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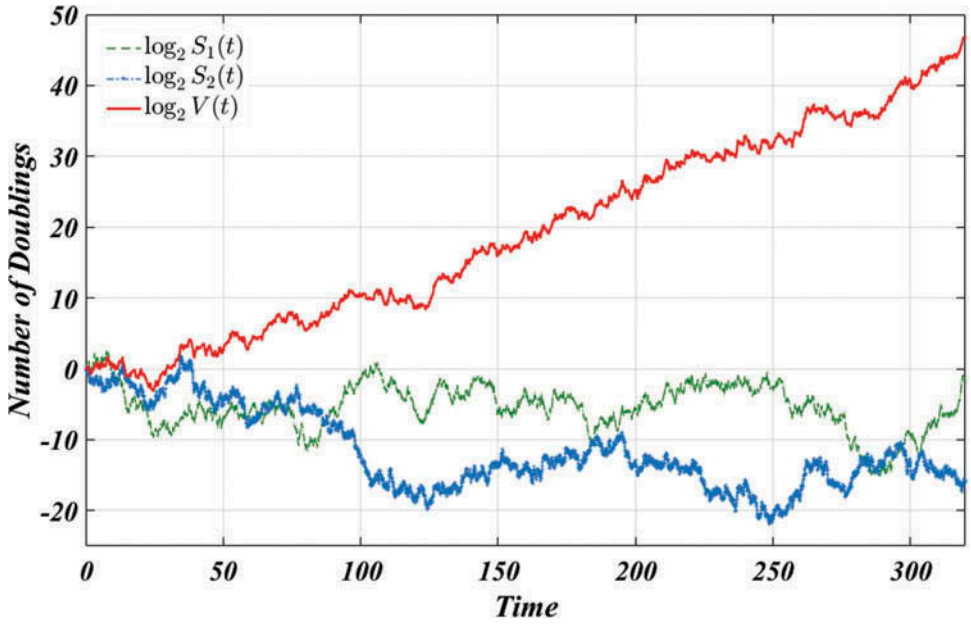

FINANCIAL ECONOMICS | LETTER

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FINANCIAL ECONOMICS | LETTER

Super-replication of the best pairs trade in hindsight

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Abstract: This paper derives a robust online equity trading algorithm that achieves the greatest possible percentage of the final wealth of the best pairs rebalancing rule in hindsight. A pairs rebalancing rule chooses some pair of stocks in the market and then perpetually executes rebalancing trades so as to maintain a target fraction of wealth in each of the two. After each discrete market fluctuation, a pairs rebalancing rule will sell a precise amount of the outperforming stock and put the proceeds into the underperforming stock.

Under typical conditions, in hindsight one can find pairs rebalancing rules that would have spectacularly beaten the market. Our trading strategy, which extends Ordentlich and Cover's "max-min universal portfolio," guarantees to achieve an acceptable percentage of the hindsight-optimized wealth, a percentage which tends to zero at a slow (polynomial) rate. This means that on a long enough investment horizon, the trader can enforce a compound-annual growth rate that is arbitrarily close to that of the best pairs rebalancing rule in hindsight. The strategy will "beat the market asymptotically" if there turns out to exist a pairs rebalancing rule that grows capital at a higher asymptotic rate than the market index.

The advantages of our algorithm over the Ordentlich and Cover strategy are twofold. First, their strategy is impossible to compute in practice. Second, in

ABOUT THE AUTHOR

Dr. Alex Garivaltis is Assistant Professor of Economics in the School of Public and Global Affairs in the College of Liberal Arts and Sciences at Northern Illinois University. He obtained his Ph.D. in Economics in 2017 at the University of Minnesota in Minneapolis, USA. Alex has many years of teaching experience, and his current research program concerns robust financial decision making in the presence of model uncertainty. He has written many papers extending important results obtained by information theorist T. Cover and his collaborators in the theories of Kelly betting, universal investment, and asymptotic capital growth in both discrete and continuous time. He has advised several master's and Ph.D. theses at Northern Illinois University, and in his spare time he makes a strenuous effort to put his theories into actual practice against "Player 2." Time (and maybe some theorems) will tell whether any of it works.

PUBLIC INTEREST STATEMENT

Anyone who trades in financial markets inevitably lives to regret some of his behavior. Practically everybody under the sun has at some point gotten the impression that his stock picking is the worst ever. We look back at the charts and wonder why we failed to do such-and-such. What were we *thinking*? The answer, of course, is that we had no clue what was going to happen. In practice, it is very hard to divine whether one's bad performance is due to stupidity or just bad luck. Market fluctuations are governed by probability laws that are themselves both mutable and unknown. Cover's *universal portfolios* are a phenomenal method for coping with this harsh reality. We ignore the probabilities altogether (for the probabilities are themselves unknown) and instead we quantify what our financial regret would be in all future scenarios. Amazingly, it then becomes possible to trade in a prescient manner that reduces or even eliminates certain types of financial regret in the long run.

considering the more modest benchmark (instead of the best all-stock rebalancing rule in hindsight), we reduce the "cost of universality" and achieve a higher learning rate.

Subjects: Game Theory Economics; Finance; Investment & Securities

Keywords: super-replication; pairs trading; correlation options; constant-rebalanced portfolios; universal portfolios; kelly criterion; robust procedures; minimax
JEL Classification: C44; D81; D83; G11; G12; G13

1. Introduction

1.1. Literature review

The theory of asymptotic portfolio growth was initiated by Kelly (1956), who considered repeated bets on horse races with odds that diverge from the true win probabilities. Kelly set forth the natural goal of optimizing the asymptotic growth rate of one's capital. This implies that one should act each period so as to maximize the expected log of his capital. By the Law of Large Numbers, the realized per-period continuously compounded growth rate converges to the expected growth rate.

