Laumen, Benjamin; Cramer, Erhard

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RESEARCH ARTICLE

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Stage life testing

Benjamin Laumen | **Erhard Cramer**

Institute of Statistics, RWTH Aachen University, Aachen, Germany

Correspondence
Erhard Cramer, Institute of Statistics, RWTH Aachen University, D-52062 Aachen, Germany.
Email: erhard.cramer@rwth-aachen.de

Abstract

In progressive censoring, items are removed at certain times during the life test. Commonly, it is assumed that the removed items are used for further testing. In order to take into account information about these additional testing in inferential procedures, we propose a two-step model of stage life testing with one fixed stage-change time which incorporates information about both the removed items (further tested under different conditions) and those remaining in the current life test. We show that some marginal distributions in our model correspond either to progressive censoring with a fixed censoring time or to a simple-step stress model. Furthermore, assuming a cumulative exposure model, we establish exact inferential results for the distribution parameters when the lifetimes are exponentially distributed. An extension to Weibull distributed lifetimes is also discussed.

KEYWORDS

accelerated life testing, exact confidence intervals, exponential distribution, likelihood inference, progressive censoring with fixed censoring times, simple step-stress model, Weibull distribution

1 | INTRODUCTION

Reliability experiments are generally used to control the lifetime of components as a relevant quality characteristic. Suppose that an experimenter places $n \in \mathbb{N}$ identical objects simultaneously on a life test. Then, due to unintentional events, censoring, or the experimental design, it may happen that not all failure times are observed during the monitoring process. For instance, the experimenter may be interested in removing items intentionally because of restrictions regarding, for example, experimental time, costs, and material resources. The intentional withdrawing of objects from an ongoing life test enables the release of units for other experiments. In order to model such a situation, progressive censoring schemes have been proposed where the most popular ones are progressive Type-II censoring introduced by Herd (1956) and Cohen (1963) and progressive Type-I censoring (see Cohen, 1963). A comprehensive discussion of probabilistic properties and inferential results is provided by Balakrishnan and Aggarwala (2000), Balakrishnan (2007), and more recently by Balakrishnan and Cramer (2014). Recently, Cohen's progressive Type-I censoring with fixed censoring times (for short, PC-FCT) has been extensively discussed by Laumen and Cramer (2019) after falling into oblivion for some time. A schematic representation of this model is given in Figure 1 for censoring times $\tau_1 < \cdots < \tau_m$ and an initially intended censoring plan $(R_1^0, \ldots, R_m^0)$. The experimental design of a PC-FCT life test requires that, if possible, R_j^0 objects are randomly withdrawn from the life test at time τ_j, $1 \leq j \leq m$. The life test is terminated when either the last remaining object in the life test fails or the last item is removed from the test. Notice that we do not assume that the experiment is terminated at time τ_m as has been commonly done in progressive Type-I censoring (for details, see Laumen & Cramer, 2019). The PC-FCT model will be of great use in the following discussion.

A common argument used to justify progressive censoring is that the intentionally removed objects are utilized for other tests (cf., eg, Balakrishnan & Aggarwala, 2000, p. 3, Balakrishnan, Han, & Iliopoulos, 2011, p. 336). However, so far, it

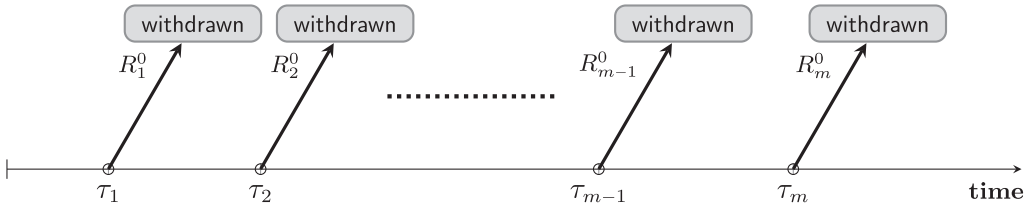

FIGURE 1 Design of a Type-I progressively censored life test with fixed censoring times $\tau_1 < \cdots < \tau_m$ and initially intended censoring plan $(R_1^0, \ldots, R_m^0)$

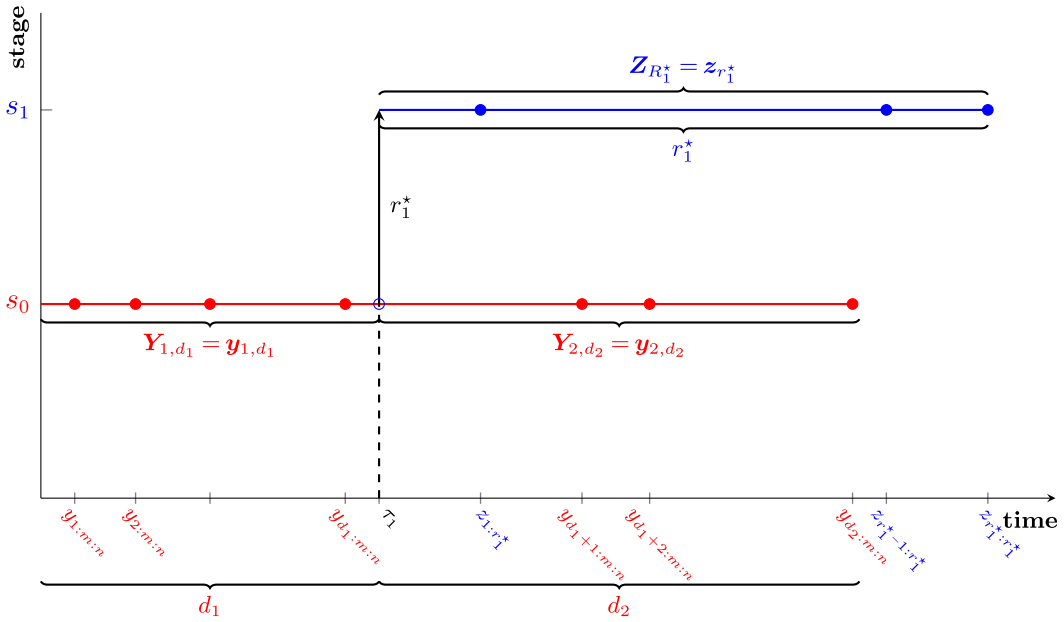

FIGURE 2 Generation process of stage life testing order statistics (alternative representation) [Colour figure can be viewed at wileyonlinelibrary.com]

seems so that no model exists to incorporate information from these additional tests in the analysis of the failure times. In order to include such an information in the statistical analysis, we connect the notion of progressive censoring with ideas from accelerated life testing. We adapt the idea of step-stress testing and assume that the "removed" objects are tested on a different stage whereas the remaining items are still tested under standard conditions. The basic situation with one censoring time τ_1 and two stages s_0, s_1 is depicted in Figure 2. It is inspired by simple step-stress testing (see below).

Therefore, the sample is split at change-time τ_1 in these items which are tested with the initial conditions (state s_0), and those which are tested further on stage s_1 under possibly different load (which may be higher or lower). This new life testing model is called stage life testing (SLT). In order to model the change in the load, we make use of the commonly used cumulative exposure model introduced by Sedyakin (1966). It connects the lifetime distributions of the objects on the different stages of the life times. Further discussions in this direction are provided by, for example, Bagdonavičius (1978) and Nelson and Meeker (1978). For comprehensive surveys on accelerated life testing, we refer to Meeker and Hahn (1985), Meeker and Escobar (1998), Bagdonavičius and Nikulin (2002), and Nelson (2004). It should be noted that the term "stage life test" has previously been used in another context by Fairbanks (1988). He presented a two-stage life test for the exponential parameter using hybrid censoring. A combination of a single-stage hybrid censoring test (cf. Epstein, 1954) and a two-stage test (cf. Bulgren & Hewett, 1973) have also been considered.

A special type of accelerated life testing is step-stress testing proposed by DeGroot and Goel (1979), and Nelson (1980). DeGroot and Goel (1979) introduced the tampered random variable model, where a change to the higher stress level results by multiplying the remaining lifetime of the unit with an unknown factor. Furthermore, Nelson (1980) discussed the model of step-stress testing in the context of the cumulative exposure model. The cumulative exposure model relates the lifetime distribution of items at one stress level to the lifetime distribution at the preceding stress level by shifting the lifetime distribution. In step-stress testing, the experimenter can choose different operating conditions at the various stress levels of the life test. Before the life test starts, the experimenter decides at which stress-change time the stress will be increased. That is, the surviving items at each stress-change time are put on a higher stress level.

The simple step-stress model is a special case of step-stress testing with only two stress levels and one stress-change time. Initially, n identical items are put on the initial stress level s_0. The stress level is changed to s_1 (generally $s_1 > s_0$) at the prefixed stress-change time τ_1. If all n failures are observed before τ_1 on stress level s_0 then the life test is terminated at the nth failure time without changing the stress level. Otherwise, all surviving items are put on the increased stress level

s_1 at τ_1 until failure. The life test continues until all n items have failed. Further work on simple step-stress testing has been done by, for example, Miller and Nelson (1983), Bai, Kim, and Lee (1989), Balakrishnan, Kundu, Ng, and Kannan (2007), Balakrishnan and Xie (2007a, 2007b), Kateri and Balakrishnan (2008), Kateri, Kamps, and Balakrishnan (2009), Kateri, Kamps, and Balakrishnan (2011), Han and Ng (2013), and Mitra, Ganguly, Samanta, and Kundu (2013). Reviews are provided by Balakrishnan, Burkschat, Cramer, and Hofmann (2008) and Kundu and Ganguly (2017).

In the following, we illustrate the cumulative exposure model in the context of a simple step-stress test with stress levels s_0 and s_1 and a stress-change time τ_1. The (absolutely continuous) lifetime distribution functions on stress levels s_0 and s_1 are denoted by F_0 and F_1, respectively. Under the cumulative exposure model, changing the stress from s_0 to s_1 entails the transition of the lifetime distribution on stress level s_1 from $F_0(t)$ to $F_1(t + v_1 - \tau_1)$ where v_1 is the solution of the equation

$$F_0(\tau_1) = F_1(v_1). \tag{1.1}$$

Hence, the cumulative distribution function of a test unit is given by

$$F_{0,1}(t) = \begin{cases} F_0(t), & 0 \le t \le \tau_1 \\ F_1(t + v_1 - \tau_1), & \tau_1 < t < \infty \end{cases}, \tag{1.2}$$

and the corresponding probability density function is given by

$$f_{0,1}(t) = \begin{cases} f_0(t), & 0 \le t \le \tau_1 \\ f_1(t + v_1 - \tau_1), & \tau_1 < t < \infty \end{cases}.$$

This paper is organized as follows. In Section 2, we introduce the SLT model formally and provide a generation procedure for SLT order statistics (SLTOSs). Then, we establish fundamental distributional results for SLTOSs in Section 3. In Section 4, we address maximum likelihood estimation (MLE) in the SLT model under various lifetime distributions. We start with an exponential distribution on both stages in Section 4.1. The MLEs are derived and the corresponding density functions are established. Furthermore, exact confidence intervals for the parameters are constructed after verifying the required stochastic monotonicity of the estimators. In Section 4.2, we consider the combination of a Weibull and an exponential distribution for the stages of the SLT model. In Section 5, we provide an illustrative example. Finally, we present the results of a simulation study in Section 6. In Section 7, we summarize the present approach and sketch some possible extensions for future work. The present work is based on the PhD thesis by Laumen (2017).

2 | SLT MODEL

First, we introduce the SLT model. The situation is depicted in Figure 2. A formal definition of the random variables is presented in Procedure 2.3.

