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Multi-index constrained transportation problem with bounds on availabilities, requirements and commodities

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ABSTRACT

In this paper, we consider a multi-index constrained transportation problem (CTP) of axial constraints with bounds on destination requirements, source availabilities, and multiple types of commodities. The specified problem is converted into a related transportation problem by adding a source, a destination, and a commodity, making it equivalent to a standard axial sum problem. This related problem is transformed into a multi-index transportation problem that can be solved easily. The provided solution method is very useful for transporting heterogeneous commodities. A transportation model may sometimes have various capacity constraints on the flow between pairs of origins and destinations. Moreover, budgetary, political, and emergency situations may impair or enhance the flow between origins and destinations, making it critical for a manager to reevaluate allocations. These considerations have motivated us to explore the multi-index CTP with impaired and enhanced flow. We present several numerical examples to demonstrate the proposed algorithms.

1. Introduction

In the classical transportation problem (TP), a commodity is transported from each of m sources to each of n destinations. The sum of the amounts available at the sources is equal to the sum of the demands at the various destinations. The goal is to determine the amounts of the commodity to be transported over all routes so that the total transportation cost is minimized. Many researchers have extensively studied such cost-time transportation problems. Appa [2] discussed variants of the transportation problem. Brigden [5] addressed transportation with mixed constraints. Klingman and Russel [21] solved a specialized method of the TP with additional linear constraints. Adlakha et al. [1] studied the branching method for fixed-charge transportation problems.

In the literature, the TP is generally presented as a two-dimensional linear transportation problem that can be solved by application of the Modified Distribution Method (MODI). MODI is an efficient method of checking the optimality of the initial feasible solution by determining the unused route with the largest negative improvement index. Once the largest index is identified, it becomes necessary to trace only one closed path. This path helps determine the maximum number of units that can be shipped via the best unused route. Haley [12,13] introduced the solid transportation problem with m origins, n destinations and p types of commodities as an extension of the MODI method. Corban [7] developed a solution procedure for three-dimensional transportation problems with axial constraints. Solid transportation problems under different uncertainty environments have been studied by several authors ([3,23]). Khurana and Adlakha [17] presented the multi-index fixed charge bi-criterion transportation problem. Khurana [16] studied three-dimensional trans-shipment problems. Recently, Sinha et al. [26] studied profit-maximizing solid transportation problems.

There may be situations when, due to external factors, the route capacity must be fixed, leading to a constrained transportation problem. Wagner [30] studied transportation problems with capacity constraints. Charnes and Klingman [6] discussed transportation with limits on the node requirements. Mishra and Dass [24] approached solid transportation with limits on the availability, demand and transportation modes. Verma and Puri [29] addressed transportation with limits on rows and columns, followed by a study of capacitated transportation with limits on rim by Dhaiya and Verma [8]. Dangalchev [9] gave an algorithm to solve non-linear non-convex capacitated transportation problems. Gupta and Arora [11] studied bottleneck capacitated transportation with limits on rim. Khurana and Verma [18] studied two-dimensional capacitated trans-shipment transportation with limits on rim.

There are situations in which the total flow of transportation over particular routes may need to be restricted or enhanced. For example,

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in emergency situations, the demand of fire trucks, police cars or ambulances might have to be increased while the supply of consumer goods might have to be curtailed. This might compel some factories to restrict or enhance their production to meet the required demand. Khanna et al. [15] discussed controlling flow in two-dimensional TPs. Bandopadhyaya and Puri [4] discussed impaired flow multi-index TPs with axial constraints. Thirwani et al. [27] and Thirwani [28] discussed restricted flow and enhanced flow, respectively, in fixed charge bi-criterion TPs. A problem of restricted and enhanced flow in linear fractional TPs was presented by Khurana and Arora [20]. Thereafter, Khurana and Arora [19] developed an algorithm for solving fixed charge bi-criterion indefinite quadratic transportation problems with enhanced flow. Kowalski et al. [22] studied a fast and simple branching algorithm for solving small scale fixed-charge transportation problem. Hu et al. [14] investigated transit time in decision making of transportation service procurement. Nomani et al. [25] presented a weighted model based on goal programming to solve multi-objective transportation problems. Recently, Di et al. [10] gave a transportation network design for maximizing flow-based accessibility.

In this paper, we provide a solution method for a constrained transportation problem (CTP) with axial constraints with bounds on source availabilities, destination demands and various commodities. We discuss solution methods for the CTP with impaired and enhanced flows and extend the methods to handle situations of unbalanced constrained transportation problems. The maximum number of iterations required in our proposed procedure to solve the constrained solid transportation problem would be ${}^{mnp}C_{m+n+p}$(m, n, p defined after problem **P1**). Note that the best polynomial running time for two-dimensional constrained transportation problems is O(m log n (m + n log n)) (Orlin, 1988 [31]).

2. The three-dimensional constrained transportation problem with axial constraints

The "Three Axial Sums" problem deals with transportation of various commodities from a set of different warehouses to different markets, where total availabilities, demands and quantity of different commodities to be transported are specified. We discuss the 3-dimensional CTP having bounds on availabilities, demand and different commodities. The problem can be written as

P1: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$a_i \le \sum_J \sum_K x_{ijk} \le A_i, \quad i \in I \tag{1}$$

$$b_j \le \sum_I \sum_K x_{ijk} \le B_j, \quad j \in J \tag{2}$$

$$e_k \le \sum_I \sum_J x_{ijk} \le E_k, \quad k \in K \tag{3}$$

$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K \tag{4}$$

where $I = \{1, 2, ...,m\}$, $J = \{1, 2, ...,n\}$, $K = \{1, 2, ...,p\}$ are respectively set of origins, destinations and commodities;a_i and $A_i$are the minimum and maximum availability at the ith origin, respectively, b_j and $B_j$are the minimum and maximum demand at the jth destination respectively, e_k and E_k are the minimum and maximum availability of the kth commodity and $c_{ij}$is the unit cost of transportation from the ith origin to the jth destination. Let l_{ijk} and u_{ijk}be the lower and upper bounds on the (i, j)th route for the kth type of commodity.

To solve problem **P1**, a related solid problem is formulated with a dummy supply point, a dummy destination and an extra commodity. The related 3-dimensional transportation problem is given as follows:

P2: Minimize $\sum_{I'} \sum_{J'} \sum_{K'} c'_{ijk}y_{ijk}$subject to

$$\sum_{J'} \sum_{K'} y_{ijk} = A'_i, \quad i \in I' = I \cup \{m+1\}$$
$$\sum_{I'} \sum_{K'} y_{ijk} = B'_j, \quad j \in J' = J \cup \{n+1\}$$
$$\sum_{I'} \sum_{J'} y_{ijk} = E'_k, \quad k \in K' = K \cup \{p+1\}$$

where

$$A'_i = \begin{cases} A_i, & i \in I \\ \sum_J B_j, & i = m+1 \end{cases};$$

$$B'_j = \begin{cases} B_j, & j \in J \\ \sum_I A_i, & j = n+1 \end{cases};$$

$$E'_k = \begin{cases} E_k, & k \in K \\ \sum_I A_i + \sum_J B_j - \sum_K E_k, & k = p+1 \end{cases}$$

$$l_{ijk} \le y_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K$$
$$0 \le \sum_K y_{in+1k} + \sum_J y_{ijp+1} + y_{in+1p+1} \le A_i - a_i, \quad i \in I$$
$$0 \le \sum_K y_{m+1jk} + \sum_i y_{ijp+1} + y_{m+1jp+1} \le B_j - b_j, \quad j \in J$$
$$0 \le \sum_j y_{m+1jk} + \sum_i y_{in+1k} + y_{m+1n+1k} \le E_k - e_k, \quad k \in K$$
$$\sum_k y_{m+1n+1k} \ge 0 \quad k \in K'$$
$$\left.\begin{aligned} & c'_{ijk} = c_{ijk}, && i \in I,\ j \in J,\ k \in K \\ & c'_{ijp+1} = M, && i \in I,\ j \in J \\ & c'_{in+1k} = c'_{m+1jk} = 0, && i \in I,\ j \in J,\ k \in K \\ & c'_{in+1p+1} = c'_{m+1jp+1} = c'_{m+1n+1k} = 0, && i \in I',\ j \in J',\ k \in K' \end{aligned}\right\} \tag{5}$$

Note 1**:** In case of balanced constrained transportation problem, $\sum_I A_i = \sum_J B_j = \sum_K E_k$, thus $E'_k = \sum_K E_k$ *for* $k = p+1$ in problem (**P2**) above.

The problem **P2** is a "Three Axial Sum" problem that can be reformulated as a transformed multi-index TP by using the following definitions by Haley [13]:

$$c''_{ijk} = c'_{ijk}\ (i \le m+1,\ j \le n+1,\ k \le p+1),\ \ c''_{m+2n+2p+2} = 0$$
$$c''_{ijp+2} = 0\ (i \le m+1,\ j \le n+1),\ \ c''_{in+2p+2} = M\ (i \le m+1),$$
$$c''_{in+2k} = 0\ (i \le m+1,\ k \le p+1),\ \ c''_{m+2jp+2} = M\ (j \le n+1),$$
$$c''_{m+2jk} = 0\ (j \le n+1,\ k \le p+1),\ \ c''_{m+2n+2k} = M\ (k \le p+1)$$
$$Let\ R = \underset{i,j,k}{Max}(A'_i, B'_j, E'_k)$$
$$A_{jk} = R\ (j \le n+1; k \le p+1),\ \ B_{ki} = R\ (k \le p+1; i \le m+1),$$
$$E_{ij} = R\ (i \le m+1; j \le n+1),$$
$$A_{n+2k} = (m+1)R - E'_k\ (k \le p+1),$$
$$A_{jp+2} = (m+1)R - B'_j\ (j \le n+1),$$
$$B_{p+2i} = (n+1)R - A'_i\ (i \le m+1),$$
$$B_{km+2} = (n+1)R - E'_k\ (k \le p+1),$$
$$E_{m+2j} = (p+1)R - B'_j\ (j \le n+1),$$
$$E_{in+2} = (p+1)R - A'_i\ (i \le m+1),$$
$$A_{n+2p+2} = B_{p+2m+2} = E_{m+2n+2} = R$$

Let **P3** denote the problem with these modified constraints and the objective function as minimize$\sum_{i=1}^{m+2} \sum_{j=1}^{n+2} \sum_{k=1}^{p+2} c'_{ijk}y_{ijk}$. Problem **P3** can be conveniently solved by Haley's [12] method.

Remark 1. The number of non-zero variables x_{ijk} in a basic feasible solution for a solid transportation problem is $mnp-(m-1)(n-1)(p-1)$.

Remark 2. For a solid transportation problem, first we find a basic feasible solution, then the shadow costs u_{jk}, v_{ki}, w_{ij} for basic cells are evaluated so that $u_{jk} + v_{ki} + w_{ij} = c_{ijk}$. There are $mn + np + pm$ shadow costs and only $mn + np + pm - m - n - p - 1$ non-zero x_{ijk}'s. So, $m + n + p - 1$ shadow costs are set at zero.

Definition. - *M- Feasible Solution for constrained transportation problem:*

A feasible solution $\{y_{ijk}\}$, $i\in I'$, $j\in J'$, $k\in K'$ to (**P2**) is called a M-feasible solution if $y_{ijp+1}=0,\ i\in I,\ j\in J$

Theorem 1. There is a one-to-one correspondence between a feasible solution of problem **P1** and an M-feasible solution of problem **P2.** The proof of Theorem 1 is provided in the Appendix.

Theorem 2. The values of the objective function of **P1** at a feasible solution and of **P2** at its corresponding feasible solution are same.

Proof. Let $\{x_{ijk}\}_{I\times J\times K}$ be corresponding feasible solutions of **P1** and $\{y_{ijk}\}_{I'\times J'\times K'}$ be corresponding feasible solution of problem **P2** with objective function value as Z. Then,

$$
\begin{aligned}
Z &= \textstyle\sum_{I'}\sum_{J'}\sum_{K'} c_{ijk}{}'y_{ijk}\\
&= \textstyle\sum_I\sum_J\sum_K c'_{ijk}y_{ijk} + \sum_I\sum_K c'_{in+1k}y_{in+1k} + \sum_J\sum_K c'_{m+1jk}y_{m+1jk}\\
&\quad + \textstyle\sum_I\sum_J c'_{ijp+1}y_{ijp+1} +\\
&\quad \textstyle\sum_J c'_{m+1jp+1}y_{m+1jp+1} + \sum_I c'_{in+1p+1}y_{in+1p+1} + \sum_K c'_{m+1n+1k}y_{m+1n+1k}\\
&\quad + c'_{m+1n+1p+1}y_{m+1n+1p+1}\\
&= \textstyle\sum_I\sum_J\sum_K c'_{ijk}y_{ijk} + \sum_I\sum_J c'_{ijp+1}y_{ijp+1}\\
&\quad + c'_{m+1n+1p+1}y_{m+1n+1p+1} \quad (\text{from}\,(5))\\
&= \textstyle\sum_I\sum_J\sum_K c_{ijk}y_{ijk} (\text{byM}-\text{feasibilityandas}\ c'_{ijk}=c_{ijk}\ i\in I, j\in J, k\in K
\end{aligned}
$$

$=$ value of the objective function **P1** at$\{x_{ijk}\}_{I\times J\times K}$.

