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Journal of Risk and Financial Management

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Suggested Citation: Ivanov, Roman V. (2022) : The risk measurement under the variance-gamma process with drift switching, Journal of Risk and Financial Management, ISSN 1911-8074, MDPI, Basel, Vol. 15, Iss. 1, pp. 1-27,
<https://doi.org/10.3390/jrfm15010022>

This Version is available at:

<https://hdl.handle.net/10419/258746>

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Article

The Risk Measurement under the Variance-Gamma Process with Drift Switching

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Abstract: The paper discusses an extension of the variance-gamma process with stochastic linear drift coefficient. It is assumed that the linear drift coefficient may switch to a different value at the exponentially distributed time. The size of the drift jump is supposed to have a multinomial distribution. We have obtained the distribution function, the probability density function and the lower partial expectation for the considered process in closed forms. The results are applied to the calculation of the value at risk and the expected shortfall of the investment portfolio in the related multivariate stochastic model.

Keywords: variance-gamma process; drift switching; exponential distribution; hypergeometric function; lower partial expectation; value at risk; expected shortfall

1. Introduction

The variance-gamma distribution determines a well-known model of the financial data evolution. The symmetric variant of the variance-gamma distribution was studied in Madan and Seneta (1990); Madan and Milne (1991). Madan et al. (1998) defined the variance-gamma process as the time-changed Brownian motion with drift. A number of modern research papers confirms statistically the idea to use the variance-gamma distribution for the financial index modeling by the statistical analysis of the stock market data. Daal and Madan (2005); Finlay and Seneta (2006) approve the variance-gamma model for the exchange rate simulation. Linders and Stassen (2016); Moosbrucker (2006); Rathgeber et al. (2016) model with the variance-gamma distribution the Dow Jones index returns. Mozumder et al. (2015) consider the S&P500 index options in the variance-gamma model. Luciano and Schoutens (2016) model the S&P500, the Nikkei225 and the Eurostoxx50 financial indices by the variance-gamma process. Luciano et al. (2016); Wallmeier and Diethelm (2012) confirm the use of the variance-gamma distribution for the modeling of the US and the Swiss stock markets, respectively. Groups of various financial indices are modeled by the multivariate variance-gamma distribution in Nitithumbundit and Chan (2020). Flora and Vargioulou (2020) find that the variance-gamma process is the best fit for the carbon price dynamics. Göncü et al. (2016) show that the variance-gamma model fits well with the financial data of developed markets. Hoyyi et al. (2021) argue for the use of the variance-gamma distribution for the return modeling on the Southeast Asian stock markets.

There is a number of theoretical papers on the variance-gamma process which mostly investigate the computation problems connected to it. Madan et al. (1998); Ano and Ivanov (2016); Ivanov (2021) establish closed-form results for the option prices and the values of the monetary risk measures. Avramidis et al. (2003); Fu (2007) suggest special Monte-Carlo type procedures. The Fourier transform methodology for the variance-gamma model was developed in Carr and Madan (1999); Almendral and Oosterlee (2007). A calculation method based on the Mellin transform is proposed in Aguilar (2020). The triple Mellin-Barnes integral representation for the European call price is given in Febrer



Citation: Ivanov, Roman V. 2022. The Risk Measurement under the Variance-Gamma Process with Drift Switching. *Journal of Risk and Financial Management* 15: 22. <https://doi.org/10.3390/jrfm15010022>

Academic Editor: Thanasis Stengos

Received: 9 December 2021

Accepted: 5 January 2022

Published: 7 January 2022

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and Guerra (2021). It is assumed in the all studies above that the drift coefficient of the underlying variance-gamma distribution is constant. However, the drift coefficient may switch because of Federal Reserve announcements. This problem is considered in the papers by Cook and Hahn (1989), by Bernanke and Blinder (1992) and by Nakamura and Steinsson (2018). The calculation of the value at risk and the expected shortfall in better economically based models helps to improve the backtesting results, see Christoffersen and Pelletier (2004). Furthermore, the computation of these two basic monetary risk measures can allow us to pass to the advanced risk measurement, see the works by Barone Adesi (2016) and by Keçeci et al. (2016).

If we suggest that the drift coefficient might change, then the usual choice for its time modeling is the exponential distribution. This type of the drift switching was discussed for the Brownian motion particularly in the works by Shiryaev (1963); Beibel and Lerche (1997); Novikov and Shiryaev (2009) for the quickest detection problem. The calculation of option prices in discrete time models under the regime switching is discussed in Rémillard et al. (2017). The geometric Brownian motion with the drift and volatility switching is considered for various computational problems in the papers by Fuh et al. (2012); Yao et al. (2006); Zhang and Zhou (2009). Some limit results for jump-diffusions with regime switching are presented in Yin and Zhu (2010).

2. Materials and Methods

We discuss in this paper an extension of the variance-gamma process assuming that the linear drift coefficient may spontaneously change. From the economical point of view such modeling can take into account a possible fluctuation of the target federal funds rate. Changes in the federal funds rate can affect other short-term interest rates, longer-term interest rates, foreign exchange rates, stock prices and the prices of goods and services as well. Probably the most known works on this topic are the papers by Cook and Hahn (1989) and by Bernanke and Blinder (1992). Among the modern works in this direction let us mention the paper by Nakamura and Steinsson (2018) where the impact of Federal Reserve announcements on the whole economy is studied.

Similarly to Shiryaev (1963), the time of the linear drift change is supposed to be exponentially distributed. The distribution of the linear drift coefficient at the time of the change is suggested to be multinomial. Section 4 summarizes the properties of the new process, introduces its distribution and probability density functions. Furthermore, we introduce in Section 4 a formula for the lower partial expectation. Section 5 applies the results to the computation of the value at risk and the expected shortfall monetary risk measures. The paper proceeds the direction of research of the works by Madan et al. (1998); Ano and Ivanov (2016); Ivanov (2018); Ivanov and Temnov (2016) where analytical results were obtained in the variance-gamma model and its generalizations.

3. Setup and Notations

3.1. Setup

Let $\gamma_t = \gamma_t(a)$, $t \geq 0$, $a > 0$, be a gamma process with unit mean from zero. It is the purely discontinuous Lévy process which has the probability density function

$$f(\gamma_t, x) = \frac{a^{at} x^{at-1} e^{-ax}}{\Gamma(at)}, \quad t > 0, x > 0, \quad (1)$$

where $\Gamma(u)$, $u > 0$, is the gamma function. The variance-gamma process X_t^{vg} is defined as

$$X_t^{vg} = \mu t + \theta \gamma_t + \sigma B_{\gamma_t}, \quad (2)$$

where $\mu, \theta \in \mathbb{R}, \sigma > 0$ are constants, $B_s, s \geq 0$, is the Brownian motion with $B_0 = 0$ and γ_t is the independent with the Brownian motion gamma process with the density (1). The variance-gamma process has the mean

$$EX_t^{vg} = (\mu + \theta)t, \tag{3}$$

the variance

$$\text{Var}X_t^{vg} = \left(\frac{\theta^2}{a} + \sigma^2\right)t \tag{4}$$

and the characteristic function

$$\varphi_{X_t^{vg}}(u) = e^{iu\mu t} \left(1 - \frac{iu\theta}{a} + \frac{(\sigma u)^2}{2a}\right)^{-at}, \tag{5}$$

see for example Madan et al. (1998).

Throughout this paper, we discuss a generalization of the process (2). It is assumed that the linear drift rate can decrease by a jump at a moment $\tau \geq 0$ from μ to some value between μ and $\mu_1, \mu_1 \leq \mu$. That is, we consider the process

$$X_t = \int_0^t \mu(s)ds + \theta\gamma_t + \sigma B_{\gamma_t} \tag{6}$$

with $\theta \in \mathbb{R}, \sigma > 0$, the Brownian motion $B_s, s \geq 0, B_0 = 0$, the independent with the Brownian motion gamma process γ_t with the density (1), $\gamma_0 = 0$, and

$$\mu(s) = \mu I_{\{\tau > s\}} + \hat{\mu} I_{\{\tau \leq s\}}, \quad s \geq 0,$$

where the random variable $\hat{\mu}$ has the multinomial distribution with

$$\hat{\mu} \in \{\mu_1, \mu_2, \dots, \mu_m, \mu\}$$

and

$$\mu_1 \leq \mu_2 \leq \dots \leq \mu_m \leq \mu.$$

It is assumed that

$$P(\hat{\mu} = \mu_j) = p_j \geq 0, \quad P(\hat{\mu} = \mu) = p \geq 0, \quad p + \sum_{j=1}^m p_j = 1.$$

The random variable τ is suggested to be exponentially distributed with a parameter $\lambda > 0$. It can be noticed easily that then

$$\int_0^t \mu(s)ds = (\mu\tau + \hat{\mu}(t - \tau))I_{\{\tau \leq t\}} + \mu t I_{\{\tau > t\}}. \tag{7}$$

It is supposed also that $\tau, \hat{\mu}$ are independent between each other and with B_s and γ_t .

3.2. Special Notations

We set

$$\text{sgn}(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0 \end{cases}$$

and use notations

$$\Psi(u), u \in \mathbb{R}, \quad B(u_1, u_2), u_1 > 0, u_2 > 0, \quad K_{u_1}(u_2), u_1 \in \mathbb{R}, u_2 > 0$$

for the normal distribution function, the beta function and the MacDonald function (the modified Bessel function of the second kind), respectively. The hypergeometric Gauss function is denoted as

$$F(u_1, u_2, u_3; u_4), u_1, u_2, u_3 \in \mathbb{R}, u_4 < 1.$$

Furthermore, the degenerate Appell functions (or the Humbert series) which is the double sum

$$\Phi(u_1, u_2, u_3; u_4, u_5) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(u_1)_{m+n} (u_2)_m}{m! n! (u_3)_{m+n}} u_4^m u_5^n$$

with $u_1, u_2, u_3, u_5 \in \mathbb{R}$ and $|u_4| < 1$, where $(u)_l, l \in \mathbb{N} \cup \{0\}$, is the Pochhammer’s symbol, is exploited. For more information on the special mathematical functions above, see the monographs by Bateman and Erdélyi (1953); Srivastava and Karlsson (1985) and the handbook by Gradshteyn and Ryzhik (2007).

Basing on the definitions of the special functions above, let us introduce supplementary functions

$$\mathfrak{L}(u_1, u_2), \quad \mathfrak{J}_=(\alpha, u_1, u_2), \quad \mathfrak{J}_>(\alpha, u_1, u_2, u_3), \quad \mathfrak{J}_<(\alpha, u_1, u_2, u_3).$$

