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Hyperbolic discounting and the time-consistent solution of three canonical environmental problems

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Abstract

In this paper I propose a time-consistent method of discounting hyperbolically and apply it to three canonical environmental problems: (i) optimal renewable resource use, (ii) the tragedy of the commons, and (iii) economic growth and pollution. I show that, irrespective of potentially high initial discount rates, time-consistent hyperbolic discounting leads always to a steady state of maximum yield, or, if the environment enters the utility function, a steady state where the Green Golden Rule applies. While (asymptotic) extinction is a real threat under exponential discounting it is impossible under time-consistent hyperbolic discounting. This result is also confirmed for open-access resources. In a model of economic growth and pollution, hyperbolic discounting establishes the Golden Rule of capital accumulation and the modified Green Golden Rule.

1 | INTRODUCTION

Economic scholars agree that the appropriate choice of the discount rate is of preeminent importance in the evaluation of policies that matter for the distant future. Environmental policies are perhaps the best example for decision problems in which the costs accrue from

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today onward and most of the benefits occur in the distant future. For example, any potential disagreement in the assessment of the material effects of global warming is dwarfed by the impact of the discounting method on the implied net present value in monetary terms (Dasgupta, 2008; Nordhaus, 2007; Weitzman, 2007). Conventional economic analysis is based on exponential discounting, that is, it applies a constant discount rate. In environmental economics the discounting debate focuses mostly on the social discount rate (SDR; of future income or consumption). Weitzman (2001) showed that, if experts disagree about the “correct” constant SDR, the discount rate implied by the average discount factor declines over time. For these reasons (and others) a panel of leading economists in environmental economics recently came to the conclusion that the discounting of costs and benefits of long-horizon projects should best be done at a hyperbolic rate (Arrow et al., 2014).

The SDR is conventionally derived from the Euler equation of intertemporal utility maximization such that $SDR = \rho + \eta g_c$, in which ρ is the pure rate of time preference (i.e., the utility discount rate), η is the inverse of the elasticity of intertemporal substitution, and g_c is the consumption growth rate. In principle, the SDR may decline because the elasticity of intertemporal substitution increases over time or because consumption growth declines over time. Another reason for a declining SDR is that the rate of pure time preference declines over time. In a setting where the elasticity of intertemporal substitution is assumed to be constant and the economy is at its steady state or there is no long-run growth, a declining rate of time preference is also the only way to generate a declining SDR. Drupp et al. (2018) extend the Weitzman (2001) approach by collecting expert assessments of the individual components of the SDR and report disagreement on the pure rate of time preference. Applying Weitzman's (2001) aggregation, this finding implies in the aggregate a declining rate of pure time preference. In the applications studied in this paper, this means that the social planner faces a declining pure rate of time preference since the elasticity of intertemporal substitution is constant and there is no growth.

An important concern, however, is that hyperbolic discounting could lead to suboptimal and inconsistent decision making. In fact, the introduction of hyperbolic discounting in the economics of psychology is motivated by the need to explain preference reversals and time-inconsistent behavior (Frederick et al., 2002). Consequently, scholars tend to believe that hyperbolic discounting inevitably implies an inconsistency problem.¹ Applying hyperbolic discounting to environmental decision making would then entail a continuous reversal of previous plans and suboptimal outcomes (Winkler, 2006). For example, Hepburn et al. (2010) show that naive social planners who apply hyperbolic discounting without being aware of the time-inconsistency problem can manage renewable resources into extinction.

Alternatively, it has been suggested that sophisticated social planners, who are aware of the time-inconsistency problem, could achieve consistent decisions by using Markovian strategies in a dynamic game with (not yet existent) future planners (Gerlagh & Liski, 2018). A feedback solution, however, is not entirely satisfying either. Closed-form solutions, which would be needed for a straightforward implementation, are usually obtained only under quite restrictive assumptions, there are usually (infinitely) many feedback strategies such that researchers confine the discussion to a specific strategy (frequently the linear strategy), and the achieved

¹For example, Angeletos et al. (2001, p. 54) write that “When a household has a hyperbolic discount function, the household will have dynamically inconsistent preferences, so the problem of allocating consumption over time cannot be treated as a straightforward optimization problem.” Recent studies of time-inconsistent hyperbolic discounting include Kang and Ye (2019) and Liu et al. (2020).

Nash equilibrium is not Pareto-optimal, which leads to potentially large welfare losses compared with the open-loop strategy. It would thus be desirable to combine the feature of Pareto-optimality of the open-loop strategy under commitment with the feature of time-consistency of the feedback strategy. This is the inspiration and starting point of the present paper.

In this paper, I suggest a different solution to the problem and investigate environmental decision making under *time-consistent* hyperbolic discounting. I use the term time-consistent hyperbolic discounting to define a discounting method that leads in ordinary (i.e., “open loop”) decision making to time-consistent solutions. Since the open-loop solution is time-consistent, the Pareto-optimal solution does not require any commitment of the social planner or generations of social planners and there is no need to compute feedback strategies. Many economists know the result from Strotz’s (1956) seminal paper that only exponential discounting leads to time-consistent decisions if the discount factor is a function of the algebraic distance ($t_0 - t$) between planning time t_0 and payoff time t . The “if”-clause, however, has sometimes been forgotten in the following literature such that the conventional wisdom evolved that nonexponential discounting necessarily entails time inconsistency. Here, I apply a form of hyperbolic discounting to which the theorem of multiplicative separability in decision time and payoff time applies (see Burness, 1976; Drouhin, 2009, 2020). As a result, decisions are time-consistent.

Time-consistent hyperbolic discounting got so far relatively little attention in environmental economics. A notable exception is Heal (2005) who discusses multiplicatively separable discounting (labeled as logarithmic discounting) and praises its attractive properties (pp. 1125 and 1138). Notwithstanding this prominent “advertisement” there are, to the best of my knowledge, no applications of time-consistent hyperbolic discounting in the context of renewable resources. Here, I fill this lacuna and solve three canonical problems of renewable resource use for the case of time-consistent hyperbolic discounting: the optimal use of a renewable resource, the noncooperative use of an open-access resource, and the problem of optimal growth and pollution.²

Time-consistent hyperbolic discounting, in contrast to time-inconsistent behavior, leads to quite benign environmental outcomes. At the asymptotic steady state, it establishes the social optimum for closed-access resource management (maximum sustainable yield and Green Golden Rule). Compared with exponential discounting, time-consistent hyperbolic discounting leads to a more conservative resource use. In particular, there exist situations in which exponential discounting leads to extinction while hyperbolic discounting leads always to sustainable and socially optimal outcomes.

It is perhaps less of a surprise that hyperbolic discounting implies a conservative resource use close to the steady state since, by design, the impact of discounting vanishes asymptotically at the steady state. However, the result that hyperbolic discounting turns out to be good for the environment when the economy starts far off the steady state, that is, that the (asymptotic) steady state is globally saddlepoint-stable, is remarkable. After all, far off the steady state, hyperbolic discount rates exceed (by far) the present-value equivalent constant discount rate. Seemingly, they should thus lead to overexploitation of the resource. The intuition for the nonobvious result of sustainability lies in the inherent transitional dynamics of the resource stock and the discount rate. The first-order conditions exclude extinction in finite time as an optimal outcome (intuitively, utility would jump to minus infinity at the point of extinction of the resource). Thus, only asymptotic extinction could be an optimal outcome. When the discount rate declines hyperbolically, however,

²Strulik and Trimborn (2018) investigate health behavior and health outcomes in a life cycle model when individuals have time-consistent hyperbolic preferences.

there exists always a finite point in time at which the discount rate falls short of any positive natural growth rate of the resource, a condition that ensures sustainability. With further declining discount rate, any resource-management strategy converges toward the long-run social optimum. For conventional exponential discounting, however, asymptotic extinction is a real threat, in particular for common-access resources. I explain these features with the help of phase diagrams and numerical implementations of the models.

The conclusion that hyperbolic discounting can be time-consistent and that its application leads to the long-run social optimum has policy relevance since a discounting rule can be easily implemented as an institution. In fact, declining SDRs are already applied in the evaluation of public projects in the United Kingdom and France (HM Treasury, 2003; Lebegue, 2005). Concrete economic and environmental policies, in contrast, are harder to govern by institutions and may change from day to day, the specific problem at hand, and the responsible policymakers (see Acemoglu et al., 2005, on the importance of institutions vs. policy). This conclusion seems to be of particular importance for the management of open-access resources where noncooperative strategies are likely. It is thus reassuring that the implementation of a hyperbolic discounting rule, given its multiplicative separability, will lead to time-consistent decisions. Future generations of social planners or future incarnations of the current social planner will stick to the present plan unless the constraints or parameters of the problem change. This means that there is no need for commitment or sophistication of strategies by the current policymaker.

The paper is organized as follows. In Section 2, I introduce time-consistent hyperbolic discounting. In Section 3, I apply the method to the optimal management of a renewable resource. In Section 4, I discuss the exploitation of an open-access resource under hyperbolic discounting and in Section 5, I discuss the problem of optimal economic growth and pollution. I discuss transitional dynamics qualitatively using phase diagrams as well as quantitatively with numerical examples. Section 6 concludes the paper.

