

Witthayapraphakorn, Aphisak; Charnsethikul, Peerayuth

## Article

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# Benders decomposition with special purpose method for the sub problem in lot sizing problem under uncertain demand

Aphisak Witthayapraphakorn\*, Peerayuth Charnsethikul

Industrial Engineering Department, Kasetsart University, 50 Ngam Wong Wan Road, Ladyao, Chatuchak, Bangkok 10900, Thailand

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## ABSTRACT

We propose herein the application of Benders decomposition with stochastic linear programming instead of the mix integer linear programming (MILP) approach to solve a lot sizing problem under uncertain demand, particularly in the case of a large-scale problem involving a large number of simulated scenarios. In addition, a special purpose method is introduced to solve the sub problem of Benders decomposition and reduce the processing time. Our experiments show that Benders decomposition combined with the special purpose method (BCS) requires shorter processing times compared to the simple MILP approach in the case of large-scale problems. Furthermore, our BCS approach shows a linear relationship between the processing time and the number of scenarios, whereas the MILP approach shows a quadratic relationship between those variables, indicating that our approach is suitable in solving such problems.

## 1. Introduction

In the field of operations research, a lot sizing problem is an important problem that aims to find a production approach with the lowest total cost. In this case, the total cost comprises the production, inventory, out-of-stock and machine set-up costs. Given a certain demand, finding a solution to this problem is generally easy because it only involves deterministic costs; however, customer demand is uncertain in real-world scenarios. This type of lot sizing problem with the introduction of demand uncertainty is referred to as the lot sizing problem under uncertain demand. In this study, we particularly consider the deterministic (production and machine set-up costs) and stochastic (inventory and out-of-stock costs that might occur in different situations) costs to find robust solutions (number of products produced in each period) that minimise the total cost, including the deterministic and stochastic costs. In practice, decision-making for dynamic lot sizing under certain demands does not include risky cost assessments because of additive inventory/back order/lost sale costs caused by demand uncertainty effects that generally exist in practice and can be represented by a large set of possible scenarios. Thus, with the advancement of both computer hardware and software, especially for Big Data Analytics, a more advanced quantitative technique can be developed to fulfil these requirements. Stochastic linear programming [1] is one of the several techniques that can achieve this through the simulation of possible scenarios that are transformed into a mathematical

model. Mix integer linear programming (MILP) can then be applied to obtain the solution. Although this technique provides a robust solution to the lot sizing problem, it involves a large number of simulated scenarios, which generate a considerably large mathematical model for MILP. Therefore, this technique has a major drawback in that it requires a considerably long processing time to obtain the solution.

In this study, we apply Benders decomposition [2,3] with stochastic linear programming instead of directly using MILP to alleviate the abovementioned problems. In particular, we separate the primal problem into a master problem and a sub problem. Furthermore, we develop a special-purpose method to solve the sub problem to reduce the processing time in the case of large-scale problems.

## 2. Literature review

A lot sizing problem was defined by Wagner and Whitin [4] as a mathematical model to find a production approach with the lowest total cost under a certain demand. However, as previously mentioned, the customer demand in the real world is uncertain. Whybark and Williams [5] discussed the possible consequences of uncertain factors for a Material Requirements Planning system, with uncertain demand being one of the factors in their study. Several studies have also considered various solutions to the lot sizing problem under uncertain demand. For example, Sox [6] presented a solution to the dynamic lot sizing problem under random demand and non-stationary cost

\* Corresponding author.

E-mail addresses: [aphisak.wi@up.ac.th](mailto:aphisak.wi@up.ac.th) (A. Witthayapraphakorn), [fengprc@ku.ac.th](mailto:fengprc@ku.ac.th) (P. Charnsethikul).

conditions by transforming the problem into a mixed-integer nonlinear programming model and applying a method based on the Wagner–Whitin algorithm to find an optimised solution. The two primary results of this study are increasing the computational efficiency and demonstrating that the lot sizes used in the rolling-horizon implementation of this algorithm are bounded below by the optimal lot sizes for a stochastic dynamic programming formulation. Bertsimas [7] proposed a robust optimisation approach for inventory theory by transforming the problem into a mathematical model to find a solution: a linear programming or mix integer programming (MIP) model in the case of no fixed costs and fixed costs, respectively. Bertsimas further demonstrated the optimality of base-stock policies in the proposed framework when the optimal stochastic policy is base stock and in other instances, where the optimal stochastic policy is not known. Brandimarte [8] recommended a solution for the capacitated lot sizing problem (CLSP) under uncertain demand using a scenario tree model based on which the CLSP can be transformed into an MIP model to obtain the solution. In addition, he proposed a heuristic approach based on the fix-and-relax strategy as an alternative method. The computational complexity significantly increases as the size of the scenario tree increases when the nested Benders decomposition approach is used to find a solution. Furthermore, Tempelmeier and Herpers [9] used the ABC<sub>β</sub> heuristic to find a solution to the dynamic CLSP with random demand under a fill rate constraint condition. This heuristic is as flexible as the deterministic variant. Although our numerical study shows that the Silver–Meal criterion dominates over the other cost criteria in the same manner as in the case of deterministic demand conditions, the least unit cost criterion does not, which is later transformed into a column generation heuristic [10] by incorporating a column generation technique. The quality of the solutions is compared with the results obtained by applying the ABC<sub>β</sub> heuristic of Tempelmeier and Herpers [9] alone. Using the proposed heuristic is fast and provides solutions that are superior to those of the ABC<sub>β</sub> heuristic on average. In another study, Tempelmeier and Hilger [11] proposed the transformation of the same problem into a linear programming model and solved it using a variant of the fix-and-optimize heuristic. This heuristic provides better solutions than the column-generation heuristic in cases with a small number of products and high-capacity utilisations. However, the linear programming model contains an error; therefore, it is corrected by integrating an extra binary decision variable and two additional constraints into the model [12]. Vargas [13] presented an optimised solution for the Wagner–Whitin dynamic lot-size model problem under random demand. Their study is interesting because their method can be used to resolve the lot-size problem under a normally distributed uncertainty. In addition, the results presented here are expected to provide a better performance benchmark for existing heuristics and form the basis for a new generation of production scheduling heuristics for the stochastic case of the dynamic lot-size model and its extensions. Agra et al. [14] studied the manufacturing problems in a make-to-order production approach under uncertain demand and proposed a robust optimised solution by transforming the problem into a MIP model; however, in the case of a large-scale problem, obtaining a solution might not be possible. Therefore, two additional techniques were proposed, namely the elite heuristic and tournament heuristic approaches. These heuristics are fast and provide solutions that typically have considerably small optimality gaps. Overall, the tournament heuristic was better than the elite heuristic only for the case wherein the number of elite elements selected was small ( $\hat{m} = 5$ ); otherwise, in all other cases, the elite heuristic outperformed the tournament heuristic. In 2017, Brahim et al. [15] reviewed the literature regarding the single-item dynamic lot sizing problems, then categorised and summarised them. The literature related to uncertain demand is discussed in their review.