We set for $t > 0$

$$\begin{aligned} \mathfrak{L}(u_1, u_2) &= \frac{a^{at} e^{\frac{\theta u_2}{\sigma^2}} \sqrt{2}}{\sigma \Gamma(at) \sqrt{\pi}} \left(\frac{|u_2|}{\sqrt{2a\sigma^2 + \theta^2}} \right)^{at - \frac{1}{2}} \times \\ &\times K_{at - \frac{1}{2}} \left(\frac{|u_2| \sqrt{2a\sigma^2 + \theta^2}}{\sigma^2} \right) \left[I_{\{u_1=0, u_2 \neq 0\}} + e^{u_1 u_2} I_{\{u_1 \neq 0, u_2 \neq 0\}} \right] + \\ &+ \frac{a^{at} \Gamma\left(at - \frac{1}{2}\right)}{\sigma \left(a + \frac{\theta^2}{2\sigma^2}\right)^{at - \frac{1}{2}} \Gamma(at) \sqrt{2\pi}} I_{\{u_2=0, t > \frac{1}{2a}\}} + \infty I_{\{u_2=0, t \leq \frac{1}{2a}\}}, \end{aligned} \tag{8}$$

where $u_1, u_2 \in \mathbb{R}$. Let

$$\begin{aligned} \mathfrak{J}_=(\alpha, u_1, u_2) &= \frac{2^\alpha \Gamma\left(\alpha + \frac{3}{2}\right)}{(\alpha + 1) |u_1|^{2(\alpha+1)} \sqrt{\pi}} I_{\{u_2=0\}} + \frac{e^{u_1 u_2} |u_2|^{\alpha + \frac{3}{2}}}{(\alpha + 1) \sqrt{2\pi} |u_1|^{\alpha + \frac{1}{2}}} \times \\ &\times \left[\text{sgn}(u_2) K_{\alpha + \frac{1}{2}}(|u_1 u_2|) + K_{\alpha + \frac{3}{2}}(|u_1 u_2|) \right] I_{\{u_2 \neq 0\}} \end{aligned} \tag{9}$$

for $\alpha > -1, u_2 \in \mathbb{R}$ and $u_1 \neq 0$. The function

$$\begin{aligned} \mathfrak{J}_>(\alpha, u_1, u_2, u_3) &= \\ &= \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{\sqrt{2\pi} (a - u_1)^{\alpha+1}} \left[\frac{B\left(\frac{1}{2}, \alpha + 1\right)}{\sqrt{2}} + \frac{u_2}{\sqrt{a - u_1}} F\left(\alpha + \frac{3}{2}, \frac{1}{2}, \frac{3}{2}; \frac{u_2^2}{2(u_1 - a)}\right) \right] I_{\{u_3=0\}} \\ &+ \frac{|s_>|^{\alpha + \frac{1}{2}} e^{s_>} (1 + q_>)^{\alpha+1}}{\sqrt{2\pi} (a - u_1)^{\alpha+1}} \left[B(\alpha + 1, 1) \left(|s_>| K_{\alpha + \frac{3}{2}}(|s_>|) + s_> K_{\alpha + \frac{1}{2}}(|s_>|) \right) \right. \\ &\times \Phi\left(\alpha + 1, -\alpha, \alpha + 2; \frac{1 + q_>}{2}, -s_> (1 + q_>)\right) - (1 + q_>) s_> B(\alpha + 2, 1) \times \\ &\left. \times K_{\alpha + \frac{1}{2}}(|s_>|) \Phi\left(\alpha + 2, -\alpha, \alpha + 3; \frac{1 + q_>}{2}, -s_> (1 + q_>)\right) \right] I_{\{u_3 \neq 0\}}, \end{aligned} \tag{10}$$

is defined for $a > u_1, u_2, u_3 \in \mathbb{R}$ and $\alpha > -1$ with

$$q_> = \frac{u_2}{\sqrt{u_2^2 + 2(a - u_1)}} \quad \text{and} \quad s_> = u_3 \sqrt{u_2^2 + 2(a - u_1)}.$$

Further, set

$$\begin{aligned} \mathfrak{J}_<(\alpha, u_1, u_2, u_3) &= \\ &= \frac{2^\alpha \Gamma(\alpha + \frac{3}{2})}{(\alpha + 1)(u_2^2 - 2(u_1 - a))^{\alpha+1} \sqrt{\pi}} F\left(\frac{1}{2}, \alpha + 1, \alpha + 2; \frac{2(u_1 - a)}{2(u_1 - a) - u_2^2}\right) I_{\{u_3=0\}} + \\ &+ \frac{(q_< - 1)^{\alpha+1} |s_<|^{\alpha+\frac{1}{2}} e^{s_<}}{(u_1 - a)^{\alpha+1} \sqrt{2\pi}} \left[B(\alpha + 1, 1) \Phi\left(\alpha + 1, -\alpha, \alpha + 2; \frac{1 - q_<}{2}, s_<(1 - q_<)\right) \right. \\ &\times \left(|s_<| K_{\alpha+\frac{3}{2}}(|s_<|) + s_< K_{\alpha+\frac{1}{2}}(|s_<|) \right) + (q_< - 1) s_< K_{\alpha+\frac{1}{2}}(|s_<|) B(\alpha + 2, 1) \\ &\left. \times \Phi\left(\alpha + 2, -\alpha, \alpha + 3; \frac{1 - q_<}{2}, s_<(1 - q_<)\right) \right] I_{\{u_3 \neq 0\}} \end{aligned} \tag{11}$$

for $a < u_1, u_2 \geq \sqrt{2u_1}, u_3 \in \mathbb{R}$ and $\alpha > -1$ with

$$q_< = -\frac{u_2}{\sqrt{u_2^2 - 2(u_1 - a)}} \quad \text{and} \quad s_< = u_3 \sqrt{u_2^2 - 2(u_1 - a)}.$$

Finally, let

$$z_j = \frac{\lambda \sigma}{\mu_j - \mu} - \frac{\theta}{\sigma} \tag{12}$$

and

$$w_j = \frac{\lambda}{\mu - \mu_j} \left(\theta + \frac{\lambda \sigma^2}{2(\mu - \mu_j)} \right) \tag{13}$$

for $j = 1, 2, \dots, m$.

4. Theoretical Results

In this section, we discuss the variance-gamma process with drift switching which is defined in (6). Using (7), one can observe that

$$X_t = (\mu\tau + \hat{\mu}(t - \tau)) I_{\{\tau \leq t\}} + \mu t I_{\{\tau > t\}} + \theta \gamma_t + \sigma B_{\gamma_t}.$$

At first, we compute the mean, the variance and the characteristic function of the process X_t .

4.1. Properties

We have that the mean of X_t

$$\begin{aligned} EX_t &= \theta t + E\left((\mu\tau + \hat{\mu}(t - \tau)) I_{\{\tau \leq t\}} + \mu t I_{\{\tau > t\}}\right) = \\ &= \theta t + \mu t e^{-\lambda t} + \mu E\left(\tau I_{\{\tau \leq t\}}\right) + \left(\rho \mu + \sum_{j=1}^m \rho_j \mu_j\right) \times \\ &\times \left(t(1 - e^{-\lambda t}) - E\left(\tau I_{\{\tau \leq t\}}\right)\right). \end{aligned}$$

Since

$$E(\tau I_{\{\tau \leq t\}}) = \lambda \int_0^t x e^{-\lambda x} dx = \frac{1}{\lambda} (1 - e^{-\lambda t}) - t e^{-\lambda t}, \tag{14}$$

we get that

$$EX_t = \theta t + \frac{\mu}{\lambda} (1 - e^{-\lambda t}) + \left(p\mu + \sum_{j=1}^m p_j \mu_j \right) \left(t - \frac{1}{\lambda} (1 - e^{-\lambda t}) \right) \tag{15}$$

with

$$E\hat{\mu} = p\mu + \sum_{j=1}^m p_j \mu_j. \tag{16}$$

If $\mu = \mu_1 = \dots = \mu_m$, we have (3) immediately from (15).

Further, the expectation

$$\begin{aligned} EX_t^2 &= \mu^2 E(\tau^2 I_{\{\tau \leq t\}}) + \mu E(\hat{\mu} \tau (t - \tau) I_{\{\tau \leq t\}}) + \mu \theta E(\gamma_t \tau I_{\{\tau \leq t\}}) + \\ &+ \mu \sigma E(B_{\gamma_t} \tau I_{\{\tau \leq t\}}) + \mu E(\hat{\mu} \tau (t - \tau) I_{\{\tau \leq t\}}) + E(\hat{\mu}^2 (t - \tau)^2 I_{\{\tau \leq t\}}) + \\ &+ \theta E(\hat{\mu} \gamma_t (t - \tau) I_{\{\tau \leq t\}}) + \sigma E(\hat{\mu} B_{\gamma_t} (t - \tau) I_{\{\tau \leq t\}}) + \mu^2 t^2 e^{-\lambda t} + \\ &+ \mu \theta t^2 e^{-\lambda t} + \theta t (EX_t - \theta t) + \theta^2 E\gamma_t^2 + \sigma^2 EB_{\gamma_t}^2 = \mu^2 E(\tau^2 I_{\{\tau \leq t\}}) + \\ &+ 2\mu t E\hat{\mu} E(\tau I_{\{\tau \leq t\}}) - 2\mu E\hat{\mu} E(\tau^2 I_{\{\tau \leq t\}}) + \mu \theta t E(\tau I_{\{\tau \leq t\}}) + \\ &+ t^2 E\hat{\mu}^2 (1 - e^{-\lambda t}) - 2t E\hat{\mu}^2 E(\tau I_{\{\tau \leq t\}}) + E\hat{\mu}^2 E(\tau^2 I_{\{\tau \leq t\}}) + \\ &+ \theta t^2 (1 - e^{-\lambda t}) E\hat{\mu} - \theta t E\hat{\mu} E(\tau I_{\{\tau \leq t\}}) + \mu^2 t^2 e^{-\lambda t} + \mu \theta t^2 e^{-\lambda t} + \\ &+ \theta t (EX_t - \theta t) + \theta^2 t \left(t + \frac{1}{a} \right) + \sigma^2 t = E(\tau^2 I_{\{\tau \leq t\}}) (\mu^2 - 2\mu E\hat{\mu} + E\hat{\mu}^2) + \\ &+ E(\tau I_{\{\tau \leq t\}}) (2\mu t E\hat{\mu} + \mu \theta t - 2t E\hat{\mu}^2 - \theta t E\hat{\mu}) + t^2 (1 - e^{-\lambda t}) \times \\ &\times (E\hat{\mu}^2 + \theta E\hat{\mu}) + t^2 \mu e^{-\lambda t} (\mu + \theta) + t \left(\theta EX_t + \frac{\theta^2}{a} + \sigma^2 \right). \end{aligned}$$

Because of

$$\begin{aligned} E(\tau^2 I_{\{\tau \leq t\}}) &= \lambda \int_0^t x^2 e^{-\lambda x} dx = -t^2 e^{-\lambda t} + 2 \int_0^t x e^{-\lambda x} dx = \\ &= \frac{2}{\lambda} E(\tau I_{\{\tau \leq t\}}) - t^2 e^{-\lambda t}, \end{aligned}$$

we get that the variance of X_t

$$\begin{aligned} \text{Var}X_t &= \left(\frac{2}{\lambda} E(\tau I_{\{\tau \leq t\}}) - t^2 e^{-\lambda t} \right) (\mu^2 - 2\mu E\hat{\mu} + E\hat{\mu}^2) + \\ &+ E(\tau I_{\{\tau \leq t\}}) (2\mu t E\hat{\mu} + \mu \theta t - 2t E\hat{\mu}^2 - \theta t E\hat{\mu}) + t^2 (1 - e^{-\lambda t}) \times \\ &\times (E\hat{\mu}^2 + \theta E\hat{\mu}) + t^2 \mu e^{-\lambda t} (\mu + \theta) + t \left(\theta EX_t + \frac{\theta^2}{a} + \sigma^2 \right) - (EX_t)^2, \end{aligned} \tag{17}$$

where $E(\tau I_{\{\tau \leq t\}})$, EX_t , $E\hat{\mu}$ are determined by (14), (15), (16), respectively, and

$$E\hat{\mu}^2 = p\mu^2 + \sum_{j=1}^m p_j \mu_j^2.$$

Setting $\mu = \mu_1 = \dots = \mu_m$, we establish from (17) the variance (4).
Next, the expectation

$$\begin{aligned} Ee^{iu \int_0^t \mu(s) ds} &= E\left(e^{iu(\mu\tau + \hat{\mu}(t-\tau))} I_{\{\tau \leq t\}}\right) + e^{(iu\mu - \lambda)t} = pe^{iu\mu t} (1 - e^{-\lambda t}) + \\ &+ e^{(iu\mu - \lambda)t} + \sum_{j=1}^m p_j E\left(e^{iu(\mu\tau + \mu_j(t-\tau))} I_{\{\tau \leq t\}}\right) \end{aligned}$$

and since

$$\begin{aligned} E\left(e^{iu\tau(\mu - \mu_j)} I_{\{\tau \leq t\}}\right) &= \lambda \int_0^t e^{-\lambda x + iux(\mu - \mu_j)} dx = \\ &= \lambda \int_0^t e^{-\lambda x} (\cos u(\mu - \mu_j)x + i \sin u(\mu - \mu_j)x) dx = \\ &= u(\mu - \mu_j) \int_0^t e^{-\lambda x} (i \cos u(\mu - \mu_j)x - \sin u(\mu - \mu_j)x) dx - \\ &e^{-\lambda t + iut(\mu - \mu_j)} + 1 = 1 + \frac{iu(\mu - \mu_j)}{\lambda} E\left(e^{iu\tau(\mu - \mu_j)} I_{\{\tau \leq t\}}\right) - e^{-\lambda t + iut(\mu - \mu_j)}, \end{aligned}$$

we get that

$$\begin{aligned} Ee^{iu \int_0^t \mu(s) ds} &= \\ &= pe^{iu\mu t} (1 - e^{-\lambda t}) + e^{(iu\mu - \lambda)t} + \sum_{j=1}^m p_j e^{iu\mu_j t} \left(\frac{\lambda(1 - e^{t(iu(\mu - \mu_j) - \lambda)})}{\lambda - iu(\mu - \mu_j)} \right). \end{aligned}$$