Theorem 3. For every optimal feasible solution to **P1**, there exists an optimal feasible solution to **P2**, and vice versa.

Proof. Follows obviously along the same lines as in Theorem 3 of Khurana and Verma [18]

3. The 3-dimensional impaired flow CTP with bounds on availabilities, demand and commodities

Consider the case of sudden shift of demand in the markets which may restrict the flow of commodities to various destinations. This could happen due to existing political environment, change in economic conditions or imposed restrictions on company budgets. These situations may require a government/country/company to reserve stock of some goods (e.g., weapons, medicines, grains etc.). This situation leads an impaired flow 3-dimensional CTP with bounds on source availabilities, destination demands and various types of commodities.

Consider the case of 3-dimensional transportation problem, where the impaired/restricted flow is exactly known. Let F be the specified total flow level. Thus, the flow constraint $\sum_I\sum_J\sum_K x_{ijk}=F$ is introduced into problem **P1** resulting in the following TP.

P4: Minimize $\sum_I\sum_J\sum_K c_{ijk}x_{ijk}$ subject to

$$
\begin{aligned}
&a_i \le \textstyle\sum_J\sum_K x_{ijk} \le A_i, && i\in I\\
&b_j \le \textstyle\sum_I\sum_K x_{ijk} \le B_j, && j\in J\\
&e_k \le \textstyle\sum_I\sum_J x_{ijk} \le E_k, && k\in K\\
&\textstyle\sum_I\sum_J\sum_K x_{ijk} = F\\
&l_{ijk} \le x_{ijk} \le u_{ijk} && i\in I,\ j\in J,\ k\in K
\end{aligned}
$$

where F is chosen such that $\max\{\sum_i a_i,\ \sum_j b_j,\ \sum_k e_k\} < F < \min\{\sum_i A_i,\ \sum_j B_j,\ \sum_k E_k\}$

To solve problem **P4**, a related solid problem is formulated with a dummy supply point, a dummy destination and an extra commodity. The related 3-dimensional transportation problem is given as follows

P5: Minimize $\sum_{I'}\sum_{J'}\sum_{K'} c'_{ijk}y_{ijk}$ subject to

$$
\begin{aligned}
&\textstyle\sum_{J'}\sum_{K'} y_{ijk} = A'_i, && i\in I' = I\cup\{m+1\}\\
&\textstyle\sum_{J'}\sum_{K'} y_{ijk} = A'_i, && i\in I' = I\cup\{m+1\}\ \textstyle\sum_{I'}\sum_{K'} y_{ijk} = B'_j,\\
&\quad j\in J' = J\cup\{n+1\}\\
&\textstyle\sum_{I'}\sum_{J'} y_{ijk} = E'_k, && k\in K' = K\cup\{p+1\}\\
&\textstyle\sum_I\sum_J\sum_K y_{ijk} = F
\end{aligned}
$$

where

$$
A'_i = \begin{cases} A_i, & i\in I\\ \sum_j B_j - F, & i = m+1\end{cases};
$$

$$
B'_j = \begin{cases} B_j, & j\in J\\ \sum_i A_i - F, & j = n+1\end{cases};
$$

$$
E'_k = \begin{cases} E_k, & k\in K\\ \sum_i A_i + \sum_j B_j - \sum_K E_k - F, & k = p+1\end{cases}
$$

$$
l_{ijk} \le y_{ijk} \le u_{ijk} \qquad i\in I,\ j\in J,\ k\in K
$$

By the definition of M-feasibility

$$
\begin{aligned}
&\textstyle\sum_{j=1}^n y_{ijp+1} + \sum_{k=1}^{p+1} y_{in+1k} \le A_i - a_i, \quad i\in I \ \Rightarrow \sum_{k=1}^{p+1} y_{in+1k} \le A_i - a_i\\
&\textstyle\sum_{i=1}^m y_{ijp+1} + \sum_{k=1}^{p+1} y_{m+1jk} \le B_j - b_j, \quad j\in J \Rightarrow \sum_{k=1}^{p+1} y_{m+1jk} \le B_j - b_j\\
&\textstyle\sum_{i=1}^m y_{in+1k} + \sum_{j=1}^n y_{m+1jk} + y_{m+1,n+1,k} \le E_k - e_k \qquad k\in K\\
&c'_{ijk} = c_{ijk} && i\in I,\ j\in J,\ k\in K\\
&c'_{ijp+1} = M && i\in I,\ j\in J\\
&c'_{in+1k} = c'_{m+1jk} = 0 && i\in I,\ j\in J,\ k\in K\\
&c'_{in+1p+1} = c'_{m+1jp+1} = 0 && i\in I',\ j\in J'\\
&c_{m+1n+1k} = M && k\in K'
\end{aligned}
$$

where M is a sufficiently large positive integer. The problem **P5**, being a "Three Axial Sum" problem, can be reformulated as a multi-index transportation problem by applying Haley's method [13] and as discussed in problem **P3** can thus be solved using Haley [12] method.

Definition: Corner feasible solution (cfs) of P5

A feasible solution $\{y_{ijk}\}_{i\in I',\ j\in J',\ k\in K'}$ of **P5** is a (CFS) corner feasible solution if $y_{m+1n+1k}=0\ k\in K'$.

Remark 3. We can tighten the bounds of the variables in problem **P2**.

As$l_{ijk} \le x_{ijk} \le u_{ijk} \qquad i\in I, j\in J, k\in K$, this implies that

$$
\begin{aligned}
&\textstyle\sum_{j\in J}\sum_{k\in K} l_{ijk} \le \sum_{j\in J}\sum_{k\in K} x_{ijk} \le \sum_{j\in J}\sum_{k\in K} u_{ijk} && i\in I, \quad (I)\\
&\textstyle\sum_{i\in I}\sum_{k\in K} l_{ijk} \le \sum_{i\in I}\sum_{k\in K} x_{ijk} \le \sum_{i\in I}\sum_{k\in K} u_{ijk} && j\in J \quad (II)
\end{aligned}
$$

and

$$
\sum_{i\in I}\sum_{j\in J} l_{ijk} \le \sum_{i\in I}\sum_{j\in J} x_{ijk} \le \sum_{i\in I}\sum_{j\in J} u_{ijk} \qquad k\in K \quad (III)
$$

Inequalities (1) and (I) together imply that

$$
a'_i = \max\left\{a_i, \sum_{j\in J}\sum_{k\in K} l_{ijk}\right\} \le \sum_J\sum_K x_{ijk} \le \min\left\{A_i, \sum_{j\in J}\sum_{k\in K} u_{ijk}\right\}
$$

$$
= A'_i \qquad i\in I
$$

Inequalities (2) and (II) imply that

$$
b'_j = \max\left\{b_j, \sum_{i\in I}\sum_{k\in K} l_{ijk}\right\} \le \sum_I\sum_K x_{ijk} \le \min\left\{B_j, \sum_{i\in I}\sum_{k\in K} u_{ijk}\right\} = B'_j \quad j\in J
$$

Inequalities (3) and (III) imply that

$$
e'_k = \max\left\{e_k, \sum_{i\in I}\sum_{j\in J} l_{ijk}\right\} \le \sum_I\sum_J x_{ijk} \le \min\left\{E_k, \sum_{i\in I}\sum_{j\in J} u_{ijk}\right\} = E'_k \quad k\in K
$$

Thus, the problem **P1** reduces to following problem:
P1a: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$a'_i \le \sum_J \sum_K x_{ijk} \le A'_i, \quad i \in I$$
$$b'_j \le \sum_I \sum_K x_{ijk} \le B'_j, \quad j \in J$$
$$e'_k \le \sum_I \sum_J x_{ijk} \le E'_k, \quad k \in K$$
$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K$$

And the related **P2a** of **P1a** is
P2a: Min $\sum_{I'} \sum_{J'} \sum_{K'} c'_{ijk}t_{ijk}$ subject to

$$\sum_{J'} \sum_{K'} t_{ijk} = A''_i, \quad i \in I' = I \cup \{m+1\}$$
$$\sum_{I'} \sum_{K'} t_{ijk} = B''_j, \quad j \in J' = J \cup \{n+1\}$$
$$\sum_{I'} \sum_{J'} t_{ijk} = E''_k, \quad k \in K' = K \cup \{p+1\}$$

where

$$A''_i = \begin{cases} A'_i, & i \in I \\ \sum_J B'_j, & i = m+1 \end{cases};$$

$$B''_j = \begin{cases} B'_j, & j \in J \\ \sum_I A'_i, & j = n+1 \end{cases};$$

$$E''_k = \begin{cases} E'_k, & k \in K \\ \sum_i A'_i + \sum_j B'_j - \sum_K E'_k, & k = p+1 \end{cases}$$

$$l_{ijk} \le t_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K$$
$$0 \le \sum_K t_{in+1k} + \sum_J t_{ijp+1} + t_{in+1p+1} \le A'_i - a_i \quad i \in I$$
$$0 \le \sum_K t_{m+1jk} + \sum_i t_{ijp+1} + t_{m+1jp+1} \le B'_j - b_j \quad j \in J$$
$$0 \le \sum_j t_{m+1jk} + \sum_i t_{in+1k} + t_{m+1n+1k} \le E'_k - e_k \quad k \in K$$
$$\sum_k t_{m+1n+1k} \ge 0 \quad k \in K'$$
$$c'_{ijk} = c_{ijk} \quad i \in I,\ j \in J,\ k \in K$$
$$c'_{ijp+1} = M \quad i \in I,\ j \in J$$
$$c'_{in+1k} = c'_{m+1jk} = 0 \quad i \in I,\ j \in J,\ k \in K$$
$$c'_{in+1p+1} = c'_{m+1jp+1} = c'_{m+1n+1k} = 0 \quad i \in I',\ j \in J',\ k \in K'$$

Note 2**:** Bounds on$t_{m+1jp+1}$, $t_{in+1p+1}$ and $t_{m+1n+1k}$, for $i \in I,\ j \in J,\ k \in K$ are tighter compared to $B_j - b_j,\ A_i - a_i,$ and$E_k - e_k$.

Note 3**:** A non-feasible extreme point of P5 is never an optimal solution of **P5** since$c_{m+1n+1p+1} = M$

Lemma 1. For all feasible solutions to **P4** there exists a corner feasible solution to **P5**, and vice versa.

Theorem 4. The objective function values of **P4** at a feasible solution $\{x_{ijk}\}_{i \in I,\ j \in J,\ k \in K}$and **P5** at its corresponding corner feasible solution $\{y_{ijk}\}_{i \in I',\ j \in J',\ k \in K'}$are equal.

Lemma 2. For every optimal solution among the cfs to **P5**, there exists an optimal solution to **P4.**

Theorem 5. Optimizing **P4** is equivalent to optimizing **P5**, provided **P4** has a feasible solution.

4. The 3-dimensional enhanced flow CTP with bounds on availabilities, demand and commodities

Now consider the case of a sudden rise in demand in the market whereby requiring an enhanced flow. Many circumstances may give rise to such a situation – for example, a natural calamity, election times or holidays seasons. This is a case of enhanced flow 3-dimensional transportation problem where, assume that, the flow is enhanced to a known specified level F. Thus, the flow constraint $\sum_I \sum_J \sum_K x_{ijk} = F$ is introduced resulting in the following transportation problem.