Model 2.1 (SLT). Assume that n identical objects are placed on a life test. The initial conditions are called stage s_0. At the prefixed stage-change time τ_1, $0 \le R_1^\star \le n$ of the surviving items are randomly withdrawn (if possible) and further tested on stage s_1. Notice that this may be regarded as a different life test with used components. The testing of the remaining items is continued on stage s_0. The life test terminates when all n objects have failed.

Let D_1 and D_2 denote the random number of failures occurring on stage s_0 before and after τ_1, respectively. Furthermore, $M = D_1 + D_2$ and

- $\boldsymbol{Y}_{1,D_1} = (Y_{1:M:n}, \dots, Y_{D_1:M:n})$ denote the (ordered) observations on stage s_0 before τ_1
- $\boldsymbol{Y}_{2,D_2} = (Y_{D_1+1:M:n}, \dots, Y_{D_1+D_2:M:n})$ denote the (ordered) observations on stage s_0 after τ_1;
- $\boldsymbol{Z}_{R_1^\star} = (Z_{1:R_1^\star}, \dots, Z_{R_1^\star:R_1^\star})$ denote the (ordered) observations on stage s_1 after τ_1 with $Y_{D_1:M:n} \le \tau_1 < Z_{1:R_1^\star}$.

The order statistics on stage s_0 and the order statistics on stage s_1 are represented by the random vectors $\boldsymbol{Y} = (\boldsymbol{Y}_{1,D_1}, \boldsymbol{Y}_{2,D_2})$, and $\boldsymbol{Z}$, respectively. Figure 2 illustrates this representation for the SLTOSs.

We consider two options to generate $R_1^\star$.

1. First, at τ_1, a (fixed) proportion π_1 of the surviving objects is selected for testing on stage s_1. In the following, this option is called *Type-P*.
2. The second way to generate $R_1^\star$ is similar to the censoring procedure of PC-FCT. Given a prefixed number R_1^0, it is intended to select at τ_1 as many items as possible (at most R_1^0) for testing on stage s_1. This option is called *Type-M* in the following.

Thus, the (random) number $R_1^\star$ is defined by

$$R_1^\star = \varrho(D_1) \quad \text{with}$$

$$\varrho(x) = \begin{cases} \lfloor \pi_1 \cdot \lfloor n - x \rfloor \rfloor, & \text{Type-P} \\ \min\{n - x, R_1^0\}, & \text{Type-M} \end{cases}, \quad x \in \{0, \dots, n\}, \tag{2.1}$$

where the proportion $\pi_1 \in [0, 1]$ and the number $R_1^0 \in \mathbb{N}$ are prespecified, respectively. Furthermore, the realization of $R_1^\star$ is defined by $r_1^\star = \varrho(d_1)$. Of course, there are other possibilities conceivable to define ϱ as a function of D_1 and d_1, respectively.

We use the following notation and assumptions:

- F_i denotes the absolutely continuous cumulative distribution function with density function f_i on stage s_i, $i \in \{0, 1\}$.

- The cumulative exposure model holds, that is, v_1 is defined as the solution of the equation $F_0(\tau_1) = F_1(v_1)$ (see (1.1)).

If $D_1 = n$ then the life test is terminated at the nth failure time without changing the stage, that is, all failures have occurred on stage s_0. Otherwise, at time τ_1, $R_1^\star$ of the $n - D_1$ surviving objects are randomly selected and put on the next stage s_1 until failure. The remaining $D_2 = n - D_1 - R_1^\star$ objects are further tested on the initial stage s_0 until failure.

Remark 2.2

1. SLT is an extension of PC-FCT (ie, $k = 2$, $T_1 = \tau_1$). However, the random removals are not just withdrawn from the life test, they are further tested on a different stage to get more information.
2. In the context of simple step-stress testing, the two stages s_0 and s_1 represent stress levels. In the present formulation of SLT, the stage levels are just denominations for the stages. The stages are interpreted as categories where the life testing is conducted under different conditions. However, as has been done in simple step-stress testing, the stages s_i may be connected to the model parameters, for example, by a link function (see (7.1)). But, the stage level need not necessarily be increased, that is, one may assume $s_0 > s_1$ if the testing conditions on stage s_1 reflect this load.
3. Notice that D_2 is a (deterministic) function of D_1 (and $R_1^\star$), that is, $D_2 = n - D_1 - R_1^\star = n - D_1 - \varrho(D_1)$.
4. Choosing $\pi_1 = 0$ ($R_1^0 = 0$) in the Type-P (Type-M) scenario in (2.1) yields a complete sample of order statistics based on the distribution function F_0. For $\pi_1 = 1$ (Type-P) or $R_1^0 = n$ (Type-M), we get a simple step-stress model with stress change time τ_1.

   Since these are well examined models, we assume $\pi_1 \in (0, 1)$ and $R_1^0 \in \{1, \dots, n-1\}$ in the following. This will ensure for instance that we observe at least one failure on stage s_0.

The following procedure provides an algorithm to generate SLTOSs with cumulative distribution functions F_0 and F_1. We use the quantile representation of order statistics which can be found in, for example, Reiss (1989, Theorem 1.2.5 (i)) and Balakrishnan and Cramer (2014, Theorem 2.1.1):

$$(X_{1:n}, \dots, X_{n:n}) \overset{\mathrm{d}}{=} (F^{-1}(U_{1:n}), \dots, F^{-1}(U_{n:n})),$$

where $U_{1:n}, \dots, U_{n:n}$ are order statistics based on independent and identically distributed (iid) random variables $U_1 \dots, U_n$ from a standard uniform distribution U(0, 1) (for short, $U_1, \dots, U_n \overset{\text{iid}}{\sim} \mathrm{U}(0,1)$). F^{-1} denotes the quantile function of the cumulative distribution function F.

Procedure 2.3 (Generation of SLTOSs) Let $U_1 \dots, U_n$ be standard uniformly distributed random variables on a probability space $(\Omega, \mathfrak{A}, P)$. Let F_i^{-1} be the quantile function of the continuous cumulative distribution function F_i (on stage s_i), $i \in \{0, 1\}$, with $s_0 \neq s_1$. Furthermore, let $R_1^\star = \varrho(D_1)$ be the random number of failures selected at stage-change time τ_1 for testing on stage s_1.

For $\omega \in \Omega$, the SLT sample

$$\boldsymbol{Y}(\omega) = (\boldsymbol{Y}_{1,D_1}(\omega), \boldsymbol{Y}_{2,D_2}(\omega)), \quad \boldsymbol{Z}(\omega)$$

based on $U_1(\omega) \dots, U_n(\omega)$, is generated as follows:

1. Compute the order statistics $U_{1:n}(\omega) \leq \dots \leq U_{n:n}(\omega)$;
2. Set $V_{j:n}^{(0)}(\omega) = F_0^{-1}(U_{j:n}(\omega))$ and $V_{j:n}^{(1)}(\omega) = F_1^{-1}(U_{j:n}(\omega)) - v_1 + \tau_1$, $1 \leq j \leq n$;
3. Define $\mathcal{P} = \{\alpha \in \{1, \dots, n\} \mid -\infty < V_{\alpha:n}^{(0)}(\omega) \leq \tau_1\}$ and $\mathcal{P}^c = \{1, \dots, n\} \setminus \mathcal{P}$;
4. Let $D_1(\omega) = \mid \mathcal{P} \mid$ and compute $r_1^\star = \varrho(D_1(\omega))$ by (2.1);
5. Choose randomly a without-replacement sample $\mathcal{R} \subseteq \mathcal{P}^c$ with $\mid \mathcal{R} \mid = r_1^\star$ and set $\mathcal{Q} = \mathcal{P}^c \setminus \mathcal{R}$;
6. If $d_1 = D_1(\omega) = \mid \mathcal{P} \mid > 0$ and $d_2 = \mid \mathcal{Q} \mid > 0$ and $r_1^\star = \mid \mathcal{R} \mid > 0$, then $m = d_1 + d_2$ and

$$\begin{aligned}(Y_{h:m:n}(\omega))_{h=1,\dots,d_1} &= (V_{h:n}^{(0)}(\omega))_{h\in\mathcal{P}},\\ (Y_{d_1+i:m:n}(\omega))_{i=1,\dots,d_2} &= (V_{i:n}^{(0)}(\omega))_{i\in\mathcal{Q}}, \quad \text{and}\\ (Z_{j:r_1^\star}(\omega))_{j=1,\dots,r_1^\star} &= (V_{j:n}^{(1)}(\omega))_{j\in\mathcal{R}}. \end{aligned} \tag{2.2}$$

If d_1, d_2, or $r_1^\star$ are zero then the corresponding vector in (2.2) has zero dimension meaning that no failures occur in this particular part of the life test.

Remark 2.4 In the context of simple step-stress testing, one may use Procedure 2.3 to generate order statistics from a simple step-stress test. In this case, step (4) is skipped and $\mathcal{R} = \mathcal{P}^c$ is defined in step (5) since all remaining objects are put on stress level s_1. In step (6), only the first and the third vector remain. Kateri and Balakrishnan (2008, p. 626) presented an algorithm to simulate data from a Type-II censored simple step-stress model with Weibull lifetimes.

In Procedure 2.3, the construction of SLTOSs does not involve distributional assumptions on both the random variables $U_1, \ldots, U_n$ and the selection process in step (5). By analogy with the assumptions imposed to define progressively Type-II censored order statistics (Type-II PCOSs) (see Cramer & Lenz, 2010, Assumption 2.2, Balakrishnan & Cramer, 2014, p. 5), one may postulate similar requirements for the SLT model. Using the notation from Procedure 2.3, they can be summarized as follows (for details, see Laumen, 2017):

1. $U_1, \ldots, U_n \overset{\text{iid}}{\sim} \mathrm{U}(0,1)$;
2. the without-replacement sample $\mathcal{R}$ with $|\mathcal{R}| = r_1^\star$ is drawn from $\mathcal{P}^c$ according to a (discrete) uniform distribution;
3. the random variables $U_1 \ldots, U_n$ and the SLT assignment are independent.

Clearly, the model may be also be considered under different probabilistic assumptions. For instance, one may allow for dependence of the underlying lifetimes by assuming exchangeability or postulating a particular copula function of $U_1 \ldots, U_n$.

3 | DISTRIBUTION THEORY

In this section, we present some fundamental distributional results for SLTOSs. First, we establish the joint density function of SLTOSs. Then, we obtain the conditional block independence of SLTOSs.

Suppose first $D_1 = n$. Thus, $\boldsymbol{Y} = (\boldsymbol{Y}_{1,D_1}, \boldsymbol{Y}_{2,D_2})$ are order statistics from a complete sample so that the joint density function $f_{1\ldots n}^{\boldsymbol{Y},D_1}$ of $\boldsymbol{Y}$ and D_1 is given by (cf. Arnold, Balakrishnan, & Nagaraja, 2008, p. 10)

$$f_{1\ldots n}^{\boldsymbol{Y},D_1}(\boldsymbol{x}_n, n) = n! \prod_{i=1}^{n} f_0(x_i), \quad x_1 \le \cdots \le x_n \le \tau_1.$$

Theorem 3.1 presents the joint density function of SLTOSs. This density function is essential for the likelihood inference discussed in Section 4. A proof is given in Appendix A.