Hence the characteristic function of X_t

$$\begin{aligned} \varphi_{X_t}(u) &= \left(Ee^{iu(X_t^{vg} - \mu t)}\right) \left(Ee^{iu \int_0^t \mu(s) ds}\right) = \left(1 - \frac{iu\theta}{a} + \frac{(\sigma u)^2}{2a}\right)^{-at} \times \\ &\times \left(pe^{iu\mu t} (1 - e^{-\lambda t}) + e^{(iu\mu - \lambda)t} + \sum_{j=1}^m p_j e^{iu\mu_j t} \left(\frac{\lambda(1 - e^{t(iu(\mu - \mu_j) - \lambda)})}{\lambda - iu(\mu - \mu_j)}\right)\right). \end{aligned} \tag{18}$$

When $\mu = \mu_1 = \dots = \mu_m$, we immediately have (5) from (18)

4.2. Main Results

Now we pass to the formulation of the main results of the work. Let us define the function \mathfrak{V} for $v_2 \neq 0$ as

$$\begin{aligned} \mathfrak{V} &= \mathfrak{V}(v_1, v_2, v_3, v_4, d_j, g_j, h_j, c_j, u; j = 1, \dots, v_4) = \frac{a^{at}}{\Gamma(at)} \times \\ &\times \left(\mathfrak{J}\left(at - 1, 0, -\frac{v_1}{v_2}, \frac{u - v_3 t}{v_2}\right) + \sum_{j=1}^{v_4} d_j \left\{ \mathfrak{J}\left(at - 1, 0, -\frac{v_1}{v_2}, \frac{u - g_j t}{v_2}\right) - \right. \right. \\ &\left. \left. e^{\frac{\lambda(g_j t - u)}{v_3 - g_j}} \left[\mathfrak{J}\left(at - 1, h_j, c_j, \frac{u - g_j t}{v_2}\right) - \mathfrak{J}\left(at - 1, h_j, c_j, \frac{u - v_3 t}{v_2}\right) \right] \right\} I_{\{g_j < v_3\}} \right), \end{aligned} \tag{19}$$

where

$$\mathfrak{J}(\alpha, u_1, u_2, u_3) = \mathfrak{J}_=(\alpha, u_2, u_3) + \mathfrak{J}_>(\alpha, u_1, u_2, u_3) + \mathfrak{J}_<(\alpha, u_1, u_2, u_3)$$

and $\mathfrak{J}_=, \mathfrak{J}_>, \mathfrak{J}_<$ defined in (9), (10), (11), respectively. The next theorem gives us the distribution function of the random variable X_t .

Theorem 1. *The distribution function*

$$F_{X_t}(u) = \mathfrak{V}(\theta, \sigma, \mu, m, p_j, \mu_j, w_j, z_j, u; j = 1, \dots, m). \tag{20}$$

The probability density function of X_t is determined by the theorem below.

Theorem 2. *The probability density function*

$$f_{X_t}(u) = \mathfrak{L}(0, u - \mu t) + \sum_{j=1}^m p_j \left[\mathfrak{L}(0, u - \mu_j t) - e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left\{ \mathfrak{L}\left(\frac{\lambda}{\mu - \mu_j}, u - \mu_j t\right) - \mathfrak{L}\left(\frac{\lambda}{\mu - \mu_j}, u - \mu t\right) \right\} \right] I_{\{\mu_j < \mu\}}, \tag{21}$$

where $\mathfrak{L}(u_1, u_2)$ is set by (8).

Further, let

$$\begin{aligned} &\mathfrak{G}(v_1, v_2, v_3, v_4, v_5, d_{1j}, d_{2j}, d_{3j}, d_{4j}, d_{5j}, d_{6j}, d_{7j}, d_{8j}, u; j = 1, \dots, v_5) = \\ &= \frac{a^{at}}{\Gamma(at)} \left(\mathfrak{J}\left(at, 0, -\frac{v_1}{v_2}, \frac{u - v_3 t}{v_2}\right) + v_4 \mathfrak{J}\left(at - 1, 0, -\frac{v_1}{v_2}, \frac{u - v_3 t}{\sigma}\right) + \right. \\ &+ \sum_{j=1}^{v_5} d_{1j} I_{\{d_{2j} < v_3\}} \left\{ d_{3j} \mathfrak{J}\left(at - 1, 0, -\frac{v_1}{v_2}, \frac{u - v_3 t}{v_2}\right) + \right. \\ &+ d_{4j} \mathfrak{J}\left(at - 1, d_{5j}, d_{6j}, \frac{u - v_3 t}{v_2}\right) + d_{7j} \mathfrak{J}\left(at - 1, 0, -\frac{v_1}{v_2}, \frac{u - d_{2j} t}{\sigma}\right) + \\ &\left. \left. + \mathfrak{J}\left(at, 0, -\frac{v_1}{v_2}, \frac{u - d_{2j} t}{v_2}\right) + d_{8j} \mathfrak{J}\left(at - 1, d_{5j}, d_{6j}, \frac{u - d_{2j} t}{v_2}\right) \right\} \right) \end{aligned} \tag{22}$$

and set

$$c = t \left(\mu \left[p + e^{-\lambda t} \sum_{j=1}^m p_j \right] + (1 - e^{-\lambda t}) \sum_{j=1}^m p_j \mu_j \right), \tag{23}$$

$$c_{1j} = (\mu_j - \mu) e^{-\lambda t} \left(t + \frac{1}{\lambda} \right) - t \mu_j, \tag{24}$$

$$c_{2j} = e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left(u + \frac{\mu - \mu_j}{\lambda} \right), \tag{25}$$

$$c_{3j} = t \mu_j + \frac{\mu - \mu_j}{\lambda}, \tag{26}$$

$$c_{4j} = -e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left(u + \frac{\mu - \mu_j}{\lambda} \right). \tag{27}$$

The next theorem determines the value of the lower partial expectation $LPE_{X_t}(u)$ of X_t which is defined as

$$LPE_{X_t}(u) = E\left(X_t I_{\{X_t \leq u\}}\right).$$

Theorem 3. *The lower partial expectation*

$$\begin{aligned} LPE_{X_t}(u) &= \\ &= \mathfrak{G}(\theta, \sigma, \mu, c, m, p_j, \mu_j, c_{1j}, c_{2j}, w_j, z_j, c_{3j}, c_{4j}, u; j = 1, \dots, m). \end{aligned} \tag{28}$$

5. Risk Measurement

The aim of this section is to calculate in our model the values of the basic monetary risk measures. That is, to compute the value at risk and the expected shortfall for the returns which are modeled with the random variable X_t defined in (6).

5.1. Risk Measures

The value at risk was prescribed to financial institutions for the estimation of the portfolio losses by the regulations Basel I and Basel II. It is defined as the low quantile of the distribution of the losses with minus sign. If the losses are modeled by a random variable ζ , then the value at risk $VaR_\alpha(\zeta)$ is defined as

$$VaR_\alpha(\zeta) = -u_\alpha, \quad \text{where } u_\alpha = \inf\{u \in \mathbb{R} : F_\zeta(u) \geq \alpha\}.$$

An improvement of the value at risk, the expected shortfall monetary risk measure was recommended to banks by the Basel III regulation. In comparison to the value at risk, the expected shortfall takes into account the size of possible losses. The expected shortfall is defined as

$$ES_\alpha(\zeta) = -\frac{1}{\alpha} [L_\zeta(u_\alpha) + u_\alpha(\alpha - F_\zeta(u_\alpha))],$$

where $L_\zeta(u)$, $u \in \mathbb{R}$, is the lower partial expectation of ζ , that is

$$L_\zeta(u) = E\left(\zeta I_{\{\zeta \leq u\}}\right).$$

If the distribution of ζ is continuous, the expected shortfall coincides with the conditional value at risk monetary measure which was primarily introduced in [Rockafellar and Uryasev \(2000\)](#).

The quantile u_α can be found as the solution of the equation

$$F_\zeta(u) - \alpha = 0$$

with a one of the root-finding algorithms, see [Brent \(1973\)](#); [Press et al. \(2007\)](#); [Stoer and Bulirsch \(2002\)](#) for details. The computation of the distribution function of losses in analytical or semi-analytical forms in various models is given in [Armenti et al. \(2018\)](#); [Drapeu et al. \(2014\)](#); [Ivanov \(2018\)](#). [Chun et al. \(2012\)](#); [Mafusalov and Uryasev \(2016\)](#) estimate the value at risk and the expected shortfall using the Monte Carlo simulations. Nonparametric methods for the calculation of these two risk measures are presented in [Cai and Wang \(2008\)](#); [Chen and Tang \(2005\)](#); [Scaillet \(2005\)](#).

5.2. Model and Results

We assume that there are n assets A_1, \dots, A_n in the investment portfolio whose dynamics is determined as

$$A_{l,t} - A_{l,0} = \int_0^t \mu_l(s) ds + \theta_l \gamma_t + \sigma_l B_{\gamma_t}^l,$$

where $\theta_l \in \mathbb{R}$, $\sigma_l > 0$, γ_t is the gamma process and $B_{l,s}$, $l = 1, 2, \dots, n$ are the Brownian motions which are correlated with coefficients ρ_{lj} . It is suggested that the Brownian motions are independent with the gamma process. Similarly to Section 2 we assume that

$$\mu_l(s) = \mu_l I_{\{\tau > s\}} + \hat{\mu}_l I_{\{\tau \leq s\}}, \quad s \geq 0,$$

where the random variable $\hat{\mu}_l$ has a multinomial distribution with

$$\hat{\mu}_l \in \{\mu_{l1}, \mu_{l2}, \dots, \mu_{lm_l}, \mu_l\}, \quad \mu_{l1} \leq \mu_{l2} \leq \dots \leq \mu_{lm_l} \leq \mu_l,$$

$$P(\hat{\mu}_l = \mu_{lj}) = p_{lj} \geq 0, \quad P(\hat{\mu}_l = \mu_l) = p_l \geq 0, \quad p_l + \sum_{j=1}^{m_l} p_{lj} = 1$$

for $l = 0, 1, \dots, n$ and τ has an exponential distribution.

It is supposed also that two low risk assets are included in the portfolio. Namely, A with the dynamics

$$A_t - A_0 = \int_0^t r(s) ds,$$

$$r(s) = r I_{\{\tau > s\}} + \hat{r} I_{\{\tau \leq s\}}, \quad s \geq 0,$$

$$\hat{r} \in \{r_1, r_2, \dots, r_{m_r}, r\}, \quad r_1 \leq r_2 \leq \dots \leq r_{m_r} \leq r,$$

$$P(\hat{r} = r_j) = p_{rj} \geq 0, \quad P(\hat{r} = r) = p_r \geq 0, \quad p_r + \sum_{j=1}^m p_{rj} = 1$$

and A_0 with the variation

$$A_{0,t} - A_{0,0} = \int_0^t \mu_0(s) ds + \theta_0 \gamma_t,$$

where $\theta_0 \in \mathbb{R}$.