P6: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$\sum_J \sum_K x_{ijk} \ge a_i \quad i \in I$$
$$\sum_I \sum_K x_{ijk} \ge b_j \quad j \in J$$
$$\sum_I \sum_J x_{ijk} \ge e_k \quad k \in K$$
$$\sum_I \sum_J \sum_K x_{ijk} = F$$
$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K$$

where F is chosen such that $F > \max\{\sum_i a_i,\ \sum_j b_j,\ \sum_k e_k\}$

A related solid problem is formulated to solve **P6**, with a dummy supply point, a dummy destination, and an extra commodity. The related 3-dimensional transportation problem is given as follows

P7: Minimize $\sum_{I'} \sum_{J'} \sum_{K'} c'_{ijk}y_{ijk}$ subject to

$$\sum_{J'} \sum_{K'} y_{ijk} = A'_i, \quad i \in I' = I \cup \{m+1\}$$
$$\sum_{I'} \sum_{K'} y_{ijk} = B'_j, \quad j \in J' = J \cup \{n+1\}$$
$$\sum_{I'} \sum_{J'} y_{ijk} = E'_k, \quad k \in K' = K \cup \{p+1\}$$
$$\sum_I \sum_J \sum_K y_{ijk} = F$$

where

$$A'_i = \begin{cases} \sum_{j=1}^{n} \sum_{k=1}^{p} u_{ijk} & i \in I \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} u_{ijk} - F, & i = m+1 \end{cases};$$

$$B'_j = \begin{cases} \sum_{i=1}^{m} \sum_{k=1}^{p} u_{ijk} & j \in J \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} u_{ijk} - F, & j = n+1 \end{cases};$$

$$E'_k = \begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ijk} & j \in J \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} u_{ijk} - F, & k = p+1 \end{cases}$$

$$l_{ijk} \le y_{ijk} \le u_{ijk} \quad \forall\ i \in I,\ j \in J,\ k \in K$$
$$\sum_{j=1}^{n} y_{ijp+1} + \sum_{k=1}^{p+1} y_{in+1k} \le \sum_{j=1}^{n} \sum_{k=1}^{p} u_{ijk} - a_i \quad \forall\ i \in I \quad \Rightarrow \sum_{k=1}^{p+1} y_{in+1k} \le \sum_{j=1}^{n} \sum_{k=1}^{p} u_{ijk} - a_i \quad \forall\ i \in I$$
$$\sum_{i=1}^{m} y_{ijp+1} + \sum_{k=1}^{p+1} y_{m+1jk} \le \sum_{i=1}^{m} \sum_{k=1}^{p} u_{ijk} - b_j \quad \forall\ j \in J \quad \Rightarrow \sum_{k=1}^{p+1} y_{m+1jk} \le \sum_{i=1}^{m} \sum_{k=1}^{p} u_{ijk} - b_j \quad \forall\ j \in J$$
$$\sum_{i=1}^{m} y_{in+1k} + \sum_{j=1}^{n} y_{m+1jk} + y_{m+1,n+1,k} \le \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ijk} - e_k \quad \forall\ k \in K$$

(By the definition of M-feasibility)

$$c'_{ijk} = c_{ijk} \quad i \in I,\ j \in J,\ k \in K$$
$$c'_{ijp+1} = M \quad i \in I,\ j \in J$$
$$c'_{in+1k} = c'_{m+1jk} = 0 \quad i \in I,\ j \in J,\ k \in K$$
$$c'_{in+1p+1} = c'_{m+1jp+1} = 0 \quad i \in I',\ j \in J'$$
$$c'_{m+1n+1k} = M \quad k \in K'$$

where M is a sufficiently large positive integer. Theorems and remarks for the 3-dimensional CTP with enhanced flow can be developed along the similar lines as for the 3-dimensional CTP with restricted flow.

5. Unbalanced 3-dimensional CTP

Consider the following balanced 3-dimensional CTP
P6: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$\sum_J \sum_K x_{ijk} = a_i, \quad i \in I$$
$$\sum_I \sum_K x_{ijk} = b_j, \quad j \in J$$
$$\sum_I \sum_J x_{ijk} = e_k, \quad k \in K$$
$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad \forall\ i \in I,\ j \in J,\ k \in K$$

In the above problem, for any feasible solution we have$\sum_I a_i = \sum_J b_j = \sum_K e_k$.

The problem **P6** becomes unbalanced constrained transportation problem (UCTP) if any of the above availability, demand or commodity constraints are in the form of inequalities. We discuss several forms of UCTP and demonstrate the solution procedure by modifying the UCTP to a problem like **P1**.

Case I: High storage costs at some sources may cause a manager to over supply to destinations. Alternatively, an anticipation of political unrest or war may cause a manger to keep emergencies stocks at the sources. In such cases, the following UCTP is formulated:

P7: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$\sum_J \sum_K x_{ijk} \le a_i, \quad i \in I \tag{6}$$

$$\sum_I \sum_K x_{ijk} \ge b_j, \quad j \in J \tag{7}$$

$$\sum_I \sum_J x_{ijk} = e_k, \quad k \in K \tag{8}$$

$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K \tag{9}$$

We assume here u_{ijk} is finite for all $i \in I,\ j \in J,\ k \in K$.

Inequality (8) implies that

$$\sum_{j\in J} \sum_{k\in K} l_{ijk} \le \sum_{j\in J} \sum_{k\in K} x_{ijk} \le \sum_{j\in J} \sum_{k\in K} u_{ijk} \quad i \in I, \tag{10}$$

$$\sum_{i\in I} \sum_{k\in K} l_{ijk} \le \sum_{i\in I} \sum_{k\in K} x_{ijk} \le \sum_{i\in I} \sum_{k\in K} u_{ijk} \quad j \in J \tag{11}$$

and

$$\sum_{i\in I} \sum_{j\in J} l_{ijk} \le \sum_{i\in I} \sum_{j\in J} x_{ijk} \le \sum_{i\in I} \sum_{j\in J} u_{ijk} \quad k \in K \tag{12}$$

Using (10) and (11), problem **P7** reduces to be like problem **P1** as follows:

P8: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$a'_i \le \sum_J \sum_K x_{ijk} \le A'_i, \quad i \in I$$
$$b'_j \le \sum_I \sum_K x_{ijk} \le B'_j, \quad j \in J$$
$$e'_k \le \sum_I \sum_J x_{ijk} \le E'_k, \quad k \in K$$
$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K$$

where $a'_i = \sum_{j\in J} \sum_{k\in K} l_{ijk}$, $A'_i = a_i$, $b'_j = b_j$, $B'_j = \sum_{i\in I} \sum_{k\in K} u_{ijk}$, $e'_k = e_k$, $E'_k = e_k$

Case II: Consider the case where, along with conditions on availability and demand constraints as in Case I, additional constraints restricting the amount of different types of commodities to be shipped are imposed. Then the problem takes the following form:

P9: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$\sum_J \sum_K x_{ijk} \le a_i, \quad i \in I$$
$$\sum_I \sum_K x_{ijk} \ge b_j, \quad j \in J$$
$$\sum_I \sum_J x_{ijk} \le e_k, \quad k \in K$$
$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad \forall\ i \in I,\ j \in J,\ k \in K$$

Problem **P9** can easily be modified to a problem like **P1**. Using 9), ((10) and (11) the above can be modified as follows:

P10: Minimize $\sum_I \sum_J \sum_K c_{ijk}x_{ijk}$ subject to

$$a'_i \le \sum_J \sum_K x_{ijk} \le A'_i, \quad i \in I$$
$$b'_j \le \sum_I \sum_K x_{ijk} \le B'_j, \quad j \in J$$
$$e'_k \le \sum_I \sum_J x_{ijk} \le E'_k, \quad k \in K$$
$$l_{ijk} \le x_{ijk} \le u_{ijk} \quad i \in I,\ j \in J,\ k \in K$$

where $a'_i = \sum_{j\in J} \sum_{k\in K} l_{ijk}$, $A'_i = a_i$, $b'_j = b_j$, $B'_j = \sum_{i\in I} \sum_{k\in K} u_{ijk}$, $e'_k = \sum_{i\in I} \sum_{j\in J} l_{ijk}$, $E'_k = e_k$

which can be solved easily.

A variety of other cases may arise. For example, (1) high storage cost at a source may prompt a manager to supply excess amounts of all k-types of commodity to some destinations; (2) a manage may wish to have some inventory at the sources to handle crisis; (3) demand at some destinations may be greater than the availability at sources, creating a need for increasing production at sources and still not meeting demand of some destinations. Alternatively, a manger might try to meet demand at a destination by over-producing. All these specified problems may be transformed to **P1,** on similar lines as discussed above, and hence may be solved.

6. Numerical illustrations: a laptop manufacturing company

Consider the case of company XYZ that produces various types of laptops. These laptops are produced at New York ($i = 1$), California ($i = 2$) and Chicago ($i = 3$). Once produced the laptops are shipped to Washington DC ($j = 1$), Maryland ($j = 2$) and Philadelphia ($j = 3$). There are (k) types of laptops, namely, type I ($k = 1$) with 8 in screen, type II ($k = 2$) with 12 in screen and type III ($k = 3$) with 16 in screen which are being transported.

Let O1, O2 and O3 denote factories at New York, California, and Chicago. Let D1, D2 and D3 denote the distribution centers at Washington DC, Maryland, and Philadelphia respectively. The handling cost or the shipping cost (c_{ijk}) paid per laptop for type I is $1, type II is $2, and type III is $2 when the goods are transported from New York to Washington DC (O1 to D1). Similarly, the shipping cost paid per laptop for type I is $3, type II is $4, and type III is $1 when the goods are transported from O1 to D2 while the costs in case of transportation from O1 to D3 are $2, $4 and $6 respectively. The shipping cost per laptop for different types from O2 to D1 are $4, $2 and $2; from O2 to D2 are $3, $3 and $2; and from O2 to D3 are $2, $1 and $3. lastly, the shipping

Table 1
Cost of the 3-dimensional transportation with limits on rim (P1).

	j=1	j=2	j=3	ai	Ai
i=1	1 / 2 / 2	3 / 4 / 1	2 / 4 / 6	30	40
i=2	4 / 2 / 2	3 / 3 / 2	2 / 1 / 3	15	30
i=3	4 / 5 / 3	5 / 4 / 2	2 / 2 / 4	10	20
bj	15	20	25		
Bj	25	25	40		

with e_1=20, E_1=30, e_2=30, E_2=40, e_3=15, E_3=20

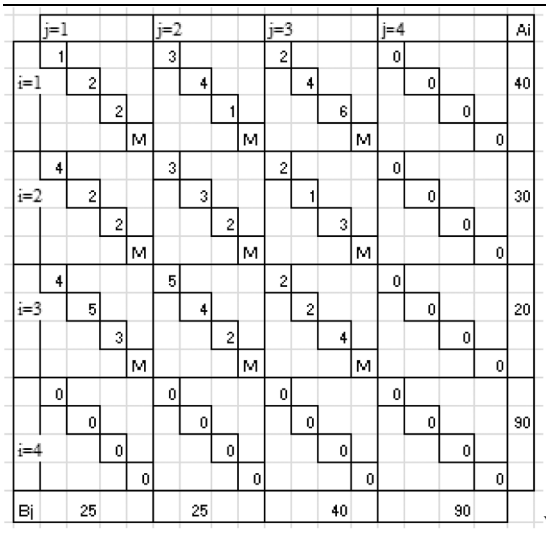
Table 2
Lower and upper bounds on laptops of different types.

x_{111}	1	12	x_{211}	0	7	x_{311}	1	3
x_{112}	2	10	x_{212}	1	10	x_{312}	0	9
x_{113}	0	8	x_{213}	1	8	x_{313}	2	11
x_{121}	2	10	x_{221}	2	5	x_{321}	3	7
x_{122}	0	7	x_{222}	0	9	x_{322}	0	6
x_{123}	2	10	x_{223}	1	7	x_{323}	1	8
x_{131}	0	8	x_{231}	2	6	x_{331}	1	4
x_{132}	3	5	x_{232}	1	10	x_{332}	0	10
x_{133}	2	6	x_{233}	0	4	x_{333}	1	9

cost from O3 to D1 for different types are \$4, \$5 and \$3; from O3 to D2 are \$5, \$4 and \$2; while from O3 to D3 the shipping costs are \$2, \$2 and \$4.

Assume that at O1 there is a min and max limit of 30 and 40 laptops of all types in a month while O2 the limits are 15 and 30 laptops of all types in a month. At O3 the limits are 10 and 20 laptops of all types in a month. These give us availability constraints. The minimum and maximum requirements of laptops of all types at D1 are 15 and 25 respectively, while for D2 are 20 and 25 respectively and for D3 are 25 and 40 respectively. These give us requirement constraints. The minimum and maximum number of laptops transported of type I are 20 and 30 respectively, while for Type II it is 30 and 40 and for Type III it is 15 and 20 respectively. These give us commodity constraints.

Table 3
Cost of the related 3-dimensional transportation (P2).



with E_1=30, E_2=40, E_3=20, E_4=90

Table 4
Cost of the transformed multi-index TP (P3).