The Kelly rule was used by *Beat the Dealer* author Edward O. Thorp (1966) to properly size his bets at the Nevada blackjack tables. For example, imagine a situation where you have a 50.5% chance of winning the next hand. What percentage of your net worth should you bet? The classical mean-variance (Markowitz, 1952) theory has no answer, except to say that it depends on your particular appetite for risk. For instance, the extreme choices of betting 0% or betting 100% are both undominated in the mean-variance plane. The Kelly criterion gives a much more satisfactory answer: bet $50.5\% - 49.5\% = 1\%$ of your wealth. This achieves the (optimum) capital growth rate of 0.005% per hand played in this (favorable) situation. By the rule of 72, you would expect to double your wealth after approximately $72/0.005 = 14,400$ hands.

Thus, it became clear to many people that the log-optimal portfolio theory should replace mean-variance as the dominant decision criterion. Breiman (1961) proved that the Kelly rule outperforms any "essentially different strategy" by an exponential factor, and it has the shortest mean waiting time to reach a distant wealth goal. Thorp's (2017) biography discusses his use of log-optimal portfolios in his money management career on Wall Street. Cover's (1987) survey and his information theory textbook (2006) are excellent primers of the theory of asymptotic growth.

Cover and Gluss (1986) were the first to exhibit an online trading algorithm that could achieve the Kelly growth rate even when starting in total ignorance of the return process. Assuming finitely supported returns, they applied Blackwell's (1956) approachability theorem to get a trading strategy that grows wealth at the same asymptotic rate as the best rebalancing rule (or fixed-fraction betting scheme) in hindsight. Thus began a whole host of so-called "universal trading strategies" that, under mild conditions, "beat the market asymptotically" for highly arbitrary (e.g. nonstationary or serially correlated) return processes.

Cover (1991) gave the first simple and intuitive universal portfolio, at the same time removing the restriction to finitely supported returns. Jamshidian (1992) transplanted Cover's (1991) idea into a continuous-time market with several correlated stocks whose Itô processes have unknown, time-varying parameters that satisfy some asymptotic stability conditions. Cover and Ordentlich (1996) gave the "universal portfolio with side information," along with more perspicuous proofs of the main (1991) regret bounds. For example, Thorp's infamous "count" in Blackjack is a canonical source of side information.

Ordentlich and Cover (1998) super-replicated the final wealth of the best rebalancing rule in hindsight at time-0, although they did not use the terminology of financial derivatives so thoroughly. It seems that their paper was not inspired so much by derivative pricing as it was by Shtarkov's (1987) "universal source code" in information theory. Properly interpreted, the universal source code amounts to a robust scheme for betting on repeated horse races with unknown (and perhaps nonstationary) win probabilities.

More recently, Iyengar (2005) has studied universal investment for discrete-time markets with two assets and proportional transaction costs. Stoltz and Lugosi (2005) extended the game-theoretic notion of internal regret to the case of online portfolio selection problems. DeMarzo, Kremer, and Mansour, (2006) used discrete-time online trading algorithms to derive no-arbitrage bounds for the prices of derivative securities. Györfi, Lugosi, and Udina (2006) gave universal procedures that find and exploit hidden complicated dependencies of asset prices on the past evolution of the market. Kozat and Singer (2011) investigated semiconstant rebalanced portfolios that may (to avoid transaction costs) opt out of rebalancing altogether in selected investment periods.

1.2. Contribution

This paper offers a workaround for two practical problems encountered by would-be practitioners of Ordentlich and Cover (1998) max-min universal portfolio. First, for markets with many assets, the practitioner must wait a tremendously long time for his bankroll to "pull away" from the market averages. Second, the online portfolio weights are impossible to calculate in practice, since the Ordentlich-Cover algorithm requires large-scale computation of multilinear forms. The cleverest methods of computation either exhaust the computer's memory or else they require eons of CPU time. Ordentlich and Cover (1998) max-min universal portfolio is only viable for markets with two or three stocks, at best. Naturally, we want procedures that work for a market with, say, 500 assets.

Accordingly, we take up the more modest goal of performing well (at the end of the investment horizon) relative to the best pairs rebalancing rule in hindsight. Our notion of a "pairs rebalancing rule" allows for the degenerate possibility of keeping 100% of wealth in either of the two stocks. Thus, the best pairs rebalancing rule in hindsight will do at least as well as the best performing stock in the market. Our use of a less aggressive benchmark leads to a computable trading strategy that "learns" more quickly, although in the long run its "understanding" of market dynamics will be somewhat less subtle than that of the original Ordentlich and Cover (1998) strategy.