Theorem 3.1 *Let $\boldsymbol{Y}$, $\boldsymbol{Z}$ be SLTOSs as generated by Procedure* 2.3 *and let F_i be an absolutely continuous cumulative distribution function with density function f_i $i \in \{0, 1\}$. Furthermore let $\tau_1 \in \mathbb{R}$.*

Then the joint density function $f_{1\ldots n}^{\boldsymbol{Y},\boldsymbol{Z},D_1}$ of ($\boldsymbol{Y}$, $\boldsymbol{Z}$) and D wrt the product of the n dimensional Lebesgue measure and the one dimensional counting measure is given by

$$f_{1\ldots n}^{\boldsymbol{Y},\boldsymbol{Z},D_1}(\boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}, \boldsymbol{z}, d_1) = \binom{n}{d_1} d_1! d_2! r_1^\star! \prod_{h=1}^{d_1+d_2} f_0(y_{h:m:n}) \prod_{j=1}^{r_1^\star} f_1(z_{j:r_1^\star} + v_1 - \tau_1), \tag{3.1}$$

for $\boldsymbol{y}_{1,d_1} = (y_{1:m:n}, \ldots, y_{d_1:m:n})$, $\boldsymbol{y}_{2,d_2} = (y_{d_1+1:m:n}, \ldots, y_{d_1+d_2:m:n})$ and $\boldsymbol{z} = (z_{1:r_1^\star}, \ldots, z_{r_1^\star:r_1^\star})$, where $d_2 = n - d_1 - r_1^\star$, and $r_1^\star = \varrho(d_1)$.

Remark 3.2

1. The marginal density function $f^{\boldsymbol{Y}_{1,D_1},\boldsymbol{Y}_{2,D_2},D_1}$ of $\boldsymbol{Y}_{1,D_1}, \boldsymbol{Y}_{2,D_2}$ and D_1 corresponds to the joint density function of order statistics from PC-FCT with one censoring time τ_1 (see Laumen & Cramer, 2019, Theorem 2.7). Thus, SLT can be seen as an extension of PC-FCT by considering only the observations on stage s_0 (cf. Remark 2.2).
2. The marginal density function $f^{\boldsymbol{Y}_{1,D_1},\boldsymbol{Z},D_1}$ of $\boldsymbol{Y}_{1,D_1}, \boldsymbol{Z}$, and D_1 is the joint density function of order statistics from a simple step-stress model under progressive Type-I censoring (cf. Balakrishnan & Cramer, 2014, p. 496; Gouno, Sen, & Balakrishnan, 2004; Han, Balakrishnan, Sen, & Gouno, 2006). Hence, when we consider only the failures observed on stage s_0 before τ_1 and those on stage s_1 as in a usual simple step-stress model, we obtain the connection between SLT and simple step-stress testing (cf. Remark 2.2).
3. It can be seen from the proof of Theorem 3.1 given in Appendix A that the distribution function of ($\boldsymbol{Y}$, $\boldsymbol{Z}$, D_1) and its marginals are obtained without using the assumption of absolute continuity. The cumulative distribution can be calculated under the weaker assumption of continuity of F_0 and F_1 (see Equation (A.2)). The assumption of absolute continuity is only necessary to establish a density function as given in Equation (3.1).

The conditional block independence of SLTOSs is presented in Lemma 3.3. It is directly obtained from the proof of Theorem 3.1. The result is helpful for the derivation of the density functions of the MLEs of the scale parameters when the lifetimes are exponentially distributed (see Section 4.1.1).

Lemma 3.3 *Let $\boldsymbol{Y}$, $\boldsymbol{Z}$ be SLTOSs defined by Procedure* 2.3 *with an absolutely continuous cumulative distribution function F_i and a density function f_i on stage s_i $i \in \{0, 1\}$. Furthermore let $s \neq s$ and $\tau_1 \in \mathbb{R}$.*

Conditionally on D = d the SLTOSs are block independent that is the random vectors

$$(Y_{1:m:n}, \ldots, Y_{d_1:m:n}), \quad (Y_{d_1+:m:n}, \ldots, Y_{d_1+d_2:m:n}), \text{ and } \quad (Z_{1:r_1^\star}, \ldots, Z_{r_1^\star:r_1^\star}),$$

are independent with

$$(Y_{1:m:n}, \dots, Y_{d_1:m:n}) \overset{\mathrm{d}}{=} (Y^{(1)}_{1:d_1}, \dots, Y^{(1)}_{d_1:d_1}),$$

$$(Y_{d_1+1:m:n}, \dots, Y_{d_1+d_2:m:n}) \overset{\mathrm{d}}{=} (Y^{(2)}_{1:d_2}, \dots, Y^{(2)}_{d_2:d_2}),$$

and

$$(Z_{1:r_1^\star}, \dots, Z_{r_1^\star:r_1^\star}) \overset{\mathrm{d}}{=} (Z^{(1)}_{1:r_1^\star}, \dots, Z^{(1)}_{r_1^\star:r_1^\star}),$$

with $d_2 = n - d_1 - r_1^\star$ *and* $r_1^\star = \varrho(d_1)$ $Y^{(1)}_{1:d_1}, \dots, Y^{(1)}_{d_1:d_1}$ *and* $Y^{(2)}_{1:d_2}, \dots, Y^{(2)}_{d_2:d_2}$ *denote order statistics from an absolutely continuous cumulative distribution function* F *but right truncated and left truncated at* τ_1 *respectively Furthermore* $Z^{(1)}_{1:r_1^\star}, \dots, Z^{(1)}_{r_1^\star:r_1^\star}$ *denote order statistics from an absolutely continuous cumulative distribution function* F *but left truncated at* τ_1 *and shifted by* ν_1-τ_1

Proof From the representation of the joint density function given in Theorem 3.1 or rather Equation (3.1) and the probability $P(D_1 = d_1)$ given in (A.3), we get

$$f^{\boldsymbol{Y},\boldsymbol{Z}|D_1}_{1\dots n}\left(\boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}, \boldsymbol{z} \mid d_1\right) = \frac{f^{\boldsymbol{Y},\boldsymbol{Z},D_1}_{1\dots n}\left(\boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}, \boldsymbol{z}, d_1\right)}{P\left(D_1 = d_1\right)}$$

$$= d_1! \prod_{h=1}^{d_1} \frac{f_0\left(y_{h:m:n}\right)}{F_0(\tau_1)} d_2! \prod_{i=1}^{d_2} \frac{f_0\left(y_{d_1+i:m:n}\right)}{1 - F_0(\tau_1)}$$

$$\times r_1^\star! \prod_{j=1}^{r_1^\star} \frac{f_1\left(z_{j:r_1^\star} + \nu_1 - \tau_1\right)}{1 - F_1(\nu_1)},$$

for $\boldsymbol{y}_{1,d_1} = (y_{1,1:n}, \dots, y_{1,d_1:n})$, $\boldsymbol{y}_{2,d_2} = (y_{2,1:n}, \dots, y_{2,d_2:n})$ and $\boldsymbol{z} = (z_{1:r_1^\star}, \dots, z_{r_1^\star:r_1^\star})$. This proves the result. ∎

4 | MLE IN SLT

First, it is worth mentioning that the MLEs of the parameters on stage s_0 always exist since $d_1 + d_2 = n - r_1^\star > 0$ by assumption in the SLT model. The MLEs of the parameters on stage s_1 do not exist when there are no failures observed on stage s_1 (ie, $r_1^\star = 0$). In particular, the probability to observe at least one failure on stage s_1 is given in Lemma 4.1.

Lemma 4.1 *Let*

$$n^\star = \begin{cases} \left\lfloor n - \frac{1}{\pi_1} \right\rfloor, & \text{Type-P} \\ n - 1, & \text{Type-M} \end{cases}. \tag{4.1}$$

Then $P(R_1^\star > 0) = P(\varrho(D_1) > 0) = \sum_{d_1=0}^{n^\star} P(D_1 = d_1)$ *where* $P(D_1 = d_1)$ *is given in* (A.3). *In particular*

$$P_{\text{Type-P}}(R_1^\star > 0) \le P_{\text{Type-M}}(R_1^\star > 0) = 1 - F_0^n(\tau_1). \tag{4.2}$$

Proof The condition $R_1^\star = \varrho(D_1) > 0$ is equivalent to

$$\begin{cases} \lfloor \pi_1 \cdot \lfloor n - D_1 \rfloor \rfloor \ge 1 \iff \\ \quad D_1 \le \left\lfloor n - \frac{1}{\pi_1} \right\rfloor, \text{ Type-P} \\ n - D_1 \ge 1 \iff \\ \quad D_1 \le n - 1, \text{ Type-M} \end{cases} \iff D_1 \le n^\star. \tag{4.3}$$

The inequality in (4.2) follows directly since $n^\star(\text{Type - P}) \le n^\star(\text{Type-M})$. ∎

4.1 | Exponential distributions

The probability density function and the cumulative distribution function of the exponential distribution $\mathrm{Exp}(\vartheta)$ are given by

$$f(x) = \frac{1}{\vartheta} \mathrm{e}^{-x/\vartheta}\, 1_{(0,\infty)}(x), \quad F(x) = (1 - \mathrm{e}^{-x/\vartheta})\, 1_{(0,\infty)}(x),$$
$$\vartheta > 0, x \in \mathbb{R}.$$

Suppose that the lifetime distributions on stages s_0 and s_1 are exponentials with means ϑ_0 and ϑ_1, respectively. Then, $\nu_1 = \tau_1 \frac{\vartheta_1}{\vartheta_0}$. Furthermore, from (3.1), the likelihood function for the data $(\boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}) = (y_{1:m:n}, \dots, y_{d_1+d_2:m:n})$ and $\boldsymbol{z} = (z_{1:r_1^\star}, \dots, z_{r_1^\star:r_1^\star})$ is given by

$$\mathrm{L}(\vartheta_0, \vartheta_1 \mid \boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}, \boldsymbol{z}) = \frac{c_1}{\vartheta_0^m \vartheta_1^{r_1^\star}} \exp\left\{ -r_1^\star \tau_1 \left(\frac{1}{\vartheta_0} - \frac{1}{\vartheta_1}\right) - \frac{1}{\vartheta_0} \sum_{i=1}^{m} y_{i:m:n} - \frac{1}{\vartheta_1} \sum_{j=1}^{r_1^\star} z_{j:r_1^\star} \right\}, \tag{4.4}$$

with $c_1 = \binom{n}{d_1} d_1! d_2! r_1^\star!$, where $m = d_1 + d_2$ and $d_2 = n - d_1 - r_1^\star$. Notice that d_1 is implicitly known from the data. This yields the log-likelihood function

$$\ell(\vartheta_0, \vartheta_1 \mid \boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}, \boldsymbol{z}) = \log(c_1) - m \log(\vartheta_0) - \frac{1}{\vartheta_0}\left(\sum_{i=1}^{m} y_{i:m:n} + r_1^\star \tau_1\right) - \frac{1}{\vartheta_1}\left(\sum_{j=1}^{r_1^\star} z_{j:r_1^\star} - r_1^\star \tau_1\right).$$

The MLEs can be obtained by standard derivations which, for brevity, are omitted. Therefore, the MLE of ϑ_0 is given by

$$\widehat{\vartheta}_0 = \frac{1}{D_1 + D_2}\left(\sum_{i=1}^{D_1+D_2} Y_{i:m:n} + R_1^\star \tau_1\right). \tag{4.5}$$

Given $R_1^\star > 0$, the MLE of ϑ_1 is obtained as

$$\widehat{\vartheta}_1 = \frac{1}{R_1^\star}\left(\sum_{j=1}^{R_1^\star} Z_{j:R_1^\star} - R_1^\star \tau_1\right). \tag{4.6}$$

It should be noted that the MLE $\widehat{\vartheta}_0$ depends only on the observations on stage s_0 whereas the MLE $\widehat{\vartheta}_1$ depends only on the observations on stage s_1. Hence, from Lemma 3.3, the MLEs $\widehat{\vartheta}_0$ and $\widehat{\vartheta}_1$ are independent given $D_1 = d_1$.