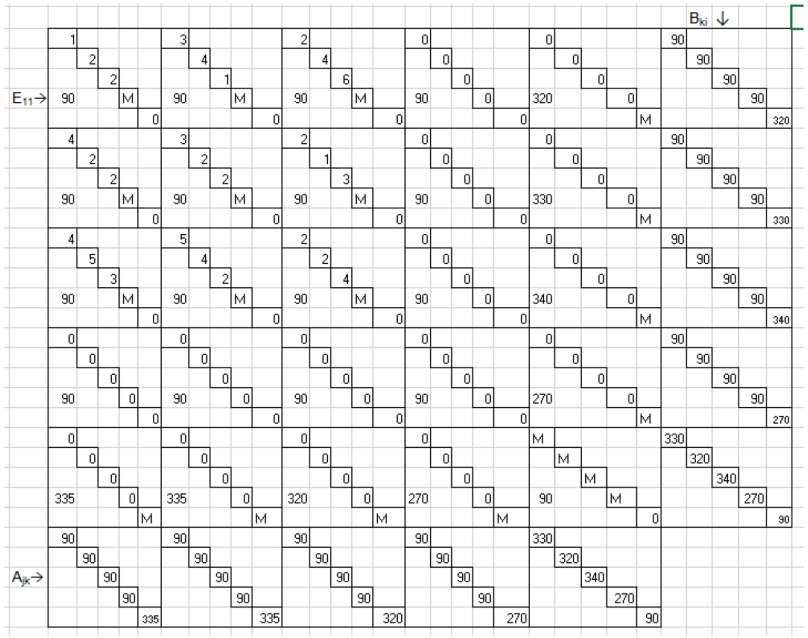


Table 5
Cost matrix of the related impaired flow 3-dimensional TP (P5).

	j=1			j=2			j=3			j=4			Ai
	1			3			2			0			
i=1	2			4			4			0			40
		2			1			6			0		
			M			M			M			0	
	4			3			2			0			
i=2	2			3			1			0			30
		2			2			3			0		
			M			M			M			0	
	4			5			2			0			
i=3	5			4			2			0			20
		3			2			4			0		
			M			M			M			0	
	0			0			0			M			
	0			0			0			M			15
i=4	0			0			0			M			
		0			0			0			M		
Bj	25			25			40			15			

with $E_1=30$, $E_2=40$, $E_3=20$, $E_4=15$

The above data is conveniently presented in the cost matrix given in Table 1 and Figure 1 below. The last 2 rows and 2 columns list the minimum and maximum availability and requirements.

Assume that the bounds on number of laptops of Type I to be transported from O1 to D1 are (1, 12), for Type II (2, 10) and (0, 8) for Type III. Similarly, the limits on laptops of Type I, II and III to be transported from O1 to D2 are (2, 10), (0, 7) and (2, 10) respectively while the bounds from O1 to D3 for different types of laptops are (0, 8), (3, 5) and (2, 6) respectively. Table 2 provides the bounds for rest of the variables wherex_{ijk}represents the number of type k (I, II, III) of laptops transported from the origin Oi (i = 1, 2, 3) to distribution center Dj (j = 1, 2, 3

The manager wishes to determine the number of laptops to be shipped so that the total shipping cost is minimized. The problem can be stated as a 3-dimensional (3 × 3 × 3) capacitated linear transportation with limits on availabilities, demand and different commodities as in Example 1 below.

Example 1. The following 3-dimensional CTP with bounds on availabilities, demand and different commodities is given.

P1: Minimize $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} c_{ijk}x_{ijk}$ subject to

$30 \le \sum_{j=1}^{3}\sum_{k=1}^{3} x_{1jk} \le 40$, $15 \le \sum_{j=1}^{3}\sum_{k=1}^{3} x_{2jk} \le 30$,

$10 \le \sum_{j=1}^{3}\sum_{k=1}^{3} x_{3jk} \le 20$ (Availabilityconstraints)

$15 \le \sum_{i=1}^{3}\sum_{k=1}^{3} x_{i1k} \le 25$, $20 \le \sum_{i=1}^{3}\sum_{k=1}^{3} x_{i2k} \le 25$,

$25 \le \sum_{i=1}^{3}\sum_{k=1}^{3} x_{i3k} \le 40$ (Requirementconstraints)

$20 \le \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij1} \le 30$, $30 \le \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij2} \le 40$,

$15 \le \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij3} \le 20$ (Commodityconstraints)

The bounds on the number of laptops Type I, II and III transported can be formulated mathematically as follows.

$1 \le x_{111} \le 12,\ 2 \le x_{112} \le 10,\ 0 \le x_{113} \le 8$ $fromO1toD1$;
$1 \le x_{111} \le 12,\ 2 \le x_{112} \le 10,\ 0 \le x_{113} \le 8$ $fromO1toD2$;
$0 \le x_{131} \le 8,\ 3 \le x_{132} \le 5,\ 2 \le x_{133} \le 6,$ $fromO1toD3$;
$0 \le x_{211} \le 7,\ 1 \le x_{212} \le 10,\ 1 \le x_{213} \le 8,$ $fromO2toD1$;
$2 \le x_{221} \le 5,\ 0 \le x_{222} \le 9,\ 1 \le x_{223} \le 7,$ $fromO2toD2$;
$2 \le x_{231} \le 6,\ 1 \le x_{232} \le 10,\ 0 \le x_{233} \le 4,$ $fromO2toD3$;
$1 \le x_{311} \le 3,\ 0 \le x_{312} \le 9,\ 2 \le x_{313} \le 11,$ $fromO3toD1$;
$3 \le x_{321} \le 7,\ 0 \le x_{322} \le 6,\ 1 \le x_{323} \le 8,$ $fromO3toD2$;
$1 \le x_{331} \le 4,\ 0 \le x_{332} \le 10,\ 1 \le x_{333} \le 9$ $fromO3toD3$.

To solve the above TP, we need to transform it to the related TP as follows

P2: Minimize $\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} c'_{ijk}y_{ijk}$ subject to

$\sum_{j=1}^{4}\sum_{k=1}^{4} y_{ijk}$=40, 30, 20, 90 i = 1, 2, 3, 4
$\sum_{i=1}^{4}\sum_{k=1}^{4} y_{ijk}$=25, 25, 40, 90 j = 1, 2, 3, 4
$\sum_{i=1}^{4}\sum_{j=1}^{4} y_{ijk}$=30, 40, 20, 90 k = 1, 2, 3, 4

$1 \le y_{111} \le 12,\ 2 \le y_{112} \le 10,\ 0 \le y_{113} \le 8,\ 2 \le y_{121} \le 10,\ 0 \le y_{122} \le 7,$
$2 \le y_{123} \le 10,\ 0 \le x_{131} \le 8,\ 3 \le y_{132} \le 5,$
$2 \le y_{133} \le 6,\ 0 \le y_{211} \le 7,\ 1 \le y_{212} \le 10,\ 1 \le y_{213} \le 8,\ 2 \le y_{221} \le 5,$
$0 \le y_{222} \le 9,\ 1 \le y_{223} \le 7,\ 2 \le y_{231} \le 6,$
$1 \le y_{232} \le 10,\ 0 \le y_{233} \le 4,\ 1 \le x_{311} \le 3,\ 0 \le y_{312} \le 9,\ 2 \le y_{313} \le 11,$
$3 \le y_{321} \le 7,\ 0 \le y_{322} \le 6,\ 1 \le y_{323} \le 8,$
$1 \le y_{331} \le 4,\ 0 \le y_{332} \le 10,\ 1 \le y_{333} \le 9$
$y_{141} + y_{142} + y_{143} + y_{144} \le 10$
$y_{241} + y_{242} + y_{243} + y_{244} \le 15$
$y_{341} + y_{342} + y_{343} + y_{344} \le 10$
$y_{411} + y_{412} + y_{413} + y_{414} \le 10$
$y_{421} + y_{422} + y_{423} + y_{424} \le 5$
$y_{431} + y_{432} + y_{433} + y_{434} \le 15$
$y_{141} + y_{241} + y_{341} + y_{411} + y_{421} + y_{431} + y_{441} \le 10$
$y_{142} + y_{242} + y_{342} + y_{412} + y_{422} + y_{432} + y_{442} \le 10$
$y_{143} + y_{243} + y_{343} + y_{413} + y_{423} + y_{433} + y_{443} \le 5$
$y_{441} + y_{442} + y_{443} + y_{444} \ge 0$

By the definition of M-feasibility, in all the above inequalities, $y_{ij4} = 0$, $\forall\ i \in I = \{1, 2, 3\}$, $j \in J = \{1, 2, 3\}$ and the costs c_{ijk} are provided below.

The above related problem can be solved by changing it into

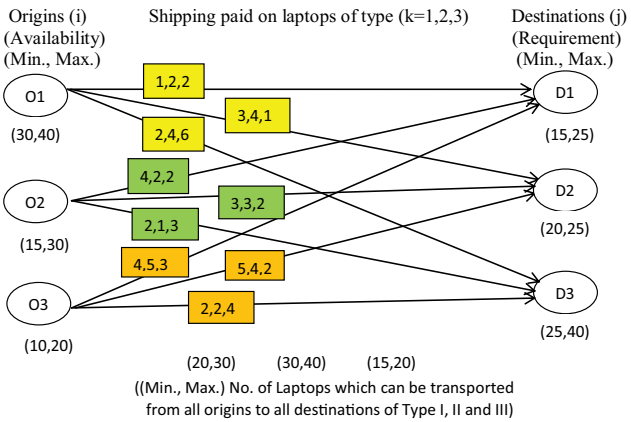


Fig. 1. Flow chart for cost matrix of Table 1.

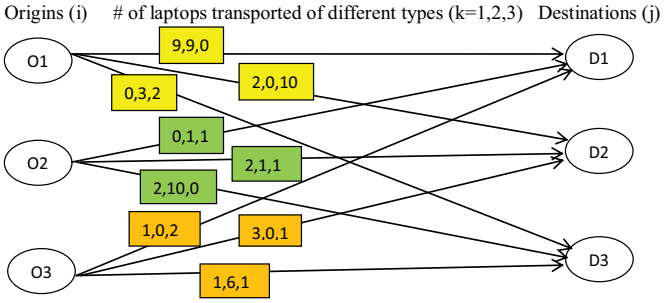


Fig. 2. Optimal solution of Example 1.

transformed multi-index transportation **P3** whose cost matrix is provided below Table 3).

By solving transformed transportation problem **P3** we can get the solution of the 3-dimensional

TP with bounds on availabilities, demand and different commodities.

Therefore, optimal feasible solution of the 3-dimensional TP with bounds on availabilities, demand and different commodities is obtained from the solution of the transformed transportation problem by solving cost matrix given in Table 4 and solution of transformed problem **P2** is given as follows

$x_{111} = 9,\ x_{112} = 9,\ x_{115} = 72,\ x_{121} = 2,\ x_{123} = 10,\ x_{125} = 78,\ x_{132} = 3,$
$x_{133} = 2,\ x_{135} = 185,\ x_{144} = 5,$
$x_{145} = 85,\ x_{151} = 79,\ x_{152} = 78,\ x_{153} = 78,\ x_{154} = 85,\ x_{212} = 1,$
$x_{213} = 1,\ x_{215} = 88,\ x_{221} = 2,\ x_{222} = 1,$
$x_{223} = 1,\ x_{225} = 86,\ x_{231} = 2,\ x_{232} = 10,\ x_{235} = 78,\ x_{244} = 12,$
$x_{245} = 78,\ x_{251} = 86,\ x_{252} = 78,\ x_{253} = 88,$
$x_{254} = 78,\ x_{311} = 1,\ x_{313} = 2,\ x_{315} = 87,\ x_{321} = 3,\ x_{323} = 1,\ x_{325} = 86,$
$x_{331} = 1,\ x_{332} = 6,\ x_{333} = 1,$
$x_{335} = 82,\ x_{344} = 5,\ x_{345} = 85,\ x_{351} = 80,\ x_{352} = 84,\ x_{353} = 86,$
$x_{354} = 90,\ x_{414} = 2,\ x_{415} = 88,\ x_{424} = 5,$
$x_{425} = 85,\ x_{434} = 15,\ x_{435} = 75,\ x_{441} = 5,\ x_{442} = 10,\ x_{443} = 2,$
$x_{444} = 51,\ x_{445} = 22,\ x_{451} = 85,\ x_{452} = 80,$
$x_{453} = 88,\ x_{454} = 17,\ x_{511} = 80,\ x_{512} = 80,\ x_{513} = 87,\ x_{514} = 88,$
$x_{521} = 83,\ x_{522} = 89,\ x_{523} = 78,\ x_{524} = 85,$
$x_{531} = 87,\ x_{532} = 71,\ x_{533} = 87,\ x_{534} = 75,\ x_{541} = 80,\ x_{542} = 80,$
$x_{543} = 88,\ x_{544} = 22,\ x_{555} = 90$

with objective function value $z = 141$. The optimal basic feasible solution is described in the flow chart (Figure 2) as below.