To assess the risks of the portfolio, it is enough to discuss the process

$$\begin{aligned} \bar{X}_t = x \Delta A_t + \sum_{l=0}^n x_l \Delta A_{l,t} &= x \int_0^t r(s) ds + \sum_{l=0}^n x_l \int_0^t \mu_l(s) ds + \\ &+ \left(\sum_{l=0}^n x_l \theta_l \right) \gamma_t + \sum_{l=1}^n x_l \sigma_l B_{\gamma_t}^l. \end{aligned}$$

Set

$$\bar{\theta} = \sum_{l=0}^n x_l \theta_l, \quad \bar{\mu} = xr + \sum_{l=0}^n x_l \mu_l, \quad \bar{\sigma} = \sqrt{\sum_{l,j=1}^n \rho_{lj} x_l x_j \sigma_l \sigma_j}.$$

Then

$$\begin{aligned} \bar{X}_t \stackrel{Law}{=} &\left(\bar{\mu} \tau + \left(x \hat{r} + \sum_{l=0}^n x_l \hat{\mu}_l \right) (t - \tau) \right) I_{\{\tau \leq t\}} + \\ &+ \bar{\mu} t I_{\{\tau > t\}} + \bar{\theta} \gamma_t + \left(\sum_{l=1}^n x_l \sigma_l N^l \right) \sqrt{\gamma_t}, \end{aligned}$$

where N^1, \dots, N^n are the standard normal random variables correlated with the same coefficients as the related Brownian motions. Since

$$\sum_{l=1}^n x_l \sigma_l N^l \stackrel{Law}{=} \bar{\sigma} N,$$

we have that

$$\bar{X}_t \stackrel{Law}{=} (\bar{\mu}\tau + M(t - \tau)) I_{\{\tau \leq t\}} + \bar{\mu}t I_{\{\tau > t\}} + \bar{\theta}\gamma_t + \bar{\sigma}B_{\gamma_t}$$

with

$$M = x\hat{r} + \sum_{l=0}^n x_l \hat{\mu}_l.$$

Let $r_0 = r, p_{r0} = p_r$ and $\mu_{l0} = \mu_l, p_{l0} = p_l$ for $l = 1, \dots, n$. Set

$$M_{j i_0 i_1 \dots i_n} = x r_j + \sum_{l=0}^n x_l \mu_{l i_l}$$

and

$$\bar{p}_{j i_0 i_1 \dots i_n} = p_{r j} \prod_{l=0}^n p_{l i_l}.$$

Then we have that

$$M \in \{M_{j i_0 i_1 \dots i_n}, j \in \{0, 1, \dots, m_r\}, i_l \in \{0, 1, \dots, m_l\}, l = 1, \dots, n\}$$

and

$$P(M = M_{j i_0 i_1 \dots i_n}) = \bar{p}_{j i_0 i_1 \dots i_n}.$$

Next, let us dispose the values of M in the increasing order

$$\bar{M}_1 \leq \bar{M}_2 \leq \dots \leq \bar{M}_{(m_r+1)(m_0+1)(m_1+1)\dots(m_n+1)-1} \leq \bar{M}$$

and set

$$P(M = \bar{M}_j) = \bar{p}_j \geq 0, \quad P(M = \bar{M}) = \bar{p} \geq 0,$$

$$\bar{p} + \sum_{j=1}^{(m_r+1)(m_0+1)(m_1+1)\dots(m_n+1)-1} \bar{p}_j = 1.$$

Let

$$\bar{K} = (m_r + 1)(m_0 + 1)(m_1 + 1) \dots (m_n + 1) - 1,$$

similarly to (12) and (13)

$$\bar{w}_j = \frac{\lambda}{\bar{M} - \bar{M}_j} \left(\bar{\theta} + \frac{\lambda \bar{\sigma}^2}{2(\bar{M} - \bar{M}_j)} \right),$$

$$\bar{z}_j = \frac{\lambda \bar{\sigma}}{\bar{M}_j - \bar{M}} - \frac{\bar{\theta}}{\bar{\sigma}}$$

and analogously to (23)–(27)

$$\begin{aligned} \bar{c} &= t \left(\bar{M} \left[\bar{p} + e^{-\lambda t} \sum_{j=1}^{\bar{K}} \bar{p}_j \right] + (1 - e^{-\lambda t}) \sum_{j=1}^{\bar{K}} \bar{p}_j \bar{M}_j \right), \\ \bar{c}_{1j} &= (\bar{M}_j - \bar{M}) e^{-\lambda t} \left(t + \frac{1}{\lambda} \right) - t \bar{M}_j, \\ \bar{c}_{2j} &= e^{\frac{\lambda(\bar{M}_j t - u_\alpha)}{\bar{M} - \bar{M}_j}} \left(u_\alpha + \frac{\bar{M} - \bar{M}_j}{\lambda} \right), \\ \bar{c}_{3j} &= t \bar{M}_j + \frac{\bar{M} - \bar{M}_j}{\lambda}, \\ \bar{c}_{4j} &= -e^{\frac{\lambda(\bar{M}_j t - u_\alpha)}{\bar{M} - \bar{M}_j}} \left(u_\alpha + \frac{\bar{M} - \bar{M}_j}{\lambda} \right). \end{aligned}$$

The following proposition issues immediately from Theorems 1 and 3.

Corollary 1. *The value at risk $\text{VaR}_\alpha(\bar{X}_t)$ is determined as the solution u_α of the equation*

$$\mathfrak{V}(\bar{\theta}, \bar{\sigma}, \bar{M}, \bar{K}, \bar{p}_j, \bar{M}_j, \bar{w}_j, \bar{z}_j, u; j = 1, \dots, \bar{K}) - \alpha = 0$$

with minus sign. The expected shortfall $\text{ES}_\alpha(\bar{X}_t)$ is calculated with respect to the formula

$$\text{ES}_\alpha(\bar{X}_t) = \mathfrak{E}(\bar{\theta}, \bar{\sigma}, \bar{M}, \bar{c}, \bar{K}, \bar{p}_j, \bar{M}_j, \bar{c}_{1j}, \bar{c}_{2j}, \bar{w}_j, \bar{z}_j, \bar{c}_{3j}, \bar{c}_{4j}, u_\alpha; j = 1, \dots, \bar{K}).$$

6. Discussion

The considered model extends the variance-gamma one under the assumption that the linear drift rate could drop down at some random time moment. From the risk measurement point of view, the formula for the distribution function of the underlying process allows us to compute the value at risk. The knowledge of the lower partial expectation permits the calculation of the expected shortfall. Various approximate computations can be made using the formula for the probability density function. Further investigations may include the calculation of risk-adjusted performance measures, downside and upside betas in the discussed model. To generalize the model, we can assume a smaller dependence between the gamma processes. Different types of the distribution of the jump time might be studied as well.

7. Conclusions

We have established the analytical formulas for the value at risk and the expected shortfall monetary risk measures in the model driven by the variance-gamma process with the probability of the linear drift rate negative jump. The time of the rate jump has been modeled by the exponential distribution. The size of the jump has been modeled by the multinomial distribution. The obtained results in particular can be considered as a step to the computation of various complicated risk characteristics in more general stochastic models. From the modeling point of view, taking into account the impact of the federal funds rate changes leads us to a model which distinguishes better the real financial market dynamics. It is expected that the new model can give better results for the value at risk backtesting, see [Christoffersen and Pelletier \(2004\)](#) for details. In general, the value at risk and the expected shortfall computation not only meets the requirements of regulators and the Basel Committee on Banking Supervision (see for example [Ming Chen \(2018\)](#)), but also allows us to construct more complex and economically sound models (see in particular [Keçeci et al. \(2016\)](#); [Barone Adesi \(2016\)](#)) and also to control the insurance market (see [Ramsey and Goodwin \(2019\)](#)).

8. Proofs

Proof of Theorem 1. Since the probability

$$P(X_t \leq u) = E\left(E\left(I_{\{X_t \leq u\}} | Y_t\right)\right),$$

where

$$Y_t = \theta\gamma_t + \sigma B_{\gamma_t}, \tag{29}$$

we might start from the computation of the function

$$\begin{aligned} g(Y_t) &= E\left(I_{\{X_t \leq u\}} | Y_t\right) = E\left(I_{\{X_t \leq u, \tau \leq t\}} | Y_t\right) + E\left(I_{\{X_t \leq u, \tau > t\}} | Y_t\right) = \\ &= E\left(I_{\{\tau \leq t, \hat{\mu}(t-\tau) \leq u - \mu\tau - Y_t\}} | Y_t\right) + e^{-\lambda t} I_{\{Y_t \leq u - \mu t\}}. \end{aligned} \tag{30}$$

One can observe that the conditional expectation

$$\begin{aligned} E\left(I_{\{\tau \leq t, \hat{\mu}(t-\tau) \leq u - \mu\tau - Y_t\}} | Y_t\right) &= E\left(I_{\{\tau \leq t, \hat{\mu}(t-\tau) \leq u - \mu\tau - Y_t, \hat{\mu} = \mu\}} | Y_t\right) + \\ &+ \sum_{j=1}^m E\left(I_{\{\tau \leq t, \hat{\mu}(t-\tau) \leq u - \mu\tau - Y_t, \hat{\mu} = \mu_j\}} | Y_t\right) = P\left(1 - e^{-\lambda t}\right) I_{\{Y_t \leq u - \mu t\}} + \\ &+ \sum_{j=1}^m P_j E\left(I_{\{\tau \leq t, \mu_j(t-\tau) \leq u - \mu\tau - Y_t\}} | Y_t\right). \end{aligned} \tag{31}$$

The conditional probability

$$\begin{aligned} P(\tau \leq t, (\mu - \mu_j)\tau \leq u - \mu_j t - Y_t | Y_t) &= \\ = P\left(\tau \leq \min\left\{t, \frac{u - \mu_j t - Y_t}{\mu - \mu_j}\right\} | Y_t\right) I_{\{Y_t \leq u - \mu_j t, \mu_j < \mu\}} + \\ + (1 - e^{-\lambda t}) I_{\{Y_t \leq u - \mu t, \mu_j = \mu\}} &= (1 - e^{-\lambda t}) I_{\{Y_t \leq u - \mu t\}} + \\ + \left(1 - e^{\lambda \frac{Y_t + \mu_j t - u}{\mu - \mu_j}}\right) I_{\{u - \mu t < Y_t \leq u - \mu_j t, \mu_j < \mu\}}. \end{aligned} \tag{32}$$

Now we have from (31) and (32) that

$$\begin{aligned} E\left(I_{\{\tau < t, \hat{\mu}(t-\tau) \leq u - \mu\tau - Y_t\}} | Y_t\right) &= (1 - e^{-\lambda t}) I_{\{Y_t \leq u - \mu t\}} + \\ + \sum_{j=1}^m P_j \left[\left(1 - e^{\lambda \frac{Y_t + \mu_j t - u}{\mu - \mu_j}}\right) I_{\{Y_t \leq u - \mu_j t\}} - \right. \\ \left. - \left(1 - e^{\lambda \frac{Y_t + \mu_j t - u}{\mu - \mu_j}}\right) I_{\{Y_t \leq u - \mu t\}} \right] I_{\{\mu_j < \mu\}} \end{aligned} \tag{33}$$

and therefore it follows from (30) and (33) that

$$\begin{aligned} g(Y_t) &= I_{\{Y_t \leq u - \mu t\}} + \sum_{j=1}^m P_j \left[\left(1 - e^{\lambda \frac{Y_t + \mu_j t - u}{\mu - \mu_j}}\right) I_{\{Y_t \leq u - \mu_j t\}} + \right. \\ &\left. + e^{\lambda \frac{Y_t + \mu_j t - u}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu t\}} \right] I_{\{\mu_j < \mu\}}. \end{aligned} \tag{34}$$

Because of $P(X_t \leq u) = Eg(Y_t)$, we have that

$$P(X_t \leq u) = P(Y_t \leq u - \mu t) + \sum_{j=1}^m p_j \left[P(Y_t \leq u - \mu_j t) - e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} E e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu_j t\}} + e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} E e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu t\}} \right] I_{\{\mu_j < \mu\}}.$$

Now it is required to compute the expectation

$$V = V(A, D) = E e^{AY_t} I_{\{Y_t \leq D\}}$$

for $A \in \left\{0, \frac{\lambda}{\mu - \mu_j}\right\}$ and $D \in \mathbb{R}$. Then

$$P(X_t \leq u) = V(0, u - \mu t) + \sum_{j=1}^m p_j \left[V(0, u - \mu_j t) - e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} V\left(\frac{\lambda}{\mu - \mu_j}, u - \mu_j t\right) + e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} V\left(\frac{\lambda}{\mu - \mu_j}, u - \mu t\right) \right] I_{\{\mu_j < \mu\}}. \tag{35}$$

Set

$$v(\gamma_t) = E\left(e^{AY_t} I_{\{Y_t \leq D\}} \mid \gamma_t\right) = E\left(e^{A(\theta \gamma_t + \sigma B_{\gamma_t})} I_{\{\theta \gamma_t + \sigma B_{\gamma_t} \leq D\}} \mid \gamma_t\right).$$