4.1.1 | Density functions of the MLEs

In the SLT model, we consider two options to generate the number of objects $r_1^\star$ that are effectively put on stage s_1 at stage-change time τ_1 (cf. Remark 2.2). This means that we get two different representations for the density functions of the MLEs $\widehat{\vartheta}_0$ and $\widehat{\vartheta}_1$, respectively, depending on the way of generating $r_1^\star$. These density functions are given in Theorem 4.2. We use the following notation

$$\gamma(t;a,b) = \frac{1}{a^b\Gamma(b)} t^{b-1}\mathrm{e}^{-t/a}1_{(0,\infty)}(t), \quad t\in\mathbb{R}, \tag{4.7}$$

for the density function of a gamma distribution with parameters $a,b>0$ where $\Gamma(\cdot)$ denotes the gamma function.

Theorem 4.2 *Let $n^\star$ be defined as in* (4.1). *Then*

1. *The density function $f_{\vartheta_0}^{\widehat{\vartheta}_0}$ of $\widehat{\vartheta}_0$ is given by*

$$f_{\vartheta_0}^{\widehat{\vartheta}_0}(t) = \sum_{d_1=0}^{n} f_{\vartheta_0,d_1}(t), \tag{4.8}$$

where

$$f_{\vartheta_0,d_1}(t) = \sum_{i=0}^{d_1}(-1)^i\binom{n}{d_1}\binom{d_1}{i}\mathrm{e}^{-(n-d_1+i)\tau_1/\vartheta_0} \times\gamma\left(t-\frac{(n-d_1+i)\tau_1}{n-\varrho(d_1)};\frac{\vartheta_0}{n-\varrho(d_1)},n-\varrho(d_1)\right),$$

for $t\in\mathbb{R}$ with ϱ as in (2.1).

2. *The conditional density function $f_{\vartheta_0,\vartheta_1}^{\widehat{\vartheta}_1|R_1^\star>0}$ of $\widehat{\vartheta}_1$ is given by*

$$f_{\vartheta_0,\vartheta_1}^{\widehat{\vartheta}_1|R_1^\star>0}(t) = \sum_{d_1=0}^{n^\star}\frac{q_{\vartheta_0,d_1}}{\sum_{i=0}^{n^\star}q_{\vartheta_0,i}}\gamma\left(t;\frac{\vartheta_1}{\varrho(d_1)},\varrho(d_1)\right), \quad t\in\mathbb{R}, \tag{4.9}$$

with

$$q_{\vartheta_0,j} = \binom{n}{j}[1-\mathrm{e}^{-\tau_1/\vartheta_0}]^j[\mathrm{e}^{-\tau_1/\vartheta_0}]^{n-j}, \quad j=0,\dots,n^\star, \text{ and} \tag{4.10}$$

$$r_1^\star = \varrho(d_1) = \begin{cases}\lfloor\pi_1\cdot(n-d_1)\rfloor, & \text{Type-P}\\ \min\{n-d_1,R_1^0\}, & \text{Type-M}\end{cases}, \quad d_1=0,\dots,n^\star. \tag{4.11}$$

Note that the density function $f_{\vartheta_0}^{\widehat{\vartheta}_0}$ is a generalized mixture of shifted gamma densities whereas the conditional density function $f_{\vartheta_0,\vartheta_1}^{\widehat{\vartheta}_1|R_1^\star>0}$ is a mixture of gamma densities.

Proof

1. The MLE $\widehat{\vartheta}_0$ in (4.5) has the same structure as the MLE $\widehat{\vartheta}$ established in Laumen and Cramer (2019, equation (3.2)) for PC-FCT. Its distribution can be directly taken from Theorem 3.1 in Laumen and Cramer (2019) (with $k=2$, $T_1=\tau_1$) since it depends only on the random variables $\boldsymbol{Y}_{1,D_1}, \boldsymbol{Z}_{R_1^\star}$, and D_1 which are progressively censored failure times with one fixed censoring time τ_1. Therefore, we find directly the density function $f_{\vartheta_0}^{\widehat{\vartheta}_0}$ of $\widehat{\vartheta}_0$ given in (4.8) for the two scenarios Type-P and Type-M, respectively.
2. First, using Equation (A.3), we see that $q_{\vartheta 0,j}$ has the representation given in (4.10). Then, with (4.3), we get for $t\in\mathbb{R}$

$$P_{\vartheta_0,\vartheta_1}^{\widehat{\vartheta}_1}(\widehat{\vartheta}_1\le t, R_1^\star>0) = \sum_{d_1=0}^{n^\star}P_{\vartheta_0,\vartheta_1}\left(\sum_{j=1}^{R_1^\star}(Z_{j:R_1^\star}-\tau_1)\le R_1^\star t | D_1=d_1\right)q_{\vartheta_0,d_1}.$$

Recalling (2.1), we have that $R_1^\star$ given $D_1=d_1$ equals $r_1^\star=\varrho(d_1)$ as specified in (4.11). Hence, using Lemma 3.3, we know that

$$(Z_{1:r_1^\star}-\tau_1,\dots,Z_{r_1^\star:r_1^\star}-\tau_1)\mid D_1=d_1 \overset{\mathrm{d}}{=} (Z_{1:r_1^\star}^{(1)},\dots,Z_{r_1^\star:r_1^\star}^{(1)}),$$

where $Z_{1:r_1^\star}^{(1)},\dots,Z_{r_1^\star:r_1^\star}^{(1)}$ are order statistics based on the exponential distribution $\mathrm{Exp}(\vartheta_1)$. Notice that, due to the cumulative exposure model,

$$\frac{f_1(x+v_1-\tau_1)}{1-F_1(v_1)} = \frac{f_1(x+v_1-\tau_1)}{1-F_0(\tau_1)} = \frac{f_1(x)}{1-F_1(\tau_1)} = \frac{1}{\vartheta_1}\mathrm{e}^{-(x-\tau_1)/\vartheta_1}, \quad x>\tau_1.$$

Thus, we get with $r_1^\star=\varrho(d_1)$

$$P_{\vartheta_0,\vartheta_1}^{\widehat{\vartheta}_1}(\widehat{\vartheta}_1\le t, R_1^\star>0) = \sum_{d_1=0}^{n^\star}q_{\vartheta_0,d_1}\int_0^{\varrho(d_1)t}\gamma(s;\vartheta_1,\varrho(d_1))ds.$$

Dividing this expression by the probability

$$P_{\vartheta_0}(R_1^\star>0) = P_{\vartheta_0}(D_1\le n^\star) = \sum_{i=0}^{n^\star}P_{\vartheta_0}(D_1=i) = \sum_{i=0}^{n^\star}q_{\vartheta_0,i}$$

we arrive at the desired conditional density function $f_{\vartheta_0,\vartheta_1}^{\widehat{\vartheta}_1|R_1^\star>0}$ given in (4.9). ∎

Corollary 4.3

1. *The mean and variance of $\widehat{\vartheta}_0$ are given by*

$$E_{\vartheta_0}(\widehat{\vartheta}_0) = \vartheta_0 + \mathrm{bias}_{\vartheta_0}(\widehat{\vartheta}_0) \quad \text{and}$$
$$\mathrm{Var}_{\vartheta_0}(\widehat{\vartheta}_0) = \mathrm{MSE}_{\vartheta_0}(\widehat{\vartheta}_0) - (\mathrm{bias}_{\vartheta_0}(\widehat{\vartheta}_0))^2,$$

where

$$\text{bias}_{\vartheta_0}\left(\widehat{\vartheta}_0\right) = \sum_{d_1=0}^{n}\sum_{i=0}^{d_1}(-1)^i\binom{n}{d_1}\binom{d_1}{i} \times \mathrm{e}^{-(n-d_1+i)\tau_1/\vartheta_0}\frac{(n-d_1+i)\tau_1}{n-\varrho(d_1)},$$

$$\text{MSE}_{\vartheta_0}\left(\widehat{\vartheta}_0\right) = \sum_{d_1=0}^{n}\sum_{i=0}^{d_1}(-1)^i\binom{n}{d_1}\binom{d_1}{i} \times \mathrm{e}^{-(n-d_1+i)\tau_1/\vartheta_0}\left(\frac{\vartheta_0^2}{n-\varrho(d_1)}+\frac{(n-d_1+i)^2\tau_1^2}{(n-\varrho(d_1))^2}\right)$$

are bias and mean squared error of $\widehat{\vartheta}_0$ respectively with ϱ as in 2.1

2. *The conditional mean and conditional variance of $\widehat{\vartheta}_1$ are given by*

$$E_{\vartheta_0,\vartheta_1}\left(\widehat{\vartheta}_1 \mid R_1^\star > 0\right) = \vartheta_1 \quad \text{and}$$

$$\text{Var}_{\vartheta_0,\vartheta_1}\left(\widehat{\vartheta}_1 \mid R_1^\star > 0\right) = \vartheta_1^2\sum_{d_1=0}^{n^\star}\frac{q_{\vartheta_0,d_1}}{\sum_{i=0}^{n^\star}q_{\vartheta_0,i}}\frac{1}{\varrho(d_1)}$$

with ϱ as in 2.1 *Notice that conditionally on $R_1^\star > 0$, $\widehat{\vartheta}_1$ is an unbiased estimator of ϑ for any $\vartheta > 0$*

4.1.2 | Confidence intervals for the scale parameters

In order to construct confidence intervals for the scale parameters ϑ_0 and ϑ_1 by the method of pivoting the survival function (cf. Casella & Berger, 2002, pp. 430–435, Balakrishnan, Cramer, & Iliopoulos, 2014, Hahn, Meeker, & Escobar, 2017), we need the survival functions of $\widehat{\vartheta}_0$ and $\widehat{\vartheta}_1$, respectively. These survival functions are given in Corollary 4.4 for both options Type-P and Type-M. We present alongside with the survival functions the limits that are also necessary for the construction of the confidence intervals. The proofs of the following results are straightforward using elementary calculations. $\overline{\Gamma}(\cdot;a,b)$ denotes the survival function of a gamma distribution with density function $\gamma(\cdot;a,b)$ given in (4.7).

Corollary 4.4

1. *The survival function of $\widehat{\vartheta}_0$ is given by*

$$P_{\vartheta_0}\left(\widehat{\vartheta}_0 > t\right) = \sum_{d_1=0}^{n}\sum_{i=0}^{d_1}(-1)^i\binom{n}{d_1}\binom{d_1}{i}\mathrm{e}^{-(n-d_1+i)\tau_1/\vartheta_0} \times \overline{\Gamma}\left(t-\frac{(n-d_1+i)\tau_1}{n-\varrho(d_1)};\frac{\vartheta_0}{n-\varrho(d_1)},n-\varrho(d_1)\right),$$

for $t\in\mathbb{R}$ with

$$\lim_{\vartheta_0\to 0+}P_{\vartheta_0}(\widehat{\vartheta}_0 > t) = 0 \quad \text{and} \quad \lim_{\vartheta_0\to+\infty}P_{\vartheta_0}(\widehat{\vartheta}_0 > t) = 1, \quad t\in\mathbb{R}.$$

2. *The conditional survival function of $\widehat{\vartheta}_1$ is given by*

$$P_{\vartheta_0,\vartheta_1}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0) = \sum_{d_1=0}^{n^\star}\frac{q_{\vartheta_0,d_1}}{\sum_{i=0}^{n^\star}q_{\vartheta_0,i}}\overline{\Gamma}\left(t;\frac{\vartheta_1}{\varrho(d_1)},\varrho(d_1)\right), \quad t\in\mathbb{R},$$

with

$$\lim_{\vartheta_1\to 0+}P_{\vartheta_0,\vartheta_1}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0) = 0 \quad \text{and}$$

$$\lim_{\vartheta_1\to+\infty}P_{\vartheta_0,\vartheta_1}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0) = 1, \quad t\in\mathbb{R}.$$

Notice that the conditional survival function of $\widehat{\vartheta}_1$ in Corollary 4.4(2) depends on ϑ_0 since the conditional density function $f^{\widehat{\vartheta}_1|R_1^\star>0}_{\vartheta_0,\vartheta_1}$ of $\widehat{\vartheta}_1$ depends on ϑ_0.