Example 2. Consider the case of impaired flow when the stocks of laptops at the factories is restricted to a total of 75. This results in an

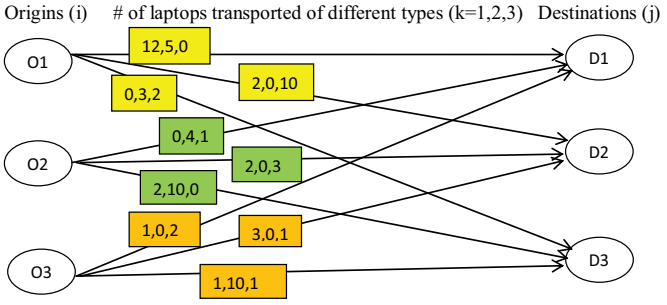


Fig. 3. Optimal solution of Example 2.

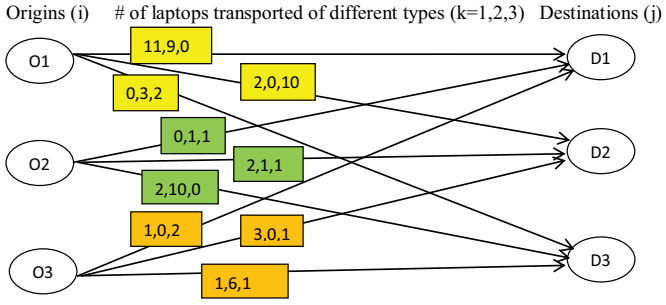


Fig. 4. Optimal solution of Example 3.

Table 6
Cost of the transformed multi-index transportation problem.

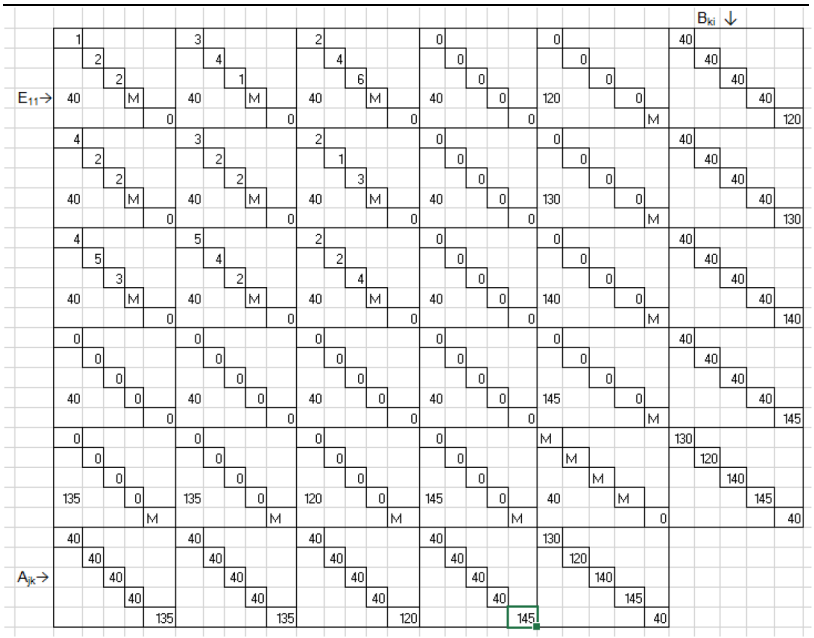


Table 7
Data for the test problems.

Test problem #.	c_{ijk} written in form of $i.j$ as column and k as row	Lower and upper bounds of availability	Lower and upper bounds of demand	Lower and upper bounds of commodity	Lower bounds (LB) and upper bounds (UB) of decision variables x_{ijk} LB UB	Lower bounds (LB) and upper bounds (UB) of decision variables x_{ijk} LB UB
6.1	1 2 3 1.1 1 2 2 1.2 3 4 1 1.3 2 4 6 1.4 2 2 1 2.1 4 2 2 2.2 3 3 2 2.3 2 1 3 2.4 3 2 1 3.1 4 5 3 3.2 5 4 2 3.3 2 2 4 3.4 2 1 1 4.1 1 2 2 4.2 3 1 2 4.3 2 1 3 4.4 2 1 2	$a_1 = 10, A_1 = 25$ $a_2 = 15, A_2 = 40$ $a_3 = 15, A_3 = 20$ $a_4 = 10, A_4 = 25$	$b_1 = 15, B_1 = 35$ $b_2 = 20, B_2 = 25$ $b_3 = 10, B_3 = 35$ $b_4 = 15, B_4 = 40$	$e_1 = 25, E_1 = 30$ $e_2 = 15, E_2 = 20$ $e_3 = 20, E_3 = 40$	x_{111} 1 10 x_{112} 2 12 x_{113} 0 9 x_{121} 1 11 x_{122} 0 7 x_{123} 2 10 x_{131} 0 8 x_{132} 3 5 x_{133} 2 6 x_{141} 3 10 x_{142} 2 7 x_{143} 1 9 x_{211} 0 7 x_{212} 1 10 x_{213} 1 8 x_{221} 2 5 x_{222} 0 11 x_{223} 1 7 x_{231} 2 6 x_{232} 1 10 x_{233} 0 4 x_{241} 1 6 x_{242} 0 11 x_{243} 0 8	x_{311} 1 7 x_{312} 0 9 x_{313} 2 11 x_{321} 3 7 x_{322} 0 7 x_{323} 1 8 x_{331} 1 4 x_{332} 0 10 x_{333} 0 12 x_{341} 3 10 x_{342} 2 11 x_{343} 1 8 x_{411} 2 11 x_{412} 1 7 x_{413} 2 7 x_{421} 1 8 x_{422} 1 10 x_{423} 0 8 x_{431} 3 10 x_{432} 0 10 x_{433} 1 7 x_{441} 1 6 x_{442} 0 10 x_{443} 2 12
6.2	1 2 3 4 1.1 15 18 7 15 1.2 9 14 11 6 1.3 10 4 12 4 1.4 8 10 13 10 1.5 3 11 5 8 2.1 19 20 17 12 2.2 15 21 14 10 2.3 20 24 18 11 2.4 16 19 23 12 2.5 11 12 17 8 3.1 14 11 8 10 3.2 24 20 13 11 3.3 17 16 15 12 3.4 11 13 7 13 3.5 8 11 10 16 4.1 24 28 31 15 4.2 14 19 21 20 4.3 18 21 14 12 4.4 9 10 20 15 4.5 12 11 11 12 5.1 10 8 13 15 5.2 12 6 11 19 5.3 8 15 12 10 5.4 12 4 20 12 5.5 11 14 15 17	$a_1 = 10, A_1 = 25$ $a_2 = 15, A_2 = 40$ $a_3 = 12, A_3 = 20$ $a_4 = 14, A_4 = 25$ $a_5 = 16, A_5 = 30$	$b_1 = 15, B_1 = 35$ $b_2 = 14, B_2 = 25$ $b_3 = 11, B_3 = 35$ $b_4 = 17, B_4 = 30$ $b_5 = 10, B_5 = 25$	$e_1 = 15, E_1 = 30$ $e_2 = 10, E_2 = 25$ $e_3 = 20, E_3 = 30$ $e_4 = 14, E_4 = 40$	x_{114} 1 11 x_{124} 0 10 x_{134} 0 7 x_{144} 0 12 x_{151} 1 7 x_{152} 2 8 x_{153} 1 11 x_{154} 0 10 x_{214} 0 8 x_{224} 1 9 x_{234} 1 10 x_{244} 2 11 x_{251} 1 12 x_{252} 0 10 x_{253} 2 5 x_{254} 1 11 x_{314} 0 11 x_{324} 0 6 x_{334} 0 7 x_{344} 0 8 x_{351} 0 10 x_{352} 0 6 x_{353} 0 5 x_{354} 0 6 x_{414} 0 11 x_{424} 1 8	x_{434} 1 7 x_{444} 0 8 x_{451} 0 7 x_{452} 0 8 x_{453} 0 11 x_{454} 0 8 x_{511} 3 12 x_{512} 2 10 x_{513} 1 7 x_{514} 2 12 x_{521} 0 10 x_{522} 0 11 x_{523} 1 8 x_{524} 2 10 x_{531} 0 8 x_{532} 0 11 x_{533} 1 12 x_{534} 0 8 x_{541} 0 7 x_{542} 0 10 x_{543} 1 9 x_{544} 0 11 x_{551} 0 7 x_{552} 0 8 x_{553} 2 11 x_{554} 0 10
Test problem #.	c_{ijk} written in form of $i.j$ as column and k as row	Lower and upper bounds of availability	Lower and upper bounds of demand	Lower and upper bounds of commodity	Lower bounds (LB) and upper bounds (UB) of decision variables	Lower bounds (LB) and upper bounds (UB) of decision variables
6.3	1 2 3 4 5 1.1 15 18 7 15 12 1.2 9 14 11 6 10 1.3 10 4 12 14 5 1.4 8 10 13 10 6 1.5 3 11 5 8 12 1.6 11 7 8 9 10 2.1 9 20 17 12 12 2.2 15 21 14 10 13 2.3 20 24 18 11 17 2.4 16 19 21 12 21 2.5 11 12 17 8 22 2.6 12 14 16 21 17 3.1 14 11 8 10 11 3.2 24 20 13 11 12 3.3 17 16 15 12 14 3.4 11 13 7 13 15 3.5 8 11 10 16 16	$a_1 = 10, A_1 = 25$ $a_2 = 15, A_2 = 40$ $a_3 = 15, A_3 = 25$ $a_4 = 15\ A_4 = 25$ $a_5 = 20, A_5 = 35$ $a_6 = 20, A_6 = 45$	$b_1 = 15, B_1 = 35$ $b_2 = 15, B_2 = 25$ $b_3 = 10, B_3 = 35$ $b_4 = 20, B_4 = 30$ $b_5 = 10, B_5 = 40$ $b_6 = 20, B_6 = 35$	$e_1 = 15, E_1 = 30$ $e_2 = 10, E_2 = 25$ $e_3 = 20, E_3 = 30$ $e_4 = 25, E_4 = 40$ $e_5 = 15, E_5 = 40$	x_{115} 0 12 x_{125} 1 18 x_{135} 0 10 x_{145} 0 15 x_{155} 0 18 x_{161} 0 10 x_{162} 0 12 x_{163} 0 17 x_{164} 0 19 x_{165} 0 20 x_{215} 1 12 x_{225} 0 11 x_{235} 1 12 x_{245} 0 15 x_{255} 0 12 x_{261} 0 12 x_{262} 0 15 x_{263} 0 17	x_{415} 0 13 x_{425} 0 14 x_{435} 0 15 x_{445} 0 12 x_{455} 0 8 x_{461} 0 10 x_{462} 0 12 x_{463} 0 13 x_{464} 0 14 x_{465} 0 15 x_{515} 1 7 x_{525} 2 12 x_{535} 0 13 x_{545} 0 10 x_{555} 0 20 x_{561} 0 10 x_{562} 0 11 x_{563} 0 9

(*continued on next page*)

Table 7 (*continued*)