Then

$$\begin{aligned} v(\gamma_t) &= \int_{-\infty}^D \frac{1}{\sigma \sqrt{2\pi \gamma_t}} e^{Ax - \frac{(x - \theta \gamma_t)^2}{2\sigma^2 \gamma_t}} dx = \\ &= e^{\frac{A(A\sigma^2 + 2\theta)\gamma_t}{2}} \int_{-\infty}^D \frac{1}{\sigma \sqrt{2\pi \gamma_t}} e^{-\frac{(x - \gamma_t(A\sigma^2 + \theta))^2}{2\sigma^2 \gamma_t}} dx = \\ &= e^{\frac{A(A\sigma^2 + 2\theta)\gamma_t}{2}} \Psi\left(\frac{D - \gamma_t(A\sigma^2 + \theta)}{\sigma \sqrt{\gamma_t}}\right) = \\ &= \Psi\left(-\frac{\theta}{\sigma} \sqrt{\gamma_t} + \frac{D}{\sigma \sqrt{\gamma_t}}\right) I_{\{A=0\}} + \\ &+ e^{\frac{\lambda}{\mu - \mu_j} \left(\frac{\lambda \sigma^2}{2(\mu - \mu_j)} + \theta\right) \gamma_t} \Psi\left(\left(\frac{\lambda \sigma}{\mu_j - \mu} - \frac{\theta}{\sigma}\right) \sqrt{\gamma_t} + \frac{D}{\sigma \sqrt{\gamma_t}}\right) I_{\{A = \frac{\lambda}{\mu - \mu_j}\}} \end{aligned} \tag{36}$$

and since $V = Ev(\gamma_t)$, we have from (35) and (36) that

$$\begin{aligned} P(X_t \leq u) &= E\left[\Psi\left(-\frac{\theta}{\sigma} \sqrt{\gamma_t} + \frac{u - \mu t}{\sigma \sqrt{\gamma_t}}\right)\right] + \\ &+ \sum_{j=1}^m p_j I_{\{\mu_j < \mu\}} \left\{ E\left[\Psi\left(-\frac{\theta}{\sigma} \sqrt{\gamma_t} + \frac{u - \mu_j t}{\sigma \sqrt{\gamma_t}}\right)\right] - \right. \\ &- e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} E\left[e^{\frac{\lambda}{\mu - \mu_j} \left(\frac{\lambda \sigma^2}{2(\mu - \mu_j)} + \theta\right) \gamma_t} \Psi\left(\left(\frac{\lambda \sigma}{\mu_j - \mu} - \frac{\theta}{\sigma}\right) \sqrt{\gamma_t} + \frac{u - \mu_j t}{\sigma \sqrt{\gamma_t}}\right)\right] - \\ &\left. - e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} E\left[e^{\frac{\lambda}{\mu - \mu_j} \left(\frac{\lambda \sigma^2}{2(\mu - \mu_j)} + \theta\right) \gamma_t} \Psi\left(\left(\frac{\lambda \sigma}{\mu_j - \mu} - \frac{\theta}{\sigma}\right) \sqrt{\gamma_t} + \frac{u - \mu t}{\sigma \sqrt{\gamma_t}}\right)\right] \right\}. \end{aligned} \tag{37}$$

Now it is easy to see from (37) that we need to calculate the integral

$$\mathfrak{J} = \mathfrak{J}(\alpha, b, h, p) = \int_0^\infty x^\alpha e^{-(a-b)x} \Psi\left(h\sqrt{x} + \frac{p}{\sqrt{x}}\right) dx \tag{38}$$

for $\alpha > -1, a > 0, p \in \mathbb{R}$ and the vector $(h, b) \in \left\{ \left(-\frac{\theta}{\sigma}, 0\right), (z_j, w_j) \right\}$, where $j = 1, 2, \dots, m$. Then

$$\begin{aligned} & \frac{\Gamma(at)P(X_t \leq u)}{b^{at}} = \\ & = \mathfrak{J}\left(at - 1, 0, -\frac{\theta}{\sigma}, \frac{u - \mu t}{\sigma}\right) + \sum_{j=1}^m p_j I_{\{\mu_j < \mu\}} \left\{ \mathfrak{J}\left(at - 1, 0, -\frac{\theta}{\sigma}, \frac{u - \mu_j t}{\sigma}\right) \right. \\ & \left. - e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left[\mathfrak{J}\left(at - 1, w_j, z_j, \frac{u - \mu_j t}{\sigma}\right) - \mathfrak{J}\left(at - 1, w_j, z_j, \frac{u - \mu t}{\sigma}\right) \right] \right\}. \end{aligned} \tag{39}$$

Further, we derive the value of $\mathfrak{J} = \mathfrak{J}(\alpha, b, h, p)$ with respect to relations between the parameters.

Case 1. $a = b$.

Case 1.1. $h \geq 0$. Then obviously $\mathfrak{J} = \infty$.

Case 1.2. $h < 0, p = 0$. We have that

$$\begin{aligned} \mathfrak{J} &= \int_0^\infty x^\alpha \Psi(h\sqrt{x}) dx = -\frac{h}{2(\alpha + 1)\sqrt{2\pi}} \int_0^\infty x^{\alpha + \frac{1}{2}} e^{-\frac{h^2 x}{2}} dx = \\ &= \frac{2^\alpha \Gamma(\alpha + \frac{3}{2})}{(\alpha + 1)|h|^{2(\alpha + 1)}\sqrt{\pi}}. \end{aligned}$$

Case 1.3. $h < 0, p \neq 0$. Integrating by parts, one can obtain that

$$\begin{aligned} \mathfrak{J} &= -\frac{1}{\alpha + 1} \int_0^\infty x^{\alpha + 1} d\Psi\left(h\sqrt{x} + \frac{p}{\sqrt{x}}\right) = \\ &= \frac{1}{2(\alpha + 1)\sqrt{2\pi}} \int_0^\infty \left(p x^{\alpha - \frac{1}{2}} - h x^{\alpha + \frac{1}{2}}\right) e^{-\frac{(hx+p)^2}{2x}} dx. \end{aligned}$$

Formula 3.471.9 from Gradshteyn and Ryzhik (2007) includes the identity

$$\int_0^\infty x^{\vartheta_1 - 1} e^{-\frac{\vartheta_2}{x} - \vartheta_3 x} dx = 2 \left(\frac{\vartheta_2}{\vartheta_3}\right)^{\frac{\vartheta_1}{2}} K_{\vartheta_1}\left(2\sqrt{\vartheta_2 \vartheta_3}\right), \tag{40}$$

where $\vartheta_1 \in \mathbb{R}, \vartheta_2 > 0$ and $\vartheta_3 > 0$. Using (40), we get that

$$\begin{aligned} \mathfrak{J} &= \frac{e^{-hp}}{(\alpha + 1)\sqrt{2\pi}} \left(\frac{p|p|^{\alpha + \frac{1}{2}}}{|h|^{\alpha + \frac{1}{2}}} K_{\alpha + \frac{1}{2}}(|ph|) - \frac{h|p|^{\alpha + \frac{3}{2}}}{|h|^{\alpha + \frac{3}{2}}} K_{\alpha + \frac{3}{2}}(|ph|) \right) = \\ &= \frac{e^{-hp}|p|^{\alpha + \frac{1}{2}}}{(\alpha + 1)\sqrt{2\pi}|h|^{\alpha + \frac{1}{2}}} \left(p K_{\alpha + \frac{1}{2}}(|ph|) + |p| K_{\alpha + \frac{3}{2}}(|ph|) \right). \end{aligned}$$

Case 2. $a \neq b, p = 0$. Then

$$\mathfrak{J} = \frac{1}{|a - b|^{\alpha + 1}} \tilde{\mathfrak{J}},$$

where $\tilde{h} = \frac{h}{\sqrt{|a-b|}}$ and $\tilde{\mathfrak{J}} = \int_0^\infty x^\alpha e^{-\text{sign}(a-b)x} \Psi(\tilde{h}\sqrt{x}) dx$.

Case 2.1. $a > b$. According to Case 2.2 at p. 208 of [Ano and Ivanov \(2016\)](#)

$$\mathfrak{J} = \frac{\Gamma(\alpha + \frac{3}{2})}{\sqrt{2\pi}(a-b)^{\alpha+1}} \left(\frac{B(\frac{1}{2}, \alpha+1)}{\sqrt{2}} + \frac{h}{\sqrt{a-b}} F\left(\alpha + \frac{3}{2}, \frac{1}{2}, \frac{3}{2}; \frac{h^2}{2(b-a)}\right) \right).$$

Case 2.2. $a < b, \tilde{h} > -\sqrt{2}$. Then $\mathfrak{J} = \infty$ with respect to Case 2.1 at p. 208 of [Ano and Ivanov \(2016\)](#).

Case 2.3. $a < b, \tilde{h} < -\sqrt{2}$. We have that

$$\begin{aligned} \tilde{\mathfrak{J}} &= \int_0^\infty x^\alpha e^x \Psi(\tilde{h}\sqrt{x}) dx = \int_0^\infty x^\alpha e^x \left(\int_{-\infty}^{\tilde{h}} \frac{\sqrt{x}}{\sqrt{2\pi}} e^{-\frac{v^2 x}{2}} dv \right) dx = \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tilde{h}} \left(\int_0^\infty x^{\alpha+\frac{1}{2}} e^{-(\frac{v^2}{2}-1)x} dx \right) dv. \end{aligned} \tag{41}$$

Using 3.382.4 from [Gradshteyn and Ryzhik \(2007\)](#), we get from (41) that

$$\tilde{\mathfrak{J}} = \frac{\Gamma(\alpha + \frac{3}{2})}{\sqrt{2\pi}} \int_{-\infty}^{\tilde{h}} \left(\frac{v^2}{2} - 1 \right)^{-\alpha-\frac{3}{2}} dv.$$

Set

$$u = \left(\frac{v^2}{2} - 1 \right)^{-1}.$$

Then

$$v = -\sqrt{\frac{2}{u} + 2}, \quad dv = u^{-\frac{3}{2}} (2 + 2u)^{-\frac{1}{2}} du$$

and hence

$$\begin{aligned} \tilde{\mathfrak{J}} &= \frac{\Gamma(\alpha + \frac{3}{2})}{2\sqrt{\pi}} \int_0^{\frac{2}{\tilde{h}^2-2}} u^\alpha (1+u)^{-\frac{1}{2}} du = \\ &= \frac{2^\alpha \Gamma(\alpha + \frac{3}{2})}{(\alpha+1)(\tilde{h}^2-2)^{\alpha+1} \sqrt{\pi}} F\left(\frac{1}{2}, \alpha+1, \alpha+2; \frac{2}{2-\tilde{h}^2}\right), \end{aligned}$$

where the last identity follows from 3.194.1 in [Gradshteyn and Ryzhik \(2007\)](#). Therefore

$$\mathfrak{J} = \frac{2^\alpha \Gamma(\alpha + \frac{3}{2})}{(\alpha+1)(h^2-2(b-a))^{\alpha+1} \sqrt{\pi}} F\left(\frac{1}{2}, \alpha+1, \alpha+2; \frac{2(b-a)}{2(b-a)-h^2}\right).$$

Case 3. $a \neq b, p \neq 0$. We have that

$$\mathfrak{J} = \frac{1}{|a-b|^{\alpha+1}} \tilde{\mathfrak{J}},$$

where

$$\tilde{h} = \frac{h}{\sqrt{|a-b|}}, \quad \tilde{p} = p\sqrt{|a-b|}$$

and

$$\tilde{\mathfrak{J}} = \int_0^\infty x^\alpha e^{-\text{sign}(a-b)x} \Psi\left(\tilde{h}\sqrt{x} + \frac{\tilde{p}}{\sqrt{x}}\right) dx.$$

Case 3.1. $a > b$. With respect to (21) of [Ano and Ivanov \(2016\)](#),

$$\begin{aligned} \mathfrak{J} = & \frac{|s|^{\alpha+\frac{1}{2}} e^s (1+q)^{\alpha+1}}{\sqrt{2\pi}(a-b)^{\alpha+1}} \left[\mathbf{B}(\alpha+1, 1) \left(|s| \mathbf{K}_{\alpha+\frac{3}{2}}(|s|) + \right. \right. \\ & \left. \left. + s \mathbf{K}_{\alpha+\frac{1}{2}}(|s|) \right) \Phi \left(\alpha+1, -\alpha, \alpha+2; \frac{1+q}{2}, -s(1+q) \right) - \right. \\ & \left. (1+q) s \mathbf{B}(\alpha+2, 1) \mathbf{K}_{\alpha+\frac{1}{2}}(|s|) \Phi \left(\alpha+2, -\alpha, \alpha+3; \frac{1+q}{2}, -s(1+q) \right) \right], \end{aligned} \tag{42}$$

where

$$q = \frac{\tilde{h}}{\sqrt{\tilde{h}^2 + 2}} \quad \text{and} \quad s = \tilde{p} \sqrt{\tilde{h}^2 + 2}.$$

Case 3.2. $a < b, \tilde{h} > -\sqrt{2}$. We conclude that $\tilde{\mathfrak{J}} = \infty$ by Case 3.1 at p. 210 of [Ano and Ivanov \(2016\)](#).