The stochastic monotonicity of an estimator is an essential condition to construct exact confidence intervals by the pivoting method. Hence, we establish the stochastic monotonicity of $\widehat{\vartheta}_0$ and $\widehat{\vartheta}_1$ in Theorem 4.6. Before we can prove the stochastic monotonicity of $\widehat{\vartheta}_1$ (given $R_1^\star > 0$), we need the following auxiliary result which can be found in, for example, Marshall and Olkin (2007, pp. 310, 311).

Lemma 4.5 *The survival function $\overline{\Gamma}(t;\vartheta,\beta)$ of a gamma distribution is increasing in $\vartheta > 0$ for every fixed $\beta t > 0$*

Theorem 4.6

1. *The MLE $\widehat{\vartheta}_0$ in* 4.5 *is stochastically increasing in $\vartheta > 0$ that is for all $t > 0$*

$$P_{\vartheta_0}(\widehat{\vartheta}_0 > t) \le P_{\vartheta_0'}(\widehat{\vartheta}_0 > t), \quad \vartheta_0 < \vartheta_0'.$$

2. *Conditionally on $R_1^\star > 0$ the MLE $\widehat{\vartheta}_1$ in* 4.6 *is stochastically increasing in $\vartheta > 0$ for all $\vartheta > 0$ that is for all $t > 0$ and $\vartheta > 0$*

$$P_{\vartheta_0,\vartheta_1}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0) \le P_{\vartheta_0,\vartheta_1'}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0), \quad \vartheta_1 < \vartheta_1'.$$

Proof

1. The result follows directly from Theorem 3.5 in Laumen and Cramer (2019) with $k = 2$ and $T_1 = \tau_1$.
2. Let $t > 0$ and $0 < \vartheta_1 < \vartheta_1'$ Then, for all $\vartheta_0 > 0$, we get by Lemma 4.5

$$P_{\vartheta_0,\vartheta_1}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0) = \sum_{d_1=0}^{n^\star}\frac{q_{\vartheta_0,d_1}}{\sum_{i=0}^{n^\star}q_{\vartheta_0,i}}\overline{\Gamma}\left(t;\frac{\vartheta_1}{\varrho(d_1)},\varrho(d_1)\right) \le \sum_{d_1=0}^{n^\star}\frac{q_{\vartheta_0,d_1}}{\sum_{i=0}^{n^\star}q_{\vartheta_0,i}}\overline{\Gamma}\left(t;\frac{\vartheta_1'}{\varrho(d_1)},\varrho(d_1)\right) = P_{\vartheta_0,\vartheta_1'}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0).$$

■

The results established in Corollary 4.4 and Theorem 4.6 enable us to construct exact confidence intervals for ϑ_0. Thus, in order to get an exact confidence interval $CI_{\vartheta_0} = [\vartheta_{0,L}, \vartheta_{0,U}]$ for ϑ_0 with level of significance $\alpha \in (0, 1)$, we have to solve the equations

$$P_{\vartheta_{0,L}}(\widehat{\vartheta}_0 > \widehat{\vartheta}_{0,obs}) = \frac{\alpha}{2} \quad \text{and}$$
$$P_{\vartheta_{0,U}}(\widehat{\vartheta}_0 > \widehat{\vartheta}_{0,obs}) = 1 - \frac{\alpha}{2} \tag{4.12}$$

for $\vartheta_{0,L}$ and $\vartheta_{0,U}$, respectively. Furthermore, its not possible to construct exact confidence intervals for ϑ_1 using this method since the conditional survival function of $\widehat{\vartheta}_1$ depends on ϑ_0, too. However, proceeding as in Kundu, Kannan, and Balakrishnan (2004), approximate confidence intervals can be established by replacing ϑ_0 by the value of its MLE $\widehat{\vartheta}_{0,obs}$. Hence, we get approximate confidence intervals $CI_{\vartheta_1} = [\vartheta_{1,L}, \vartheta_{1,U}]$ for ϑ_1 with $\alpha \in (0, 1)$ by solving the equations

$$P_{\widehat{\vartheta}_{0,obs},\vartheta_{1,L}}(\widehat{\vartheta}_1 > \widehat{\vartheta}_{1,obs} | R_1^\star > 0) = \frac{\alpha}{2} \quad \text{and}$$
$$P_{\widehat{\vartheta}_{0,obs},\vartheta_{1,U}}(\widehat{\vartheta}_1 > \widehat{\vartheta}_{1,obs} | R_1^\star > 0) = 1 - \frac{\alpha}{2} \tag{4.13}$$

for $\vartheta_{1,L}$ and $\vartheta_{1,U}$, respectively. As can be seen from the simulations presented in Section 6, this construction yields confidence intervals with a coverage probability close to the desired level of significance.

4.2 | Cumulative exposure model with Weibull and exponential distribution

In this section, we consider a cumulative exposure model for Weibull and exponential lifetimes. The probability density function and the cumulative distribution function of the Weibull distribution $\mathrm{Wei}(\vartheta, \beta)$ are given by

$$f(x) = \frac{\beta}{\vartheta} x^{\beta-1} \mathrm{e}^{-x^\beta/\vartheta}\, 1_{(0,\infty)}(x), \quad F(x) = (1 - \mathrm{e}^{-x^\beta/\vartheta})\, 1_{(0,\infty)}(x),$$
$$\vartheta > 0, \beta > 0, x \in \mathbb{R}.$$

We assume that the lifetimes on stage s_0 are $\mathrm{Wei}(\vartheta_0, \beta)$-distributes whereas they are $\mathrm{Exp}(\vartheta_1)$-distributed on stage s_1. Therefore, $\nu_1 = \tau_1^\beta \frac{\vartheta_1}{\vartheta_0}$.

Using Equation (3.1), the likelihood function for the observed data $(\boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}) = (y_{1:m:n}, \ldots, y_{d_1+d_2:m:n})$ and $\boldsymbol{z} = (z_{1:r_1^\star}, \ldots, z_{r_1^\star:r_1^\star})$ is given by

$$\begin{aligned}
&\mathrm{L}(\vartheta_0, \beta, \vartheta_1 \mid \boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}, \boldsymbol{z}_{r_1^\star}) \\
&= c_1 \frac{\beta^m}{\vartheta_0^m} \frac{1}{\vartheta_1^{r_1^\star}} \exp\left\{ -r_1^\star \left\{ \frac{\tau_1^\beta}{\vartheta_0} - \frac{\tau_1}{\vartheta_1} \right.\right. \\
&\quad \left.\left. - \frac{1}{\vartheta_0} \sum_{i=1}^m y_{i:m:n} \right\}^\beta - \frac{1}{\vartheta_1} \sum_{j=1}^{r_1^\star} z_{j:r_1^\star} \right\} \\
&\quad \times \prod_{i=1}^m (y_{i:m:n})^{\beta-1},
\end{aligned}$$

where $m = d_1 + d_2$, $d_2 = n - d_1 - r_1^\star$, and $c_1 = \binom{n}{d_1} d_1! d_2! r_1^\star!$. The corresponding log-likelihood function is given by

$$\begin{aligned}
&\ell(\vartheta_0, \beta, \vartheta_1 \mid \boldsymbol{y}_{1,d_1}, \boldsymbol{y}_{2,d_2}, \boldsymbol{z}_{1,r_1^\star}) = \log(c_1) + m \log(\beta) - m \log(\vartheta_0) \\
&- r_1^\star \log(\vartheta_1) - r_1^\star \left(\frac{\tau_1^\beta}{\vartheta_0} - \frac{\tau_1}{\vartheta_1} \right) - \frac{1}{\vartheta_0} \sum_{i=1}^m (y_{h,i:n})^\beta - \frac{1}{\vartheta_1} \sum_{j=1}^{r_1^\star} z_{j:r_1^\star} \\
&\quad + (\beta - 1) \sum_{i=1}^m \log(y_{i:m:n}).
\end{aligned}$$

Notice that the MLEs of ϑ_0 and β always exist by construction of the SLT model and that the MLE of ϑ_1 does not exist when $r_1^\star = 0$. Given $R_1^\star > 0$, the MLE of ϑ_1 is given by

$$\widehat{\vartheta}_1 = \frac{1}{R_1^\star} \left(\sum_{j=1}^{R_1^\star} Z_{j:R_1^\star} - R_1^\star \tau_1 \right) \tag{4.14}$$

as in (4.6). The MLE of ϑ_0 is given by

$$\widehat{\vartheta}_0(\widehat{\beta}) = \frac{1}{M} \left(\sum_{i=1}^M (Y_{i:M:n})^{\widehat{\beta}} + R_1^\star \tau_1^{\widehat{\beta}} \right), \tag{4.15}$$

depending on the MLE of β. The MLE $\widehat{\beta}$ can be determined by solving the equation

$$\begin{aligned}
&\frac{1}{\beta} - \frac{r_1^\star \tau_1^\beta \log(\tau_1) + \sum_{i=1}^m (y_{i:m:n})^\beta \log(y_{i:m:n})}{r_1^\star \tau_1^\beta + \sum_{i=1}^m (y_{i:m:n})^\beta} \\
&\quad + \frac{1}{m} \sum_{i=1}^m \log(y_{i:m:n}) = 0
\end{aligned} \tag{4.16}$$

for β. This equation can only be solved numerically, for example, with the Newton–Raphson method (cf. Ortega & Rheinboldt, 1970, pp. 181–189). A simple option to choose an initial value is given by $\beta = 1$. It should be noted that it follows directly from Balakrishnan and Kateri (2008, eq. (3.11); case progressive Type-I censoring) that Equation (4.16) has a unique solution for $\beta > 0$.

Note that the MLEs $\widehat{\vartheta}_0$ and $\widehat{\beta}$ depend only on the observations on stage s_0. Hence, analogously to the case of two exponential distributions, the MLEs on each stage can be obtained only with the observed data on the corresponding stage. The MLE $\widehat{\vartheta}_1$ given in (4.14) has the same form as in Section 4.1 (Equation (4.6)). However, the density function of $\widehat{\vartheta}_1$ is slightly different from that one given in Theorem 4.2(2) since F_0 is the cumulative distribution function of a Weibull distribution.

Theorem 4.7 *The conditional density function $f_{\vartheta_0,\beta,\vartheta_1}^{\widehat{\vartheta}_1 | R_1^\star > 0}$ of $\widehat{\vartheta}_1$ is given by*

$$f_{\vartheta_0,\beta,\vartheta_1}^{\widehat{\vartheta}_1 | R_1^\star > 0}(t) = \sum_{d_1=0}^{n^\star} \frac{q_{\vartheta_0,\beta,d_1}}{\sum_{i=0}^{n^\star} q_{\vartheta_0,\beta,i}} \gamma\left(t; \frac{\vartheta_1}{\varrho(d_1)}, \varrho(d_1) \right), \quad t \in \mathbb{R},$$

TABLE 1 Two samples from an exponential distribution with $n = 16$, $\vartheta_0 = 40$, $\vartheta_1 = 20$, and $\tau_1 = 15$

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$v^{(0)}_{j:16}$	2.55	7.77	9.01	9.23	12.34	18.61	19.39	27.54	34.03	39.67	43.25	46.47	58.76	91.46	95.53	115.78
$v^{(1)}_{j:16}$	8.78	11.39	12.00	12.11	13.67	16.81	17.19	21.27	24.51	27.34	29.12	30.73	36.88	53.23	55.26	65.39

with $r_1^\star = \varrho(d_1)$ *as in* (4.11) *and*

$$q_{\vartheta_0,\beta,j} = \binom{n}{j}[1 - e^{-\tau_1^\beta/\vartheta_0}]^j[e^{-\tau_1^\beta/\vartheta_0}]^{n-j}, \quad j = 0, \ldots, n^\star.$$

As in Corollary 4.3, it can be shown that, conditionally on $R_1^\star > 0$, the MLE $\widehat{\vartheta}_1$ is an unbiased estimator for both options Type-P and Type-M.