2 | WELFARE AND DISCOUNTING

Consider a social planner who maximizes welfare understood as utility from consumption $u(c)$ experienced over an infinite time horizon and discounted to the present.

$$W(t_0) = \int_{t_0}^{\infty} D(t_0, t) \cdot u(c(t)) dt, \quad (1)$$

with the discount factor $D(t_0, t)$ depending on calendar time t and decision time t_0 . As a reference case we consider conventional exponential discounting at a constant rate such that $D(t, t_0) = e^{-\rho(t-t_0)}$. The main focus, however, is on hyperbolic discounting where the discount factor is assumed to be given by the following multiplicatively separable function of decision time and calendar time:

$$D(t_0, t) = \left(\frac{1 + \alpha t_0}{1 + \alpha t} \right)^{\beta}, \quad \alpha \geq 0, \quad \beta > 1. \quad (2)$$

The social planner faces a discount factor of one at any decision time t_0 and the discount factor declines hyperbolically with calendar time. The parameters α and β are useful to calibrate discounting. The discount rate is defined as $\rho(t) = -(dD/dt)/D$ and it is obtained as

$$\rho(t) = \begin{cases} \frac{\alpha\beta}{(1+\alpha t)} & \text{for hyperbolic discounting,} \\ \bar{\rho} & \text{for exponential discounting.} \end{cases} \quad (3)$$

Notice that the hyperbolic discount rate does not depend on decision time t_0 . It declines in calendar time t and a larger α or β implies a higher discount rate at any time. Since the discount rate is independent from decision time, decisions are time-consistent. The crucial feature that provides time-consistency is that the discount factor is multiplicatively separable in decision time and calendar time (Burness, 1976; Drouhin, 2009, 2020).

To make inferences, welfare has to be finite, which in turn requires $\beta > 1$. To see this, assume that a steady state of constant utility exists. Then, for $\int_{t_0}^{\infty} \left(\frac{1+\alpha t_0}{1+\alpha t}\right)^\beta dt = \frac{(1+\alpha t_0)^\beta}{\alpha(\beta-1)} \left(\frac{1}{1+\alpha t}\right)^{\beta-1} \Big|_{t_0}^{\infty}$ to be finite we need $\beta > 1$. The feature that the utility integral is finite although the discount rate vanishes in the long run simplifies the analysis. It allows one to apply standard optimization techniques and avoids the use of less well-known methods, like, the catching-up criterion (Seierstad & Sydsaeter, 1986).

To alleviate the comparison between exponential and hyperbolic discounting, it has been suggested to apply the equivalent-present-value method (Myerson et al., 2001; see also Caliendo & Findley, 2014; Strulik, 2015). This requires to compute the constant rate of exponential discounting that provides the same present value of a constant infinite stream (of, e.g., utility) as the hyperbolic discounting method. It provides the $\bar{\rho}$ that solves $\int_{t_0}^{\infty} e^{-\bar{\rho}(t-t_0)} dt = \int_{t_0}^{\infty} \left(\frac{1+\alpha t_0}{1+\alpha t}\right)^\beta dt$. This leads to the solution

$$\bar{\rho} = \frac{\alpha(\beta-1)}{1+\alpha t_0}. \quad (4)$$

The condition of a finite present value of an infinite utility stream is thus equivalent to the condition that the associated equivalent constant time preference rate is positive ($\beta > 1$). The equivalence present-value formula is useful to disentangle the effect of the discounting method (constant vs. hyperbolic) from pure impatience. Here we are interested in how the *hyperbolic decline* of the discount rate affects the management of a renewable resource. Since hyperbolic discounting is time-variant, the equivalent constant discount rate declines as decision time t_0 proceeds. For practical reasons, it could be helpful to normalize the first instant of decisions as $t_0 = 0$ and calibrate parameters α and β accordingly. Long-term planning projects are often evaluated with reference to a specific initial situation and initial decision time, as, for example, the cutting of greenhouse gas emissions relative to their 1990 level. This reference point could be used to obtain a particularly interesting equivalent value of $\bar{\rho}$.

As discussed in Section 1, Weitzman (2001) has shown that individual disagreement about the correct constant discount rate leads in the aggregate to a hyperbolically declining discount rate. The Weitzman case is represented by setting $\alpha = \sigma^2/\mu$ and $\beta = \mu^2/\sigma^2$ which leads to the discount rate

$$\rho(t) = \frac{\mu}{1 + \frac{\sigma^2}{\mu} \cdot t}, \quad (5)$$

where μ is the mean and σ the standard deviation of the individual discount rates.³ Normalizing initial decision time to zero, the Weitzman's estimates of $\mu = 0.04$ and $\sigma = 0.03$ imply $\alpha = 0.0225$, $\beta = 1.77$ and an equivalent constant discount rate $\bar{\rho} = 0.0175$.

3 | OPTIMAL HARVESTING OF A RENEWABLE RESOURCE

We first consider renewable resource growth according to the Verhulst (1838) model (see, e.g., Wilen, 1985). This means that absent any harvesting (or pollution) the resource stock (or environmental quality), denoted by x , grows logistically until it reaches its carrying capacity κ .⁴ The change of the resource stock is assumed to be given by $g(x) = rx(1 - x/\kappa)$, where r denotes the maximum natural growth rate. The maximum sustainable yield from the resource is attained where $g(x)$ reaches a maximum, that is, where $g'(x) = 0$.⁵ Consumption c is equal to the harvesting rate such that the change of stock is obtained as

$$\dot{x} = rx \left(1 - \frac{x}{\kappa} \right) - c, \quad r > 0, \quad \kappa > 0. \quad (6)$$

We furthermore assume an iso-elastic utility function

$$u(c) = \frac{c^{1-\eta} - 1}{1-\eta}, \quad \eta > 0, \quad \eta \neq 1, \quad (7)$$

and $u(c) = \log(c)$ for $\eta = 1$.

The social planner maximizes (1) subject to (2) and (6), given the initial stock x_0 and the nonnegativity conditions, $x \geq 0$, $c \geq 0$. The associated Hamiltonian function is given by

$$H(x, c, \lambda, t) = \left(\frac{1 + \alpha t_0}{1 + \alpha t} \right)^\beta u(c) + \lambda [rx(1 - x/\kappa) - c],$$

in which λ is the shadow price of the resource. The solution fulfills the first-order conditions (8), the costate Equation (9), and the transversality condition (10):

$$\left(\frac{1 + \alpha t_0}{1 + \alpha t} \right)^\beta u'(c) = \lambda, \quad (8)$$

$$\lambda g'(x) = -\dot{\lambda}, \quad (9)$$

$$\lim_{t \rightarrow \infty} \lambda(t)x(t) = 0. \quad (10)$$

³In an application, Weitzman (2001) focuses on SDRs. Of course, the formal result on declining discount rates from aggregation holds for utility discounting as well. See Drupp et al. (2018) for individual disagreement on the pure rate of time preference.

⁴The $g(x)$ function needs not to be exactly quadratic. The subsequent results also hold for general $g(x)$ with $g(0) = 0$, $g''(x) < 0$ for all $x \geq 0$, and $g'(x)|_{x=x^*} = 0$, $x^* > 0$.

⁵I omit the time argument of a function whenever it is not needed for clarification.

Differentiating the first-order condition with respect to time and eliminating λ and $\dot{\lambda}$ leads to the following consumption rule:

$$\dot{c} = \frac{c}{\eta} \left(g'(x) - \frac{\alpha\beta}{1 + \alpha t} \right). \quad (11)$$

Notice that, by log-differentiating of (8), the open-loop consumption rule is independent from decision time t_0 . Thus, the consumption plan will not be revised by any future (reincarnation of the current) planner. The plan is time-consistent. This is the distinguishing feature to open-loop solutions of conventional models of hyperbolic discounting. There, the consumption rule depends on decision time and is usually revised at future decision times.

Optimal resource management is determined by resource growth (6) and consumption growth (11), together with the transversality condition $\lim_{t \rightarrow \infty} \lambda(t)x(t) = 0$. Consumption growth is high when the return on investment in the resource stock, that is, the return on foregone consumption, reflected by the first term in parenthesis in (11), is high or when the discount rate, reflected by the second term, is low.

A steady state is defined as a time path along which all variables are constant or grow at constant rates (see, e.g., Barro & Sala-i-Martin, 2004, pp. 33–34). For hyperbolic discounting, however, the discount rate is never exactly constant. It only approaches a constant (i.e., zero) asymptotically. I thus define an asymptotic steady state, as a path along which asymptotically, that is, as time goes to infinity, all variables are constant or grow at constant rates. Specifically, an asymptotic steady state fulfills $\lim_{t \rightarrow \infty} x(t) = x^*$ and $\lim_{t \rightarrow \infty} c(t) = c^*$, in which x^* and c^* are nonnegative constants. More generally, for the dynamic system $\dot{y} = f(y, t)$, in which y is the vector of time-dependent variables, y^* solves $\lim_{t \rightarrow \infty} f(y^*, t) = 0$ at the asymptotic steady state. The notion of an asymptotic steady state is borrowed from growth theory where it is popular because, naturally, in growing economies some aggregates are never exactly constant but only asymptotically (e.g., Barro & Sala-i-Martin, 2004, pp. 226–235; Jones & Manuelli, 1990). It is worth noting that the focus on an asymptotic steady state implies no limitation on the potential dynamics (compared with an exact steady state). The reason is that, irrespective from whether the discount rate is constant at all times or declining, that is, irrespective of whether an exact or an asymptotic steady state exists, the adjustment dynamics follow a saddlepath along which the growth rates of x and c vanish but are never exactly equal to zero. Thus, any steady state, asymptotic or exact, is reached only asymptotically. The only limitation is that a dynamic system can never rest initially exactly at an asymptotic steady state, which is, however, no constraint for the subsequent analyses.