1.3. Motivating example

To motivate the paper, we use a continuous-time version of "Shannon's Demon" (Poundstone, 2010) to illustrate the fact that the possibility of "beating the market asymptotically" is no contradiction to the random walk model of stock prices. For simplicity, consider two stocks $i \in \{1, 2\}$ that follow independent geometric Brownian motions. Suppose that the price processes $S_i(t)$ evolve according to

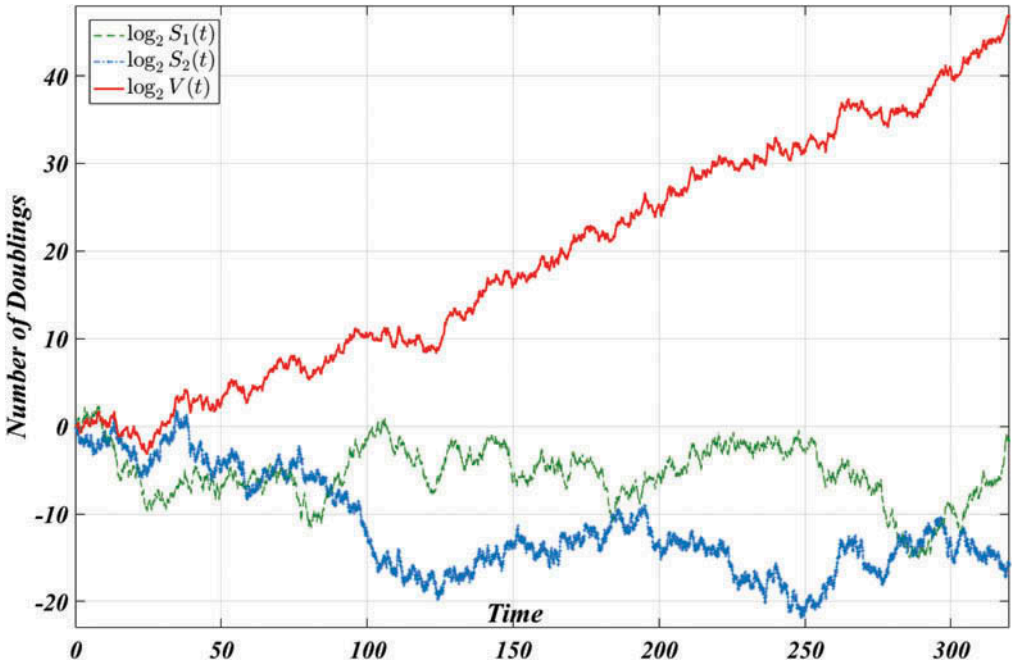
$$\frac{dS_i(t)}{S_i(t)} = \frac{\sigma^2}{2}\, dt + \sigma\, dW_i(t),$$

where $W_1(t), W_2(t)$ are independent unit Brownian motions. In Shannon's original lecture (Poundstone, 2010), at each (discrete) time step the stock price either doubled or got cut in half, each with equal probability. To match this tradition, we put $\sigma = \log 2 = 0.7$. We have

$$\frac{S_i(t)}{S_i(0)} = e^{\sigma W_i(t)} = 2^{W_i(t)}.$$

Note that $\lim_{t\to\infty} \frac{1}{t} \log\left\{\frac{S_i(t)}{S_i(0)}\right\} = 0$. This means that the stocks themselves have zero asymptotic growth; they trade "sideways."

Figure 1. Shannon's demon in continuous time.



Now, consider a gambler who continuously maintains half his wealth in each stock. This is not a buy-and-hold strategy: the rebalancing rule dictates that he sell some shares of whichever stock performed better over $[t, t+dt]$. He puts the proceeds into the underperforming stock. If the trader starts with a dollar, his wealth $V(t)$ evolves according to

$$\frac{dV(t)}{V(t)} = \frac{1}{2}\cdot\frac{dS_1(t)}{S_1(t)} + \frac{1}{2}\cdot\frac{dS_2(t)}{S_2(t)} = \frac{\sigma^2}{2}\,dt + \frac{\sigma}{2}[dW_1(t) + dW_2(t)].$$

Applying Itô's Lemma for functions of several diffusion processes (Wilmott, 2001), we get

$$V(t) = \exp\left\{\frac{\sigma^2 t}{4} + \frac{\sigma}{2}[W_1(t) + W_2(t)]\right\}.$$

We thus have $\lim_{t\to\infty} \frac{1}{t}\log V(t) = \sigma^2/4 = 12\%$. From two dead-money substrates, the gambler has manufactured continuous growth at a rate of 12% per unit time, leaving the market portfolio in the dust. Notice that this growth is merely the result of "volatility harvesting" (Poundstone, 2010) or "volatility pumping" (Luenberger, 1998). Note that the gambler has not attempted to guess which stock will outperform over the interval $[t, t+dt]$—rather, he just rebalances his portfolio after the fact. A sample path for Shannon's Demon has been simulated in Figure 1. For a pair of correlated stocks, the dynamics will be substantially the same, albeit with a lower growth rate.

It is an axiom of capital growth theory that one should seek out a pair of volatile, uncorrelated stocks. But which ones? For the Dow Jones (30) stocks, there are $\binom{30}{2} = 435$ pairs to choose from. For the S&P 500, there are $124,750$. For a badly chosen pair $\{i,j\}$ of stocks, the gambler may very well beat stocks i and j but still underperform the market as a whole. The best pair $\{i^*,j^*\}$ will only be apparent in hindsight.

2. Definitions and notation

We consider a financial market with m assets, called $j \in \{1,\dots,m\}$. For convenience, we will refer to these assets merely as "stocks," although they can be any sort of financial products whatsoever (cash, bonds, lottery tickets, arrow securities, insurance contracts, real estate, etc.). The stocks are traded in T

discrete sessions, called $t \in \{1, \ldots, T\}$. We let $x_{tj} \geq 0$ be the gross return of a \$1 investment (or "bet") on stock j in session t. $x_t := (x_{t1}, \ldots, x_{tm})$ is the gross-return vector in session t. We will require only that $x_t \in \mathbb{R}^m_+ - \{0\}$. This means that at least one of the assets must have a strictly positive gross return. In the sequel, we will adhere to the *individual sequence approach* to investment that was pioneered by Cover and Ordentlich (1996). This means that we will assume no particular dynamics for $(x_t)_{t=1}^T$; the analysis will be completely model-independent. However, to make some concrete sense of what follows, the reader may want to keep the following examples in mind.