Case 3.3. $a < b, \tilde{h} < -\sqrt{2}$. Let $D(v)$ and $H(v), v \leq V \in \mathbb{R}$, be two differentiable functions with $D(V) = \tilde{p}$ and $H(V) = \tilde{h}$. Since

$$\begin{aligned} \Psi \left(\frac{D(v)}{\sqrt{x}} + H(v) \sqrt{x} \right) &= \int_{-\infty}^v \Psi'_u \left(\frac{D(u)}{\sqrt{x}} + H(u) \sqrt{x} \right) du = \\ &= \int_{-\infty}^v \Psi'_{\frac{D(u)}{\sqrt{x}} + H(u) \sqrt{x}} \left(\frac{D(u)}{\sqrt{x}} + H(u) \sqrt{x} \right) \left(\frac{D'_u(u)}{\sqrt{x}} + H'_u(u) \sqrt{x} \right) du, \end{aligned}$$

one gets that

$$\begin{aligned} \Psi \left(\frac{\tilde{p}}{\sqrt{x}} + \tilde{h} \sqrt{x} \right) &= \\ &= \int_{-\infty}^V \frac{1}{\sqrt{2\pi}} e^{-D(v)H(v) - \frac{D^2(v)}{2x} - \frac{H^2(v)x}{2}} \left(\frac{D'_v(v)}{\sqrt{x}} + H'_v(v) \sqrt{x} \right) dv. \end{aligned}$$

Set $V = \tilde{h}$ and

$$H(v) = v, \quad D(v) = \frac{\tilde{p} \sqrt{\tilde{h}^2 - 2}}{\sqrt{v^2 - 2}}, \quad v \leq \tilde{h}.$$

Then

$$\begin{aligned} \tilde{\mathfrak{J}} = & \frac{1}{\sqrt{2\pi}} \int_0^\infty x^\alpha e^x \left[\int_{-\infty}^{\tilde{h}} e^{-\frac{\tilde{p}v\sqrt{\tilde{h}^2-2}}{\sqrt{v^2-2}} - \frac{\tilde{p}^2(\tilde{h}^2-2)}{2(v^2-2)x} - \frac{v^2x}{2}} \times \right. \\ & \left. \times \left(\sqrt{x} - \frac{\tilde{p}v\sqrt{\tilde{h}^2-2}}{(v^2-2)^{\frac{3}{2}}\sqrt{x}} \right) dv \right] dx. \end{aligned}$$

Let us consider the double integrals

$$J_1 = \int_0^\infty \int_{-\infty}^{\tilde{h}} \left(x^{\alpha+\frac{1}{2}} e^{x - \frac{\tilde{p}v\sqrt{\tilde{h}^2-2}}{\sqrt{v^2-2}} - \frac{\tilde{p}^2(\tilde{h}^2-2)}{2(v^2-2)x} - \frac{v^2x}{2}} \right) dv dx$$

and

$$J_2 = \int_0^\infty \int_{-\infty}^{\tilde{h}} \left(\frac{\tilde{p}x^{\alpha-\frac{1}{2}}v\sqrt{\tilde{h}^2-2}}{(v^2-2)^{\frac{3}{2}}} e^{x - \frac{\tilde{p}v\sqrt{\tilde{h}^2-2}}{\sqrt{v^2-2}} - \frac{\tilde{p}^2(\tilde{h}^2-2)}{2(v^2-2)x} - \frac{v^2x}{2}} \right) dv dx.$$

One can notice that both J_1 and J_2 are integrals of constant sign functions. Therefore, if the iterated integrals

$$\widehat{J}_1 = \int_{-\infty}^{\tilde{h}} \left(\int_0^{\infty} x^{\alpha+\frac{1}{2}} e^{x-\frac{\tilde{p}v\sqrt{\tilde{h}^2-2}}{\sqrt{v^2-2}}-\frac{\tilde{p}^2(\tilde{h}^2-2)}{2(v^2-2)x}-\frac{v^2x}{2}} dx \right) dv$$

and

$$\widehat{J}_2 = \int_{-\infty}^{\tilde{h}} \left(\int_0^{\infty} \frac{\tilde{p}x^{\alpha-\frac{1}{2}}v\sqrt{\tilde{h}^2-2}}{(v^2-2)^{\frac{3}{2}}} e^{x-\frac{\tilde{p}v\sqrt{\tilde{h}^2-2}}{\sqrt{v^2-2}}-\frac{\tilde{p}^2(\tilde{h}^2-2)}{2(v^2-2)x}-\frac{v^2x}{2}} dx \right) dv$$

exist, then $J_1 = \widehat{J}_1$, $J_2 = \widehat{J}_2$ and the Fubini's theorem can be exploited to J_1 and J_2 , that is

$$\tilde{\mathcal{J}} = \frac{1}{\sqrt{2\pi}} (\widehat{J}_1 - \widehat{J}_2). \tag{43}$$

Set $s = \tilde{p}\sqrt{\tilde{h}^2-2}$. Then we have according to 3.471.9 from [Gradshteyn and Ryzhik \(2007\)](#) that

$$\widehat{J}_1 = 2|s|^{\alpha+\frac{3}{2}} K_{\alpha+\frac{3}{2}}(|s|) \int_{-\infty}^{\tilde{h}} (v^2-2)^{-\alpha-\frac{3}{2}} e^{-\frac{sv}{\sqrt{v^2-2}}} dv$$

and

$$\widehat{J}_2 = 2s|s|^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(|s|) \int_{-\infty}^{\tilde{h}} v(v^2-2)^{-\alpha-2} e^{-\frac{sv}{\sqrt{v^2-2}}} dv.$$

Next, let us make the change of variables

$$v \rightarrow y, \quad y = -\frac{v}{\sqrt{v^2-2}}.$$

Then

$$v = -\frac{y\sqrt{2}}{\sqrt{y^2-1}}, \quad v^2-2 = \frac{2}{y^2-1}, \quad dv = \frac{\sqrt{2}}{(y^2-1)^{\frac{3}{2}}} dy$$

and one can get that

$$\widehat{J}_1 = 2^{-\alpha}|s|^{\alpha+\frac{3}{2}} K_{\alpha+\frac{3}{2}}(|s|) \int_1^{\frac{|\tilde{h}|}{\sqrt{\tilde{h}^2-2}}} (y^2-1)^{\alpha} e^{sy} dy$$

and

$$\widehat{J}_2 = -2^{-\alpha}s|s|^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(|s|) \int_1^{\frac{|\tilde{h}|}{\sqrt{\tilde{h}^2-2}}} y(y^2-1)^{\alpha} e^{sy} dy.$$

Set

$$q = \frac{|\tilde{h}|}{\sqrt{\tilde{h}^2-2}} \quad \text{and} \quad u = \frac{y-1}{q-1}.$$

Then

$$\begin{aligned} \widehat{J}_1 &= 2^{-\alpha}(q-1)|s|^{\alpha+\frac{3}{2}} e^s K_{\alpha+\frac{3}{2}}(|s|) \int_0^1 \left((q-1)^2 u^2 + 2(q-1)u \right)^{\alpha} e^{s(q-1)u} du \\ &= (q-1)^{\alpha+1} |s|^{\alpha+\frac{3}{2}} e^s K_{\alpha+\frac{3}{2}}(|s|) \int_0^1 u^{\alpha} \left(\frac{(q-1)u}{2} + 1 \right)^{\alpha} e^{s(q-1)u} du \end{aligned} \tag{44}$$

and

$$\begin{aligned} \widehat{J}_2 &= -2^{-\alpha}(q-1)s|s|^{\alpha+\frac{1}{2}}e^sK_{\alpha+\frac{1}{2}}(|s|)\left(\int_0^1((q-1)^2u^2+2(q-1)u)^\alpha e^{s(q-1)u}du\right. \\ &\quad \left.+ (q-1)\int_0^1u((q-1)^2u^2+2(q-1)u)^\alpha e^{s(q-1)u}du\right) = \\ &= -(q-1)^{\alpha+1}s|s|^{\alpha+\frac{1}{2}}e^sK_{\alpha+\frac{1}{2}}(|s|)\left(\int_0^1u^\alpha\left(\frac{(q-1)u}{2}+1\right)^\alpha e^{s(q-1)u}du\right. \\ &\quad \left.+ (q-1)\int_0^1u^{\alpha+1}\left(\frac{(q-1)u}{2}+1\right)^\alpha e^{s(q-1)u}du\right). \end{aligned} \tag{45}$$

Applying 3.385 of Gradshteyn and Ryzhik (2007) to (44) and (45) and keeping in mind (43), we establish now that

$$\begin{aligned} \mathfrak{J} &= \frac{(q-1)^{\alpha+1}|s|^{\alpha+\frac{1}{2}}e^s}{(b-a)^{\alpha+1}\sqrt{2\pi}}\left[B(\alpha+1,1)\Phi\left(\alpha+1,-\alpha,\alpha+2;\frac{1-q}{2},s(1-q)\right)\times\right. \\ &\quad \times\left(|s|K_{\alpha+\frac{3}{2}}(|s|)+sK_{\alpha+\frac{1}{2}}(|s|)\right)+(q-1)sK_{\alpha+\frac{1}{2}}(|s|)B(\alpha+2,1)\times \\ &\quad \left.\times\Phi\left(\alpha+2,-\alpha,\alpha+3;\frac{1-q}{2},s(1-q)\right)\right]. \end{aligned}$$

Now we have from Cases 1 to 3 that

$$\begin{aligned} \mathfrak{J}\left(\alpha,0,-\frac{\theta}{\sigma'},\frac{u-\bar{\mu}t}{\sigma}\right) &= \\ &= \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{a^{\alpha+1}\sqrt{2\pi}}\left(\frac{B\left(\frac{1}{2},\alpha+1\right)}{\sqrt{2}}-\frac{\theta}{\sigma\sqrt{a}}F\left(\alpha+\frac{3}{2},\frac{1}{2},\frac{3}{2};-\frac{\theta^2}{2\sigma^2a}\right)\right)I_{\{u=\bar{\mu}t\}}+ \\ &\quad +\frac{|s|^{\alpha+\frac{1}{2}}e^s(1+q)^{\alpha+1}}{a^{\alpha+1}\sqrt{2\pi}}\left[B(\alpha+1,1)\left(|s|K_{\alpha+\frac{3}{2}}(|s|)+sK_{\alpha+\frac{1}{2}}(|s|)\right)\times\right. \\ &\quad \times\Phi\left(\alpha+1,-\alpha,\alpha+2;\frac{1+q}{2},-s(1+q)\right)-(1+q)sB(\alpha+2,1)K_{\alpha+\frac{1}{2}}(|s|)\times \\ &\quad \left.\times\Phi\left(\alpha+2,-\alpha,\alpha+3;\frac{1+q}{2},-s(1+q)\right)\right]I_{\{u\neq\bar{\mu}t\}}, \end{aligned} \tag{46}$$

with

$$q = -\frac{\theta}{\sqrt{\theta^2+2\sigma^2a}} \quad \text{and} \quad s = \frac{(u-\bar{\mu}t)\sqrt{\theta^2+2\sigma^2a}}{\sigma^2},$$

where $\bar{\mu} \in \{\mu_j, \mu\}$, $j = 1, 2, \dots, m$.