Proposition 1. *Assume a social planner maximizes welfare (1) subject to (6) and (7). Then, the unique solution path under hyperbolic discounting, according to (2), leads asymptotically to steady-state consumption of the maximum sustainable yield.*

The proof consists of a series of Steps 1–5. In Step 1 we first establish the existence of a unique positive steady state and show that it is associated with maximum sustainable yield. By setting $\dot{c} = \dot{x} = 0$ in (6) and (11) and applying the nonnegativity constraints, we obtain a trivial steady state ($c = 0, x = 0$) and by considering $t \rightarrow \infty$ and $\lim_{t \rightarrow \infty} \rho(t) = 0$ we obtain the unique asymptotic steady state at

$$x = x^* \equiv \frac{\kappa}{2}, \quad c = c^* \equiv \frac{r\kappa}{4}. \tag{12}$$

Maximization of long-run consumption w.r.t. (6) leads to the solution $c = r\kappa/4$, the well-known solution of maximum sustainable yield. Step 2, to assess transitional dynamics and global stability in a phase diagram, we compute the $\dot{x} = 0$ -isocline from (6) as $c = rx(1 - x/\kappa)$. It is shown by the hump-shaped curve in Figure 1. The resource declines above the isocline and grows below it, which leads to the drawn arrows of motion. The $\dot{c} = 0$ -isocline is a vertical line through $x\kappa(r - \rho(t))/(2r) \equiv \tilde{x}(t)$. Consumption grows to the left of the isocline and declines to the right of it. The resulting arrows of motion are also depicted in Figure 1. From the arrows of motion follows that, if the intersection of isoclines exists, any movement is either toward the intersection, or toward the abscissa or ordinate where $x = 0$ or $c = 0$ is reached in finite time. The unconventional feature of the model is that the \dot{c} -isocline is time-variant, which entails that the proof needs to continue in a number of steps.

In Step 3, we begin with concluding that, as time proceeds, the \dot{c} -isocline moves to the right and converges toward the horizontal line at $x^* = \lim_{t \rightarrow \infty} \kappa(r - \rho(t))/(2r) = \kappa/2$. This is so because the hyperbolic discount rate vanishes as time proceeds. The intersection identifies (x^*, c^*) as the unique positive steady-state, which is however reached only asymptotically as explained above. Suppose the isocline associated with initial time $t(0) = 0$ is situated in the positive quadrant, as shown by the isocline that goes through $\tilde{x}(0)$ in Panel A in Figure 1. The initial isocline and the associated arrows of motion are drawn by dotted lines. As time proceeds the isocline moves to the right and converges toward the asymptotic isocline that identifies the steady state. The asymptotic isocline and the associated arrows of motion are drawn by solid lines. At any point of time, we have a new isocline and a new set of arrows of motion, which are, however, isomorph to the initial and final set. This feature allows us to identify the unique optimal solution trajectory associated with the initial situation $x(0)$.

The key observation for Step 4 is that any set of arrows of motion in Panel A allows for convergence toward the asymptotic steady state at x^* . The two other possibilities of motion are to follow paths that hit the abscissa or ordinate in finite time. When a trajectory hits the ordinate, the resource becomes extinct, $x = 0$, and in this instant $c = 0$ and $\dot{c} = -\infty$. This movement violates the Weierstrass–Erdmann condition. In other words, the first-order condition (8) and the continuity conditions on the costate variable are violated, which can also be

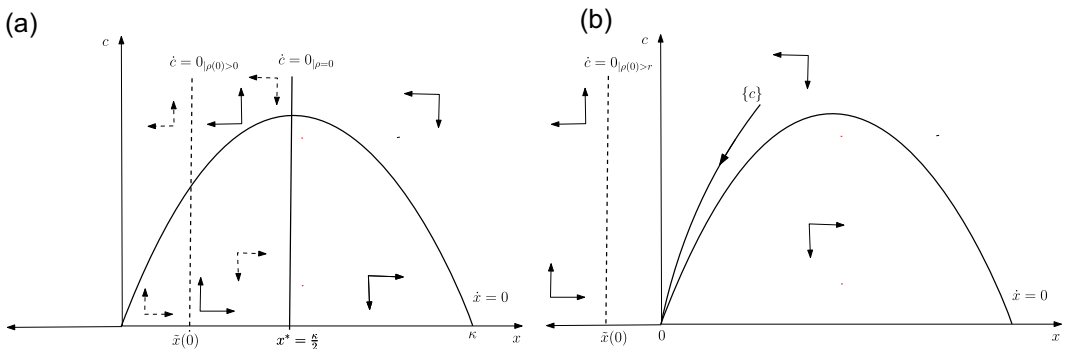


FIGURE 1 Hyperbolic discounting and optimal renewable resource use. *Note:* The solid arrows of motion are associated with the solid $\dot{c} = 0$ -line; the dashed arrows of motion are associated with the dashed $\dot{c} = 0$ -line. See text for details

seen by the fact the consumption rule (11) is violated. If, on the other hand, a trajectory hits the abscissa in finite time, we have $c = 0$ and $x > 0$. With no consumption, the resource is at its unconstrained maximum, at $x^{\max} = \kappa$ with $g'(x)(x^{\max}) = -r$ (the slope of the parabola when its right arm hits the abscissa). This situation violates the transversality condition. To see this, solve (9) for $\lambda(t)$ and insert the solution into (10). This provides the rewritten transversality condition:

$$\lim_{t \rightarrow \infty} \lambda(0) e^{-\int_0^t g'(x(v)) dv} x(t) = 0. \quad (13)$$

At intersection with the abscissa, however, we obtain $\lim_{t \rightarrow \infty} \lambda(0) e^{-\int_0^t g'(x(v)) dv} x(t) > 0$ since $g'(x)$ is negative and x is positive. For the trajectory that converges to the steady state, $g'(x) = 0$ applies only in infinite time and $g'(x) > 0$ for adjustment from below. Thus, the trajectory that converges asymptotically toward the steady state (from below) fulfills the transversality condition. It is also intuitively plausible that paths leading to zero consumption in finite time are characterized as suboptimal and only the unique path associated with infinite positive consumption is identified as optimal.

Finally, in Step 5 we consider situations where the initial isocline lies to the left of the ordinate, as in Panel B of Figure 1. This situation occurs when the initial discount rate is sufficiently high such that $\rho(0) > r$, that is, for $\alpha\beta > r$. Since one objective of hyperbolic discounting is to represent (very) high initial discount rates, this situation is relevant and potentially problematic. Recalling that both c and x are constrained to be nonnegative, any trajectory needs to follow the arrows of motion as depicted in Panel B. We have already shown that time paths that converge to either $c = 0$ or $x = 0$ in finite time can be excluded. However, asymptotic extinction, as shown by the trajectory toward the origin, is consistent with the first and transversality conditions. For the latter feature, recall (13) and notice that $g'(x) > 0$ along the path to $x = 0$ at the origin. The key observation is that it needs infinite time to arrive at the origin. This can be seen from (11) that shows that \dot{c} declines in sync with c , such that the change in consumption declines to zero as consumption declines to zero.

To complete the proof, notice that the right-shifting isocline crosses the abscissa in finite time. Specifically, the abscissa is crossed at time $t^{\text{cross}} = \beta/r - 1/\alpha$ when $\tilde{x}(t) = 0$. In many applications a positive value of t^{cross} may not exist and then Panel B is never relevant. However if $t^{\text{cross}} > 0$ exists, its value is finite. At this finite point of time, extinction has not materialized (because it could occur only asymptotically without violating the first-order condition). At this finite point of time, dynamics as discussed with Panel A become relevant. As shown, they imply convergence toward the asymptotic steady state of maximum yield.