Example 1 (Log-Normal Random Walk) Put $x_{tj} := \exp(\nu_j + \sigma_j \varepsilon_{tj})$, where $\varepsilon_{tj} : \sim N(0, 1)$, $\text{Corr}(\varepsilon_{tj}, \varepsilon_{tk}) = \rho_{jk}$, and the vectors $\varepsilon_t := (\varepsilon_{t1}, \ldots, \varepsilon_{tm})$ are independent across time.

Example 2 (Kelly (1956) **Horse Race)** Assume that m horses run T races, $1 \leq t \leq T$. Prior to race t, a bookie sets the (gross) odds at $\mathcal{O}_{tj}$ on horse j. This means that a \$1 bet on the winning horse yields a gross payoff of $\mathcal{O}_{tj}$ dollars. Any money bet on the other horses is lost. If horse j_t is the winner of race t, then the gross-return vector is $x_t = \mathcal{O}_{tj_t} e_{j_t} = \left(0, \ldots, \underset{j_t}{\mathcal{O}_{tj_t}}, \ldots, 0\right)$, where $e_1, \ldots, e_m$ are the unit basis vectors for $\mathbb{R}^m$.

We will consider constant rebalancing rules (or constant-rebalanced portfolios) called $b \in \mathbb{R}^m$, where $b_j \geq 0$ and $\sum_{j=1}^m b_j = 1$. At the start of trading session (race) t, the gambler distributes his wealth among the m stocks, putting the fraction b_j of wealth into stock j. To adhere to a constant rebalancing rule, the gambler must generally trade every period. For, at the end of trading session t, the gambler now has the fraction

$$\frac{b_j x_{tj}}{\langle b, x_t \rangle}$$

of wealth in stock j, where $\langle b, x_t \rangle$ is the inner product $\sum_{j=1}^m b_j x_{tj}$. If stock j outperformed the portfolio in session t, then the trader must sell some shares of stock j to restore the balance. Likewise, he must buy additional shares of stock j if it underperformed the portfolio as a whole. He can refrain from adjusting his holdings in stock j only if $x_{tj} = \langle b, x_t \rangle$, e.g. only when stock j's performance is identical to that of the portfolio as a whole. After T plays, the wealth of the constant-rebalanced portfolio b is

$$W_b(x_1, \ldots, x_T) := \langle b, x_1 \rangle \langle b, x_2 \rangle \cdots \langle b, x_T \rangle,$$

where we have assumed that the gambler starts with \$1 and keeps reinvesting all his capital. The final wealth is just the product of the growth factors $\langle b, x_t \rangle$ from each trading session. Note that the degenerate rebalancing rule $b = e_j$ amounts to buying and holding stock j, as it keeps 100% of wealth in stock j. For the Kelly horse race, the rebalancing rule b amounts to a fixed-fraction betting scheme that bets the fixed fraction b_j of wealth on horse j in every race. In the sequel, we will let $x^t := (x_1, \ldots, x_t)$ be the return history after t trading sessions, with transition law $x^{t+1} = (x^t, x_{t+1})$. For the Kelly horse race, this amounts to the win history $j^t := (j_1, \ldots, j_t) \in \{1, \ldots, m\}^t$.

Definition 1 Given the return history x^t, the **best rebalancing rule in hindsight** is the rebalancing rule $b^*(x^t)$ that would have yielded the most final wealth:

$$b^*(x^t) := \underset{b \in \Delta_m}{\arg\max} \langle b, x_1 \rangle \langle b, x_2 \rangle \cdots \langle b, x_t \rangle,$$

where $\Delta_m := \left\{ b \in \mathbb{R}^n_+ : \sum_{j=1}^m b_j = 1 \right\}$ is the portfolio simplex.

The number $D(x_1, \ldots, x_T) := \max_{b \in \Delta_m} \prod_{t=1}^T \langle b, x_t \rangle$ can be regarded as the payoff of a path-dependent financial derivative ("Cover's Derivative") of the m underlying stocks. In the terminology of the exotic option literature (cf. Wilmott 1998), this would be called a "correlation" or "rainbow" option. In the continuous-time context of several correlated stocks in geometric Brownian motion, Cover's Derivative has been priced and replicated by the author (Garivaltis 2018a), under the assumption of continuous rebalancing and levered hindsight optimization. By contrast, the present paper deals with discrete-time, unlevered rebalancing, and super-replication under total model uncertainty. In this extreme generality, there is no way to guarantee the solvency of leveraged rebalancing rules. Thus, neither the hindsight-optimization nor the super-replicating strategy will be permitted to use leverage.

Definition 2 A **trading strategy** must specify a portfolio vector $\theta_t := (\theta_{t1}, \ldots, \theta_{tm})$ as a function of the history x^{t-1}. In session t, the strategy bets the fraction $\theta_{tj} = \theta_{tj}(x^{t-1})$ of wealth on stock j, where $\sum_{j=1}^m \theta_{tj}(x^{t-1}) = 1$ and $\theta_{tj}(x^{t-1}) \geq 0$. For simplicity, we write $\theta(x^{t-1}) = (\theta_{t1}(x^{t-1}), \ldots, \theta_{tm}(x^{t-1}))$. We write h^0 for the empty history, and $\theta(h^0)$ for the initial portfolio vector.