As previously mentioned, Benders decomposition [2,3] is a technique for solving complex or large-scale problems by dividing them into two problems, namely the master problem and the sub problem. Each problem is then solved to obtain the final solution. Benders

decomposition has been adopted in several previous studies. Pishvae et al. [16] enhanced Benders decomposition to an accelerated Benders decomposition to obtain solutions to problems in medical supply chains under uncertainty and compared its efficiency in terms of the processing time with MILP in the CPEX programme. Khatami et al. [17] applied Benders decomposition to stochastic programming to obtain a solution for the reverse supply chain network with demand and return uncertainties using Cholesky's factorisation and k-means clustering methods to reduce the number of scenarios. Wheatley et al. [18] presented a method using Benders decomposition to solve an inventory-location problem under uncertain demand (Poisson-distributed) with service constraints and an exact method for resolving the sub problem. Sadeghian and Ardehali [19] adopted a double Benders decomposition approach to find an economic dispatch scheduling solution for integrated heat and power systems to maximise economic profits and minimise environmental emissions. Hemmati et al. [20] proposed the application of Benders decomposition to reduce the time required to obtain a solution for the sustainable energy hub design problem under uncertainty in the case when simulation leads to a large MILP model. Furthermore, Kergosien et al. [21] proposed the application of Benders decomposition together with the Tabu search heuristic to solve problems comprising the uniform parallel machine scheduling problem and the multi-trip travelling salesman problem with only one salesman. In 2017, the aforementioned research related to Benders decomposition was also included in a review [22].

Based on the literature review, Benders decomposition is generally used to solve a problem involving uncertainty to reduce the problem size. However, to the best of our knowledge, Benders decomposition has never been applied to obtain a solution for a lot sizing problem under uncertain demand using stochastic linear programming. Benders decomposition is applied to solve similarly structured problems; hence, Bahu and Zions [24] applied Benders decomposition for the multi-item scheduling problem to find the solution of decision variables in the sub problem. Moreover, they applied the principle of complementary slackness to decrease the time required to find solutions. Hence, in this study, we focused on the application of Benders decomposition with stochastic linear programming to reduce the processing time required to obtain a solution for the lot sizing problem under uncertain demand in the case that the resulting problem from the simulation is a large-scale one.

### 3. Research method

#### 3.1. Lot sizing problem model

The lot sizing problem model employed herein is the Wagner's basic production model [4]. The details of this model are as follows:

**Model 1:** Lot Sizing Model

$$\text{Minimize } Z = \sum_t c_t x_t + \sum_t g_t I_t^- + \sum_t h_t I_t^+ + \sum_t e_t P_t \tag{1}$$

$$x_1 - I_1^+ + I_1^- = D_1 \tag{2}$$

$$x_t + (I_{t-1}^+ - I_{t-1}^-) - (I_t^+ - I_t^-) = D_t, 2 \leq t \leq T \tag{3}$$

$$x_t \leq MP_t \tag{4}$$

$$x_t, I_t^+, I_t^- \geq 0 \tag{5}$$

$$P_t \in \{0, 1\} \tag{6}$$

where,

Decision variables:

$x_t$  is the number of products produced in time period  $t$ ;

$P_t$  is the production status in time period  $t$ ;

$I_t^+$  is the number of inventory in time period  $t$ ;

$I_t^-$  is the number of out-of-stock products in time period  $t$ ;

Parameters:

$c_t$  is the production cost per unit in time period  $t$ ;  
 $D_t$  is the demand in time period  $t$ ;  
 $g_t$  is the out-of-stock cost per unit in time period  $t$ ;  
 $h_t$  is the inventory cost per unit in time period  $t$ ;  
 $e_t$  is the machine set-up cost in time period  $t$ ;  
 $T$  is the last time period; and

$M$  represents Big  $M$  (i.e. the convergence of a large value to infinity).

Model 1 can be explained as follows: Eq. (1) sets a target for the total cost optimisation, whereas Eqs. (2) and (3) specify that a sum of manufactured products and inventory or out-of-stock products must be equal to demand. Furthermore, Eq. (4) specifies a constraint that a product can be made when a machine is set up, whereas Eq. (5) specifies that the number of products, inventory and out-of-stock products must be greater than zero. Finally, Eq. (6) indicates that a production status of '1' indicated 'In Production', whereas '0' indicates 'Not in Production'.

### 3.2. Problem statements

The problem model used can be defined as follows:

- Only one type of product is manufactured.
- The production cost is equal in every scenario.
- Demand is uncertain.
- The number of products manufactured is equal in every scenario.
- An unlimited number of products can be manufactured in every time period, and this number must always be greater than or equal to zero.
- The number of products that is manufactured can be higher or lesser than the demand.
- The number of inventory and out-of-stock products varies according to the demand.
- The cost of inventory and out-of-stock products varies according to the scenarios.

This problem requires a robust solution to minimise the total cost in every scenario. Therefore, as many scenarios as possible must be simulated to find a suitable production approach for all scenarios. Thus, this problem is translated into a mathematical model as follows:

**Model 2:** Lot Sizing Model under Uncertain Demand (modified from Chapter 16 in Wagner [23])

$$\text{Minimize } Z = \sum_t c_t x_t + \sum_{t,s} g_{t,s} I_{t,s}^- + \sum_{t,s} h_{t,s} I_{t,s}^+ + \sum_t e_t P_t \quad (7)$$

$$x_t - I_{1,s}^+ + I_{1,s}^- = D_{1,s} \quad \forall s \quad (8)$$

$$x_t + (I_{t-1,s}^+ - I_{t-1,s}^-) - (I_{t,s}^+ - I_{t,s}^-) = D_{t,s} \quad 2 \leq t \leq T, \forall s \quad (9)$$

$$x_t \leq MP_t \quad (10)$$

$$x_t, I_{t,s}^+, I_{t,s}^- \geq 0 \quad (11)$$

$$P_t \in \{0, 1\} \quad (12)$$

where,

Decision variables:

$x_t$  is the number of products produced in time period  $t$ ;  
 $P_t$  is the production status in time period  $t$ ;  
 $I_{t,s}^+$  is the number of inventory in time period  $t$  under scenario  $s$ ;  
 $I_{t,s}^-$  is the number of out-of-stock products in time period  $t$  under scenario  $s$ ;

Parameters:

$c_t$  is the production cost per unit in time period  $t$ ;  
 $D_{t,s}$  is the demand in time period  $t$  under scenario  $s$ ;  
 $g_{t,s}$  is the out-of-stock cost per unit in time period  $t$  under scenario  $s$ ;  
 $h_{t,s}$  is the inventory cost per unit in time period  $t$  under scenario  $s$ ;  
 $e_t$  is the machine set-up cost in time period  $t$ ;

$T$  is the last time period; and

$M$  represents Big  $M$  (i.e., the convergence of large value to infinity).

Model 2 can be explained as follows: Eq. (7) sets a target for the total cost optimisation in every scenario, whereas Eqs. (8) and (9) specify that the demand in scenario  $s$  must be equal to the sum of the manufactured products and inventory or out-of-stock products in scenario  $s$ . Furthermore, Eq. (10) specifies a constraint that a product can be produced when the machine is set up, whereas Eq. (11) specifies that the number of products, inventory and out-of-stock products must be greater than zero. Finally, Eq. (12) specifies that the production status of '1' indicates 'In Production', whereas '0' indicates 'Not in Production'.