Furthermore, a confidence interval for ϑ_1 can be obtained by proceeding as in Section 4.1.2 for exponentially distributed lifetimes. In particular, the conditional survival function

$$P_{\vartheta_0,\beta,\vartheta_1}(\widehat{\vartheta}_1 > t \mid R_1^\star > 0) = \sum_{d_1=0}^{n^\star} \frac{q_{\vartheta_0,\beta,d_1}}{\sum_{i=0}^{n^\star} q_{\vartheta_0,\beta,i}} \overline{\Gamma}\left(t; \frac{\vartheta_1}{\varrho(d_1)}, \varrho(d_1)\right), \quad t \in \mathbb{R},$$

is seen to be stochastically increasing in ϑ_1 with limits as in Corollary 4.4. Notice that the conditional survival function of ϑ_1 depends on ϑ_0 and β, too. In order to compute an approximate confidence interval $CI_{\vartheta_1} = [\vartheta_{1,L}, \vartheta_{1,U}]$ for ϑ_1 with level $\alpha \in (0, 1)$, we have to solve the equations

$$P_{\widehat{\vartheta}_{0,obs},\widehat{\beta}_{obs},\vartheta_{1,L}}(\widehat{\vartheta}_1 > \widehat{\vartheta}_{1,obs} \mid R_1^\star > 0) = \frac{\alpha}{2},$$
$$P_{\widehat{\vartheta}_{0,obs},\widehat{\beta}_{obs},\vartheta_{1,U}}(\widehat{\vartheta}_1 > \widehat{\vartheta}_{1,obs} \mid R_1^\star > 0) = 1 - \frac{\alpha}{2}$$

for $\vartheta_{1,L}$ and $\vartheta_{1,U}$, respectively.

5 | ILLUSTRATIVE EXAMPLE

To get a better understanding of the SLT model, we present a detailed illustrative example by using the generation procedure for SLTOSs (cf. Procedure 2.3). In Table 1, we show two samples generated according to step (2) of Procedure 2.3 when the lifetimes are exponentially distributed. Furthermore, the resulting SLTOSs for the two scenarios Type-P and Type-M are given in Tables 2 and 3, respectively. The sets $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ occurring in the generation process for both options (Type-P and Type-M) are given by

Type-P : $\mathcal{P} = \{1, 2, 3, 4, 5\}, \mathcal{Q} = \{6, 7, 8, 10, 11, 14\}$, and $\mathcal{R} = \{9, 12, 13, 15, 16\}$

so that $d_1 = |\mathcal{P}| = 5$, $d_2 = |\mathcal{Q}| = 6$, and $r_1^\star = |\mathcal{R}| = 5$,

Type-M : $\mathcal{P} = \{1, 2, 3, 4, 5\}, \mathcal{Q} = \{10, 11, 15\}$, and $\mathcal{R} = \{6, 7, 8, 9, 12, 13, 14, 16\}$

so that $d_1 = |\mathcal{P}| = 5$, $d_2 = |\mathcal{Q}| = 3$, and $r_1^\star = |\mathcal{R}| = 8$.

TABLE 2 Stage life testing sample with $p_1 = 0.5$ (Type-P)

j	1	2	3	4	5	6
$y_{j:11:16}$	2.55	7.77	9.01	9.23	12.34	
$y_{5+j:11:16}$	18.61	19.39	27.54	39.67	43.25	91.46
$z_{j:5}$	24.51	30.73	36.88	55.26	65.39	

TABLE 3 Stage life testing sample with $R_1^0 = 8$ (Type-M)

j	1	2	3	4	5	6	7	8
$y_{j:8:16}$	2.55	7.77	9.01	9.23	12.34			
$y_{5+j:8:16}$	39.67	43.25	95.53					
$z_{j:8}$	16.81	17.19	21.27	24.51	30.73	36.88	53.23	65.39

Based on these values (cf. Tables 2 and 3), the MLEs are given by

$$\text{Type-P}: \quad \widehat{\vartheta}_0 = \frac{1}{5+6}(280.82 + 5 \cdot 15) = 32.35$$
$$\text{and} \quad \widehat{\vartheta}_1 = \frac{1}{5}(212.77 - 5 \cdot 15) = 27.55,$$
$$\text{Type-M}: \quad \widehat{\vartheta}_0 = \frac{1}{5+3}(219.35 + 8 \cdot 15) = 42.42$$
$$\text{and} \quad \widehat{\vartheta}_1 = \frac{1}{8}(266.01 - 8 \cdot 15) = 18.25.$$

The corresponding densities (with true values ϑ_0, ϑ_1) are depicted in Figure 3. The confidence intervals with level of significance $\alpha = 0.05$ are obtained as

$$\text{Type-P}: \quad CI_{\vartheta_0} = [19.23, 63.74]$$
$$\text{and} \quad CI_{\vartheta_1} = [13.08, 92.61],$$
$$\text{Type-M}: \quad CI_{\vartheta_0} = [23.53, 98.25]$$
$$\text{and} \quad CI_{\vartheta_1} = [10.11, 42.40].$$

6 | SIMULATION STUDY

6.1 | Exponential distributions

6.1.1 | MLEs

For illustration, we simulated $N = 10^6$ samples from the exponential distribution with $\vartheta_0 = 1.0$ on stage s_0 and $\vartheta_1 = 0.5$ on stage s_1, and sample size $n = 12$. The same samples were then used to generate the SLTOSs with option Type-P and Type-M, respectively. Based on the generated samples, we derived the mean of the estimates $\bar{\widehat{\vartheta}}_0$ and $\bar{\widehat{\vartheta}}_1$ via $\bar{\widehat{\vartheta}} = \frac{1}{N}\sum_{i=1}^N \widehat{\vartheta}^{(i)}$ and the sample coefficient of variation $CV_{\widehat{\vartheta}_0}$ and $CV_{\widehat{\vartheta}_1}$ defined by

$$CV_{\widehat{\vartheta}} = CV(\bar{\widehat{\vartheta}}) = \frac{SD_{\widehat{\vartheta}}}{\bar{\widehat{\vartheta}}}, \quad \text{where}$$

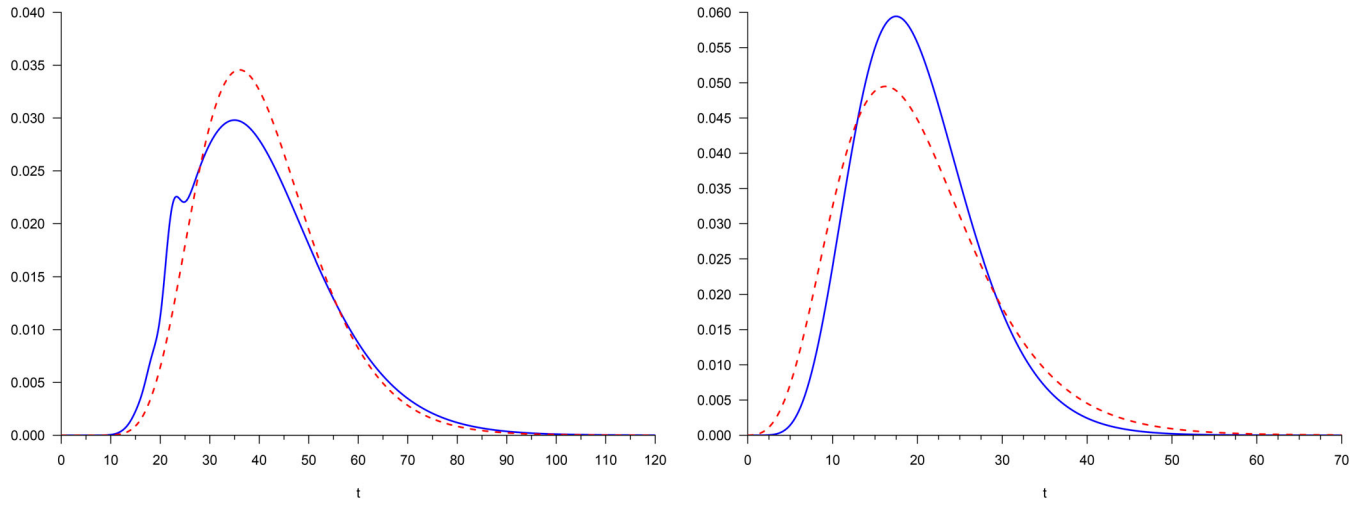

FIGURE 3 Plots of density functions $f_{\vartheta_0}^{\hat{\vartheta}_0}$ (left) and $f_{\vartheta_0,\vartheta_1}^{\hat{\vartheta}_1|R_1^\star>0}$ (right) and for Type-M (solid blue line) and Type-P (dashed red line) with true values $\vartheta_0 = 40$ and $\vartheta_1 = 20$ (see caption of Table 1) [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 4 Maximum likelihood estimations for $\vartheta_0 = 1.0$ and $\vartheta_1 = 0.5$ with $\tau_1 = 0.5$, $n = 12$, and $N = 10^6$ (stage life testing for exponential distributions)

Model	π_1	R_1^0	$\bar{\hat{\vartheta}}_0$	$\bar{\hat{\vartheta}}_1$	E_{ϑ_0}	$CV_{\hat{\vartheta}_0}$	$CV_{\hat{\vartheta}_1}$	CV_{ϑ_0}	CV_{ϑ_1}	$\overline{m}$	$\overline{r_1^\star}$	N_1
Type-P	0.25		1.0089	0.4996	1.0087	0.3073	0.8766	0.3073	0.8764	10.55	1.45	986,446
	0.50	–	1.0253	0.5001	1.0250	0.3426	0.5677	0.3424	0.5672	8.61	3.39	999,701
	0.75		1.0611	0.5001	1.0609	0.3964	0.4630	0.3971	0.4621	6.91	5.09	999,701
Type-M		3	1.0004	0.5000	1.0002	0.3327	0.5783	0.3329	0.5778	9.00	3.00	999,982
	—	6	1.0179	0.5001	1.0177	0.3811	0.4197	0.3818	0.4193	6.21	5.79	999,982
		9	1.1438	0.5000	1.1433	0.4640	0.3852	0.4645	0.3848	4.83	7.17	999,982

$$SD_{\hat{\vartheta}} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\hat{\vartheta}^{(i)} - \bar{\hat{\vartheta}})^2}$$

denotes the standard deviation. Furthermore, we calculated the mean number of the observed objects on each stage (ie, $\overline{m}$, $\overline{r_1^\star}$). Moreover, we computed the expectation $E_{\vartheta_0} = E_{\vartheta_0}(\hat{\vartheta}_0)$ and the coefficient of variation CI_{ϑ_0} and CI_{ϑ_1} defined by

$$CV_\vartheta = \frac{\sqrt{Var_\vartheta(\hat{\vartheta})}}{E_\vartheta}.$$

Since $\hat{\vartheta}_1$ is unbiased, we omit the expectation $E_{\vartheta_1}(\hat{\vartheta}_1) = \vartheta_1$. Furthermore, the counter $N_1 \le N$ denotes the number of samples where at least one failure has been observed on stage s_1, that is, N_1 is the number of samples with $r_1^\star > 0$. Note that we used the same samples for each design of the life test. From the results presented in Table 4, we conclude:

- The expectations E_{ϑ_0} are close to the means $\bar{\hat{\vartheta}}_0$;
- The values $CV_{\hat{\vartheta}_0}$ and $CV_{\hat{\vartheta}_1}$ are close to the theoretical values CI_{ϑ_0} and CI_{ϑ_1}, respectively;
- The higher the means $\overline{m}$, the closer the means $\bar{\hat{\vartheta}}_0$ are to the true value $\vartheta_0 = 1.0$ and the smaller are the values of $CV_{\hat{\vartheta}_0}$ for both options;
- The higher the means $\overline{r_1^\star}$, the smaller are the values of $CV_{\hat{\vartheta}_1}$ for both scenarios;
- The means $\bar{\hat{\vartheta}}_1$ are more or less identical to the true value $\vartheta_1 = 0.5$. This underlines that $\hat{\vartheta}_1$ is an unbiased estimator for both options;
- The difference between the two options Type-P and Type-M is reflected by the different means $\overline{m}$ and $\overline{r_1^\star}$ on stage s_0 and s_1, respectively;
- The values of N_1 show that the MLE $\hat{\vartheta}_1$ exists more often for Type-M than for Type-P. This illustrates Lemma 4.1. The theoretical probabilities as well as the respective frequencies are given in Table 5.