Note that every trading strategy $\theta(\bullet)$ induces a betting scheme for the Kelly horse race. After observing the win history $j^{t-1} := (j_1, \ldots, j_{t-1})$, $\theta(\bullet)$ prescribes that one should be the fraction $\theta_{tj}(\mathcal{O}_{1j_1} e_{j_1}, \ldots, \mathcal{O}_{t-1 j_{t-1}} e_{j_{t-1}})$ of wealth on horse j in race t, where $\mathcal{O}_{rj_r}$ was the gross odds on the winner of race r. The betting scheme induced by the Ordentlich and Cover (1998) universal portfolio is known in information theory as the universal source code (Shtarkov, 1987).

Definition 3 The **final wealth function** W_θ induced by the trading strategy $\theta(\bullet)$ is defined by

$$W_\theta(x_1, \ldots, x_T) := \langle \theta(h^0), x_1 \rangle \langle \theta(x_1), x_2 \rangle \cdots \langle \theta(x_1, \ldots, x_{T-1}), x_T \rangle = \prod_{t=1}^T \langle \theta(x^{t-1}), x_t \rangle$$

Definition 4 A **super-hedge** (or **super-replicating strategy**) for a derivative payoff $D(\bullet)$ is a pair (p, θ), where $\theta(\bullet)$ is a trading strategy and p is an initial deposit of money, such that

$$p \cdot W_\theta(x_1, \ldots, x_T) \geq D(x_1, \ldots, x_T)$$

for all $x_1, \ldots, x_T \in \mathbb{R}^m_+ - \{0\}$.

In the words of an undated memo by Eric Benhamou of Goldman Sachs, "A super-hedge is defined as a portfolio that will generate greater or equal cash-flows in any outcome. A super-hedge guarantees to make no loss as the super-hedge more than offsets the derivative security." The concept is due to Bensaid, Lesne, and Scheinkman (1992). Note that in our context, we demand that the final wealth $p \cdot W_\theta$ dominate the derivative payoff literally *everywhere*, and not merely with probability 1.

Definition 5 The **super-hedging price** (or **super-replicating cost**) p^* of $D(\bullet)$ is the minimum initial deposit needed to super-replicate $D(\bullet)$, i.e. $p^*[D] := \inf\{p \geq 0 : (p, \theta) \text{ is a super-hedge for some } \theta\}$.

Under this terminology, the cost of super-replicating the final wealth of the best rebalancing rule in hindsight (Ordentlich & Cover, 1998) is

$$p(T, m) := \sum_{n_1 + \ldots + n_m = T} \binom{T}{n_1, \ldots, n_m} (n_1/T)^{n_1} \ldots (n_m/T)^{n_m}$$

where the sum is taken over all solutions of the equation $n_1 + \cdots + n_m = T$ in non-negative integers.

Definition 6 A **pairs rebalancing rule** is a rebalancing rule $b \in \mathbb{R}^n_+$ whose support has at most two stocks, e.g. $\#\text{supp}(b) = \#\{j : b_j > 0\} \leq 2$. More generally, an **s-stock rebalancing rule** is defined by the condition $\#\text{supp}(b) \leq s$.

Definition 7 The **best pairs rebalancing rule in hindsight** is the pairs rebalancing rule that would have yielded the greatest final wealth, given $x_1, \ldots, x_T$. The final wealth of the best pairs rebalancing rule in hindsight is

$$D^{(2)}(x_1, \ldots, x_T) := \max_{\#\text{supp}(c) \leq 2} \langle c, x_1 \rangle \cdots \langle c, x_T \rangle = \max_{(i,j):1 \leq i < j \leq m} \max_{0 \leq b \leq 1} \prod_{t=1}^{T} \{b x_{ti} + (1-b) x_{tj}\}.$$

In general, $D^{(s)}(x_1, \ldots, x_T)$ will denote the final wealth of the best s-stock rebalancing rule in hindsight. Ordentlich and Cover (1998) corresponds to the special case $s := m$.

For the Kelly horse race, a pairs rebalancing rule is a fixed-fraction betting scheme that, each race, bets all its money on the same two horses $\{i, j\}$ in the same fixed proportions $(b, 1-b)$. Thus, if three or more distinct horses wind up winning over the T races, every pairs rebalancing rule will eventually go bankrupt, just as soon as a horse other than i or j wins a race.

3. Super-replication

Lemma 1 For any trading strategy $\theta(\bullet)$, we have

$$\sum_{(j_1, \ldots j_T) \in \{1, \ldots, m\}^T} W_\theta(e_{j_1}, \ldots, e_{j_T}) = 1.$$

Proof In the definition of a final wealth function, start by substituting $x_T := e_{j_T}$ and summing both sides over all $j_T = 1, \ldots, m$. Since the coordinates of any portfolio vector sum to 1, we get $\sum_{j_T=1}^{m} W_\theta(x^{T-1}, e_{j_T}) = \sum_{j_T=1}^{m} W_\theta(x^{T-1}) \langle \theta(x^{T-1}), e_{j_T} \rangle = W_\theta(x^{T-1})$, where $W_\theta(x^{T-1})$ is the final wealth function for the first $T-1$ investment periods. Next, substitute $x_{T-1} := e_{j_{T-1}}$ and sum over all $j_{T-1} = 1, \ldots, m$. We get $W_\theta(x^{T-2})$, and so on down the line. After summing over the indices $j_T, j_{T-1}, \ldots, j_2$, we finally substitute $x_1 := e_{j_1}$ and get $\sum_{j_1=1}^{m} W_\theta(e_{j_1}) = \sum_{j_1=1}^{m} \langle \theta(h^0), e_{j_1} \rangle = 1$, which is the desired result.