### 3.3. Applying benders decomposition

We applied herein two models of Benders decomposition: (i) a sub problem model in which at least one type of decision variable is substituted by a constant in the primal problem, after which the problem is transformed into the dual problem; and (ii) Benders relaxed master problem (also called Benders optimality cut), a model derived from the application of the properties of the weak duality theorem in linear programming, which is used to obtain a solution by adapting the strong duality theorem, and uses convolution of target value in the primal problem and Benders optimality cut for verification. The applications of these two models are described below.

#### 3.3.1. Application of the sub problem model

The application of the sub problem model begins with the substitution of a decision variable  $x_t$  by a constant in Model 2, which is the primal model. Model 2 can then be transformed to a dual model as follows:

**Model 3:** Sub Problem Model

$$\text{Maximize } W = \sum_t \sum_s (D_{t,s} - x_t^*) y_{t,s} \quad (13)$$

$$-h_{t,s} \leq y_{t,s} - y_{t+1,s} \leq g_{t,s} \quad 1 \leq t \leq T - 1, \forall s \quad (14)$$

$$-h_{T,s} \leq y_{T,s} \leq g_{T,s} \quad \forall s \quad (15)$$

where,

Decision variables:

$x_t^*$  substitutes  $x_t$  by a constant in Model 2, and

$y_{t,s}$  is a decision variable generated from the transformation of Model 2 into a dual problem.

Model 3 is a dual model obtained using Model 2 after substituting  $x_t$  with a constant to become  $x_t^*$ .  $x_t^*$  can then be moved from the left-hand side to the right-hand side in Eqs. (8) and (9). Model 3 will be used to find the value of  $y_{t,s}$ , which will, in turn, be used in the Benders optimality cut model.

#### 3.3.2. Application of the benders optimality cut model

After obtaining the value for  $y_{t,s}$  using Model 3, the weak duality theorem is applied to create the Benders optimality cut model as follows:

**Model 4:** Benders Optimality Cut Model

$$\text{Minimize } T \quad (16)$$

$$T \geq \sum_t \sum_s D_{t,s} y_{t,s}^* - \sum_t [ \sum_s y_{t,s}^* - c_t ] x_t + \sum_t e_t P_t \quad \forall j \quad (17)$$

$$x_t \leq MP_t, \forall t \quad (18)$$

$$x_t \geq 0, P_t \in \{0, 1\} \quad \forall t \quad (19)$$

where,

$y_{t,s,j}^*$  is the value of  $y_{t,s}$  obtained from Model 3 in iteration  $j$ .

Model 4 is obtained by substituting the value of  $y_{t,s}$  obtained from Model 3 in iteration  $j$  for  $y_{t,s,j}^*$  in Eq. (17) to find the value of  $x_t$  until such a value is obtained when  $T$  in Model 4 and  $Z$  in Model 1 are equal based

on the strong duality theorem.

### 3.4. Creation of a special-purpose method for $y_{t,s}$ in model 3

This special-purpose method is based on the idea of exchanging advance optimisation's classroom to solve the lot sizing problem under uncertain demand in case of no set-up costs. The three primary approaches to solve this problem are: 1) linear programming in primal problem, 2) linear programming in dual problem and 3) Benders decomposition. The Benders decomposition approach requires the most time in solving the problem because it uses the linear programming approach to solve the sub problem, which involves many decision variables nearby in the primal problem. In the specific structure of the sub problem, it is interesting to study another approach in problem-solving instead of using the linear programming method.

However, this study aimed to create a special-purpose method to find the solution to the required sub problem instead of using the linear programming method that the sub-problem structure is only shown in Model 3. The concept of this special-purpose method can be explained below.

In finding a value for the decision variables that can maximise the value of the objective equation, we generally consider the coefficient of decision variables in the objective equation if those variables are independent. The underlying idea is that when a coefficient is positive, the decision variables will have a value closer to the upper bound; otherwise, they will have a value at the lower bound. Based on this principle, we can immediately obtain the required value if the decision variables in Eq. (13) are made independent of each other.

Therefore, the structure of Model 3 is modified as follows to make the decision variables in Eq. (13) independent of each other:

**Step 1.** Define variable  $u_{t,s}$ .

**Step 2.** Define  $y_{t,s} = y_{t+1,s} + u_{t,s}$  and substitute it in Eq. (14), which will change the equation's structure as follows:

$$-h_{t,s} \leq y_{t+1,s} + u_{t,s} - y_{t+1,s} \leq g_{t,s}$$

$$-h_{t,s} \leq u_{t,s} \leq g_{t,s}$$

Finally, the structure of Eq. (14) is changed as follows:

$$-h_{t,s} \leq u_{t,s} \leq g_{t,s}$$

**Step 3.** Substitute  $y_{t,s} = y_{t+1,s} + u_{t,s}$  in Eq. (13) to change the equation's structure as follows:

$$\text{Maximize } W = \sum_t \sum_s (D_{t,s} - x_t^*) y_{t,s}$$

Separate the equation as:

$$\text{Maximize } W = \sum_s (D_{1,s} - x_1^*) y_{1,s} + \sum_s (D_{2,s} - x_2^*) y_{2,s} + \dots + \sum_s (D_{T,s} - x_T^*) y_{T,s}$$

Substitute  $y_{t,s} = y_{t+1,s} + u_{t,s}$  in the equation to obtain:

$$\text{Maximize } W = \sum_s (D_{1,s} - x_1^*) (u_{1,s} + y_{2,s}) + \sum_s (D_{2,s} - x_2^*) (u_{2,s} + y_{3,s}) + \dots + \sum_s (D_{T,s} - x_T^*) y_{T,s}$$

Continuously substitute  $y_{t,s} = y_{t+1,s} + u_{t,s}$  in the equation until

$$\text{Maximize } W = \sum_s (D_{1,s} - x_1^*) (u_{1,s} + \dots + u_{T-1,s} + y_{T,s}) + \sum_s (D_{2,s} - x_2^*) (u_{2,s} + \dots + u_{T-1,s} + y_{T,s}) + \dots + \sum_s (D_{T,s} - x_T^*) y_{T,s}$$

Then, rearrange the equation as

$$\begin{aligned} \text{Maximize } W &= \sum_s (D_{1,s} - x_1^*) u_{1,s} \\ &+ \left[ \sum_s (D_{1,s} - x_1^*) + \sum_s (D_{2,s} - x_2^*) \right] u_{2,s} \\ &+ \dots + \left[ \sum_s (D_{1,s} - x_1^*) + \dots + \sum_s (D_{T-1,s} - x_{T-1}^*) \right] u_{T-1,s} \\ &+ \left[ \sum_s (D_{1,s} - x_1^*) + \dots + \sum_s (D_{T,s} - x_T^*) \right] y_{T,s} \end{aligned}$$

Arrange the mathematical equation as

$$\text{Maximize } W = \sum_s \sum_{t=1}^{T-1} \left[ \sum_{j=1}^t (D_{j,s} - x_j^*) \right] u_{t,s} + \sum_s \left[ \sum_{t=1}^T (D_{t,s} - x_t^*) \right] y_{T,s}$$

Finally, the structure of Eq. (13) is presented as follows:

$$\text{Maximize } W = \sum_s \sum_{t=1}^{T-1} \left[ \sum_{j=1}^t (D_{j,s} - x_j^*) \right] u_{t,s} + \sum_s \left[ \sum_{t=1}^T (D_{t,s} - x_t^*) \right] y_{T,s}$$