Further, since if $w_j > 0$, then $z_j < 0$ and

$$\frac{z_j^2}{w_j} = \frac{\frac{\theta^2}{\sigma^2} + \frac{2\lambda\theta}{\mu-\mu_j} + \frac{\lambda^2\sigma^2}{(\mu-\mu_j)^2}}{\frac{\lambda\theta}{\mu-\mu_j} + \frac{\lambda^2\sigma^2}{2(\mu-\mu_j)^2}} \geq 2,$$

we have that

$$\begin{aligned} \mathfrak{J}\left(\alpha, w_j, z_j, \frac{u - \bar{\mu}t}{\sigma}\right) &= \\ = \mathfrak{J}_= \left(\alpha, w_j, z_j, \frac{u - \bar{\mu}t}{\sigma}\right) &+ \mathfrak{J}_> \left(\alpha, w_j, z_j, \frac{u - \bar{\mu}t}{\sigma}\right) + \mathfrak{J}_< \left(\alpha, w_j, z_j, \frac{u - \bar{\mu}t}{\sigma}\right) \end{aligned} \tag{47}$$

with

$$\begin{aligned} \mathfrak{J}_= \left(\alpha, w_j, z_j, \frac{u - \bar{\mu}t}{\sigma}\right) &= \frac{2^\alpha \Gamma\left(\alpha + \frac{3}{2}\right)}{(\alpha + 1) |z_j|^{2(\alpha+1)} \sqrt{\pi}} I_{\{a=w_j, u=\bar{\mu}t\}} + \\ &+ \frac{e^{\frac{z_j(u-\bar{\mu}t)}{\sigma}} |u - \bar{\mu}t|^{\alpha+\frac{3}{2}}}{(\alpha + 1) \sqrt{2\pi} |z_j|^{\alpha+\frac{1}{2}} \sigma^{\alpha+\frac{3}{2}}} \left[\operatorname{sgn}(u - \bar{\mu}t) \mathbf{K}_{\alpha+\frac{1}{2}}\left(\left|\frac{z_j(u - \bar{\mu}t)}{\sigma}\right|\right) + \right. \\ &\left. + \mathbf{K}_{\alpha+\frac{3}{2}}\left(\left|\frac{z_j(u - \bar{\mu}t)}{\sigma}\right|\right) \right] I_{\{a=w_j, u \neq \bar{\mu}t\}}, \end{aligned}$$

$$\begin{aligned} \mathfrak{J}_> \left(\alpha, w_j, z_j, \frac{u - \bar{\mu}t}{\sigma}\right) &= \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{\sqrt{2\pi} (a - w_j)^{\alpha+1}} \left[\frac{\mathbf{B}\left(\frac{1}{2}, \alpha + 1\right)}{\sqrt{2}} + \right. \\ &+ \left. \frac{z_j}{\sqrt{a - w_j}} \mathbf{F}\left(\alpha + \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{z_j^2}{2(w_j - a)}\right) \right] I_{\{a>w_j, u=\bar{\mu}t\}} + \\ &+ \frac{|s_>|^{\alpha+\frac{1}{2}} e^{s_>} (1 + q_>)^{\alpha+1}}{\sqrt{2\pi} (a - w_j)^{\alpha+1}} \left[\mathbf{B}(\alpha + 1, 1) \left(|s_>| \mathbf{K}_{\alpha+\frac{3}{2}}(|s_>|) + s_> \mathbf{K}_{\alpha+\frac{1}{2}}(|s_>|) \right) \right. \\ &\times \Phi\left(\alpha + 1, -\alpha, \alpha + 2; \frac{1 + q_>}{2}, -s_> (1 + q_>)\right) - (1 + q_>) s_> \mathbf{B}(\alpha + 2, 1) \times \\ &\left. \times \mathbf{K}_{\alpha+\frac{1}{2}}(|s_>|) \Phi\left(\alpha + 2, -\alpha, \alpha + 3; \frac{1 + q_>}{2}, -s_> (1 + q_>)\right) \right] I_{\{a>w_j, u \neq \bar{\mu}t\}}, \end{aligned}$$

$$q_> = \frac{z_j}{\sqrt{z_j^2 + 2(a - w_j)}}, \quad s_> = \frac{(u - \bar{\mu}t) \sqrt{z_j^2 + 2(a - w_j)}}{\sigma},$$

$$\begin{aligned} \mathfrak{J}_< \left(\alpha, w_j, z_j, \frac{u - \bar{\mu}t}{\sigma}\right) &= \frac{2^\alpha \Gamma\left(\alpha + \frac{3}{2}\right)}{(\alpha + 1) (z_j^2 - 2(w_j - a))^{\alpha+1} \sqrt{\pi}} \times \\ &\times \mathbf{F}\left(\frac{1}{2}, \alpha + 1, \alpha + 2; \frac{2(w_j - a)}{2(w_j - a) - z_j^2}\right) I_{\{a<w_j, u=\bar{\mu}t\}} + \\ &\frac{(q_< - 1)^{\alpha+1} |s_<|^{\alpha+\frac{1}{2}} e^{s_<}}{(w_j - a)^{\alpha+1} \sqrt{2\pi}} \left[\mathbf{B}(\alpha + 1, 1) \Phi\left(\alpha + 1, -\alpha, \alpha + 2; \frac{1 - q_<}{2}, s_< (1 - q_<)\right) \right. \\ &\times \left(|s_<| \mathbf{K}_{\alpha+\frac{3}{2}}(|s_<|) + s_< \mathbf{K}_{\alpha+\frac{1}{2}}(|s_<|) \right) + (q_< - 1) s_< \mathbf{K}_{\alpha+\frac{1}{2}}(|s_<|) \mathbf{B}(\alpha + 2, 1) \times \\ &\left. \times \Phi\left(\alpha + 2, -\alpha, \alpha + 3; \frac{1 - q_<}{2}, s_< (1 - q_<)\right) \right] I_{\{a<w_j, u \neq \bar{\mu}t\}}, \end{aligned}$$

$$q_{<} = -\frac{z_j}{\sqrt{z_j^2 - 2(w_j - a)}}, \quad s_{<} = \frac{(u - \bar{\mu}t)\sqrt{z_j^2 - 2(w_j - a)}}{\sigma},$$

where $\bar{\mu} \in \{\mu_j, \mu\}$, $j = 1, 2, \dots, m$.

Finally, we get (20) from (19), (39), (46) and (47). \square

Proof of Theorem 2. With respect to (35), the distribution function

$$F_{X_t}(u) = P(X_t \leq u) = V(0, u - \mu t) + \sum_{j=1}^m p_j \left[V(0, u - \mu_j t) - e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} V\left(\frac{\lambda}{\mu - \mu_j}, u - \mu_j t\right) + e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} V\left(\frac{\lambda}{\mu - \mu_j}, u - \mu t\right) \right] I_{\{\mu_j < \mu\}}, \tag{48}$$

where $V(A, D) = \mathbb{E}v(\gamma_t; A, D)$ with

$$v(\gamma_t; A, D) = \Psi\left(-\frac{\theta}{\sigma}\sqrt{\gamma_t} + \frac{D}{\sigma\sqrt{\gamma_t}}\right) I_{\{A=0\}} + e^{\frac{\lambda}{\mu - \mu_j}\left(\frac{\lambda\sigma^2}{2(\mu - \mu_j)} + \theta\right)\gamma_t} \Psi\left(\left(\frac{\lambda\sigma}{\mu_j - \mu} - \frac{\theta}{\sigma}\right)\sqrt{\gamma_t} + \frac{D}{\sigma\sqrt{\gamma_t}}\right) I_{\left\{A = \frac{\lambda}{\mu - \mu_j}\right\}},$$

see (36). We have for $A \in \{0, \frac{\lambda}{\mu - \mu_j}\}$ that

$$V'_u(A, u - \bar{\mu}t) = \mathbb{E}v'_u(\gamma_t; A, u - \bar{\mu}t) = \frac{1}{\sigma\sqrt{2\pi}} \mathbb{E}\gamma_t^{-\frac{1}{2}} e^{-\frac{(u - \bar{\mu}t - \theta\gamma_t)^2}{2\sigma^2\gamma_t}} I_{\{A=0\}} + \frac{1}{\sigma\sqrt{2\pi}} \mathbb{E}\gamma_t^{-\frac{1}{2}} e^{\frac{\lambda}{\mu - \mu_j}\left(\frac{\lambda\sigma^2}{2(\mu - \mu_j)} + \theta\right)\gamma_t} e^{-\frac{(u - \bar{\mu}t - (\theta + \lambda\sigma^2/(\mu - \mu_j))\gamma_t)^2}{2\sigma^2\gamma_t}} I_{\left\{A = \frac{\lambda}{\mu - \mu_j}\right\}}. \tag{49}$$

Since

$$\int_0^\infty x^{\chi_1 - 1} e^{-\frac{\chi_2}{x} - \chi_3 x} dx = 2 \left(\frac{\chi_2}{\chi_3}\right)^{\frac{\chi_1}{2}} K_{\chi_1}(2\sqrt{\chi_2\chi_3})$$

for $\chi_1 \in \mathbb{R}$, $\chi_2 > 0$, $\chi_3 > 0$ with respect to formula 3.471.9 from Gradshteyn and Ryzhik (2007), we get for $u \neq \bar{\mu}t$ that the expectations

$$\begin{aligned} \mathbb{E}\gamma_t^{-\frac{1}{2}} e^{-\frac{(u - \bar{\mu}t - \theta\gamma_t)^2}{2\sigma^2\gamma_t}} &= \frac{a^{at} e^{\frac{\theta(u - \bar{\mu}t)}{\sigma^2}}}{\Gamma(at)} \int_0^\infty x^{at - \frac{3}{2}} e^{-\frac{(u - \bar{\mu}t)^2}{2\sigma^2 x} - \left(a + \frac{\theta^2}{2\sigma^2}\right)x} dx = \\ &= \frac{2a^{at} e^{\frac{\theta(u - \bar{\mu}t)}{\sigma^2}}}{\Gamma(at)} \left(\frac{|u - \bar{\mu}t|}{\sqrt{2a\sigma^2 + \theta^2}}\right)^{at - \frac{1}{2}} K_{at - \frac{1}{2}}\left(\frac{|u - \bar{\mu}t|\sqrt{2a\sigma^2 + \theta^2}}{\sigma^2}\right) \end{aligned} \tag{50}$$

and

$$\begin{aligned} \mathbb{E}\gamma_t^{-\frac{1}{2}} e^{\frac{\lambda}{\mu - \mu_j}\left(\frac{\lambda\sigma^2}{2(\mu - \mu_j)} + \theta\right)\gamma_t} e^{-\frac{(u - \bar{\mu}t - (\theta + \lambda\sigma^2/(\mu - \mu_j))\gamma_t)^2}{2\sigma^2\gamma_t}} &= \\ \frac{a^{at} e^{\frac{(\theta + \lambda\sigma^2/(\mu - \mu_j))(u - \bar{\mu}t)}{\sigma^2}}}{\Gamma(at)} \int_0^\infty x^{at - \frac{3}{2}} e^{-\frac{(u - \bar{\mu}t)^2}{2\sigma^2 x} - \left(a + \frac{\theta^2}{2\sigma^2}\right)x} dx &= \\ \frac{2a^{at} e^{\frac{(\theta + \lambda\sigma^2/(\mu - \mu_j))(u - \bar{\mu}t)}{\sigma^2}}}{\Gamma(at)} \left(\frac{|u - \bar{\mu}t|}{\sqrt{2a\sigma^2 + \theta^2}}\right)^{at - \frac{1}{2}} K_{at - \frac{1}{2}}\left(\frac{|u - \bar{\mu}t|\sqrt{2a\sigma^2 + \theta^2}}{\sigma^2}\right). \end{aligned} \tag{51}$$

Now we have from (48)–(51) that if $u \notin \{\mu_1 t, \mu_2 t, \dots, \mu_m t, \mu t\}$, then the probability density function

$$f_{X_t}(u) = F'_{X_t}(u) = \tilde{\mathfrak{L}}(0, u - \mu t) + \sum_{j=1}^m p_j \left[\tilde{\mathfrak{L}}(0, u - \mu_j t) - e^{-\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \tilde{\mathfrak{L}}\left(\frac{\lambda}{\mu - \mu_j}, u - \mu_j t\right) + e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \tilde{\mathfrak{L}}\left(\frac{\lambda}{\mu - \mu_j}, u - \mu t\right) \right] I_{\{\mu_j < \mu\}}, \tag{52}$$

where

$$\begin{aligned} \tilde{\mathfrak{L}}(u_1, u_2) &= \\ &= \frac{2a^{at} e^{\frac{\theta u_2}{\sigma^2}}}{\sigma \Gamma(at) \sqrt{2\pi}} \left(\frac{|u_2|}{\sqrt{2a\sigma^2 + \theta^2}} \right)^{at - \frac{1}{2}} K_{at - \frac{1}{2}} \left(\frac{|u_2| \sqrt{2a\sigma^2 + \theta^2}}{\sigma^2} \right) I_{\{u_1 = 0\}} + \\ &+ \frac{2a^{at} e^{\frac{(\theta + u_1 \sigma^2) u_2}{\sigma^2}}}{\sigma \Gamma(at) \sqrt{2\pi}} \left(\frac{|u_2|}{\sqrt{2a\sigma^2 + \theta^2}} \right)^{at - \frac{1}{2}} K_{at - \frac{1}{2}} \left(\frac{|u_2| \sqrt{2a\sigma^2 + \theta^2}}{\sigma^2} \right) I_{\{u_1 \neq 0\}} \end{aligned} \tag{53}$$

for $u_2 \neq 0$.