Proposition 2. *Assume a social planner maximizes welfare (1) subject to (6) and (7) under exponential discounting at rate $\bar{\rho} > 0$. For a sufficiently high-discount rate, $\bar{\rho} \geq r$, exponential discounting implies (asymptotic) extinction. For sufficiently low-discount rate $\bar{\rho} < r$, the solution trajectory converges toward the steady state*

$$x = x^{**} \equiv \left(\frac{r - \bar{\rho}}{2r} \right) \kappa, \quad c = c^{**} \equiv \frac{(r^2 - \bar{\rho}^2) \kappa}{4r}. \quad (14)$$

The proof of the proposition will be described in less detail since it is textbook knowledge and parts of it can be taken from the proof of Proposition 1. In case of exponential discounting, we arrive at the consumption rule

$$\dot{c} = \frac{c}{\eta}(g'(x) - \bar{\rho}). \quad (15)$$

Dynamics are now governed by (6) and (15). Setting $\dot{c} = \dot{x} = 0$ we obtain again the steady state of asymptotic extinction ($c = 0$ and $x = 0$) and the positive steady state (14). The $\dot{x} = 0$ -isocline is the same as before. The time-invariant $\dot{c} = 0$ -isocline is given by a vertical line through $x^{**} < x^*$. For the case of asymptotic extinction, we can reuse Panel B from Figure 1 with the distinction that now, for $\bar{\rho} > r$, the isocline is fixed to the left of the abscissa. Dynamics that hit the abscissa or ordinate in finite time can be ruled as nonoptimal (see Step 4 of the previous proof). Asymptotic convergence toward extinction fulfills the transversality rule (see Step 5 of the previous proof). Thus, asymptotic extinction is the optimal solution when $\bar{\rho} \geq r$. The case $\bar{\rho} < r$ can be discussed with Panel A in Figure 1 with the distinction that the $\dot{c} = 0$ -isocline does not move. It is located left of x^* since $x^{**} < x^*$. Since $g'(x) > 0$ at the steady state, movements toward the steady state fulfill the transversality condition while all other movements hit one of the axes in finite time and are not optimal (see Step 4 of the previous proof). Thus, dynamics for $\bar{\rho} < r$ are characterized by convergence toward the steady state (14). \square

Corollary 1. *Adjustment dynamics under hyperbolic discounting lead to higher steady-state consumption than under exponential discounting. Extinction is impossible under hyperbolic discounting whereas it is the unique solution for exponential discounting when $\bar{\rho} > r$.*

While the steady-state results are obvious, those on adjustment dynamics are less obvious and perhaps even surprising. Intuitively, it is tempting to believe that extinction is more frequent under hyperbolic discounting because the initial discount rate could be arbitrarily high and is allowed to exceed any extinction-ensuring exponential discount rate by an arbitrarily high factor. The main intuition for the result is that, as shown in conjunction with the proofs, extinction can be optimal only asymptotically, a feature which gives the hyperbolic discount rate always enough time to become sufficiently low for extinction to vanish as a potential solution.

We next illustrate adjustment dynamics with an example.⁶ For that purpose I set $r = 0.04$, $\kappa = 10$, and $\eta = 2$ as well as $\mu = 0.04$ and $\sigma = 0.03$ (i.e., $\alpha = 0.0225$ and $\beta = 1.777$). Using (4), the equivalent constant discount rate is obtained as $\bar{\rho} = 0.0175$ for $t_0 = 0$. The initial value is set $x(0) = \kappa/2$, that is, we consider an economy that starts with the resource endowment of the asymptotic steady state. The relaxation algorithm solves the nonlinearized system and obtains the solution trajectory that converges toward the boundary condition $c = c^*$ with a user-specified error (set to 10^{-12}).

Blue (solid) lines in Figure 2 show the optimal trajectories for the benchmark hyperbolic case. The discount rate declines continuously from 4% to 0% (lower left panel). Consumption c is initially high due to the high initial discount rate and then declining toward its steady-state

⁶For the solution of this problem and all further numerical experiments in the paper I use the relaxation algorithm of Trimborn et al. (2008).

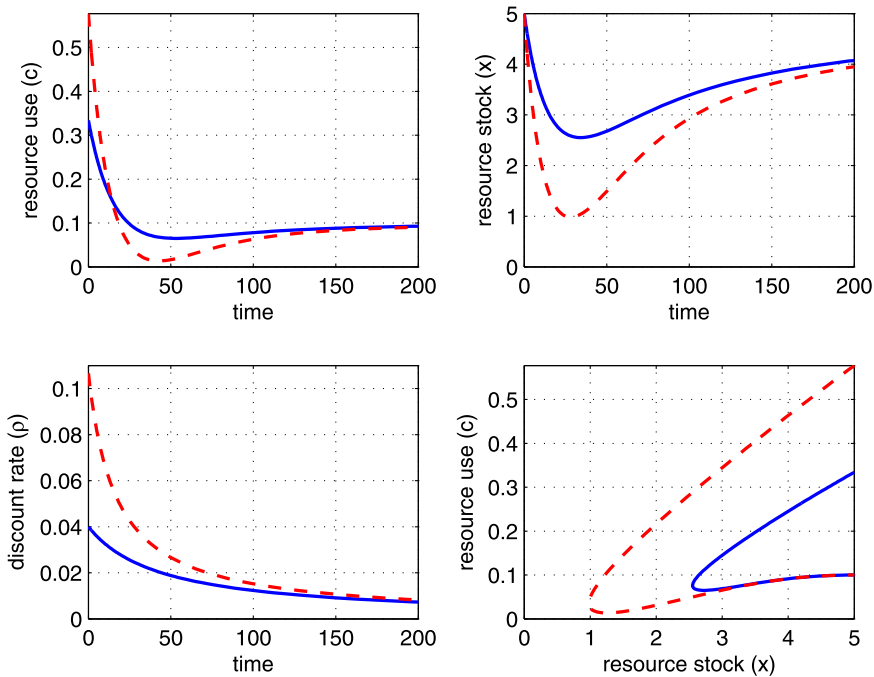


FIGURE 2 Hyperbolic discounting: adjustment paths. *Note:* Parameters: $r = 0.04$, $\kappa = 10$, and $\eta = 2$. Blue (solid) lines: benchmark case with $\mu = 0.04$ and $\sigma = 0.03$ (i.e., $\alpha = 0.0225$ and $\beta = 1.777$). Red (dashed) lines: higher initial discount rate $\alpha = 0.06$ (implying unsustainability for exponential discounting at $\bar{\rho} = \alpha(\beta - 1)$). The lower right panel shows the phase diagram

level (upper left panel). The resource stock first declines due to the high initial consumption and then recovers gradually to the level of $\kappa/2 = 5$ (upper right panel) where consumption equals the maximum sustainable yield of 0.1 (upper left panel). The lower right panel shows the trajectory in phase space. Initially the system starts to the right of the $\dot{c} = 0$ -isocline and both c and x decline. Eventually, however, the solution trajectory meets the rightward shifting $\dot{c} = 0$ -isocline and takes a u-turn. With further proceeding time, both c and x increase and converge toward the steady state. In the appendix, I show adjustment dynamics in three dimensions (3D) (with time added as the third dimension).

Now consider an increase of α to 0.06. Applying (4), the equivalent constant discount rate is obtained as $\bar{\rho} = 0.0467$. This value exceeds the maximum natural growth rate ($r = 0.04$) and thus extinction cannot be avoided when a constant discount rate is applied. The resulting trajectories for hyperbolic discounting are shown by red (dashed) lines in Figure 2. Although the social planner initially uses a very high-discount rate of almost 11%, extinction is avoided. Before the resource is completely exploited, the discount rate has fallen below the natural growth rate (lower left panel) and the resource starts growing again (upper right panel).

A distinctive feature of the nonautonomous model is that the path to the steady state depends on the initial endowment $x(0)$ whereas for conventional models of natural resource use, adjustment dynamics are along a unique stable path associated with the time-invariant saddlepoint. Figure 3 illustrates this distinctive feature in a phase diagram. It shows the benchmark case from Figure 2 for alternative initial values. Panel A shows adjustment

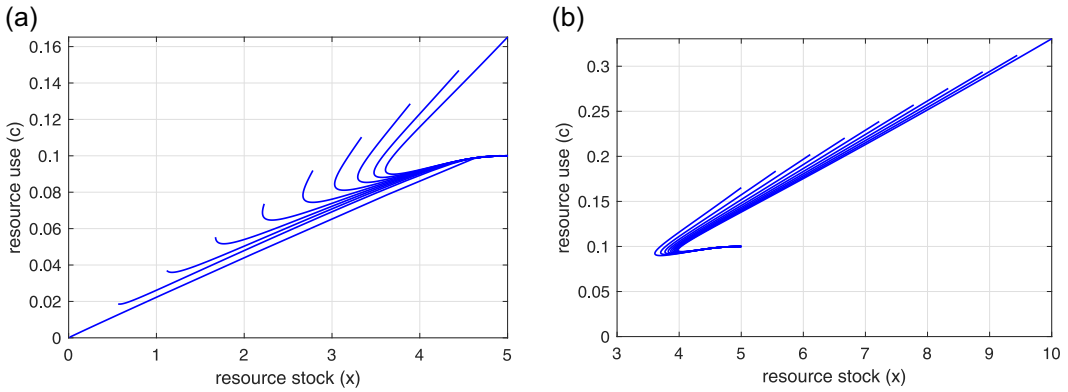


FIGURE 3 Phase diagram for alternative initial values of $x \in [0.001, \kappa]$. *Note:* Left: 10 equally spaced $x(0) \in [0.001, \kappa/2]$. Right: 10 equally spaced $x(0) \in [\kappa/2, \kappa]$. Parameters as for Figure 2 (benchmark run)

dynamics for 10 equally spaced initial values $x(0) \in [0.001, \kappa/2]$. Initially the $\dot{c} = 0$ -isocline is situated far to the left such that the initial endowment lies to right of it, and the arrows motion point downward for c and x (cf. Figure 1). With passing time, c and x decline and the $\dot{c} = 0$ -isocline moves rightward. Eventually the solution trajectory crosses the isocline and takes a u-turn. The current x now lies to the left of the isocline, the arrows of motion point toward higher values of c and x , and c and x increase and converge toward the asymptotic steady state. Clearly, the higher the initial value $x(0)$ the longer it takes until the downward moving solution trajectory meets the rightward moving isocline, that is, the longer is the downward moving path in the phase diagram. Only the trajectory starting at the lowest $x(0)$ increases right from the start. In this case, $x(0)$ lies to the left of the $\dot{c} = 0$ right from the start since for the benchmark run the initial $\dot{c} = 0$ -isocline is already situated in the positive quadrant.