**Step 4** Restructure Model 3 to obtain Model 5.

The new sub problem model, Model 5, is then obtained as follows:

$$\begin{aligned} \text{Maximize } W &= \sum_s \sum_{t=1}^{T-1} \left[ \sum_{j=1}^t (D_{j,s} - x_j^*) \right] u_{t,s} \\ &+ \sum_s \left[ \sum_{t=1}^T (D_{t,s} - x_t^*) \right] y_{T,s} \end{aligned} \tag{20}$$

$$-h_{t,s} \leq u_{t,s} \leq g_{t,s} \text{ where } 1 \leq t \leq T-1, \forall s \tag{21}$$

$$-h_{T,s} \leq y_{T,s} \leq g_{T,s} \tag{22}$$

$$y_{t,s} = y_{t+1,s} + u_{t,s} \text{ where } 1 \leq t \leq T-1, \forall s \tag{23}$$

Thus, Model 5 is obtained from Model (3), wherein decision variables  $u_{t,s}$  and  $y_{T,s}$  in Eq. (20) are related to each other at time period  $T-1$  in Eq. (23) as  $y_{T-1,s} = y_{T,s} + u_{T-1,s}$ . However, the value of  $y_{T-1,s}$  can be derived if  $u_{T-1,s}$  and  $y_{T,s}$  are known; therefore, even if  $u_{t,s}$  and  $y_{T,s}$  have a relationship at time period  $T-1$ , they are independent. The coefficient of  $u_{t,s}$  is the cumulative frequency of  $(D_{j,s} - x_j^*)$  from  $j=1$  to  $j=t$  ( $t=1, \dots, T-1$ ), whilst the coefficient of  $y_{T,s}$  is the cumulative frequency of  $(D_{t,s} - x_t^*)$  from 1 to  $T$ . Thus, in principle, finding a solution  $(y_{t,s})$  from Model 3 can be described as follows:

- If the coefficient of  $u_{t,s}$  and  $y_{T,s}$  is positive, set  $u_{t,s}$  and  $y_{T,s}$  as  $g_{t,s}$  and  $g_{T,s}$ , respectively.
- If the coefficient of  $u_{t,s}$  and  $y_{T,s}$  is less than or equal to 0, set  $u_{t,s}$  and  $y_{T,s}$  as  $-h_{t,s}$  and  $-h_{T,s}$ , respectively.

The value of  $u_{t,s}$  and  $y_{T,s}$  is then used to obtain  $y_{t,s}$  in Eq. (23), thereby acquiring a solution for Model 3.

### 3.5. Combining benders decomposition with the Special-purpose method

Benders decomposition combined with the previously introduced special-purpose method (together referred to as BCS) can be used as follows to find the solution for the lot sizing problem under uncertain demand:

**Step 1.** Set an initial answer  $x_t$ , where the solution must be in a range of possible solutions for Model 2.

**Step 2.** Substitute  $x_t$  to  $x_t^*$  in Model 2 and find the value of  $y_{t,s}$  in iteration  $j$  using the special-purpose method.

**Step 3.** Substitute the value of  $y_{t,s}$  from iteration  $j$  to  $y_{t,s,j}^*$  in Model 4 to obtain the value of  $x_t$ ,  $P_t$  and  $T$  using MILP.

**Step 4.** Substitute  $x_t$  and  $P_t$  from Step 3 to find the value of  $Z$  using Model 2, specifically Eqs. (8) to (10).  $I_{t,s}^+$  and  $I_{t,s}^-$  can be calculated as

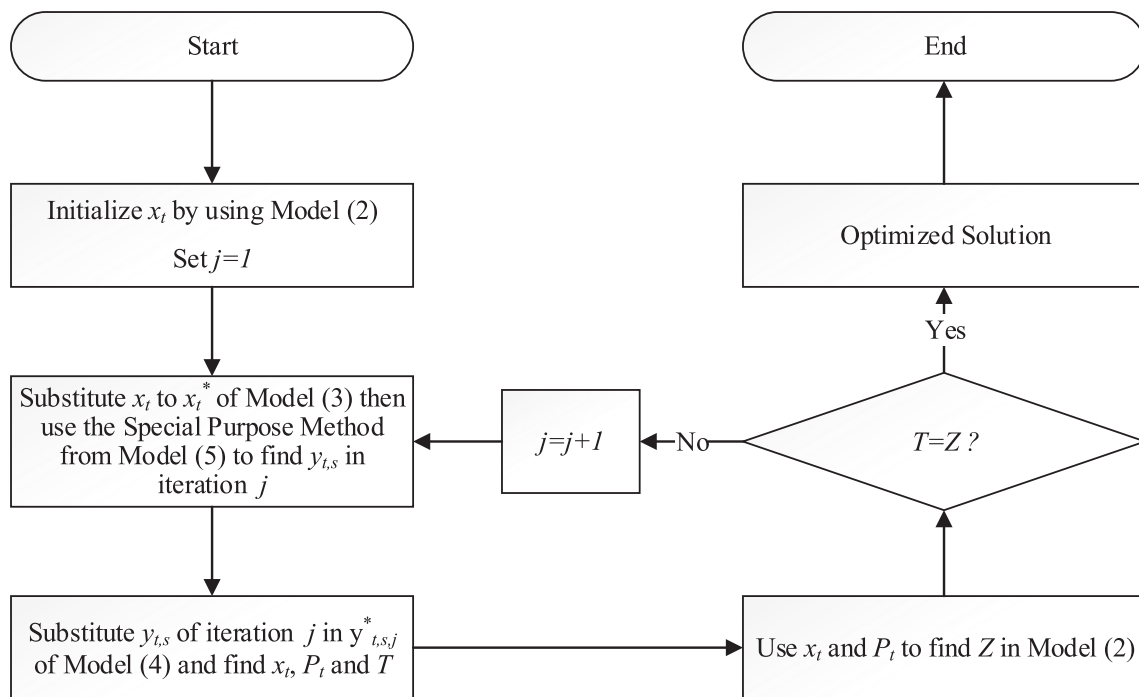


Fig. 1. Steps to find the solution for the lot sizing problem under uncertain demand using BCS.

Table 1  
Results of Experiment 1.

Scenario	Problem scale (m × n)	Processing time (s)	
		MILP	BCS
1000	5005 × 10,010	11.7061	0.2266
2000	10,005 × 20,010	130.4895	0.3933
3000	15,005 × 30,010	294.7193	0.4107
4000	20,005 × 40,010	579.9738	0.4367
5000	25,005 × 50,010	977.2036	0.4724
6000	30,005 × 60,010	-	0.5037
7000	35,005 × 70,010	-	0.6536
8000	40,005 × 80,010	-	0.6738
9000	45,005 × 90,010	-	0.7854
10,000	50,005 × 100,010	-	1.6678

Table 2  
Results of Experiment 2.