TABLE 5 Probabilities from Lemma 4.1 and respective frequencies in the setting of Table 4

Model	π_1	R_1^0	$n^\star$	$P(R_1^\star > 0)$	Frequency
Type-P	0.25	—	8	0.9864756680	0.986446
	0.50	—	10	0.9997315155	0.999701
	0.75	—	10	0.9997315155	0.999701
Type-M	—	3, 6, 9	11	0.9999862301	0.999982

TABLE 6 Confidence intervals ("nleqslv") for $\vartheta_0 = 1.0$ and $\vartheta_1 = 0.5$ with $\tau_1 = 0.5$, $n = 12$, $\pi_1 = 0.50$, $R_1^0 = 6$, and $N = 10^4$ (stage life testing for exponential distributions)

Model	α	$\overline{CI}_{\vartheta_0}$	CP_{ϑ_0}	$\bar{\hat{\vartheta}}_0$	$\overline{CI}_{\vartheta_1}$	CT_{ϑ_1}	$\bar{\hat{\vartheta}}_1$	$\overline{m}$	$\overline{r_1^\star}$	N_1
Type-P	0.01	[0.4798,3.0407]	99.06	1.0241	[0.1562,7.9395]	99.48	0.4988	8.61	3.39	9,998
	0.05	[0.5686,2.2806]	94.98	1.0241	[0.2066,2.9087]	95.84	0.4988	8.61	3.39	9,998
	0.10	[0.6226,1.9868]	89.75	1.0241	[0.2389,2.0071]	90.97	0.4988	8.61	3.39	9,998
Type-M	0.01	[0.4323,3.8828]	99.02	1.0195	[0.1996,2.6256]	99.33	0.4977	6.21	5.79	10,000
	0.05	[0.5233,2.7213]	94.70	1.0195	[0.2473,1.5352]	95.89	0.4977	6.21	5.79	10,000
	0.10	[0.5802,2.2978]	89.89	1.0195	[0.2767,1.2429]	91.16	0.4977	6.21	5.79	10,000

TABLE 7 Maximum likelihood estimations ("nleqslv") for $\vartheta_0 = 2.0$, $\beta = 4.0$ and $\vartheta_1 = 0.5$ with $\tau_1 = 1.1$, $n = 24$, and $N = 10^6$ (stage life testing for Weibull and exponential distribution)

Model	π_1	R_1^0	$\bar{\hat{\vartheta}}_0$	$\bar{\hat{\beta}}$	$\bar{\hat{\vartheta}}_1$	$CV_{\hat{\vartheta}_0}$	$CV_{\hat{\beta}}$	$CV_{\hat{\vartheta}_1}$	CV_{ϑ_1}	$\overline{m}$	$\overline{r_1^\star}$	N_1
	0.25		2.1726	4.2748	0.4994	0.3285	0.1819	0.6602	0.6596	21.49	2.51	999,741
Type-P	0.50	—	2.1720	4.3180	0.5002	0.3361	0.1999	0.4385	0.4385	18.48	5.52	999,997
	0.75		2.1708	4.3629	0.5000	0.3444	0.2272	0.3581	0.3578	15.72	8.28	999,997
		6	2.1605	4.3691	0.5003	0.3370	0.2034	0.4086	0.4086	18.01	5.99	999,999
Type-M	—	12	2.1584	4.4384	0.5000	0.3435	0.2666	0.3091	0.3089	13.21	10.79	999,999
		18	2.1858	4.3375	0.5000	0.3758	0.2880	0.3022	0.3020	12.46	11.54	999,999

TABLE 8 Confidence intervals ("nleqslv") for $\vartheta_1 = 0.5$, where $\vartheta_0 = 2.0$ and $\beta = 4.0$ with $\tau_1 = 1.1$, $n = 24$, and $N = 10^4$ (stage life testing for Weibull and exponential distribution)

Model	α	π_1	R_1^0	$\overline{CI}_{\vartheta_1}$	$\overline{L}_{\vartheta_1}$	CP_{ϑ_1}	$\bar{\hat{\vartheta}}_1$	$\overline{m}$	$\overline{r_1^\star}$	N_1
Type-P	0.01			[0.1957,2.8419]	2.6462	99.33	0.4986	18.47	5.53	10,000
	0.05	0.50	—	[0.2445,1.6216]	1.3771	95.64	0.4986	18.47	5.53	10,000
	0.10			[0.2745,1.2942]	1.0197	90.74	0.4986	18.47	5.53	10,000
Type-M	0.01			[0.2451,1.4533]	1.2082	99.50	0.4976	13.21	10.79	10,000
	0.05	–	12	[0.2900,1.0663]	0.7763	96.20	0.4976	13.21	10.79	10,000
	0.10			[0.3163,0.9308]	0.6145	91.30	0.4976	13.21	10.79	10,000

6.1.2 | Exact confidence intervals

The results of a simulation study are presented in Table 6. Based on the generated samples, we determined the confidence intervals by solving the equations in (4.12) and in (4.13) using the Newton–Raphson method with initial values $0.5 \cdot \widehat{\vartheta}_h^{(i)}$ and $1.5 \cdot \widehat{\vartheta}_h^{(i)}$, $h \in \{0, 1\}$, $i \in \{1, \ldots, N\}$. We use the package "nleqslv" from the statistical software R (cf. Hasselman, 2017). The implemented procedure is also called "nleqslv." This procedure is always used with default options (ie, method = "Newton," xtol = 10^{-8}, ftol = 10^{-8}). We derived the "mean" of the confidence intervals $\overline{CI}_{\vartheta_0}$ and $\overline{CI}_{\vartheta_1}$ via $\overline{CI}_{\vartheta} = [\overline{\vartheta}_L, \overline{\vartheta}_U]$. Moreover, we present the empirical coverage probabilities $CP_{\vartheta 0}$ and $CP_{\vartheta 1}$ which, in general, are close to the desired level of significance. Recall that we used the same samples for each design of the life test.

6.2 | Weibull distribution and exponential distribution

To illustrate the MLEs and the constructed confidence intervals, we conducted a simulation study. We applied the Newton–Raphson method by using the procedure "nleqslv" with initial value $\widetilde{\beta} = 1$ to solve Equation (4.16). With the numerical results for $\widehat{\beta}$, we determined the estimates of ϑ_0 with (4.15). The estimates of ϑ_1 are obtained from (4.14). The results for the MLEs are presented in Table 7 whereas the results for the confidence intervals are presented in Table 8.

7 | CONCLUSION AND OUTLOOK

The present *SLT* model is the first approach to incorporate the idea of further testing of withdrawn objects in a progressive Type-I censored life test. As a starting point, we have discussed this model for the case of a single stage-change time τ_1. It has been illustrated that the model can be seen as a unified approach to either progressive censoring or simple-step stress testing which are seen to be marginal cases of the present model ignoring some information. Moreover, the presented findings illustrate that exact and approximate inferential results can be obtained for exponentially and Weibull distributed lifetimes. An adaption to other lifetime distributions will be possible.

Clearly, the present model can be developed in various direction. First, it is evident that an extension to more than one stage-change times $\tau_1 < \cdots < \tau_k$ is an interesting option. This k-step SLT is currently under study. Furthermore, it would be possible to consider a link function relating the parameters

ϑ_0 and ϑ_1 to the SLT levels s_0,s_1 as it has been extensively discussed in simple-step stress testing, for example,

$$\vartheta_j = \eta_j(a, b), \quad j = 0, 1, \tag{7.1}$$

with an appropriate link functions $\eta_j, j = 0, 1$, and (unknown) parameters a,b. As an example, one might consider a log-linear link relationship $\vartheta_j = e^{a+bs_j}$, $j = 0, 1$ (see, eg, Kundu and Ganguly (2017, p. 76)). Moreover, additional censoring (Type-I, Type-II, or kinds of hybrid censoring) can be imposed on the data. For instance, one may introduce a threshold T that terminates the complete life test. On the other hand, the experimenter may stop the observation after observing a prefixed number k_i of failures on each stage. This would introduce a Type-II censoring to the data. Of course, such models can also be extended to situation of k-step SLT where, due to the complexity of the model, even more options will be conceivable.

Furthermore, other inferential approaches like Bayesian inference or the study of alternative lifetime distributions may be possible.

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ORCID

Erhard Cramer https://orcid.org/0000-0001-8354-5425

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APPENDIX (PROOFS)

In the proof of Theorem 3.1, we utilize the distributional assumptions sketched in Section 2 as well as the result presented in Lemma A.1. For integers $1 \le d < \ell < k$, we need the set $S^{<}_{d\dots k,\ell}$ of all permutations π on the set $\{d, \dots, k\}$ with $\pi(d) < \cdots < \pi(k-\ell)$ and $\pi(k-\ell+1) < \cdots < \pi(k)$.

Lemma A.1 *Let* $U_1, \dots, U_k \overset{\text{iid}}{\sim} \mathrm{U}(0,1)$ *and* $2 \le \ell \le k-1$ *Then*,

$$\sum_{\pi \in S^{<}_{1\dots k,\ell}} P\left(U_{\pi(j):k} \le t_j, 1 \le j \le k-\ell, U_{\pi(j):k} \le t_j, k-\ell+1 \le j \le k\right)$$

$$= \binom{k}{\ell} P\left(U_{j:k-\ell} \le t_j, 1 \le j \le k-\ell\right) \times P\left(U_{j-k+\ell:\ell} \le t_j, k-\ell+1 \le j \le k\right).$$

Proof of Lemma A.1 Let $B_k = \{(u_1, \dots, u_k) \in [0,1]^k \mid \exists \pi \in S_k : u_{\pi(1)} < \cdots < u_{\pi(k)}\}$ where S_k denotes the set of all permutations of $\{1, \dots, k\}$. Furthermore, for $(u_1, \dots, u_k) \in B_k$, denote by $u_{1:k} < \cdots < u_{k:k}$ the respective ordered values. Then, for $t_j \in \mathbb{R}$, $1 \le j \le k$, we get

$$\sum_{\pi \in S^{<}_{1\dots k,\ell}} P(U_{\pi(j):k} \le t_j, 1 \le j \le k-\ell, U_{\pi(j):k} \le t_j, k-\ell+1 \le j \le k)$$

$$= \sum_{\pi \in S^{<}_{1\dots k,\ell}} \int_{B_k} P(u_{\pi(j):k} \le t_j, 1 \le j \le k-\ell, u_{\pi(j):k} \le t_j, k-\ell+1 \le j \le k) d(u_1, \dots, u_k)$$

Let $(u_1, \dots, u_k) \in B_k$ be fixed but arbitrary. Then, for every $\pi \in S^{<}_{1\dots k,\ell}$, there exists exactly one permutation $\pi^* \in S^{<}_{1\dots k,\ell}$ with $(u_{\pi(j):k})_{1\le j\le k-\ell} = (u^{\pi^*}_{j:k-\ell})_{1\le j\le k-\ell}$ and $(u_{\pi(j):k})_{k-\ell+1\le j\le k} = (u^{\pi^*}_{j-k+\ell:\ell})_{k-\ell+1\le j\le k}$. Here, $u^{\pi^*}_{1:k-\ell}, \dots, u^{\pi^*}_{k-\ell:k-\ell}$ and $u^{\pi^*}_{1:\ell}, \dots, u^{\pi^*}_{\ell:\ell}$ denote the ordered values of $u_{\pi^*(1)}, \dots, u_{\pi^*(k-\ell)}$ and $u_{\pi^*(k-\ell+1)}, \dots, u_{\pi^*(k)}$, respectively.

