq N^\theta$ such that $|X^\theta| \geq l+1$ and $G[X^\theta]$ is complete, i.e. $G_{ij} = 1 \ \forall i,j \in X^\theta$. Consider an action profile $\bar{x}$ in which $x_i = \theta_i \ \forall i \in X^0 \cup X^1$ and $x_i = BR_i(\theta_i, d, \tau_i)$ $\forall i \in N \setminus (X^0 \cup X^1)$. Since every $i \in X^0 \cup X^1$ has at least l companions, θ_i is her best response action. Hence, actions in $\bar{x}$ are the best responses for all players, which implies that $\bar{x}$ is an (asymmetric) equilibrium.

(ii) Fix $(\delta, \lambda) \in R_{MM}$. Denote by $l := l_{(\delta,\lambda)}(d)$ the number of opponents a player needs in order to follow her preference. Suppose, $\forall \theta \in \{0,1\}$ $\exists X^\theta \subseteq N^\theta$ such that $|X^\theta| \geq l_{(\delta,\lambda)}(d)$ and $K[X^0, X^1] \subseteq G$, i.e. $G_{ij} = 1 \ \forall i \in X^0 \ \forall j \in X^1$. Consider an action profile $\bar{x}$ in which $x_i = \theta_i \ \forall i \in X^0 \cup X^1$ and $x_i = BR_i(\theta_i, d, \tau_i)$ $\forall i \in N \setminus (X^0 \cup X^1)$. Every player $i \in X^0 \cup X^1$ has at least l opponents, thus θ_i is her best response. Since no player has an incentive to deviate from her action in $\bar{x}$, it is an (asymmetric) equilibrium.

Proof of Lemma 3

Note that the curve $\tilde{\lambda}(m)$ defines the boundary of $R_{NP}(d_{(m)})$. For a graph with degree partition $(D_1, ..., D_K)$ there are K such curves, one for each degree. Obviously, $\tilde{\lambda}(1) \leq ... \leq \tilde{\lambda}(K)$. Among these curves, $\tilde{\lambda}(k^*)$ is the one that lies closest to a given λ.

If $\delta = \frac{1}{2}$ then neither matching nor mismatching gives players interactional advantage, hence $N = N^{NP}_{\delta,\lambda}$. In the rest of the proof we let $\delta \neq \frac{1}{2}$, which implies $\tilde{\lambda}(1) < ... < \tilde{\lambda}(K)$.

If $\lambda \geq \tilde{\lambda}(k^*)$, then $\lambda \geq \tilde{\lambda}(m)$ for $m = 1, ..., k^*$, and $\lambda < \tilde{\lambda}(m)$ for $m = k^*+1, ..., K$ (strictly, since $\tilde{\lambda}(k^*)$ is the closest to λ curve). Recall that $N^{NP}_{\delta,\lambda} = \{i \in N : (\delta, \lambda) \in R_{NP}(d_i)\} = \{i \in N : \lambda \geq |(2\delta - 1) \cdot d_i|\}$. By the definition of the degree partition, $d_i = d_{(m)}$ for some $m \in \{1, ..., K\}$ if and only if $i \in D_m$. Hence, $N^{NP}_{\delta,\lambda} = D_1 \cup ... \cup D_{k^*}$. The remaining players $D_{k^*+1} \cup ... \cup D_K$ belong to $N^{M}_{\delta,\lambda}$ if $\delta > \frac{1}{2}$, or to $N^{MM}_{\delta,\lambda}$ if $\delta < \frac{1}{2}$.

If $\lambda < \tilde{\lambda}(k^*)$, then $\lambda > \tilde{\lambda}(m)$ for $m = 1, ..., k^* - 1$, and $\lambda < \tilde{\lambda}(m)$ for $m = k^*, ..., K$. Thus, $N^{NP}_{\delta,\lambda} = D_1 \cup ... \cup D_{k^*-1}$ and the remaining players belong to either $N^{M}_{\delta,\lambda}$ or $N^{MM}_{\delta,\lambda}$.