Scenario	Problem scale (m × n)	Processing time (s)	
		MILP	BCS
500	5010 × 10,020	7.4921	0.6413
1000	10,010 × 20,020	32.0065	0.9158
1500	15,010 × 30,020	60.7807	0.9961
2000	20,010 × 40,020	157.2473	1.0638
2500	25,010 × 50,020	700.7136	1.4635
3000	30,010 × 60,020	-	1.9329
3500	35,010 × 70,020	-	2.2179
4000	40,010 × 80,020	-	2.3773
4500	45,010 × 90,020	-	3.056
5000	50,010 × 100,020	-	3.219

follows:

- Consider Eq. (8):  $I_{t,s}^+ = x_t - D_{1,s}$  and  $I_{t,s}^- = 0$  if the value of  $x_1$  is greater than or equal to  $D_{1,s}$ . However,  $I_{1,s}^- = D_{1,s} - x_1$  and  $I_{1,s}^+ = 0$  if the value of  $x_1$  is less than  $D_{1,s}$ .
- Consider Eq. (9):  $I_{t,s}^+ = x_t + I_{t-1,s}^+ - I_{t-1,s}^- - D_{t,s}$  and  $I_{t,s}^- = 0$  if the value of  $x_t + I_{t-1,s}^+ - I_{t-1,s}^-$  is greater than or equal to  $D_{t,s}$ . However,

Table 3  
Results of Experiment 3.

Scenario	Problem scale (m × n)	Processing time (s)	
		MILP	BCS
200	3015 × 6030	0.2444	0.9119
400	6015 × 12,030	4.9386	1.6481
600	9015 × 18,030	10.5739	1.7538
800	12,015 × 24,030	48.1299	1.8151
1000	15,015 × 30,030	70.6503	2.3841
1200	18,015 × 36,030	81.9115	2.6132
1400	21,015 × 42,030	88.26	3.6186
1600	24,015 × 48,030	384.4042	5.8284
1800	27,015 × 54,030	495.5283	6.8886
2000	30,015 × 60,030	-	7.3389

Table 4  
Results of Experiment 4.

Scenario	Problem scale (m × n)	Processing time (s)	
		MILP	BCS
200	4020 × 8040	0.418	3.3065
400	8020 × 16,040	4.1471	3.3792
600	12,020 × 24,040	12.0243	4.6506
800	16,020 × 32,040	36.1055	4.9598
1000	20,020 × 40,040	74.8652	5.9521
1200	24,020 × 48,040	82.477	7.788
1400	28,020 × 56,040	-	10.6545
1600	32,020 × 64,040	-	12.1189
1800	36,020 × 72,040	-	14.4019
2000	40,020 × 80,040	-	16.4084

$$I_{t,s}^- = D_{t,s} - (x_t + I_{t-1,s}^+ - I_{t-1,s}^-) \quad \text{and} \quad I_{t,s}^+ = 0 \quad \text{if the value of } x_t + I_{t-1,s}^+ - I_{t-1,s}^- \text{ is less than } D_{t,s}$$

- Substitute the value of  $x_t, P_t, I_{t,s}^+$  and  $I_{t,s}^-$  in Eq. (7) to obtain the value of  $Z$ .

Step 5. Compare the value of  $Z$  from Step 4 with the value of  $T$  from Step 3. If  $T$  is not equal to  $Z$ , take the value of  $x_t$  from Step 3 to

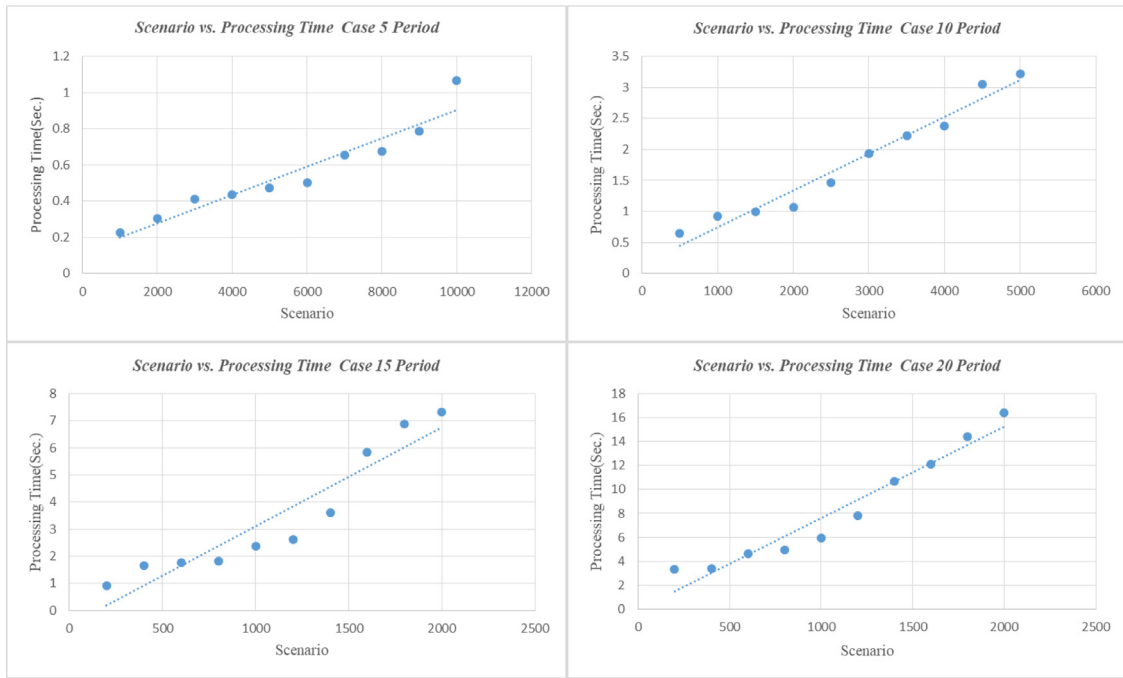


Fig. 2. Relationship between the number of scenarios and the processing time in the case of the BCS approach.

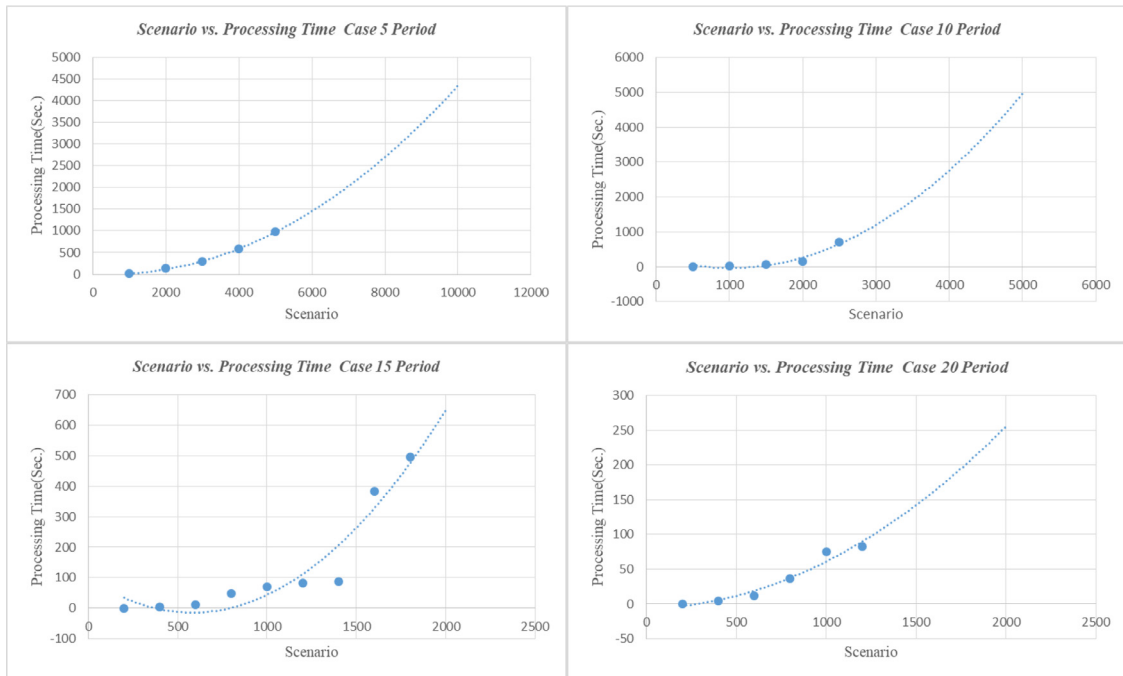


Fig. 3. Relationship between the number of scenarios and the processing time in the case of the MILP approach.

repeatedly obtain another value of  $y_{t,s}$  in Step 2 until  $T = Z$ . The value of  $x_b, P_b, I_{t,s}^+$  and  $I_{t,s}^-$  in Step 4 will be the right solution that leads to the final result when  $T = Z$ .