Test problem #.	c_{ijk} written in form of *i.j* as column and *k* as row	Lower and upper bounds of availability	Lower and upper bounds of demand	Lower and upper bounds of commodity	Lower bounds (LB) and upper bounds (UB) of decision variables x_{ijk} LB UB	Lower bounds (LB) and upper bounds (UB) of decision variables x_{ijk} LB UB
	3.6 21 5 10 8 9				x_{264} 0 12	x_{564} 0 11
	4.1 24 28 31 15 12				x_{265} 0 8	x_{565} 0 10
	4.2 14 19 11 20 13				x_{315} 1 17	x_{615} 0 10
	4.3 18 21 14 12 14				x_{325} 0 15	x_{625} 0 11
	4.4 9 10 20 15 11				x_{335} 0 11	x_{635} 0 12
	4.5 12 11 11 12 12				x_{345} 0 10	x_{645} 0 10
	4.6 11 13 14 16 18				x_{355} 1 12	x_{655} 0 8
	5.1 10 8 13 15 16				x_{361} 0 13	x_{661} 0 8
	5.2 12 6 11 19 17				x_{362} 0 12	x_{662} 0 10
	5.3 8 15 12 10 16				x_{363} 0 11	x_{663} 0 12
	5.4 12 4 9 12 14				x_{364} 0 13	x_{664} 0 8
	5.5 11 14 15 17 21				x_{365} 0 14	x_{665} 0 9
	5.6 7 12 7 12 9					
	6.1 11 15 9 14 12					
	6.2 12 12 10 15 6					
	6.3 7 11 11 16 21					
	6.4 21 17 12 9 20					
	6.5 22 8 13 7 14					
	6.6 15 9 14 8 5					
6.4	1 2 3 4 5 6	$a_1 = 10$, $A_1 = 25$	$b_1 = 15$, $B_1 = 35$	$e_1 = 15$, $E_1 = 30$	x_{116} 0 11	x_{416} 0 7
	1.1 15 18 7 15 12 8	$a_2 = 15$, $A_2 = 40$	$b_2 = 15$, $B_2 = 25$	$e_2 = 10$, $E_2 = 25$	x_{126} 1 8	x_{426} 0 8
	1.2 9 14 11 6 10 7	$a_3 = 15$, $A_3 = 25$	$b_3 = 10$, $B_3 = 35$	$e_3 = 20$, $E_3 = 30$	x_{136} 0 12	x_{436} 0 9
	1.3 10 4 12 14 5 5	$a_4 = 15$ $A_4 = 25$	$b_4 = 20$, $B_4 = 30$	$e_4 = 25$, $E_4 = 40$	x_{146} 0 14	x_{446} 0 10
	1.4 8 10 13 10 6 10	$a_5 = 20$, $A_5 = 35$	$b_5 = 10$, $B_5 = 40$	$e_5 = 15$, $E_5 = 40$	x_{156} 0 16	x_{456} 0 11
	1.5 3 11 5 8 12 8	$a_6 = 20$, $A_6 = 45$	$b_6 = 20$, $B_6 = 35$	$e6 = 30$, $E_6 = 50$	x_{166} 0 21	x_{466} 0 12
	1.6 11 7 8 9 10 11	$a_7 = 15$, $A_7 = 40$	$b_7 = 25$, $B7 = 45$		x_{171} 0 7	x_{471} 0 11
	1.7 10 11 12 5 22 7				x_{172} 0 11	x_{472} 0 12
	2.1 9 20 17 12 12 14				x_{173} 0 8	x_{473} 0 8
	2.2 15 21 14 10 13 11				x_{174} 0 21	x_{474} 0 11
	2.3 20 24 18 11 17 13				x_{175} 0 15	x_{475} 0 12
	2.4 16 19 21 12 21 14				x_{176} 0 11	x_{476} 0 8
	2.5 11 12 17 8 22 15				x_{216} 0 12	x_{516} 0 9
	2.6 12 14 16 21 17 8				x_{226} 0 13	x_{526} 0 11
	2.7 13 15 17 23 5 9				x_{236} 0 15	x_{536} 0 12
	3.1 14 11 8 10 11 11				x_{246} 0 16	x_{546} 0 13
	3.2 24 20 13 11 12 12				x_{256} 0 7	x_{556} 0 12
	3.3 17 16 15 12 14 14				x_{266} 0 11	x_{566} 0 7
	3.4 11 13 7 13 15 15				x_{271} 0 13	x_{571} 0 8
	3.5 8 11 10 16 16 16				x_{272} 0 14	x_{572} 0 6
	3.6 21 5 10 8 9 8				x_{273} 0 15	x_{573} 0 9
	3.7 9 8 11 12 11 13				x_{274} 0 7	x_{574} 0 12
	4.1 24 28 31 15 12 14				x_{275} 0 12	x_{575} 0 21
	4.2 14 19 11 20 13 21				x_{276} 0 12	x_{576} 0 11
	4.3 18 21 14 12 14 22				x_{316} 0 13	x_{616} 0 9
	4.4 9 10 20 15 11 7				x_{326} 0 14	x_{626} 0 21
	4.5 12 11 11 12 12 11				x_{336} 0 15	x_{636} 0 12
	4.6 11 13 14 16 18 12				x_{346} 0 10	x_{646} 0 7
	4.7 10 11 15 17 15 23				x_{356} 0 9	x_{656} 0 8
	5.1 10 8 13 15 16 26				x_{366} 0 8	x_{666} 0 12
	5.2 12 6 11 19 17 21				x_{371} 0 11	x_{671} 0 12
	5.3 8 15 12 10 16 14				x_{372} 0 10	x_{672} 0 10
	5.4 12 4 9 12 14 8				x_{373} 0 12	x_{673} 0 12
	5.5 11 14 15 17 21 11				x_{374} 0 14	x_{674} 0 11
	5.6 7 12 7 12 9 12				x_{375} 0 8	x_{675} 0 11
	5.7 8 11 8 11 8 7				x_{376} 0 9	x_{676} 0 21
	6.1 11 15 9 14 12 8				x_{716} 0 7	x_{771} 0 4
	6.2 12 12 10 15 6 11				x_{726} 0 8	x_{772} 0 5
	6.3 7 11 11 16 21 15				x_{736} 0 19	x_{773} 0 6
	6.4 21 17 12 9 20 10				x_{746} 0 11	x_{774} 0 7
	6.5 22 8 13 7 14 12				x_{756} 0 10	x_{775} 0 8
	6.6 15 9 14 8 5 13				x_{766} 0 12	x_{776} 0 9
	6.7 17 10 21 12 16 11					
	7.1 11 11 20 11 7 10					
	7.2 9 12 9 10 18 12					
	7.3 12 11 24 11 9 11					
	7.4 13 7 11 16 14 14					
	7.5 14 21 12 15 11 21					
	7.6 15 22 13 9 10 8					
	7.7 4 10 6 17 8 9					

Table 8
Result of test problems.

Test problems no.	Type of problem	Order of original problem	Order of transformed problem (as of [13])	Impaired flow(IF) and enhanced flow (EF) value	Optimal objective function value for IF and EF	Optimal basic feasible solution
6.1	Impaired flow	4 × 4 × 3	6 × 6 × 5	80	154	$x_{111} = 1$, $x_{112} = 2$, $x_{121} = 1$, $x_{123} = 10$, $x_{132} = 3$, $x_{133} = 2$, $x_{141} = 3$, $x_{142} = 2$, $x_{143} = 9$, $x_{212} = 1$, $x_{213} = 1$, $x_{221} = 2$, $x_{223} = 1$, $x_{231} = 2$, $x_{232} = 8$, $x_{241} = 1$, $x_{243} = 8$, $x_{311} = 1$, $x_{313} = 2$, $x_{321} = 3$, $x_{323} = 1$, $x_{331} = 1$, $x_{341} = 3$, $x_{342} = 2$, $x_{343} = 3$, $x_{411} = 6$, $x_{412} = 1$, $x_{413} = 2$, $x_{421} = 1$, $x_{422} = 1$, $x_{431} = 3$, $x_{433} = 1$, $x_{441} = 1$
	Enhanced flow			100	174	$x_{111} = 10$, $x_{112} = 2$, $x_{121} = 1$, $x_{123} = 2$, $x_{132} = 3$, $x_{133} = 2$, $x_{141} = 3$, $x_{142} = 2$, $x_{143} = 9$, $x_{212} = 1$, $x_{213} = 1$, $x_{223} = 1$, $x_{231} = 2$, $x_{232} = 1$, $x_{241} = 1$, $x_{243} = 8$, $x_{311} = 1$, $x_{313} = 2$, $x_{321} = 3$, $x_{323} = 1$, $x_{331} = 1$, $x_{341} = 3$, $x_{342} = 3$, $x_{343} = 1$, $x_{411} = 11$, $x_{412} = 1$, $x_{413} = 2$, $x_{421} = 1$, $x_{422} = 10$, $x_{431} = 3$, $x_{432} = 4$, $x_{433} = 1$, $x_{441} = 1$
6.2	Impaired flow	5 × 5 × 4	7 × 7 × 6	95	1245	$x_{111} = 1$, $x_{112} = 2$, $x_{114} = 1$, $x_{121} = 1$, $x_{123} = 2$, $x_{124} = 2$, $x_{132} = 3$, $x_{133} = 2$, $x_{134} = 1$, $x_{143} = 9$, $x_{212} = 1$, $x_{213} = 1$, $x_{221} = 2$, $x_{223} = 1$, $x_{224} = 1$, $x_{231} = 2$, $x_{232} = 1$, $x_{234} = 1$ $x_{241} = 1$, $x_{244} = 2$, $x_{251} = 1$, $x_{253} = 2$, $x_{254} = 1$, $x_{311} = 1$, $x_{313} = 2$, $x_{314} = 1$, $x_{321} = 3$, $x_{323} = 1$, $x_{141} = 3$, $x_{142} = 2$, $x_{143} = 1$, $x_{151} = 1$, $x_{152} = 2$, $x_{153} = 1$, $x_{323} = 1$, $x_{331} = 1$, $x_{341} = 3$, $x_{342} = 2$, $x_{343} = 1$, $x_{411} = 2$, $x_{412} = 1$, $x_{413} = 2$, $x_{421} = 1$, $x_{422} = 1$, $x_{424} = 1$, $x_{431} = 3$, $x_{433} = 1$, $x_{434} = 1$, $x_{441} = 1$, $x_{443} = 2$, $x_{511} = 3$, $x_{512} = 2$, $x_{513} = 1$, $x_{514} = 2$, $x_{523} = 1$, $x_{524} = 2$, $x_{533} = 1$, $x_{542} = 8$, $x_{543} = 1$, $x_{553} = 2$
	Enhanced flow			110	1299	$x_{111} = 1$, $x_{112} = 2$, $x_{114} = 1$, $x_{121} = 1$, $x_{123} = 2$, $x_{124} = 2$, $x_{132} = 4$, $x_{133} = 2$, $x_{134} = 7$, $x_{141} = 3$, $x_{142} = 2$, $x_{143} = 1$, $x_{151} = 7$, $x_{152} = 2$, $x_{153} = 1$, $x_{212} = 1$, $x_{213} = 1$, $x_{221} = 2$, $x_{223} = 1$, $x_{224} = 1$, $x_{231} = 2$, $x_{232} = 1$, $x_{234} = 1$ $x_{241} = 1$, $x_{244} = 2$, $x_{251} = 1$, $x_{253} = 2$, $x_{254} = 1$, $x_{311} = 1$, $x_{313} = 2$, $x_{314} = 1$, $x_{321} = 3$, $x_{323} = 1$, $x_{323} = 1$, $x_{331} = 1$, $x_{341} = 3$, $x_{342} = 2$, $x_{343} = 1$, $x_{411} = 2$, $x_{412} = 1$, $x_{413} = 2$, $x_{421} = 1$, $x_{422} = 1$, $x_{424} = 1$, $x_{431} = 3$, $x_{433} = 1$, $x_{434} = 1$, $x_{441} = 1$, $x_{443} = 2$, $x_{511} = 3$, $x_{512} = 2$, $x_{513} = 1$, $x_{514} = 2$, $x_{523} = 1$, $x_{524} = 2$, $x_{533} = 1$, $x_{542} = 10$, $x_{543} = 1$, $x_{553} = 2$
Test problems no.	Type of problem	Order of original problem	Order of transformed problem (as of [13])	Impaired flow(IF) and enhanced flow (EF) value	Optimal objective function value for IF and EF	Optimal basic feasible solution
6.3	Impaired flow	6 × 6 × 5	8 × 8 × 7	115	1451	$x_{111} = 1$, $x_{112} = 2$, $x_{114} = 1$, $x_{121} = 1$, $x_{123} = 2$, $x_{124} = 2$, $x_{125} = 1$, $x_{132} = 3$, $x_{133} = 2$, $x_{141} = 3$, $x_{142} = 2$, $x_{143} = 1$, $x_{151} = 1$, $x_{152} = 2$, $x_{153} = 1$, $x_{212} = 1$, $x_{213} = 1$, $x_{215} = 1$, $x_{221} = 2$, $x_{223} = 1$, $x_{224} = 1$, $x_{231} = 2$, $x_{232} = 1$, $x_{234} = 1$, $x_{235} = 1$, $x_{241} = 1$, $x_{244} = 2$, $x_{251} = 1$, $x_{253} = 2$, $x_{254} = 1$, $x_{311} = 1$, $x_{313} = 2$, $x_{315} = 1$, $x_{321} = 3$, $x_{323} = 1$, $x_{331} = 1$, $x_{341} = 3$, $x_{342} = 2$, $x_{343} = 1$, $x_{411} = 2$, $x_{412} = 1$, $x_{413} = 2$, $x_{421} = 1$, $x_{422} = 1$, $x_{424} = 1$, $x_{431} = 3$, $x_{433} = 1$, $x_{434} = 1$, $x_{441} = 1$, $x_{443} = 2$, $x_{511} = 3$, $x_{512} = 2$, $x_{513} = 1$, $x_{514} = 2$, $x_{515} = 1$, $x_{523} = 1$, $x_{524} = 2$, $x_{525} = 2$, $x_{533} = 1$, $x_{543} = 1$, $x_{553} = 2$, $x_{563} = 1$, $x_{564} = 1$, $x_{644} = 1$, $x_{654} = 1$, $x_{662} = 3$, $x_{664} = 8$, $x_{665} = 9$
	Enhanced flow			130	1493	$x_{111} = 1$, $x_{112} = 2$, $x_{114} = 1$, $x_{121} = 1$, $x_{123} = 2$, $x_{124} = 2$, $x_{125} = 1$, $x_{132} = 5$, $x_{133} = 2$, $x_{141} = 3$, $x_{142} = 2$, $x_{143} = 1$, $x_{151} = 7$, $x_{152} = 2$, $x_{153} = 1$, $x_{212} = 1$, $x_{213} = 1$, $x_{215} = 1$, $x_{221} = 2$, $x_{223} = 1$, $x_{224} = 1$, $x_{231} = 2$, $x_{232} = 1$, $x_{234} = 1$, $x_{235} = 1$, $x_{241} = 1$, $x_{244} = 2$, $x_{251} = 1$, $x_{253} = 2$, $x_{254} = 1$, $x_{311} = 1$, $x_{313} = 2$, $x_{315} = 1$, $x_{321} = 3$, $x_{323} = 1$, $x_{331} = 1$, $x_{341} = 3$, $x_{342} = 2$, $x_{343} = 1$, $x_{362} = 7$, $x_{411} = 2$, $x_{412} = 1$, $x_{413} = 2$, $x_{421} = 1$, $x_{422} = 1$, $x_{424} = 2$, $x_{431} = 3$, $x_{433} = 1$, $x_{434} = 1$, $x_{441} = 1$, $x_{443} = 2$, $x_{511} = 3$, $x_{512} = 2$, $x_{513} = 1$, $x_{514} = 2$, $x_{515} = 1$, $x_{523} = 1$, $x_{524} = 2$, $x_{525} = 2$, $x_{533} = 1$, $x_{542} = 2$, $x_{543} = 1$, $x_{553} = 2$, $x_{654} = 7$, $x_{664} = 4$, $x_{665} = 9$
6.4	Impaired flow	7 × 7 × 6	9 × 9 × 8	145	1675	$x_{111} = 1$, $x_{112} = 2$, $x_{114} = 1$, $x_{121} = 1$, $x_{123} = 2$, $x_{124} = 2$, $x_{125} = 1$, $x_{132} = 3$, $x_{133} = 2$, $x_{141} = 3$, $x_{142} = 2$, $x_{143} = 1$, $x_{151} = 1$, $x_{152} = 2$, $x_{153} = 1$, $x_{212} = 1$, $x_{213} = 1$, $x_{215} = 1$, $x_{221} = 2$, $x_{223} = 1$, $x_{224} = 1$, $x_{231} = 2$, $x_{232} = 1$, $x_{234} = 1$, $x_{235} = 1$, $x_{241} = 1$, $x_{244} = 2$, $x_{251} = 1$, $x_{253} = 2$, $x_{254} = 1$, $x_{275} = 2$, $x_{311} = 1$, $x_{313} = 2$, $x_{315} = 1$, $x_{321} = 3$, $x_{323} = 1$, $x_{331} = 1$, $x_{341} = 3$, $x_{342} = 2$, $x_{343} = 1$, $x_{411} = 2$, $x_{412} = 1$, $x_{413} = 2$, $x_{421} = 1$, $x_{422} = 1$, $x_{424} = 1$, $x_{431} = 3$, $x_{433} = 1$, $x_{434} = 1$, $x_{441} = 1$, $x_{443} = 2$, $x_{511} = 3$, $x_{512} = 2$, $x_{513} = 1$, $x_{514} = 2$, $x_{515} = 1$,