Panel B illustrates solution trajectories for 10 equally spaced initial values $x(0) \in [\kappa/2, \kappa]$. Transitional dynamics always undershoot, in the sense that they temporarily fall below the asymptotic steady state and then adjust from below. Notice that during a particular time window we observe a phase where x is below its steady state and the dynamics are moving away from the steady state. In the conventional resource model with monotonous adjustment dynamics, this would indicate a movement toward extinction. Here, however, the observer needs not to panic. The temporarily divergent trajectory will take a u-turn when the discount rate becomes sufficiently small.

Finally, we extend the model by considering utility experienced from the resource stock capturing, for example, an interest in species diversity. The welfare functional (1) is replaced by

$$W(t_0) = \int_{t_0}^{\infty} D(t_0, t)u(c, x) dt, \quad \frac{\partial u(c, x)}{\partial j} > 0, \quad \frac{\partial^2 u(c, x)}{\partial j^2} < 0, \quad j = c, x \tag{16}$$

and maximization with respect to (2) and (6) provides the costate equation

$$\left(\frac{1 + \alpha t_0}{1 + \alpha t} \right)^\beta \frac{\partial u(x, c)}{\partial x} \lambda + \lambda g'(x) = -\dot{\lambda}, \tag{17}$$

which replaces (9). The first-order condition (8) remains valid. Eliminating λ we obtain from (8) and (17):

$$-\frac{d[\partial u(c, x)/\partial c]/dt}{\partial u(c, x)/\partial c} = \left\{ g'(x) + \frac{\partial u(c, x)/\partial x}{\partial u(c, x)/\partial c} - \frac{\alpha\beta}{1 + \alpha t} \right\}. \tag{18}$$

Proposition 3. Assume a social planner maximizes welfare (16) subject to (6). Time-consistent hyperbolic discounting (2) establishes the Green Golden Rule as the asymptotic steady-state solution of optimal renewable resource management.

For the proof, we exploit that consumption and the resource stock stay constant at the steady state such that the left-hand side of (18) is zero. Then, at the asymptotic steady state, $\lim_{t \rightarrow \infty} \alpha\beta/(1 + \alpha t) = 0$, such that $\lim_{t \rightarrow \infty} g'(x) = \frac{\partial u(c, x)/\partial x}{\partial u(c, x)/\partial c} = dc/dx$. The slope of the resource growth function is tangent to the planners indifference curve at the highest achievable utility \bar{u}^* , see Figure 4. A resource management strategy that establishes this welfare maximizing solution has been defined as the Green Golden Rule (Heal, 2005).

To proceed further, assume that utility is separable in c and x and isoelastic, $u = c^{1-\eta}/(1 - \eta) + x^{1-\psi}/(1 - \psi)$. Then, (18) implies that the $\dot{c} = 0$ -isocline is given by $c = (2rx/\kappa + \rho(t) - r)^{1/\eta} x^{\psi/\eta}$ with consumption increasing above and falling below the isocline. The $\dot{x} = 0$ -isocline remains as for the simple model. We obtain the phase diagram shown in Figure 4. Since x is nonnegative, extinction is no threat. As time proceeds and $\rho(t) = \alpha\beta/(1 + \alpha t)$ declines, the $\dot{c} = 0$ -isocline shifts to the right toward an intersection with the resource curve $g(x)$, where $g(x)$ is tangent to the highest attainable indifference curve \bar{u}^* where the Green Golden Rule holds. Thus, hyperbolic discounting generalizes the Green Golden Rule. For exponential discounting the Green Golden Rule applies only for the knife-edge case of $\bar{\rho} = 0$. For hyperbolic discounting the Green Golden Rule applies asymptotically for any discounting function $\rho(t)$.

4 | HYPERBOLIC DISCOUNTING AND THE TRAGEDY OF THE COMMONS

In this section we explore the use of a renewable resource when property rights are not defined (or cannot be enforced). Suppose that the resource x is harvested by $i = 1, \dots, n$ agents. The agents are symmetric, that is, they share all parameters but follow their own resource

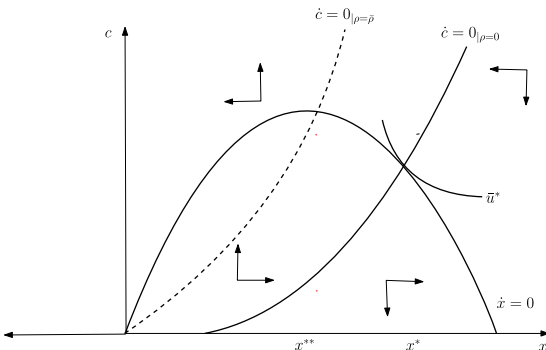


FIGURE 4 Hyperbolic discounting and the Green Golden Rule

management calculus. Here we are interested in the noncooperative solution (since the cooperative solution coincides with the one for the simple model from Section 3). To simplify the problem, we return to the assumption that the resource does not enter the utility function. The scenario describes a game in which n social planners are the players who face a common discounting rule but who cannot commit to any particular policy. As discussed in Section 1, a discounting rule can be imagined as the implementation of social preferences in form of a hard-to-change institution while the actual policy (the “daily” use of the resource) cannot be contracted. We are interested in time-consistent policies, such that the planners play Markovian (feedback) strategies $c^i(x, t)$.⁷ Summarizing, planner i maximizes

$$\int_{t_0}^{\infty} D(t_0, t) u(c^i(x)) dt. \quad (19)$$

$$\dot{x} = rx \left(1 - \frac{x}{\kappa} \right) - \sum_{j=1}^n c^j(x), \quad r > 0, \kappa > 0. \quad (20)$$

Let the Hamiltonians be defined as $H^i(x, c^i, \lambda^i, t) = D(t_0, t) u(c^i) + \lambda^i \left[rx \left(1 - \frac{x}{\kappa} \right) - c^i - \sum_{j=1, j \neq i}^n c^j(x) \right]$ and assume that there exist n continuous differentiable strategies $c^i(x)$, $i = 1, \dots, n$ and n continuous functions λ^i and that the maximized Hamiltonians are continuously differentiable and concave for all i . Then, the Markovian feedback strategies fulfill the following first-order conditions and costate equations:

$$D(t_0, t) \frac{\partial u(c^i)}{\partial c^i} = \lambda^i, \quad (21)$$

$$\lambda^i \left[g'(x) - \sum_{j=1, j \neq i}^n \frac{\partial c^j(x)}{\partial x} \right] = -\dot{\lambda}^i. \quad (22)$$

The feedback terms $\partial c^j(x)/\partial x$ indicate that planners take the response of the other players into account. Eliminating the costate variable and applying symmetry, (21) and (22) can be summarized as

$$-\frac{d[\partial u(c)/\partial c]/dt}{\partial u(c)/\partial c} = g'(x) - (n-1) \frac{\partial c(x)}{\partial x} - \rho(t), \quad (23)$$

in which $c(x) = c^i(x)$ for all $i = 1, \dots, n$.

Equation (23) is hard to interpret since it involves the solution of a partial differential equation. For a special case, however, we can obtain a closed-form solution for the optimal harvesting strategy. Suppose the utility function is iso-elastic, $u(c) = c^{1-\eta}/(1-\eta)$, and consider the case that $\eta = 2$. Fortunately, this assumption has empirical relevance. A recent meta-analysis of 2735 published estimates of the intertemporal elasticity of substitution found the world average of η at 2.0 (Havranek et al., 2015). As shown in the appendix, for $\eta = 2$ we obtain the optimal consumption strategy

⁷These kinds of “fish war” games are based on Levhari and Mirman (1980). See Long (2010, Chapter 3) for a recent survey of common-access differential games. Here, I again omit the time argument whenever it is not needed for clarification.

$$c(x, t) = \left(\frac{r + \rho(t)}{n + 1} \right) x(t), \quad (24)$$

with $\rho(t) = \alpha\beta/(1 + \alpha t)$ in the case of hyperbolic discounting (2) and $\rho(t) = \bar{\rho}$ in the case of exponential discounting.⁸ As shown in (24), the consumed share of the resource is large if the resource regenerates quickly (for high r) or if social planners face a high discount rate. Comparing discounting methods we see that under exponentially discounting individuals consume a constant share of the resource at all times whereas under hyperbolic discounting individuals consume a declining share of the resource.

The consumed share of the resource declines in the number of competing players but the totally consumed resource, nc , increases in the number of players. Two players consume the share $2/3(r + \rho)$, three players consume the share $3/4(r + \rho)$, and so forth. The feature that the total use of the resource increases in the number of players is an expression of non-cooperative behavior. For any player, the opportunity cost of harvesting a unit less is that the other players are induced to consume more when there are more resources. Formally, this feature is reflected by the $\partial c^j/\partial x$ term in (22). This feedback effect drives down the value of the resource λ for every player and increases consumption. The increased consumption due to noncooperative behavior leads to overuse of the resource and potentially to extinction, that is, the tragedy of the commons (Hardin, 1968). Extinction, however, depends on the method of discounting.