**Remark.** Condition  $T = Z$  might be replaced by  $Z - T < \beta$ , where  $\beta$  is tending to 0 ( $\beta$  is used to construct a stopping condition whilst finding solutions using the special-purpose method).

The abovementioned steps are summarised as a flowchart in Fig. 1.

### 3.6. Using MILP and genetic algorithm (GA) in MATLAB to compare with BCS

- The MILP approach is used to solve Model 2 using `intlinprog()` in MATLAB to consider all decision variables ( $x_b, P_b, I_{t,s}^+$  and  $I_{t,s}^-$ ).
- The GA approach is used to solve Model 2 using `ga()` in MATLAB to construct the solutions of  $x_b$ , where  $P_b, I_{t,s}^+$  and  $I_{t,s}^-$  are varied by  $x_b$ , and the crossover fraction in `ga()` is set equal to 0.5. If the value of  $x_t$  is more than zero,  $P_t$  is equal to one. However, if this is not true,  $P_t$  is equal to zero. This approach to find  $I_{t,s}^+$  and  $I_{t,s}^-$  is similar to the technique of Step 4 in Section 3.5.

**Table 5**  
Results of Experiment 5.

Scenario	Problem scale (m × n)	Processing time (s)		% GA difference from BCS
		BCS	GA	
1000	40,040 × 80,080	32.73	46.62	3.96%
2000	80,040 × 160,080	36.12	223.74	4.86%
3000	120,040 × 240,080	42.47	240.39	5.60%
4000	160,040 × 320,080	44.63	303.33	8.62%
5000	200,040 × 400,080	48.90	312.73	2.49%
6000	240,040 × 480,080	51.02	362.49	2.23%
7000	280,040 × 560,080	55.47	466.73	8.88%
8000	320,040 × 640,080	58.60	719.96	4.37%
9000	360,040 × 720,080	60.89	1122.20	2.53%
10,000	400,040 × 800,080	68.47	1823.90	8.95%

3.7. Creating the test problem in MATLAB

The scenario is randomised under the uniform distribution condition using the randi() function to construct the test problem. These random scenarios are disposed in Model 2 for problem testing.

4. Experiment design and results

Our experiments involved comparing our proposed BCS approach with that of the MILP to directly solve Model 2. However, this comparison was only in terms of efficiency. In terms of effectiveness, both techniques had similar solutions considering that our BCS approach was based on Benders decomposition and a linear programming technique that will always result in an optimised solution.

The efficiency was measured in terms of the processing time for both methods by adjusting two experimental factors, namely the number of scenarios, which would reveal the relationship between the processing time and the number of scenarios when the latter increases, and the number of time periods, which would show the relationship between the processing time and the number of scenarios when the number of time periods increases. The four following experiments were conducted based on this:

- Experiment 1 spans five time periods with the number of simulations involving 1000–10,000 scenarios.
- Experiment 2 spans 10 time periods with the number of simulations involving 500–5000 scenarios.
- Experiment 3 spans 15 time periods with the number of simulations involving 200–2000 scenarios.
- Experiment 4 spans 20 time periods with the number of simulations involving 200–2000 scenarios.

In addition to the abovementioned four patterns, the study results showed the problem testing in a very large-scale problem to represent the efficiency of the BCS approach by comparing it with GA instead of

MILP. The problem cannot find the solution if BCS is compared with MILP. Moreover, the solutions from solving by BCS and GA were used to compare the quality.

- Experiment 5 spans 40 time periods with the number of simulations involving 1000–10,000 scenarios.

All experiments were performed on MATLAB R2016a on a computer with the following specifications: Intel Core i7-7700 CPU 3.60 GHz and RAM 16GB.

Tables 1–4 show the results from our four experiments.

**Remark.**  $n$  = variables and  $m$  = constrains

The results in Tables 1–4 clearly show that in the case of a large number of scenarios, our BCS approach required a shorter processing time compared to the MILP approach to obtain the same result. Furthermore, when the problem structure is large at any point, the MILP approach cannot be used to find the solution, but our approach can. However, note that when testing with a small number of scenarios, the MILP approach requires a slightly shorter processing time than our BCS approach. We plotted a graph to visualise the relationship between the processing time and the increasing number of scenarios for BCS and MILP (Figs. 2 and 3).

Figs. 2 and 3 depict that a linear relationship with  $R^2 > 0.88$  existed between the number of scenarios and the processing time in the case of the BCS approach. The relationship in the case of the MILP approach showed a quadratic trend with  $R^2 > 0.9$ , implying that in the case of a large number of scenarios for the lot sizing problem under uncertain demand, the BCS approach will perform more efficiently than the MILP approach.

Thus, in summary, we can conclude from the obtained results that the BCS approach is suitable for the lot sizing problem under uncertain demand in the case of a large-scale problem because of the addition of more scenarios. This result can be attributed to the fact that the BCS approach only considers two decision variables of the MILP approach, namely  $x_t$  and  $P_t$ . Therefore, adding more scenarios has an insignificant impact considering that the increasing number of scenarios will be reflected in variables  $I_{t,s}^+$  and  $I_{t,s}^-$ , which in turn, do not affect the decision variables that are considered in the case of the subsequent MILP.

However, this is not the case when applying the MILP approach to directly solve a problem based on Model 2 because it considers all decision variables ( $x_t, P_t, I_{t,s}^+$  and  $I_{t,s}^-$ ). Therefore, adding more scenarios directly affects the performance because  $I_{t,s}^+$  and  $I_{t,s}^-$  in Model 2 will consequently increase, thereby directly affecting all the decision variables. In addition, the MILP approach cannot obtain the solution when the number of scenarios is considerably high because of many decision variables. Table 5 presents the solutions from Experiment 5.