Proof of Theorem 6

(i) First, note that $N^{NP}_{\delta,\lambda} \neq \emptyset$ for games $(\delta, \lambda) \in \tilde{R}_M(d_{(1)}, d_{(K)})$ and take some $i \in N^{NP}_{\delta,\lambda}$. According to Proposition 3, in any equilibrium $x_i = \theta_i$, thus at most one symmetric equilibrium is possible.

Necessity. Suppose the contrary holds: $i, j \in N^{NP}_{\delta,\lambda}$ and $\theta_i \neq \theta_j$. In any equilibrium $x_i = \theta_i$ and $x_j = \theta_j$, hence $x_i \neq x_j$, which contradicts the definition of a symmetric equilibrium.

Sufficiency. Suppose the condition of the theorem holds. Without loss of generality, let $\theta_i = 0 \ \forall i \in N^{NP}_{\delta,\lambda}$. Then $\bar{x} = (0, ..., 0)$ is an equilibrium action profile, since no player has an incentive to deviate: $x_i = 0$ is the unique best response for $i \in N^{NP}_{\delta,\lambda}$ (Proposition 3), and for $i \in N^{M}_{\delta,\lambda}$ see the proof of part (i) of Theorem 1.

(ii) Fix a network $(G, \bar{\theta})$ with degree partition $(D_1, ..., D_K)$ and a game $(\delta, \lambda) \in \tilde{R}_M(d_{(1)}, d_{(K)})$. The proof is analogous to the proof of part (ii) of Theorem 1 with the only difference: now $N^{NP}_{\delta,\lambda} \neq \emptyset$.

Necessity. Let $\bar{x} = (x_1, ..., x_n)$ be an asymmetric equilibrium and set $S^\theta := \{i \in N \mid x_i = \theta\}$ for $\theta = 0, 1$. Since $\bar{x}$ is asymmetric, both S^0 and S^1 are nonempty and thus form a partition of N. We need to prove that this partition satisfies the conditions of the theorem.

Since $i \in N^{NP}_{\delta,\lambda}$ implies $i \in S^\theta \cap N^\theta$ for some $\theta \in \{0, 1\}$, the first condition must hold: $S^\theta \cap N^{1-\theta} \cap N^{NP}_{\delta,\lambda} = \emptyset$. Moreover, $(\delta, \lambda) \in R_M(d_i)$ for all players $i \in N^{M}_{\delta,\lambda}$, thus the second and third conditions can be proved as in Theorem 1.

Sufficiency. Assume, there exists a partition $\{S^0, S^1\}$ of N satisfying the conditions of the theorem, and consider an action profile $\bar{x} = (x_1, ..., x_n)$ such that $x_i = 0$ for $i \in S^0$

and $x_i = 1$ for $i \in S^1$. Since $\{S^0, S^1\}$ is a partition of N, both S^0 and S^1 are nonempty, and thus $\bar{x}$ is an asymmetric action profile. We need to prove that it is an equilibrium.

The first condition implies that $x_i = \theta_i$ for any $i \in N^{NP}_{\delta,\lambda}$, that is, players from $N^{NP}_{\delta,\lambda}$ have no incentive to deviate from their actions in $\bar{x}$. Neither do players from $N^{M}_{\delta,\lambda}$, as $(\delta, \lambda) \in R_M(d_i)$ for every $i \in N^{M}_{\delta,\lambda}$ and thus the same reasoning as in the proof of Theorem 1 applies.

Proof of Theorem 7

(i) Note that $N^{MM}_{\delta,\lambda} \neq \emptyset$ for games $(\delta, \lambda) \in \tilde{R}_{MM}(d_{(1)}, d_{(K)})$ and take some $i \in N^{MM}_{\delta,\lambda}$. Suppose a symmetric equilibrium exists. Then all i's neighbors choose the same action in this equilibrium: $\forall j \in N_i(G)\ x_j = x^*$ with some $x^* \in \{0, 1\}$. The rest of the proof coincides with the proof of part (i) of Theorem 3.

(ii) The proof builds on Theorem 3 and is analogous to the proof of Theorem 6, part (ii).

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