$$= \sum_{\pi^* \in S^{<}_{1\dots k,\ell}} \int_{B_k} P(u^{\pi^*}_{j:k-\ell} \le t_j, 1 \le j \le k-\ell, u^{\pi^*}_{j-k+\ell:\ell} \le t_j, k-\ell+1 \le j \le k) d(u_1, \dots, u_k)$$

$$= \sum_{\pi^* \in S^{<}_{1\dots k,\ell}} P(U^{\pi^*}_{j:k-\ell} \le t_j, 1 \le j \le k-\ell, U^{\pi^*}_{j-k+\ell:\ell} \le t_j, k-\ell+1 \le j \le k)$$

Since $(U^{\pi^*}_{j:k-\ell})_{1\le j\le k-\ell}$ and $(U^{\pi^*}_{j-k+\ell:\ell})_{k-\ell+1\le j\le k}$ depend on different random variables, we get

$$= \sum_{\pi^* \in S^{<}_{1\dots k,\ell}} P\left(U^{\pi^*}_{j:k-\ell} \le t_j, 1 \le j \le k-\ell\right) \times P\left(U^{\pi^*}_{j-k+\ell:\ell} \le t_j, k-\ell+1 \le j \le k\right)$$

Applying the iid assumption, that is, $U_1, \dots, U_k \overset{\text{iid}}{\sim} \mathrm{U}(0,1)$, and $\mid S^{<}_{1\dots k,\ell} \mid = \binom{k}{\ell}$, we arrive finally at the desired representation

$$= \binom{k}{\ell} P\left(U_{j:k-\ell} \le t_j, 1 \le j \le k-\ell\right) \times P\left(U_{j-k+\ell:\ell} \le t_j, 1 \le j \le \ell\right).$$

■

Proof of Theorem 3.1 The case $d_1 = n$ has been discussed above. Hence, let $d_1 < n$ and $U_1, \dots, U_n \overset{\text{iid}}{\sim} \mathrm{U}(0,1)$. We consider the

probability

$$q(x_1, \dots, x_n) = P(Y_{1:m:n} \le x_1, \dots, Y_{d_1:m:n} \le x_{d_1}, Y_{d_1+1:m:n} \le x_{d_1+1}, \dots, Y_{n-r_1^\star:m:n} \le x_{n-r_1^\star}, Z_{1:r_1^\star} \le x_{n-r_1^\star+1}, \dots, Z_{r_1^\star:r_1^\star} \le x_n, D_1 = d_1), \quad \text{(A.1)}$$

for $x_1 \le \cdots \le x_{d_1} \le \tau_1 < x_{d_1+1} \le \cdots \le x_{n-r_1^\star}$ and $\tau_1 < x_{n-r_1^\star+1} \le \cdots \le x_n$, where either $r_1^\star = \lfloor \pi_1 \cdot (n-d_1) \rfloor$ or $r_1^\star = \min\{n-d_1, R_1^0\}$. From the definition of D_1, D_2, and $r_1^\star$, it follows that.

1. a. d_1 ordered observations are located in the interval $(-\infty, \tau_1]$ on stage s_0;
   b. $n - d_1 - r_1^\star$ ordered observations are located in the interval (τ_1, ∞) on stage s_0;
2. $r_1^\star$ ordered observations are located in the interval (τ_1, ∞) on stage s_1.

Inspired by Procedure 2.3, we model the selection process in step (5) by a random permutation of the integers $d_1+1, \dots, n$. In particular, Υ denotes the random variable selecting one particular permutation from $S^<_{d_1+1 \dots n, r_1^\star}$ according to a discrete uniform distribution, that is, $P(\Upsilon = \pi) = 1/\binom{n-d_1}{r_1^\star}$, $\pi \in S^<_{d_1+1 \dots n, r_1^\star}$. Then,

$$\begin{aligned} q(x_1, \dots, x_n) &= P(F_0^{-1}(U_{j:n}) \le x_j, \\ &1 \le j \le d_1, F_0^{-1}(U_{\Upsilon(j):n}) \le x_j, d_1+1 \le j \le n - r_1^\star, \\ &F_1^{-1}(U_{\Upsilon(j):n}) - v_1 + \tau_1 \le x_j, n - r_1^\star + 1 \le j \le n, D_1 = d_1) \\ &= \sum_{\pi \in S^<_{d_1+1 \dots n, r_1^\star}} P(F_0^{-1}(U_{j:n}) \le x_j, 1 \le j \le d_1, \\ &F_0^{-1}(U_{\pi(j):n}) \le x_j, d_1+1 \le j \le n - r_1^\star, \\ &F_1^{-1}(U_{\pi(j):n}) - v_1 + \tau_1 \le x_j, n - r_1^\star + 1 \le j \le n, \\ &D_1 = d_1 \mid \Upsilon = \pi) P(\Upsilon = \pi) \end{aligned}$$

Now, $P(\Upsilon = \pi) = 1/\binom{n-d_1}{r_1^\star}$ and $U_{1:n}, \dots, U_{n:n}$ are independent of Υ by assumption. Thus, we get

$$\begin{aligned} &= \frac{1}{\binom{n-d_1}{r_1^\star}} \sum_{\pi \in S^<_{d_1+1 \dots n, r_1^\star}} P(F_0^{-1}(U_{j:n}) \le x_j, 1 \le j \le d_1, \\ &F_0^{-1}(U_{\pi(j):n}) \le x_j, d_1+1 \le j \le n - r_1^\star, \\ &F_1^{-1}(U_{\pi(j):n}) - v_1 + \tau_1 \le x_j, n - r_1^\star + 1 \le j \le n, D_1 = d_1) \end{aligned}$$

Using that $U_{1:n}, \dots, U_{d_1:n}$ and $U_{d_1+1:n}, \dots, U_{n:n}$ are (conditionally) independent given $D_1 = d_1$, we find

$$\begin{aligned} &= \frac{1}{\binom{n-d_1}{r_1^\star}} P(U_{j:n} \le F_0(x_j), 1 \le j \le d_1 \mid D_1 = d_1) P(D_1 = d_1) \\ &\times \sum_{\pi \in S^<_{d_1+1 \dots n, r_1^\star}} P(U_{\pi(j):n} \le F_0(x_j), d_1+1 \le j \le n - r_1^\star, \\ &U_{\pi(j):n} \le F_1(x_j + v_1 - \tau_1), n - r_1^\star + 1 \le j \le n \mid D_1 = d_1) \end{aligned}$$

According to Iliopoulos and Balakrishnan (2009), $U_{d_1+1:n}, \dots, U_{n:n} \mid D_1 = d_1$ are distributed as order statistics $U^*_{1:n-d_1}, \dots, U^*_{n-d_1:n-d_1}$ from a uniform distribution $\mathrm{U}(F_0(\tau_1), 1)$. Thus, we arrive at

$$\begin{aligned} &= \frac{1}{\binom{n-d_1}{r_1^\star}} P(U_{j:n} \le F_0(x_j), 1 \le j \le d_1 \mid D_1 = d_1) P(D_1 = d_1) \\ &\times \sum_{\pi \in S^<_{1 \dots n-d_1, r_1^\star}} P(U^*_{\pi(j):n-d_1} \le F_0(x_{j+d_1}), 1 \le j \le n - d_1 - r_1^\star, \\ &U^*_{\pi(j):n-d_1} \le F_1(x_{j+d_1} + v_1 - \tau_1), n - d_1 - r_1^\star + 1 \le j \le n - d_1) \end{aligned}$$

Interpreting $U^*_{1:n-d_1}, \dots, U^*_{n-d_1:n-d_1}$ as order statistics of the iid sample $U^*_1, \dots, U^*_{n-d_1} \overset{\text{iid}}{\sim} \mathrm{U}(F_0(\tau_1), 1)$, we get with $U_j = (U^*_j - F_0(\tau_1))/(1 - F_0(\tau_1)) \sim \mathrm{U}(0,1)$

$$\begin{aligned} &= \frac{1}{\binom{n-d_1}{r_1^\star}} P\left(U_{j:n} \le F_0(x_j), 1 \le j \le d_1 \mid D_1 = d_1\right) \\ &\times P(D_1 = d_1) \sum_{\pi \in S^<_{1 \dots n-d_1, r_1^\star}} P\Bigg(U_{\pi(j):n-d_1} \le \frac{F_0(x_{j+d_1}) - F_0(\tau_1)}{1 - F_0(\tau_1)}, \\ &1 \le j \le n - d_1 - r_1^\star, \\ &U_{\pi(j):n-d_1} \le \frac{F_1(x_{j+d_1} + v_1 - \tau_1) - F_0(\tau_1)}{1 - F_0(\tau_1)}, \\ &n - d_1 - r_1^\star + 1 \le j \le n - d_1\Bigg). \end{aligned}$$

Applying Lemma A.1 and using that $(U_{1:n}/F_0(\tau_1), \dots, U_{d_1:n}/F_0(\tau_1)) \mid D_1 = d_1 \overset{d}{=} (U_{1:d_1}, \dots, U_{d_1:d_1})$ (see Iliopoulos & Balakrishnan, 2009) and $\mid S^<_{1 \dots n-d_1, r_1^\star} \mid = \binom{n-d_1}{r_1^\star}$, this yields

$$\begin{aligned} q(x_1, \dots, x_n) &= P(D_1 = d_1) P\left(U_{j:d_1} \le \frac{F_0(x_j)}{F_0(\tau_1)}, 1 \le j \le d_1\right) \\ &\times P\left(U_{j:n-d_1-r_1^\star} \le \frac{F_0(x_{d_1+j}) - F_0(\tau_1)}{1 - F_0(\tau_1)}, 1 \le j \le n - d_1 - r_1^\star\right) \\ &\times P\left(U_{j:r_1^\star} \le \frac{F_1(x_{n-r_1^\star+j} + v_1 - \tau_1) - F_0(\tau_1)}{1 - F_0(\tau_1)}, 1 \le j \le r_1^\star\right). \end{aligned} \quad \text{(A.2)}$$

Obviously, q can be written as an integral so that the joint density function of $\mathbf{Y}, \mathbf{Z}, D_1$ is given by

$$\begin{aligned} f^{\mathbf{Y},\mathbf{Z},D_1}(t_1, \dots, t_n, d_1) &= P(D_1 = d_1) d_1! \\ &\times \prod_{j=1}^{d_1} \frac{f_0(t_j)}{F_0(\tau_1)} (n - d_1 - r_1^\star)! \end{aligned}$$

$$\times \prod_{j=d_1+1}^{n-r_1^\star} \frac{f_0(t_j)}{1-F_0(\tau_1)} r_1^\star! \prod_{j=n-r_1^\star+1}^{n} \frac{f_1(t_j+v_1-\tau_1)}{1-F_0(\tau_1)}$$

where $t_1 \leq \cdots \leq t_{d1} \leq \tau_1$, $\tau_1 < t_{d_1+1} \leq \cdots \leq t_{n-r_1^\star}$, and $\tau_1 < t_{n-r_1^\star+1} \leq \cdots \leq t_n$. Using that

$$P(D_1 = d_1) = \binom{n}{d_1} F_0^{d_1}(\tau_1)(1-F_0(\tau_1))^{n-d_1}, \quad \text{(A.3)}$$

we arrive at the desired expression given in (3.1). ■