Proposition 4. *Assume n social planners harvest a renewable resource competitively without property rights and maximize welfare (19) in favor of their agent $j \in \{1, \dots, n\}$ subject to (20) and (7) with $\eta = 2$ and using Markovian strategies $c^j(x)$. Then there exists always a sustainable asymptotic steady state $x^{**} > 0$ in the case of hyperbolic discounting (2), for any number of competing agents. For exponential discounting there exists no sustainable steady state if the discount rate is sufficiently high or if the number of competing agents is sufficiently high, that is, for $r < n\bar{\rho}$.*

For the proof, insert (24) into (20), let $t \rightarrow \infty$ and solve for $\dot{x} = 0$. This provides the (asymptotic) steady state:

$$x^{**} = \frac{\kappa(r - n\rho)}{r(1 + n)}, \quad (25)$$

with $\rho = 0$ in case of hyperbolic discounting and $\rho = \bar{\rho}$ in case of exponential discounting. Hyperbolic discounting eliminates extinction as a steady-state phenomenon because $\rho(t)$ vanishes in the long run. Hyperbolic discounting, however, does not solve the overuse of the commons. For $n > 1$, the steady-state resource stock x^{**} falls short of x^* , the level that provides the maximum sustainable yield.

⁸The solution is a nondegenerate Markov strategy. Markovian strategies are decisions rules $c(x(t), t)$ in which the choice of player's current action is conditioned on current time and the state vector (Dockner et al., 2000, pp. 30 and 59–60). The presence of the state variable eliminates the commitment problem. Notice that the presence of a time-dependent discount rate does not raise a commitment problem because discounting according to (2) implies a time-consistent solution.

FIGURE 5 Hyperbolic discounting and the tragedy of the commons

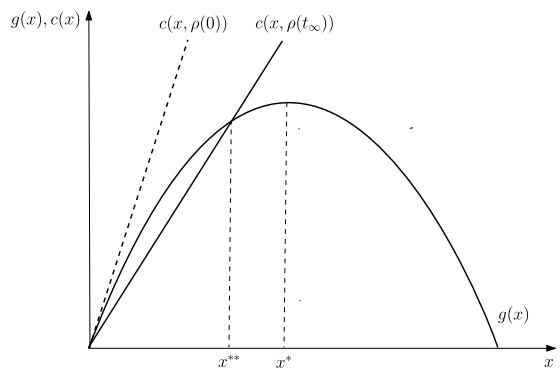


Figure 5 illustrates these results. The steady state is obtained where the linear consumption strategy $c(x)$ intersects the natural growth rate $g(x)$. As time proceeds and the discount rate declines, the consumption profile gets flatter and converges toward the intersection at x^{**} in the long run. Under exponential discounting, (asymptotic) extinction occurs for $n\bar{\rho} > r$, a relatively mild condition when there are many players. In this case, consumption policy $c(x)$ lies everywhere above the $g(x)$ curve and x and c decline at declining rates. With the intuition developed in the context of Proposition 1, it is clear that extinction is impossible for hyperbolic discounting. While x can be only exhausted asymptotically there exists always a finite time at which $n\rho(t) < r$, which guarantees long-run sustainability.

I next illustrate adjustment dynamics with an example. As for the simple model, I set $r = 0.04$, $\kappa = 10$, $\eta = 2$ and consider first the case of two players ($n = 2$). Blue (solid) lines in Figure 6 show adjustment dynamics for $\mu = 0.04$ and $\sigma = 0.03$ (implying $\alpha = 0.0225$ and $\beta = 1.777$). The resource approaches quickly the steady-state level (below the maximum yield of $\kappa/2 = 5$) as the discount rate declines. The lower right panel shows the linear consumption strategy $c(x)$.

When α rises to 0.06, implying an equivalent constant discount rate of $\bar{\rho} = 0.0467$, extinction becomes inevitable with exponential discounting. Hyperbolic discounting however leads to sustainable consumption. Results are shown by red (dash-dotted) lines in Figure 6. The initial discount rate is now above 10% and far above the equivalent constant rate. The consumption strategy $c(x)$ gets steeper (lower right panel) and agents consume more initially (upper left panel). However, as time proceeds, consumption declines at a faster rate than the resource stock due to the declining interest rate. In the long run, the system approaches the same steady state as in the benchmark case.

Finally, we increase the number of competing agents to $n = 10$ and keep the high initial discount rate ($\alpha = 0.06$). Results are shown by green (dash-dotted) lines in Figure 5. Initial consumption is now much higher because of the increased opportunity cost of foregone consumption (upper left panel and lower right panel) and the resource declines steeply. However, since the discount rate declines as well and faster, sustainability is ensured and the system converges toward the (asymptotic) steady state at $x^{**} = \kappa/11 = 0.9$. Summarizing, missing property rights for renewable resources are less of a tragedy when noncooperative social planners apply hyperbolic discounting.

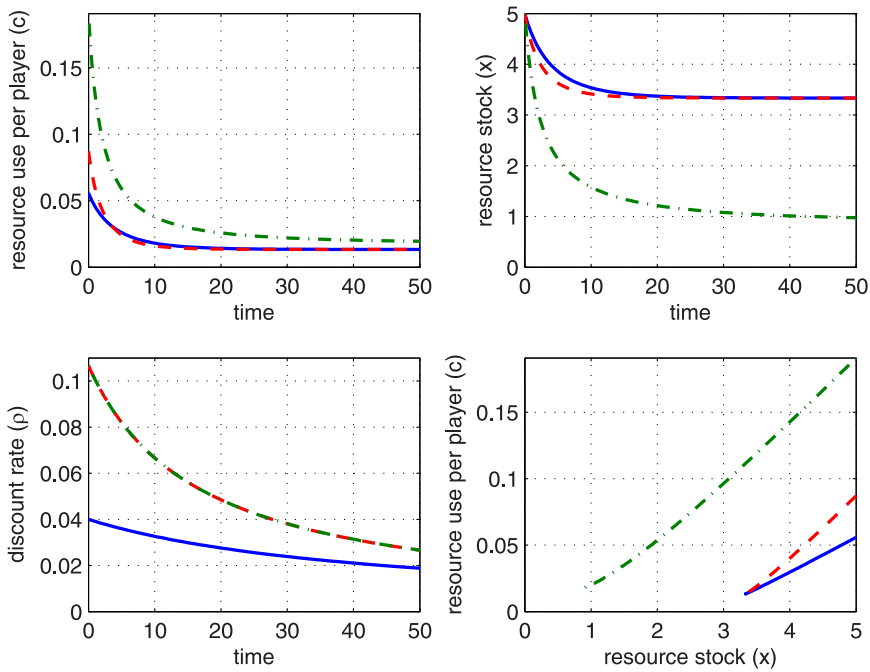


FIGURE 6 Tragedy of the commons: adjustment paths. *Note:* Parameters: $r = 0.04$, $\kappa = 10$, $\eta = 2$, and $n = 2$ players. Blue (solid) lines: $\mu = 0.04$, $\sigma = 0.03$ (i.e., $\alpha = 0.0225$ and $\beta = 1.777$). Red (dashed) lines: higher initial discount rate $\alpha = 0.06$ [implying unsustainability for exponential discounting at $\bar{\rho} = \alpha(\beta - 1)$]. Green (dash-dotted) lines: high-discount rate ($\alpha = 0.06$) and $n = 10$ players. The lower right panel shows the consumption strategy $c(x)$

5 | ECONOMIC GROWTH, POLLUTION, AND THE ENVIRONMENTAL KUZNETS CURVE

In this section we investigate the role of time-consistent hyperbolic discounting in the neo-classical growth model augmented by pollution and environmental quality based on Brock (1977) and Becker (1982). To minimize changes with regard to Section 4, we follow Bovenberg and Smulders (1995) and assume that the quality of the natural environment is conceptualized as a stock of natural capital x , which accumulates due to the regenerative capacity of nature and depreciates due to the damaging effects of pollution p :

$$\dot{x} = g(x) - p, \tag{26}$$

in which $g(x)$ is an environmental regeneration function and p is the pollution. For simplicity, we model pollution as an input in production such that aggregate output is given by $y = f(k, p)$, in which k is the aggregate capital stock.

The economy is populated by a large number of individuals (normalized to one) who derive utility from consumption and environmental quality. As in Bovenberg and Smulders (1995), environmental quality is assumed to affect utility due to, for example, the effect of air quality on health and the aesthetic value of unspoiled landscapes. Lifetime utility at decision time t_0 is given by $W(t_0) = \int_{t_0}^{\infty} D(t_0, t)u(c, x) dt$, as in (16). The social planner maximizes the lifetime

utility of the representative individual taking into account (26) and the accumulation of new capital by net investment:

$$\dot{k} = f(k, p) - c - \delta k, \delta > 0, \quad \frac{\partial f(k, p)}{\partial j} > 0, \quad \frac{\partial^2 f(k, p)}{\partial j^2} < 0, j = k, p, \quad (27)$$

in which δ is the depreciation rate. For exponential discounting it has been shown that this type of problem can lead to a steady state where a higher discount rate improves environmental quality (Smulders, 2007). The reason is that higher impatience increases consumption and reduces investment such that the economy converges toward a lower steady-state capital stock, implying less pollution and higher environmental quality.