Proposition 1 *For any derivative $D(\bullet)$, we have the bound*

$$p^*[D] \geq \sum_{(j_1, \ldots j_T) \in \{1, \ldots, m\}^T} D(e_{j_1}, \ldots, e_{j_T}).$$

Proof In the definition of super-hedging, we substitute $x_1 = e_{j_1}$, $x_2 = e_{j_2}, \cdots, x_T = e_{j_T}$ and sum the inequality over all possible indices $j_1, \ldots, j_T$. By Lemma 1, the left-hand side of the resulting inequality is equal to $p^*[D]$.

Proposition 1 says that if (p, θ) dominates $D(\bullet)$ everywhere, then in particular it dominates the derivative payoff for the special case of the Kelly horse race. However, we have ignored the odds $\mathcal{O}_{tj_t}$, and just substituted $x_t = e_{j_t}$. This amounts to an artificial situation where a one dollar bet on the winning horse gets you your dollar back; otherwise the money is lost.

Proposition 2 For unit basis vectors $x_t := e_{j_t}$, Cover's Derivative has the value

$$D^{(m)}(e_{j_1}, \ldots, e_{j_T}) = (n_1/T)^{n_1}(n_2/T)^{n_2} \cdots (n_m/T)^{n_m},$$

Where $n_i := \#\{t : j_t = i\}$ is the number of races won by horse i.

Here we have used the tacit convention that "$0^0 := 1$" for the situation where there are horses i such that $n_i = 0$.

Proof We have to solve a standard Cobb-Douglas optimization problem over the simplex:

$$\underset{b \in \Delta_m}{\text{Max}} \; b_1^{n_1} \cdots b_m^{n_m}.$$

If $n_i = 0$, then $b_i^* = 0$, e.g. in hindsight no money should have been bet on horse *i*. For all other horses we have $b_i^* > 0$, and Lagrange's multipliers give the solution

$$b_i^* = \frac{n_i}{\sum_{i=1}^m n_i} = \frac{n_i}{T}.$$

Corollary 1 For unit basis vectors $x_t := e_{j_t}$, we have $D^{(2)} = 0$ if at least three distinct horses ever won a race. Otherwise, if $j_t \in \{i, j\}$ for all t, then $D^{(2)} = (n_i/T)^{n_i}(1 - n_i/T)^{T-n_i}$.

Corollary 2

$$p^*\left[D^{(2)}\right] \geq \binom{m}{2} \sum_{n=0}^{T} \binom{T}{n} (n/T)^n (1 - n/T)^{T-n} = \binom{m}{2} p(T, 2).$$

Proof We proceed to sum $D^{(2)}$ over all horse race sequences that have at most two winning horses i, j. To this end, we let $k^T := (k_1, k_2, \ldots, k_T)$, where $k_t \in \{i, j\}$ denotes the winner of race *t*. For each $i = 1, \ldots, m$, we let $n_i(k^T)$ denote the number of races won by horse *i* in the sequence k^T.

$$\sum_{i<j} \sum_{k^T \in \{i,j\}^T} \left[n_i(k^T)/T\right]^{n_i(k^T)} \left[n_j(k^T)/T\right]^{n_j(k^T)}$$
$$= \sum_{i<j} \sum_{n=0}^{T} \binom{T}{n} (n/T)^n (1 - n/T)^{T-n} = \binom{m}{2} p(T, 2).$$

In other words, among the 2^T histories that have only horses i, j as winners, there are $\binom{T}{n}$ histories for which horse *i* wins *n* times, and for all such histories we have $D^{(2)} = (n/T)^n (1 - n/T)^{T-n}$.

Thus, we have found that the cost of achieving the best pairs rebalancing rule in hindsight is at least $\binom{m}{2} p(T, 2)$. To prove the equality $p^*\left[D^{(2)}\right] = \binom{m}{2} p(T, 2)$, we need only exhibit a super-hedge that costs $\binom{m}{2} p(T, 2)$.

Theorem 1 The (minimum) cost of achieving the best pairs rebalancing rule in hindsight is $\binom{m}{2} p(T, 2)$. To achieve the super-replication, one can proceed as follows: for every pair $\{i, j\}$ of stocks with i<j, purchase a minimum cost super-hedge at $t = 0$ for the final wealth of the best $\{i, j\}$

rebalancing rule in hindsight. This amounts to depositing $p(T,2)$ dollars into the Ordentlich and Cover (1998) strategy over stocks i and j, for an aggregate deposit of $\binom{m}{2}p(T,2)$.

This simple strategy, which is the best possible, leads to a final wealth of at least

$$\sum_{i<j}\left[\max_{0\leq b\leq 1}\ \prod_{t=1}^{T}\{bx_{ti}+(1-b)x_{tj}\}\right].$$

One of the terms (i^*,j^*) of this sum will correspond to the final wealth of the best rebalancing rule in hindsight. In practice, this will easily dominate the final wealth of the best pairs rebalancing rule in hindsight. However, in the worst-case scenario of the Kelly market, the trader's wealth will be exactly equal to that of the best pairs rebalancing rule in hindsight.