(*continued on next page*)

Table 8 (*continued*)

Test problems no.	Type of problem	Order of original problem	Order of transformed problem (as of [13])	Impaired flow(IF) and enhanced flow (EF) value	Optimal objective function value for IF and EF	Optimal basic feasible solution
						$x_{523} = 1, x_{524} = 2, x_{525} = 2, x_{533} = 1, x_{543} = 1, x_{553} = 2, x_{654} = 6, x_{664} = 5, x_{665} = 9$
	Enhanced flow			170	1722	$x_{111} = 1, x_{112} = 2, x_{114} = 1, x_{121} = 1, x_{123} = 2, x_{124} = 2, x_{125} = 1, x_{132} = 3, x_{133} = 2, x_{141} = 3, x_{142} = 2, x_{143} = 1, x_{151} = 7, x_{152} = 2, x_{153} = 1, x_{174} = 11, x_{212} = 1, x_{213} = 1, x_{215} = 1, x_{221} = 2, x_{223} = 1, x_{224} = 1, x_{231} = 2, x_{232} = 1, x_{234} = 1, x_{235} = 1, x_{241} = 1, x_{244} = 2, x_{251} = 1, x_{253} = 2, x_{254} = 1, x_{275} = 10, x_{311} = 1, x_{313} = 2, x_{315} = 1, x_{321} = 3, x_{323} = 1, x_{331} = 1, x_{341} = 3, x_{342} = 2, x_{343} = 1, x_{411} = 2, x_{412} = 1, x_{413} = 2, x_{421} = 1, x_{422} = 1, x_{424} = 1, x_{431} = 3, x_{433} = 1, x_{434} = 1, x_{441} = 1, x_{443} = 2, x_{511} = 3, x_{512} = 2, x_{513} = 1, x_{514} = 2, x_{515} = 1, x_{523} = 1, x_{524} = 2, x_{525} = 2, x_{533} = 1, x_{542} = 3, x_{543} = 1, x_{553} = 2, x_{625} = 4, x_{665} = 9, x_{771=4}$

Table 9
Complexity analysis of the capacitated solid transportation problem.

Problem	Complexity (proposed procedure)
(**P1**)	O(*mnp*, $m + n + p$)
(**P2**)	O($(m + 1)(n + 1)(p + 1)$, $m + n + p + 3$)
(**P3**)	O($(m + 2)(n + 2)(p + 2)$, $mp + np + mn + 4(m + n + p) + 12$)
(**P4**)	O(*mnp*, $m + n + p + 1$)
(**P5**)	O($(m + 1)(n + 1)(p + 1)$, $m + n + p + 4$)

impaired/restricted flow 3-dimensional CTP with bounds on availabilities, demand and different commodities (**P4**). We have taken the same cost matrix, availabilities, demands and capacities as in Example 1, with an additional flow constraint given below.

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} x_{ijk} = 75 \quad (75 < 90)$$

In this case, we find the solution of the related 3-dimensional transportation problem for impaired flow as follows:

P5: Minimize $\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} c'_{ijk}y_{ijk}$ subject to

$\sum_{j=1}^{4}\sum_{k=1}^{4} y_{ijk}$=40, 30, 20, 15 $\quad i = 1, 2, 3, 4$

$\sum_{i=1}^{4}\sum_{k=1}^{4} y_{ijk}$=25, 25, 40, 15 $\quad j = 1, 2, 3, 4$

$\sum_{i=1}^{4}\sum_{j=1}^{4} y_{ijk}$=30, 40, 20, 15 $\quad k = 1, 2, 3, 4$

$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} y_{ijk} = 75$

with same bounds, as in Problem **P2** of Example 1. The costs c_{ijk} of the related three-dimensional transportation problem for impaired flow are provided in Table 5 below.

The above related problem can be solved by changing it into transformed multi-index transportation whose cost matrix is provided below.

By solving transformed transportation problem, we can get the solution of the impaired flow-dimensional TP with bounds on availabilities, demand and different commodities (Figs. 1 and 2).

Therefore, an optimal feasible solution of the 3-dimensional TP having bounds on availabilities, demand and commodities with restriction on total flow can be obtained from the solution of the transformed transportation problem by solving cost matrix given in Table 5. The optimal basic feasible solution in terms of original variables from transformed problem **P2** with objective function value as 151 is described by flow chart in Fig. 3 below.

Example 3. Consider the case when the manager wishes to enhance the supply of laptops (the total transportation flow) due to beginning to the semester in Colleges and Universities, say.

This gives rise to 3-dimensional enhanced flow CTP with bounds on availabilities, demand and different commodities (**P4**). We have taken the same cost matrix, availabilities, demands and capacities as taken in Example 1, with an additional flow constraint as given below.

P6: Minimize $\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} c'_{ijk}y_{ijk}$subject to

$\sum_{j=1}^{3}\sum_{k=1}^{3} y_{ijk} \geq$ 30, 15, 10 $\quad i = 1, 2, 3$

$\sum_{i=1}^{3}\sum_{k=1}^{3} y_{ijk} \geq$ 15, 20, 25 $\quad j = 1, 2, 3$

$\sum_{i=1}^{3}\sum_{j=1}^{3} y_{ijk} \geq$ 20, 30, 15 $\quad k = 1, 2, 3$

$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} x_{ijk} = 70$

with upper and lower limits on the decision variables, like Example 1.

Now, to solve this enhanced flow problem, we convert the given problem into a related capacitated multi-index transportation problem as below with costs c'_{ijk} for i, j, k = 1,2,3,4 same as of Table 5.

P7: Minimize $\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} c'_{ijk}y_{ijk}$subject to

$\sum_{j=1}^{4}\sum_{k=1}^{4} y_{ijk} = 76,\ 66,\ 67,\ 139 \quad i = 1,\ 2,\ 3,\ 4$

$\sum_{i=1}^{4}\sum_{k=1}^{4} y_{ijk} = 78,\ 69,\ 62,\ 139 \quad j = 1,\ 2,\ 3,\ 4$

$\sum_{i=1}^{4}\sum_{j=1}^{4} y_{ijk} = 62,\ 76,\ 71,\ 139 \quad k = 1,\ 2,\ 3,\ 4$

$1 \le y_{111} \le 12,\ 2 \le y_{112} \le 10,\ 0 \le y_{113} \le 8,\ 2 \le y_{121} \le 10,\ 0 \le y_{122} \le 7,$

$2 \le y_{123} \le 10,\ 0 \le x_{131} \le 8,\ 3 \le y_{132} \le 5,$

$2 \le y_{133} \le 6,\ 0 \le y_{211} \le 7,\ 1 \le y_{212} \le 10,\ 1 \le y_{213} \le 8,\ 2 \le y_{221} \le 5,\ 0 \le y_{222} \le 9,$

$1 \le y_{223} \le 7,\ 2 \le y_{231} \le 6,$

$1 \le y_{232} \le 10,\ 0 \le y_{233} \le 4,\ 1 \le x_{311} \le 3,\ 0 \le y_{312} \le 9,\ 2 \le y_{313} \le 11,\ 3 \le y_{321} \le 7,$

$0 \le y_{322} \le 6,\ 1 \le y_{323} \le 8,$

$1 \le y_{331} \le 4,\ 0 \le y_{332} \le 10,\ 1 \le y_{333} \le 9$

$y_{141} + y_{142} + y_{143} + y_{144} \le 46$

$y_{241} + y_{242} + y_{243} + y_{244} \le 51$

$y_{341} + y_{342} + y_{343} + y_{344} \le 57$

$y_{411} + y_{412} + y_{413} + y_{414} \le 63$

$y_{421} + y_{422} + y_{423} + y_{424} \le 49$

$y_{431} + y_{432} + y_{433} + y_{434} \le 37$

$y_{141} + y_{241} + y_{341} + y_{411} + y_{421} + y_{431} + y_{441} \le 42$

$y_{142} + y_{242} + y_{342} + y_{412} + y_{422} + y_{432} + y_{442} \le 46$

$y_{143} + y_{243} + y_{343} + y_{413} + y_{423} + y_{433} + y_{443} \le 56$

Using definition of M-feasibility, in all the above inequalities,

$y_{ij4} = 0\ , \quad i \in I = \{1,\ 2,\ 3\},\ j \in J = \{1,\ 2,\ 3\}$

Problem **P7** is again converted into an equivalent multi-index transportation problem **P8** [13]. The optimal feasible solution of the enhanced flow 3-dimensional TP with bounds on availabilities, demand and commodities is obtained from the solution of the equivalent transformed multi-index transportation problem **P8**. The optimal solution of in terms of original variables from equivalent transformed problem **P8** with optimal objective function value as 143 as described by the flow chart in Fig. 4 below.