Table 5 illustrates that solving the problem using the BCS approach resulted in the shortest processing time in the case of all experiments. Furthermore, these solutions had the highest quality. The difference in the percentage of the quality of the results using the GA and BCS

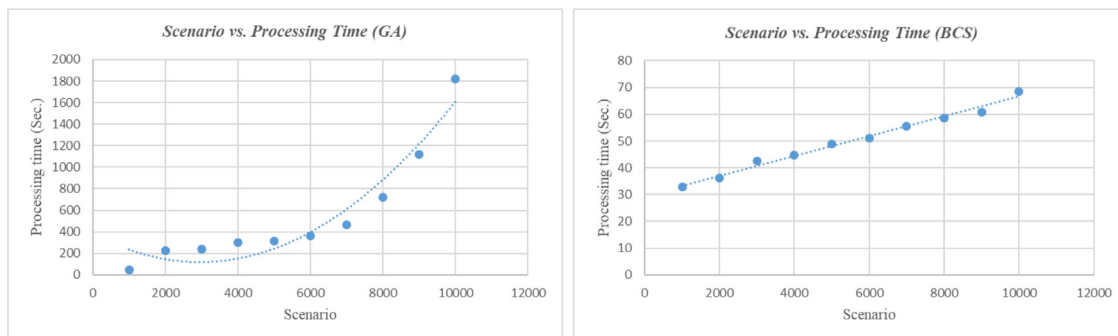


Fig. 4. Relationship between the number of scenarios and the processing time (GA and BCS).



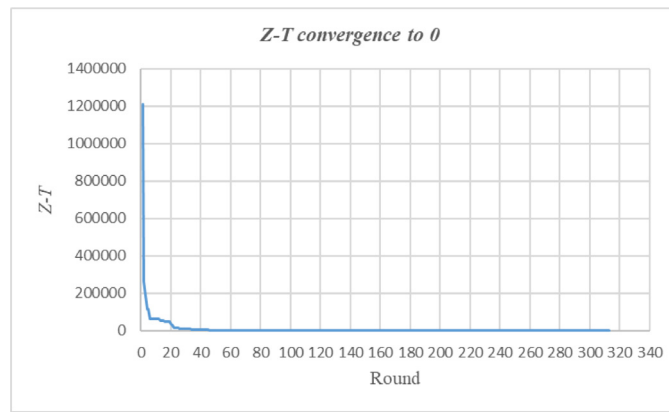


Fig. 5. Convergence of Z-T to 0.

Table 6  
Show Data Z-T convergence to 0.

Round	Z-T	Round	Z-T	Round	Z-T	Round	Z-T	Round	Z-T	Round	Z-T	Round	Z-T	Round	Z-T
1	1,211,909.56	41	5238.1	81	1169.98	121	114.21	161	39.75	201	6.57	241	0.82	281	0.09
2	264,753.57	42	5100.67	82	1022.45	122	113.86	162	39.21	202	5.39	242	0.8	282	0.09
3	185,690.7	43	5038.81	83	996.93	123	108.92	163	38.46	203	5.14	243	0.78	283	0.08
4	115,770.13	44	4949.66	84	953.94	124	108.9	164	37.58	204	5.05	244	0.77	284	0.08
5	115,770.13	45	4403.73	85	947.74	125	108.7	165	36.5	205	4.78	245	0.64	285	0.08
6	65,792.79	46	3938.73	86	907.43	126	108.66	166	36.22	206	4.25	246	0.63	286	0.08
7	65,792.79	47	3884.96	87	878.1	127	104.49	167	35.6	207	4.23	247	0.62	287	0.08
8	65,792.79	48	3860.86	88	862.87	128	102.22	168	33.99	208	4.09	248	0.54	288	0.06
9	65,792.79	49	3816.77	89	836.7	129	99.22	169	33.3	209	4.05	249	0.47	289	0.05
10	65,792.79	50	3799.05	90	434.9	130	96.36	170	33.06	210	4.03	250	0.41	290	0.05
11	65,792.79	51	3226.79	91	315.1	131	90.18	171	33.03	211	3.96	251	0.37	291	0.05
12	65,792.79	52	2894.84	92	262.79	132	89.87	172	32.39	212	3.65	252	0.34	292	0.04
13	53,383	53	2794.72	93	242.63	133	88.88	173	31.71	213	3.48	253	0.33	293	0.04
14	53,383	54	2766.76	94	222.08	134	85.1	174	31.37	214	3.43	254	0.32	294	0.04
15	53,383	55	2744.35	95	219.37	135	84.7	175	24.64	215	2.92	255	0.31	295	0.04
16	49,807.62	56	2372.98	96	218.24	136	82.77	176	21.45	216	2.86	256	0.3	296	0.03
17	49,807.62	57	2039.63	97	210.72	137	80.22	177	21.37	217	2.75	257	0.27	297	0.03
18	49,807.62	58	1989.23	98	208.82	138	79.65	178	18.45	218	2.7	258	0.27	298	0.03
19	49,807.62	59	1961.96	99	203.6	139	77	179	17.35	219	2.4	259	0.25	299	0.03
20	41,022.64	60	1895.85	100	199.05	140	75.01	180	16.35	220	2.25	260	0.25	300	0.03
21	31,269.54	61	1884.72	101	189.99	141	73.9	181	16.21	221	2.09	261	0.25	301	0.03
22	17,050.95	62	1859.97	102	187.45	142	71.23	182	15.6	222	2.03	262	0.24	302	0.03
23	16,876.84	63	1763.89	103	179.35	143	69.64	183	14.79	223	1.96	263	0.23	303	0.03
24	16,768.98	64	1732.42	104	162.07	144	68.44	184	14.53	224	1.82	264	0.23	304	0.02
25	16,429.56	65	1721.29	105	151.75	145	65.54	185	14.01	225	1.8	265	0.23	305	0.02
26	12,821.98	66	1703.08	106	149.67	146	63.41	186	12.57	226	1.77	266	0.21	306	0.02
27	12,531.12	67	1668.71	107	146.79	147	62.91	187	12.55	227	1.73	267	0.19	307	0.02
28	12,281.47	68	1441.38	108	139.02	148	62.18	188	12.4	228	1.72	268	0.19	308	0.02
29	12,147.87	69	1393.45	109	138.95	149	61.73	189	12.13	229	1.7	269	0.19	309	0.02
30	10,525.96	70	1383.49	110	137.19	150	61.63	190	12.01	230	1.64	270	0.18	310	0.01
31	10,405.84	71	1379.08	111	134.96	151	58.18	191	10.49	231	1.42	271	0.18	311	0.01
32	9644.78	72	1325.21	112	131.82	152	58.16	192	10.1	232	1.37	272	0.18	312	0.01
33	9479.33	73	1275.17	113	128	153	58.15	193	9.84	233	1.32	273	0.16	313	0
34	7993.26	74	1263.01	114	127.67	154	55.4	194	9.64	234	1.22	274	0.15		
35	7964.88	75	1231.5	115	124.27	155	53.26	195	9.45	235	1.08	275	0.14		
36	6670.76	76	1230.53	116	123.05	156	52.75	196	8.9	236	0.89	276	0.14		
37	6594.5	77	1216.24	117	120.98	157	52.18	197	8.58	237	0.86	277	0.1		
38	6530.85	78	1200.21	118	119.62	158	52.03	198	8.25	238	0.84	278	0.1		
39	6436.29	79	1188.42	119	117.46	159	41.89	199	6.06	239	0.84	279	0.09		
40	5603.16	80	1181.51	120	117.27	160	41.29	200	5.69	240	0.84	280	0.09		

approach to solve the problem was 2–9%. Fig. 4 shows the relationship between the processing times and the number of scenarios when the GA and BCS approaches were used to solve the problem.