Theorem 2 After T periods, the excess per-period continuously-compounded growth rate of the best pairs rebalancing rule in hindsight over and above that of the super-hedging trader is at most

$$\frac{\log\binom{m}{2}+\log p(T,2)}{T},$$

which tends to 0 as $T\to\infty$. Thus, the trader compounds his money at the same asymptotic rate as the best pairs rebalancing rule in hindsight.

Proof The trader takes his initial dollar and purchases $\frac{1}{\binom{m}{2}p(T,2)}$ super-hedges of the final wealth of the best pairs rebalancing rule in hindsight. From the definition of super-hedge, we have

$$\binom{m}{2}p(T,2)W_\theta(x_1,\ldots,x_T)\geq D^{(2)}(x_1,\ldots,x_T).$$

Taking logs, we have the uniform bound

$$\frac{\log D^{(2)}-\log W_\theta}{T}\leq\frac{\log\binom{m}{2}+\log p(T,2)}{T}.$$

The fact that $\lim_{T\to\infty}\frac{1}{T}\log p(T,2)=0$ follows from the upper bound $p(T,2)\leq 2\sqrt{T+1}$ (Ordentlich & Cover, 1998, Lemma 3).

It remains to write explicit formulas for the super-replicating strategy. To this end, let $W_{ij}(x^t)$ be the wealth, after x^t, that has accrued to a \$1 deposit into the Ordentlich-Cover (1998) strategy applied to the specific pair $\{i,j\}$ of stocks, with $i<j$. Alternatively, one can use the sequential-minimax universal portfolio (Garivaltis, 2018b) applied to stocks $\{i,j\}$. On account of the fact that we have made an initial deposit of $p(T,2)$ dollars into each distinct pairs strategy, our aggregate wealth after x^t will be $p(T,2)\sum_{i<j}W_{ij}(x^t)$. Let $b_{ij}(x^t)$ denote the fraction of wealth held by this strategy in stock i after x^t, where $1-b_{ij}(x^t)$ is the fraction of wealth held in stock j. How much wealth in total is put into stock k after x^t? We have

$$\sum_{i=1}^{k-1}p(T,2)W_{ik}(x^t)\left[1-b_{ik}(x^t)\right]+\sum_{i=k+1}^{m}p(T,2)W_{ki}(x^t)b_{ki}(x^t).$$

Thus, the total fraction of wealth to bet on stock k in session $t+1$ (after return history x^t) is

$$\theta_{t+1,k}(x^t) = \frac{\sum_{i=1}^{k-1} W_{ik}(x^t)\left[1 - b_{ik}(x^t)\right] + \sum_{i=k+1}^{m} W_{ki}(x^t) b_{ki}(x^t)}{\sum_{i<j} W_{ij}(x^t)}.$$

This expression accounts for the total wealth held by the $m-1$ pairs strategies (i,k) and (k,i) that have stock k in the portfolio, as a fraction of the aggregate wealth held by all $\binom{m}{2}$ strategies. The practitioner is required to keep track of the wealths and portfolio vectors of $\binom{m}{2}$ separate pairs strategies.

3.1. Generalized max-min game

Ordentlich and Cover (1998) considered a two-person zero-sum trading game between the trader (Player 1) and nature (Player 2). The trader picks an entire trading algorithm $\theta(\bullet)$ while nature simultaneously picks the returns $(x_1, \ldots, x_T)$ of all stocks in all periods. They used the payoff kernel

$$(\theta(\bullet), x_1, \ldots, x_T) \mapsto \frac{W_\theta(x_1, \ldots, x_T)}{D^{(m)}(x_1, \ldots, x_T)},$$

which is the ratio of the trader's final wealth to that of the best (full support) rebalancing rule in hindsight. In this subsection, we solve the generalized game with payoff kernel

$$(\theta(\bullet), x_1, \ldots, x_T) \mapsto \frac{w_\theta(x_1, \ldots, x_T)}{D^{(s)}(x_1, \ldots, x_T)}.$$

Theorem 3 In pure strategies, the lower value of the game is $\frac{1}{\left[\binom{m}{s} p(T,s)\right]}$ and the upper value is 1. Thus, there is no pure strategy Nash equilibrium. The trader's maximin strategy is to play a minimum-cost super-hedge for $D^{(s)}$. Nature's minimax strategy is to pick (any) particular stock j^* and have it be the best performing stock in all periods, e.g. $x_{tj^*} \geq x_{tj}$ for all t, j.

Proof Let $\theta(\bullet)$ be a minimum-cost super-hedge for $D^{(s)}$. From the definition of super-hedging, we have the uniform bound

$$\binom{m}{s} p(T,s) W_\theta(x_1, \ldots, x_T) \geq D^{(s)}(x_1, \ldots, x_T).$$

Thus, the trading strategy $\theta(\bullet)$ guarantees that the payoff is at least $\frac{1}{\binom{m}{s} p(T,s)}$. This is the best possible guarantee. For, suppose that a trading strategy $\psi(\bullet)$ guarantees that $\frac{W_\psi}{D^{(s)}} \geq g$ for all $x_1, \ldots, x_T$. Then, since $\left(\frac{1}{g}\right) W_\psi \geq D^{(s)}$, the strategy $\psi(\bullet)$ is a super-hedge for $D^{(s)}$, with an initial deposit of $\frac{1}{g}$ dollars. Since the cheapest possible super-hedge costs $\binom{m}{s} p(T,s)$, we must have $\frac{1}{g} \geq \binom{m}{s} p(T,s)$, so that $g \leq \frac{1}{\binom{m}{s} p(T,s)}$. This shows that $\frac{1}{\binom{m}{s} p(T,s)}$ is the highest possible payoff the trader can guarantee.