7. Computational results

Two test problems were initially solved for multi-index CTP with bounds on availabilities, demand and commodities. At first, the constraint of restricted flow was added to both the test problems, and then the enhanced flow constraint was added to both the test problems, and the new problems were solved. The data for the test problems are given in Table 7 below and the results are summarized in Table 8. Note that the lower and upper bounds of the decision variables which are not defined for test problems 6.2, 6.3 and 6.4 in Table 6 are same as of problem 6.1. To test the validity of the proposed solution method to the larger scale problems, we, further, solved several other problems increasing in size to $10 \times 10 \times 9$. The data and the solutions of these problems are not included here to avoid redundancy. The test problems were solved using GAMS (General Algebraic Modeling System) on a computer with Intel Pentium Processor 1.70 GHz having 8 GB RAM. We solved both the original problems and the transformed problems and noticed a computation time of less than 5 s for each problem.

The computational complexity O(m, n) pertaining to each of the solid transportation problem from P1 to P5 with m nodes and n arcs is presented below (Table 9).

8. Conclusion

We have provided a solution method for a constrained transportation problem (CTP) with axial constraints with bounds on source availabilities, destination demands, and various commodities. We also discussed solution methods for the case of impaired and enhanced flow in a CTP, which involves the addition of a flow constraint to a given CTP. We discuss an extension of the methods to handle situations of unbalanced constrained transportation problems. The solution method consists of transforming a given CTP into a related solid problem by adding a dummy supply point, a dummy destination and an extra commodity. The related problem—a "Three Axial Sum" problem—can be reformulated as a multi-index transportation problem by applying Haley's method [13] and thus can be solved using Haley's [12] method. The solution method is very useful for transporting heterogeneous commodities. The maximum number of iterations required in our proposed procedure to solve the constrained solid transportation problem is shown to be ${}^{mnp}C_{m+n+p}$, where m, n, and p represent the number of origins, destinations and commodities, respectively. Future work should extend the solution method for solving time constrained transportation problems with axial constraints with bounds on rim conditions. Further research could also explore cost-time trade-off pairs for the problem in order to identify the optimal pair according to the priority between time and cost.

Appendix

Theorem 1. There is one-to-one relationship between a feasible solution of problem **P1** and a M-feasible solution of problem **P2.**

Proof. Let $\{y_{ijk}\}$be a M-feasible solution of problem **P2**. Define $\{x_{ijk}\}$, $i \in I, j \in J, k \in K$ by the following transformation:

$$x_{ijk} = y_{ijk} \quad \forall\ i \in I,\ j \in J,\ k \in K \tag{A}$$

As $l_{ijk} \le y_{ijk} \le u_{ijk}\ \ \forall\ i \in I, j \in J, k \in K$, (A) implies that $l_{ijk} \le y_{ijk} \le u_{ijk} \quad \forall\ i \in I, j \in J, k \in K$.

Also, $\sum_{J'}\sum_{K'} y_{ijk} = A_i' \quad \forall\ i \in I \subset I'$

$\Rightarrow \sum_J \sum_K y_{ijk} + \sum_K y_{in+1k} + \sum_J y_{ijp+1} + y_{in+1p+1} = A_i \quad \forall\ i \in I \quad [as\ A_i' = A_i \quad \forall\ i \in I]$

$\Rightarrow \sum_J \sum_K y_{ijk} = A_i - (\sum_K y_{in+1k} + \sum_J y_{ijp+1} + y_{in+1p+1}) \quad \forall\ i \in I$

$\Rightarrow a_i \le \sum_J \sum_K y_{ijk} \le A_i \quad \forall\ i \in I$

Since $0 \le \sum_K y_{in+1k} + \sum_J y_{ijp+1} + y_{in+1p+1} \le A_i - a_i \quad \forall\ i \in I$, but $y_{ijk} = x_{ijk} \quad \forall\ i \in I,\ j \in J,\ k \in K$

$$\Rightarrow a_i \le \sum_J \sum_K x_{ijk} \le A_i \quad \forall\ i \in I$$

Similarly, it can be shown that

$$b_j \le \sum_I \sum_K x_{ijk} \le B_j \quad \forall\ j \in I; \quad e_k \le \sum_I \sum_J x_{ijk} \le E_k \quad \forall\ k \in K$$

So x_{ijk} defined by (A) is a feasible solution of problem **P1**.

Conversely, $\{x_{ijk}\}$, $i \in I,\ j \in J,\ k \in K$let be a feasible solution of problem **P1**.

Define

$$y_{ijk} = x_{ijk} \quad \forall\ i \in I,\ j \in J,\ k \in K \tag{B}$$

$$\sum_K y_{m+1jk} + \sum_i y_{ijp+1} + y_{m+1jp+1} = B_j - \sum_I \sum_K y_{ijk} \quad \forall\ j \in J \tag{C}$$

$$\sum_K y_{in+1k} + \sum_J y_{ijp+1} + y_{in+1p+1} = A_i - \sum_J \sum_K y_{ijk} \quad \forall\ i \in I \tag{D}$$

$$\sum_J y_{m+1jk} + \sum_I y_{in+1k} + y_{m+1n+1k} = E_k - \sum_I \sum_J y_{ijk} \quad \forall\ k \in K \tag{E}$$

and

$$\sum_{K'} y_{m+1n+1k} = \sum_I \sum_J \sum_K x_{ijk} + \sum_i \sum_j y_{ijp+1} \tag{F}$$

Since $l_{ijk} \le x_{ijk} \le u_{ijk}\ \forall\ i \in I, j \in J, k \in K$, so $l_{ijk} \le y_{ijk} \le u_{ijk}\ \forall\ i \in I, j \in J, k \in K$,

(1) and (D) together imply that $0 \le \sum_K y_{in+1k} + \sum_J y_{ijp+1} + y_{in+1p+1} \le A_i - a_i \quad \forall \quad i \in I$ and
(2) and (C) together imply that $0 \le \sum_K y_{m+1jk} + \sum_i y_{ijp+1} + y_{m+1jp+1} \le B_j - b_j \quad \forall \quad j \in J$, and
(3) and (D) together imply $0 \le \sum_j y_{m+1jk} + \sum_i y_{in+1k} + y_{m+1n+1k} \le E_k - e_k \quad \forall \quad k \in K$

In all the above inequalities, By the definition of M-feasibility, the second summation would be zero as $y_{ijp+1} = 0\ \forall\ i \in I,\ j \in J$
Clearly $\sum_{K'} y_{m+1n+1k} = \sum_I \sum_J \sum_K x_{ijk} + \sum_i \sum_j y_{ijp+1} = \sum_I \sum_J \sum_K x_{ijk} \ge 0$ (By M-feasibility) thus $\sum_k y_{m+1n+1k} \ge 0 \quad \forall\ k \in K'$
Consider, for $i \in I$

$\sum_{J'} \sum_{K'} y_{ijk} = \sum_J \sum_K y_{ijk} + \sum_K y_{in+1k} + \sum_J y_{ijp+1} + y_{in+1p+1}$
$= \sum_J \sum_K x_{ijk} + \sum_K y_{in+1k} + \sum_J y_{ijp+1} + A_i - \sum_J \sum_K x_{ijk} - \sum_K y_{in+1k} - \sum_J y_{ijp+1}$ using (B) and (D)
$= A_i = A_i' \quad \forall\ i \in I$

For $i = m + 1$

$\sum_{J'} \sum_{K'} y_{m+1jk} = \sum_J \sum_K y_{m+1jk} + \sum_K y_{m+1n+1k} + \sum_J y_{m+1jp+1} + y_{m+1n+1p+1}$
$= \sum_J \sum_K y_{m+1jk} + \sum_{K'} y_{m+1n+1k} + \sum_J (B_j - \sum_I \sum_K x_{ijk} - \sum_K y_{m+1jk} - \sum_i y_{ijp+1})$ using(1)and(C)
$= \sum_I \sum_J \sum_K x_{ijk} + \sum_J B_j - \sum_I \sum_J \sum_K x_{ijk}$ using(F)
$= \sum_J B_j$

Similarly, $\sum_{I'} \sum_{K'} y_{ijk} = B_j' \quad \forall\ j \in J$ and $\sum_{I'} \sum_{K'} y_{ijk} = \sum_I A_i, \quad j = n + 1$.
Consider, for $k \in K'$

$\sum_{I'} \sum_{J'} y_{ijk} = \sum_I \sum_J y_{ijk} + \sum_I y_{in+1k} + \sum_J y_{m+1jk} + y_{m+1n+1k}$
$= \sum_I \sum_J x_{ijk} + \sum_I y_{in+1k} + \sum_J y_{m+1jk} + E_k - \sum_I \sum_J x_{ijk} - \sum_I y_{in+1k} - \sum_J y_{m+1jk}$ using(B)and(E)
$= E_k = E_k' \quad \forall\ k \in K$

For $k = p + 1$

$\sum_{I'} \sum_{J'} y_{ijp+1} = \sum_I \sum_J y_{ijp+1} + \sum_I y_{in+1p+1} + \sum_J y_{m+1jp+1} + y_{m+1n+1p+1}$
$= \sum_I \sum_J y_{ijp+1} + \sum_I y_{in+1p+1} + y_{m+1n+1p+1} + \sum_J (B_j - \sum_I \sum_K x_{ijk} - \sum_K y_{m+1jk} - \sum_i y_{ijp+1})$ $(from\,(C))$
$= \sum_I \sum_J y_{ijp+1} + \sum_I y_{in+1p+1} + y_{m+1n+1p+1} + \sum_J B_j - \sum_I \sum_J \sum_K y_{ijk} - \sum_J \sum_K y_{m+1jk} - \sum_I \sum_J y_{ijp+1}$
$= \sum_I y_{in+1p+1} + y_{m+1n+1p+1} + \sum_J B_j - \sum_I \sum_J \sum_K y_{ijk} - \sum_J \sum_K y_{m+1jk}$
$= \sum_I y_{in+1p+1} + y_{m+1n+1p+1} + \sum_J B_j - \sum_I \sum_J \sum_K y_{ijk} - \sum_J \sum_K y_{m+1jk} + \sum_K y_{m+1n+1k} - \sum_K y_{m+1n+1k}$
$= \sum_I y_{in+1p+1} + y_{m+1n+1p+1} + \sum_J B_j - \sum_I \sum_J \sum_K y_{ijk} - \sum_{J'} \sum_K y_{m+1jk} + \sum_k y_{m+1n+1k}$
$= \sum_I y_{in+1p+1} + \sum_k y_{m+1n+1k} + y_{m+1n+1p+1} + \sum_J B_j - \sum_I \sum_J \sum_K y_{ijk} - \sum_{J'} \sum_K y_{m+1jk} - \sum_I \sum_K y_{in+1k} + \sum_I \sum_K y_{in+1k}$
$= \sum_I y_{in+1p+1} + \sum_K y_{m+1n+1k} + y_{m+1n+1p+1} + \sum_I \sum_K y_{in+1k} + \sum_J B_j - \sum_K (\sum_I \sum_J y_{ijk} + \sum_{J'} y_{m+1jk} + \sum_i y_{in+1k})$
$= \sum_I y_{in+1p+1} + \sum_{K'} y_{m+1n+1k} + \sum_I \sum_K y_{in+1k} + \sum_J B_j - \sum_K E_k$ $(\text{from}\,(E))$
$= \sum_I y_{in+1p+1} + \sum_I \sum_J \sum_K x_{ijk} + \sum_I \sum_J y_{ijp+1} + \sum_I \sum_K y_{in+1k} + \sum_J B_j - \sum_K E_k$ $(\text{from}\,(F))$
$= \sum_I \sum_J \sum_K x_{ijk} + \sum_I \sum_J y_{ijp+1} + \sum_I \sum_{K'} y_{in+1k} + \sum_J B_j - \sum_K E_k$
$= \sum_I (\sum_J \sum_K x_{ijk} + \sum_J y_{ijp+1} + \sum_{K'} y_{in+1k}) + \sum_J B_j - \sum_K E_k$ $(\text{from}\,(B)\,\text{and}\,(D))$
$= \sum_I A_i + \sum_J B_j - \sum_K E_k$

So $\{y_{ijk}\}_{I' \times J' \times K'}$ defined by equations (B) through (F) is a M-feasible solution of problem **P2**.

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