The Hamiltonian associated with the maximization problem is given by $H(x, k, c, p, \lambda, \mu) = D(t_0, t)u(c, x) + \lambda [g(x) - p] + \mu [f(k, p) - \delta k - c]$. To further simplify the exposition, I follow Smulders (2007) and impose iso-elastic functions for production and utility, $f(k, p) = k^\gamma p^\epsilon$, $u(c, x) = (cx^\phi)^{1-\eta}/(1-\eta)$. Then, as shown in the appendix, the solution leads to an augmented Ramsey rule for consumption (28) and an equation of motion for optimal pollution (29):

$$\frac{\dot{c}}{c} = \frac{1}{\eta} \left[\gamma k^{\gamma-1} p^\epsilon - \delta - \rho(t) + (1-\eta) \phi \frac{\dot{x}}{x} \right], \quad (28)$$

$$\frac{\dot{p}}{p} = \frac{1}{1-\epsilon} \left\{ \frac{\phi c p}{\epsilon y x} + g'(x) - [\gamma k^{\gamma-1} p^\epsilon - \delta] + \gamma \frac{\dot{k}}{k} \right\}. \quad (29)$$

Proposition 5. *Assume a social planner maximizes welfare (16) subject to (26) and (27) and an iso-elastic utility function. Hyperbolic discounting, according to (2), establishes the Golden Rule of capital accumulation and the modified Green Golden Rule as asymptotic steady-state outcomes. Exponential discounting is neither consistent with the Golden Rule nor with the modified Green Golden Rule.*

For the proof, first note that, at the asymptotic steady state, $\dot{c} = \dot{x} = 0$ and $\rho(t) = 0$. Then, (28) reduces to $\gamma y/k - \delta = 0$. Let s denote the savings rate such that $c = (1-s)y$ and (27), evaluated at the asymptotic steady state, becomes $\dot{k} = y - (1-s)y - \delta k = 0$, implying $sy/k - \delta = 0$. Hence, $s = \gamma$. The result that the savings rate equals the capital share in production is known as the Golden Rule of capital accumulation. That it cannot be fulfilled given exponential discounting is textbook knowledge (see, e.g., Barro & Sala-i-Martin, 2004).⁹ Using the information that $\gamma k^{\gamma-1} p^\epsilon - \delta = 0$ and noting that, at the asymptotic steady state, $\dot{p} = \dot{k} = 0$, Equation (29) reduces to

$$\frac{\phi c p}{\epsilon y x} + g'(x) = 0 \quad \Rightarrow \quad \frac{\phi(1-\gamma)p}{\epsilon x} + g'(x) = 0, \quad (30)$$

where the last equality follows from $c = (1-s)y$ and $s = \gamma$. As shown in the appendix, condition (30), in conjunction with the Golden Rule, maximizes utility at the steady state. It has

⁹The Golden Rule of capital accumulation is the savings policy that provides the highest indefinitely sustainable consumption per capita (Phelps, 1961).

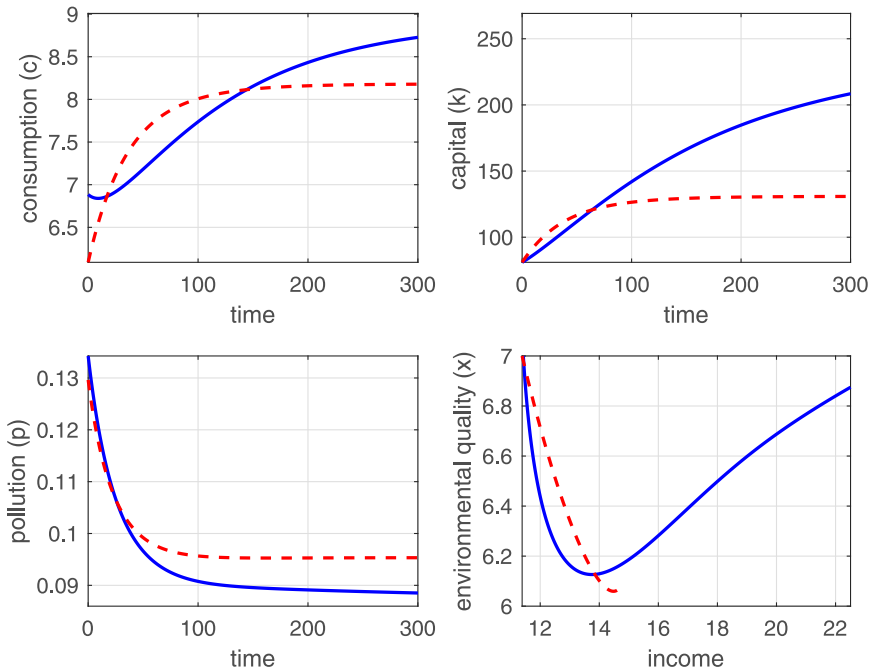


FIGURE 7 Economic growth and hyperbolic discounting. *Note:* Parameters: $r = 0.04$, $\kappa = 10$, $\eta = 2$, $\gamma = 0.6$, $\epsilon = 0.1$, $\phi = 0.3$, $k(0) = 0.3k^*$, and $x(0) = 0.7x^*$. Blue (solid) lines: $\mu = 0.04$ and $\sigma = 0.03$ (i.e., $\alpha = 0.0225$ and $\beta = 1.777$). Red (dashed) lines: Same economy with exponential discounting $\bar{\rho} = \alpha(\beta - 1) = 0.0175$

therefore been defined as the modified Green Golden Rule (Smulders, 2007). That exponential discounting cannot meet the Green Golden Rule is evident from the fact that, at the steady state supported by exponential discounting, we have (from Equation 28 with $\rho(t) = \bar{\rho}$) that $\gamma k^{\gamma-1} p^\epsilon - \delta = \bar{\rho}$, implying that the right-hand side of (30) equals $\bar{\rho} \neq 0$.

Finally, we investigate adjustment dynamics. For simplicity we retain the functional form of $g(x)$ from (6), that is,

$$\dot{x} = rx \left(1 - \frac{x}{\kappa} \right) - p. \tag{31}$$

The economy is described by the dynamic systems (27)–(29) and (31) with $f(k, p) = k^\gamma p^\epsilon$. Since four dimensions cannot be analyzed using phase diagram techniques, we start right away with a numerical analysis. I set $r = 0.04$, $\kappa = 10$, $\eta = 2$ as for the simple model from Section 3. I set the capital share γ to 0.6, capturing a broad definition of capital, and I set the pollution externality to 0.3. To start the economy at a relatively early phase of industrialization, I set the initial value for capital at 30% of steady-state level and the initial value of environmental quality to 70% of the pristine level κ . Adjustment dynamics are shown in Figure 7.

It is interesting to inspect first the adjustment dynamics implied by using the equivalent constant discount rate (of 0.0175). They are shown by red (dashed) lines in Figure 7. The upper panels show the concave adjustment paths for consumption and capital stock, a behavior, well known from the standard neoclassical growth model. Pollution is the strongest in the early phase of industrialization and declines to almost steady-state level in a century (lower left

panel). Consequently, environmental quality x falls initially, overshoots mildly, and adjusts from below toward its steady-state level. The lower right panel shows the implied association between income and environmental quality. The economy starts at the North-West corner with low income and high environmental quality and converges monotonously toward the steady state of high income.

Solid (blue) lines in Figure 7 show the adjustment dynamics for hyperbolic discounting. Individuals consume initially more due to the high initial discount rate (upper left panel). This means that they invest and pollute initially a little less (upper right and lower left panel). The discount rate, however, declines quickly and after less than 20 years, consumption falls short of the level implied by exponential discounting. Investment, in turn, exceeds the level implied by exponential discounting and after about 60 years the capital stock exceeds the level implied by exponential discounting (upper right panel). The economy becomes richer and cleaner at about the same time as pollution falls below the level implied by exponential discounting. After a century, the exponentially discounting economy basically converged to its steady state while the hyperbolically discounting economy keeps growing and becoming cleaner. As a result, an environmental Kuznets curve emerges (lower right panel).¹⁰

6 | CONCLUSION

In this paper, I proposed a time-consistent method to discount at a declining rate and applied it to three canonical environmental problems. I showed that hyperbolic discounting leads to a more conservative use of resources and the environment than exponential discounting and that it establishes the social optimum at the asymptotic steady state (consumption of maximum yield, implementation of the Golden Rule, and the Green Golden Rule). The unconditional convergence toward the social optimum occurs despite high initial discount rates that by far exceed the equivalent constant rate. The reason is that in finite time the declining discount rate falls short of any positive natural growth rate of the resource and extinction in finite time can be ruled out by the optimality conditions. This means that any trajectory irrespective of potentially very high initial discount rates converges eventually to the long-run social optimum. Under exponential discounting, however, asymptotic extinction is a real threat, in particular when property rights are not defined or hard to enforce and many parties exploit an exhaustible resource noncooperatively. For exponential discounting, the Green Golden Rule (the long-run social optimum) applies only for the knife-edge case of $\bar{\rho} = 0$ whereas, for hyperbolic discounting, any discounting function $\rho(t)$ converges toward the Green Golden Rule.