$$(\theta(\bullet), x_1, \ldots, x_T) \mapsto \frac{W_\theta(x_1, \ldots, x_T)}{D^{(s)}(x_1, \ldots, x_T)}.$$

To show that the upper value is 1, assume that nature chooses a specific return path $(x_1, \ldots, x_T)$ with the property that a certain stock j^* is the best performer in all periods. Then, the best s-stock rebalancing rule in hindsight is a degenerate rebalancing rule that keeps 100% of its wealth in stock j^* at all times (and 0% in the other $s-1$ stocks). This also happens to be

the best trading strategy of *any kind* that could be played against the specific path $(x_1, \ldots, x_T)$. Thus, this specific path guarantees that $\frac{W_\theta}{D^{(s)}} \leq 1$ for all $\theta(\bullet)$.

Theorem 4 To fill the duality gap, nature randomizes over Kelly horse race sequences $x_t := e_{j_t}$ that have at most s distinct winners $(\#\{j_1, \ldots, j_T\} \leq s)$. It plays the particular sequence $(e_{j_1}, \ldots, e_{j_T})$ with probability $\frac{D^{(s)}(e_{j_1}, \ldots, e_{j_T})}{\binom{m}{s} p(T,s)}$. The value of the game is $\frac{1}{\binom{m}{s} p(T,s)}$. In the mixed-strategy Nash equilibrium, Player 1 does not randomize; he continues to play a minimum-cost super-hedge $\theta(\bullet)$.

Proof First note that these are legitimate probabilities, since they are non-negative and sum to 1. For a specific Kelly sequence $(e_{j_t})_{t=1}^T$ that has at most s distinct winners, the payoff is

$$\frac{W_\theta(e_{j_1}, \ldots, e_{j_T})}{D^{(s)}(e_{j_1}, \ldots, e_{j_T})}.$$

Multiplying these payoffs by their probabilities and summing over all such sequences, we obtain an expected payoff of

$$\frac{\sum_{\#\{j_1, \ldots j_T\} \leq s} W_\theta(e_{j_1}, \ \ldots, e_{j_T})}{\binom{m}{s} p(T,s)}.$$

By Lemma 1, the numerator is at most 1. Thus, nature's mixed strategy has guaranteed that the expected payoff is at most $\frac{1}{\left[\binom{m}{s} p(T,s)\right]}$, regardless of $\theta(\bullet)$. This proves the theorem. □

4. Conclusion

This paper generalized Ordentlich and Cover's beautiful (1998) result, that the cost of super-replicating the best full support rebalancing rule in hindsight is

$$p(T,m) = \sum_{n_1 + \ldots + n_m = T} \binom{T}{n_1, \ldots, n_m} (n_1/T)^{n_1} \cdots (n_m/T)^{n_m}$$

We obtained the fact that the cost of super-replicating the best s-stock rebalancing rule in hindsight is $\binom{m}{s} p(T,s)$. For any significant number of stocks (say, the Dow Jones 30), the full support universal portfolio is impossible to compute in practice. However, it is very easy to super-replicate the best pairs trade in hindsight: one need only calculate and account for $\binom{m}{2}$ 2-stock universal portfolios. The minimum-cost super-hedge amounts to buying $\binom{m}{2}$ super-hedges for $p(T,2)$ dollars each, one for each pair of stocks $i<j$. If the realized volatility of stock prices turns out to be low, then this strategy will easily have enough final wealth with which to dominate the best pairs rebalancing rule in hindsight. However, if the realized volatility of the stock market turns out to be extremely high, then the final wealth of this strategy will not be much more than that of the best pairs rebalancing rule in hindsight. In the limiting case of the Kelly horse race market, the strategy will have a final wealth that is exactly equal to the derivative payoff.

In practice, the trader will have to pick a tolerance ε, and calculate the shortest horizon $T(\varepsilon)$ on which he can guarantee to achieve a compound growth rate that is within ε of that of the best pairs rebalancing rule in hindsight. $T(\varepsilon)$ is the smallest solution T of the inequality

$$\log\binom{m}{2} + \log p(T,2) < T\varepsilon.$$

With this horizon in hand, the trader takes his initial dollar and purchases $\frac{1}{\left[\binom{m}{2}^{p(T,2)}\right]}$ super-hedges, yielding a final wealth of at least $\frac{D^{(2)}}{\left[\binom{m}{2}^{p(T,2)}\right]}$, where $D^{(2)}$ is the wealth of the best pairs rebalancing rule in hindsight. If the realized returns $(x_t)_{t=1}^{\infty}$ are such that the best pairs rebalancing rule in hindsight sustains a higher asymptotic growth rate than the best performing stock in the market, then the trader will beat the market asymptotically. Put more concisely, we hope that

$$\liminf_{T\to\infty} \frac{1}{T}\left\{\log D^{(2)}(x_1,\ldots,x_T) - \log\left(\max_{1\le j\le m}\prod_{t=1}^{T} x_{tj}\right)\right\} > 0.$$

However, the trader's asymptotic growth rate will be somewhat lower than that achieved by the full-support universal portfolio.

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Correction
This article has been republished with minor changes. These changes do not impact the academic content of the article.

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