Fig. 4 depicts a linear relationship between the processing times and the number of scenarios when the problem was solved using the BCS approach similar to Experiments 1–4 with  $R^2$  equal to 0.98. The relationship between the processing times and the number of scenarios when the problem was solved using the GA approach was in the form of a power function with degree 2 and  $R^2$  equal to 0.98. Based on these

relationships and the solution quality in Table 5, we can confirm that solving the problem is more efficient when using the BCS approach instead of the GA approach.

The GA approach in this research was applied from a function in MATLAB; hence, the solution may not be of high quality. Thus, it will be developed to solve this problem in the future.

## 5. Conclusions

Our experimental results indicated that the application of BCS for stochastic linear programming to solve the lot sizing problem under uncertain demand, especially in the case where the problem is large (resulting from an increase in the number of scenarios), is an efficient approach in terms of the processing time compared with the MILP approach because the BCS approach had a positive, linear relationship between the processing time and the number of scenarios, whereas the MILP approach had a positive, quadratic relationship between both variables. In addition, the BCS approach can address the problem even when it is considerably large, in which case, it cannot be handled using the MILP approach. Although the MILP approach requires a shorter processing time than the BCS approach in the case of small problems, this difference is insignificant. In conclusion, our proposed BCS approach is better than the MILP approach in solving the lot sizing problem under uncertain demand.

## 6. Future research

The value of Z-T converged to zero in the case of the BCS approach. In particular, the value of Z-T gradually decreased after it approached close to zero, which is the reason why many iterations were used in solving the problem. For example, the iterations in the experiment that spans 40 time periods with the number of simulations involving 10,000 scenarios are 313. Fig. 5 shows the convergence of Z-T to zero with the values of Z-T listed in Table 6.

Fig. 5 denotes that the values of Z-T rapidly decreased initially, after which they gradually decreased when the value approached zero. The number of iterations to solve the problem was 78. This finding showed that the BCS approach can be used to improve the quality of the solution by increasing the speed of convergence to zero.

The technique of increasing the speed of Z-T convergence to zero is referred to as ‘accelerating Benders decomposition’. Vahdat and Vahdatzad [25], Pishvaei et al. [16] and Saharidis et al. [26] used different techniques in their studies to add other constraints into the Benders optimality cut model instead of the cut constraint to decrease the number of iterations during problem solving. This technique used an approach to construct the valid inequality equations specified for some problem. Moreover, Costa et al. [27] constructed more than one cut constraint for solving each iteration using the heuristic method. Poojari and Beasley [28] used the GA approach to solve the problem of Benders optimality cut model instead of using the MILP approach. His technique led to different solutions using different cut constraints.

Two approaches for improving the quality of BCS that might be undertaken in the future include adding valid inequality equations to Model 4 in the case that the model can construct the valid inequality equations and 2) constructing many cut constraints instead of constructing the valid inequality equations.

## Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.orp.2018.100096.

## Appendix

## References

- [1] Dantzig GB. Linear programming under uncertainty. *Manag Sci* 1955;1:197–206.
- [2] Benders JF. Partitioning procedures for solving mixed-variables programming problems. *Numer Math* 1962;4(3):238–52.
- [3] Lasdon LS. Optimization theory for large systems. New York: Dover Publications; 2002.
- [4] Wagner HM, Whitin T. Dynamic version of the economic lot size model. *Manag Sci* 1985;5:89–96.
- [5] Whybark DC, Williams JG. Material requirements planning under uncertainty. *Decis Sci* 1976;7:595–606.
- [6] Sox CR. Dynamic lot sizing with random demand and non-stationary costs. *Oper Res Lett* 1997;20:155–64.
- [7] Bertsimas D. A robust optimization approach to inventory theory. *Oper Res* 2006;54(1):150–68.
- [8] Brandimarte P. Multi-item capacitated lot-sizing with demand uncertainty. *Int J Prod Res* 2006;44(15):2997–3022.
- [9] Tempelmeier H, Herpers S. ABC<sub>p</sub>—a heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. *Int J Prod Res* 2010;48(17):5181–93.
- [10] Tempelmeier H. A column generation heuristics for dynamic capacitated lot sizing with random demand under a fill rate constraint. *Omega* 2011;39:627–33.
- [11] Tempelmeier H, Hilger T. Linear programming models for a stochastic dynamic capacitated lot sizing problem. *Comp Oper Res* 2015;59:119–25.
- [12] van Pelt D, Fransoo JC. A note on linear programming models for a stochastic dynamic capacitated lot sizing problem. *Comp Oper Res* 2018;89:13–6.
- [13] Vargas V. An optimal solution for the stochastic version of the Wagner–Whitin dynamic lot-size model. *Eur J Oper Res* 2009;198(2):447–51.
- [14] Agra A, Poss M, Santos M. Optimizing make-to-stock policies through a robust lot-sizing model. *Int J Prod Econ* 2018;200:302–10.
- [15] Brahimi N, Absi N, AtleNordlic SD. Single-item dynamic lot-sizing problems: an updated survey. *Eur J Oper Res* 2017;263(3):838–63.
- [16] Pishvaei MS, Razmi J, Torabi SA. An accelerated Benders decomposition algorithm for sustainable supply chain network design under uncertainty: a case study of medical needle and syringe supply chain. *Transp Res Part E* 2014;67:14–38.
- [17] Khatami M, Mahootchi M, Farahani RJ. Benders’ decomposition for concurrent redesign of forward and closed-loop supply chain network with demand and return uncertainties. *Transp Res Part E* 2015;79:1–21.
- [18] Wheatley D, Gzara F, Jewkes E. Logic-based Benders decomposition for an inventory-location problem with service constraints. *Omega* 2015;55:10–23.
- [19] Sadeghian HR, Ardehali MM. A novel approach for optimal economic dispatch scheduling of integrated combined heat and power systems for maximum economic profit and minimum environmental emissions based on Benders decomposition. *Energy* 2016;102:10–23.
- [20] Hemmati S, Ghaderi SF, Ghazizadeh MS. Sustainable energy hub design under uncertainty using Benders decomposition method. *Energy* 2018;143:1029–47.
- [21] Kergosien Y, Gendreau M, Billaut JC. A Benders decomposition-based heuristic for a production and outbound distribution scheduling problem with strict delivery constraints. *Eur J Oper Res* 2017;262:287–98.
- [22] Rahmaniani R, Crainic TG, Gendreau M, Rei W. The Benders decomposition algorithm: a literature review. *Eur J Oper Res* 2017;259:801–17.
- [23] Wagner HM. Principles of operations research, New Jersey: Prentice-Hall, Inc.; 1975.
- [24] Bahu HC, Zionts S. Multi-item scheduling by Benders’ decomposition. *J Oper Res Soc* 1987;38:1141–8.
- [25] Vahdat Vahab, Vahdatzad Mohammad Ali. Accelerated Benders’ decomposition for integrated forward/reverse logistics network design under uncertainty. *Logistics* 2017;1:11. <https://doi.org/10.3390/logistics1020011>.
- [26] Saharidis GK, Boile M, Theofanis S. Initialization of the Benders master problem using valid inequalities applied to fixed-charge network problems. *Expert Syst Appl* 2011;38:6627–36.
- [27] Costal AM, Cordeau J, Gendron B, Laporte G. Accelerating Benders decomposition with heuristic master problem solutions. *Pesq Oper* 2012;32:3–19.
- [28] Poojari CA, Beasley JE. Improving Benders decomposition using a genetic algorithm. *Eur J Oper Res* 2009;199:89–97.