It could be argued that the assumption of a discount rate that converges to zero in the long run is overly restrictive. In this case, it is straightforward to generalize the model by replacing (2) with the hybrid discount factor

$$D(t_0, t) = \left(\frac{1 + \alpha t_0}{1 + \alpha t} \right)^\beta \cdot e^{-\bar{\rho}(t-t_0)}, \alpha \geq 0, \beta > 1, \bar{\rho} > 0.$$

¹⁰See Brock and Taylor (2005) and Egli and Steger (2007). The standard environmental Kuznets curve refers to a u-shaped association between income and the pollution stock. Here, this is expressed as an inverted u-shaped association between income and environmental quality.

The implied discount rate is $\bar{\rho} + \alpha\beta/(1 + \alpha t)$. It does not depend on decision time, declines hyperbolically, and converges in the long run toward the constant positive rate $\bar{\rho}$. For positive $\bar{\rho}$ we do no longer obtain convergence toward the Green Golden Rule. The conclusion of time-consistency, however, remains valid. The open-loop consumption plan is time-consistent and will not be revised by future social planners unless the constraints or parameters of the optimization problem change.

These results are policy relevant since they show that the concern of time-inconsistent, suboptimal, and potentially disastrous outcomes of hyperbolic discounting can be resolved by the use of multiplicatively separable discount factors. A discounting rule is relatively easy to implement as an institution and then hard to change. These features are in particular important when long-run decisions are made subject to a slow-changing state variable, such as the environment.

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APPENDIX A

A.1 | Adjustment dynamics in 3D

Figure A1 shows the adjustment for the benchmark case from Figure 2 when time is added as a third dimension to the phase diagram. The $\dot{x} = 0$ -curve from Figure 2 is represented by the gray surface (the tunnel) since the isocline is time-invariant. The time-variant $\dot{c} = 0$ -isocline is represented by vertical lines, providing (in continuous abstraction) a plane that vertically intersects the tunnel. For the figure, I took $t \in [0, 500]$ and graphically represented isoclines in 10-year time steps. As time proceeds, the isocline moves from the leftmost to the rightmost position. The red line is the optimal solution trajectory. It starts with consumption above its steady-state value (because the discount rate is initially high), undershoots, and converges from below toward the steady state. Figure A2 shows a different representation of the same results. Here we look “from behind,” that is, backward from $t = 500$ to 0.

A.2 | Derivation of (24)

Using the utility function $u = c^{1-\eta}/(1 - \eta)$ with $\eta = 2$ and inserting $g'(x)$, Equation (23) simplifies to

$$\frac{\dot{c}}{c} = \frac{1}{2} \left\{ r - \frac{2rx}{\kappa} - (n - 1) \frac{\partial c}{\partial x} - \rho(t) \right\}. \tag{A1}$$

To solve the problem with the method of undetermined coefficients, assume that $c = ax$, implying $\dot{x} = a\dot{x}$ and $\partial c/\partial x = a$. Applying this information, (A1) becomes

$$\dot{c} = ax \frac{1}{2} \left[r - \frac{2rx}{\kappa} - (n - 1)a - \rho(t) \right] = a\dot{x} = a \left[rx \left(1 - \frac{x}{\kappa} \right) - nax \right], \tag{A2}$$

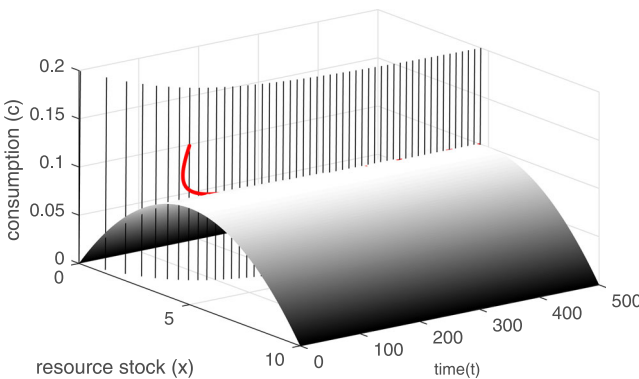
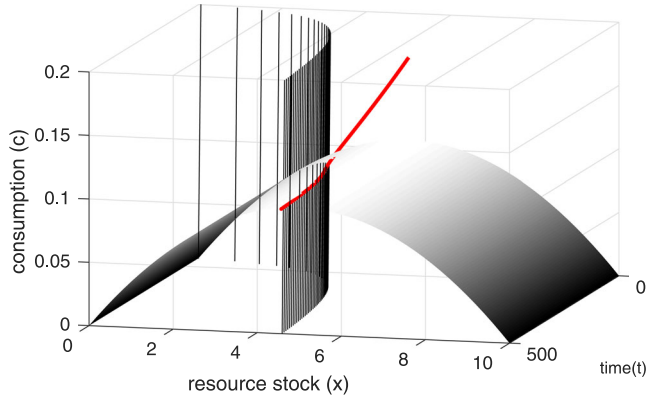


FIGURE A1 Adjustment dynamics in 3D. *Note:* Parameters from Figure 2 (benchmark case)

FIGURE A2 Another view on adjustment dynamics in 3D. *Note:* Parameters from Figure 2 (benchmark case)



where the last equality follows from (20). Thus

$$\frac{r}{2} - \frac{rx}{\kappa} - (n - 1)\frac{a}{2} - \frac{\rho(t)}{2} = r - \frac{rx}{\kappa} - na.$$

Solving for a provides $a = (r + \rho(t))/(n + 1)$ and thus (24) in the main text.

A.3 | Golden Rule and modified Green Golden Rule

The Golden Rule is defined as the utility-maximizing savings rate at the steady state. Let the savings rate be denoted by s . Inserting $c = (1 - s)y$ in (27) and using the Cobb–Douglas production function provides $\dot{k}/k = sk^{\gamma-1}p^\epsilon - \delta$. Evaluating the function at the steady state and solving for capital provides $k^* = [(p^\epsilon s)/\delta]^{1/(1-\gamma)}$. Furthermore, from (26), $g(x) = p$ at the steady state. Inserting this information and $c = (1 - s)y$ into the utility function, we obtain steady-state utility:

$$u = \frac{1}{1 - \eta} \left\{ (1 - s) \left(\frac{s}{\delta} \right)^{\frac{\gamma}{1-\gamma}} g(x)^{\frac{\epsilon}{1-\gamma}} x^\phi \right\}^{1-\eta}. \tag{A3}$$

Maximizing utility with respect to s provides the first-order condition

$$-\left(\frac{s}{\delta}\right)^{\frac{\gamma}{1-\gamma}} + (1 - s)\left(\frac{s}{\delta}\right)^{\frac{\gamma}{1-\gamma}} \frac{\gamma}{1 - \gamma} \frac{1}{s} = 0 \iff \frac{s}{1 - s} = \frac{\gamma}{1 - \gamma}$$

and thus $s = \gamma$, the Golden Rule of capital accumulation.

Maximizing utility (A3) with respect to x provides the first-order condition

$$g'(x) \frac{\gamma\epsilon}{1 - \gamma} g(x)^{\frac{\epsilon}{1-\gamma}} \frac{x^\phi}{g(x)} + \phi g(x)^{\frac{\epsilon}{1-\gamma}} \frac{x^\phi}{x} = 0.$$

Simplifying, we obtain

$$g'(x) + \frac{(1-\gamma)\phi g(x)}{\epsilon x} = 0. \quad (\text{A4})$$

A.4 | Derivation of (28) and (29)

Using the parameterized functional forms for utility and production, the first-order conditions and costate equations for a maximum of the Hamiltonian are given by

$$D(t_0, t)c^{-\eta}x^{\phi(1-\eta)} - \mu = 0, \quad (\text{A5})$$

$$-\lambda + \mu \frac{\epsilon y}{p} = 0, \quad (\text{A6})$$

$$\phi D(t_0, t)c^{1-\eta}x^{\phi(1-\eta)-1} + \lambda g'(x) = -\dot{\lambda}, \quad (\text{A7})$$

$$\mu \left(\frac{\gamma y}{k} - \delta \right) = -\dot{\mu}. \quad (\text{A8})$$

Differentiating (A5) with respect to time and eliminating μ and $\dot{\mu}$ in (A8) provides

$$-\frac{\dot{\mu}}{\mu} = \frac{\gamma y}{k} - \delta = \eta \frac{\dot{c}}{c} - (1-\eta)\phi \frac{\dot{x}}{x} + \rho(t). \quad (\text{A9})$$

Solving for \dot{c}/c provides the modified Ramsey rule (28). Inserting (A5) into (A7) and eliminating the discount factor provides

$$\phi \mu \frac{c}{x} + \lambda g'(x) = \dot{\lambda}. \quad (\text{A10})$$

Differentiating (A6) with respect to time and eliminating λ and $\dot{\lambda}$ in (A10) provides

$$\frac{\phi c p}{\epsilon y x} + g'(x) = \frac{\dot{p}}{p} - \frac{\dot{y}}{y} - \frac{\dot{\mu}}{\mu}. \quad (\text{A11})$$

Finally, using (A8) and $\dot{y}/y = \gamma \dot{k}/k + \epsilon \dot{p}/p$ we obtain (29) in the text.