Send $u_2 \rightarrow 0$ in (53). Using the asymptotics

$$K_{\chi_1}(\chi_2) \sim \frac{\Gamma(\chi_1)}{2} \left(\frac{2}{\chi_2} \right)^{\chi_1}, \quad \chi_1 > 0, \text{ as } \chi_2 \rightarrow 0,$$

we get that

$$\begin{aligned} \tilde{\mathfrak{L}}(u_1, u_2) &\sim \\ &\sim \frac{a^{at} \Gamma\left(at - \frac{1}{2}\right)}{\sigma \left(a + \frac{\theta^2}{2\sigma^2}\right)^{at - \frac{1}{2}} \Gamma(at) \sqrt{2\pi}} I_{\{t > \frac{1}{2a}\}} + \infty I_{\{t \leq \frac{1}{2a}\}} \text{ as } u_2 \rightarrow 0. \end{aligned} \tag{54}$$

We have (21) from (8) and (52)–(54). \square

Proof of Theorem 3. We have that

$$\text{LPE}_{X_t}(u) = E\left(X_t I_{\{X_t \leq u\}}\right) = E\left(E\left(X_t I_{\{X_t \leq u\}} \mid Y_t\right)\right)$$

with Y_t defined in (29). Let

$$\bar{g}(Y_t) = E\left(X_t I_{\{X_t \leq u\}} \mid Y_t\right).$$

Then

$$\begin{aligned} \bar{g}(Y_t) &= E\left(\left(\int_0^t \mu(s) ds\right) I_{\{X_t \leq u\}} \mid Y_t\right) + Y_t g(Y_t) = \\ &= \mu t e^{-\lambda t} I_{\{Y_t \leq u - \mu t\}} + E\left((\mu \tau + \hat{\mu}(t - \tau)) I_{\{X_t \leq u, \tau \leq t\}} \mid Y_t\right) + Y_t g(Y_t) = \\ &= \mu t e^{-\lambda t} I_{\{Y_t \leq u - \mu t\}} + Y_t g(Y_t) + p \mu t (1 - e^{-\lambda t}) I_{\{Y_t \leq u - \mu t\}} + \\ &+ t \sum_{j=1}^m p_j \mu_j \left[P\left(\tau \leq \min\left\{t, \frac{u - \mu_j t - Y_t}{\mu - \mu_j}\right\} \mid Y_t\right) I_{\{Y_t \leq u - \mu_j t, \mu_j < \mu\}} + \right. \\ &+ \left. (1 - e^{-\lambda t}) I_{\{Y_t \leq u - \mu t, \mu_j = \mu\}} \right] + \sum_{j=1}^m p_j (\mu - \mu_j) \times \\ &\times E\left(\tau I_{\left\{\tau \leq \min\left\{t, \frac{u - \mu_j t - Y_t}{\mu - \mu_j}\right\}\right\}} \mid Y_t\right) I_{\{Y_t \leq u - \mu_j t, \mu_j < \mu\}}, \end{aligned} \tag{55}$$

where $g(Y_t)$ is assigned by (30). Using (32), we get that

$$\begin{aligned} \bar{g}(Y_t) &= \mu t \left(p + e^{-\lambda t} \sum_{j=1}^m p_j \right) I_{\{Y_t \leq u - \mu t\}} + Y_t g(Y_t) + \\ &+ t \sum_{j=1}^m p_j \mu_j \left[\left(1 - e^{-\lambda t} \right) I_{\{Y_t \leq u - \mu t\}} + \left(1 - e^{\lambda \frac{Y_t + \mu_j t - u}{\mu - \mu_j}} \right) \times \right. \\ &\times I_{\{u - \mu t < Y_t \leq u - \mu_j t, \mu_j < \mu\}} \left. \right] + \sum_{j=1}^m p_j (\mu - \mu_j) \times \\ &\times E \left(\tau I_{\left\{ \tau \leq \min \left\{ t, \frac{u - \mu_j t - Y_t}{\mu - \mu_j} \right\} \right\}} \middle| Y_t \right) I_{\{Y_t \leq u - \mu_j t, \mu_j < \mu\}}. \end{aligned} \tag{56}$$

Next, the conditional expectation

$$\begin{aligned} &E \left(\tau I_{\left\{ \tau \leq \min \left\{ t, \frac{u - \mu_j t - Y_t}{\mu - \mu_j} \right\} \right\}} \middle| Y_t \right) I_{\{Y_t \leq u - \mu_j t\}} = \\ &= E \left(\tau I_{\{\tau \leq t\}} \middle| Y_t \right) I_{\{Y_t \leq u - \mu t\}} + E \left(\tau I_{\left\{ \tau \leq \frac{u - \mu_j t - Y_t}{\mu - \mu_j} \right\}} \middle| Y_t \right) I_{\{u - \mu t < Y_t \leq u - \mu_j t\}} = \\ &= \left(\frac{1}{\lambda} (1 - e^{-\lambda t}) - t e^{-\lambda t} \right) I_{\{Y_t \leq u - \mu t\}} + \left(\frac{1}{\lambda} + \left(\frac{\mu_j t - u}{\mu - \mu_j} - \frac{1}{\lambda} \right) \times \right. \\ &\times e^{\frac{\lambda(Y_t + \mu_j t - u)}{\mu - \mu_j}} + \frac{Y_t}{\mu - \mu_j} e^{\frac{\lambda(Y_t + \mu_j t - u)}{\mu - \mu_j}} \left. \right) I_{\{u - \mu t < Y_t \leq u - \mu_j t\}} \end{aligned} \tag{57}$$

for $\mu_j < \mu$ since

$$\lambda \int_0^\lambda x e^{-\lambda x} dx = -\lambda e^{-\lambda x} + \frac{1}{\lambda} (1 - e^{-\lambda x}).$$

Substituting (57) in (56), we get that

$$\begin{aligned} \bar{g}(Y_t) &= Y_t g(Y_t) + C_1 I_{\{Y_t \leq u - \mu t\}} + \sum_{j=1}^m p_j \left\{ t \mu_j \left(1 - e^{\lambda \frac{Y_t + \mu_j t - u}{\mu - \mu_j}} \right) \times \right. \\ &\times I_{\{u - \mu t < Y_t \leq u - \mu_j t\}} + (\mu - \mu_j) \left[C_2 I_{\{Y_t \leq u - \mu t\}} + \right. \\ &\left. \left. + \left(\frac{1}{\lambda} + \left(C_{3j} + \frac{Y_t}{\mu - \mu_j} \right) e^{\frac{\lambda(Y_t + \mu_j t - u)}{\mu - \mu_j}} \right) I_{\{u - \mu t < Y_t \leq u - \mu_j t\}} \right] \right\} I_{\{\mu_j < \mu\}} \end{aligned} \tag{58}$$

with

$$\begin{aligned} C_1 &= t \left(\mu \left[p + e^{-\lambda t} \sum_{j=1}^m p_j \right] + (1 - e^{-\lambda t}) \sum_{j=1}^m p_j \mu_j \right), \\ C_2 &= \frac{1}{\lambda} (1 - e^{-\lambda t}) - t e^{-\lambda t}, \\ C_{3j} &= \frac{\mu_j t - u}{\mu - \mu_j} - \frac{1}{\lambda}. \end{aligned}$$

It follows from (58) that

$$\begin{aligned} \bar{g}(Y_t) = & Y_t g(Y_t) + C_1 I_{\{Y_t \leq u - \mu t\}} + \sum_{j=1}^m P_j \left\{ \left[(\mu - \mu_j) \left(C_2 - \frac{1}{\lambda} \right) - t\mu_j \right] \times \right. \\ & \times I_{\{Y_t \leq u - \mu t\}} + e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left(t\mu_j - C_{3j}(\mu - \mu_j) \right) e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu t\}} - e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \times \\ & \times Y_t e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu t\}} + \left(t\mu_j + \frac{(\mu - \mu_j)}{\lambda} \right) I_{\{Y_t \leq u - \mu_j t\}} + \\ & + e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left((\mu - \mu_j) C_{3j} - t\mu_j \right) e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu_j t\}} + \\ & \left. + e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} Y_t e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu_j t\}} \right\} I_{\{\mu_j < \mu\}} \end{aligned} \tag{59}$$

and substituting $g(Y_t)$ in (59) from (34), we establish that

$$\begin{aligned} \bar{g}(Y_t) = & Y_t I_{\{Y_t \leq u - \mu t\}} + C_1 I_{\{Y_t \leq u - \mu t\}} + \sum_{j=1}^m P_j \left\{ \left[(\mu - \mu_j) \left(C_2 - \frac{1}{\lambda} \right) - t\mu_j \right] \times \right. \\ & \times I_{\{Y_t \leq u - \mu t\}} + e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left(t\mu_j - C_{3j}(\mu - \mu_j) \right) e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu t\}} + \\ & + \left(t\mu_j + \frac{(\mu - \mu_j)}{\lambda} \right) I_{\{Y_t \leq u - \mu_j t\}} + Y_t I_{\{Y_t \leq u - \mu_j t\}} + \\ & \left. + e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left((\mu - \mu_j) C_{3j} - t\mu_j \right) e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu_j t\}} \right\} I_{\{\mu_j < \mu\}}. \end{aligned}$$

Let

$$\begin{aligned} C_{4j} &= (\mu - \mu_j) \left(C_2 - \frac{1}{\lambda} \right) - t\mu_j, \\ C_{5j} &= e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left(t\mu_j - C_{3j}(\mu - \mu_j) \right), \\ C_{6j} &= t\mu_j + \frac{(\mu - \mu_j)}{\lambda}, \\ C_{7j} &= e^{\frac{\lambda(\mu_j t - u)}{\mu - \mu_j}} \left((\mu - \mu_j) C_{3j} - t\mu_j \right). \end{aligned}$$

Then since $LPE_{X_t}(u) = E\bar{g}(Y_t)$, we have that

$$\begin{aligned} LPE_{X_t}(u) = & E \left(Y_t I_{\{Y_t \leq u - \mu t\}} \right) + C_1 P(Y_t \leq u - \mu t) + \\ & + \sum_{j=1}^m P_j I_{\{\mu_j < \mu\}} \left\{ C_{4j} P(Y_t \leq u - \mu t) + C_{5j} E \left(e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu t\}} \right) + \right. \\ & \left. + C_{6j} P(Y_t \leq u - \mu_j t) + E \left(Y_t I_{\{Y_t \leq u - \mu_j t\}} \right) + C_{7j} E \left(e^{\frac{\lambda Y_t}{\mu - \mu_j}} I_{\{Y_t \leq u - \mu_j t\}} \right) \right\}. \end{aligned}$$

Using (38), we get that

$$\begin{aligned} \frac{\text{LPE}_{X_t}(u)\Gamma(at)}{a^{at}} &= \mathfrak{J}\left(at, 0, -\frac{\theta}{\sigma}, \frac{u - \mu t}{\sigma}\right) + C_1 \mathfrak{J}\left(at - 1, 0, -\frac{\theta}{\sigma}, \frac{u - \mu t}{\sigma}\right) + \\ &+ \sum_{j=1}^m \mathbb{P}_j I_{\{\mu_j < \mu\}} \left\{ C_{4j} \mathfrak{J}\left(at - 1, 0, -\frac{\theta}{\sigma}, \frac{u - \mu t}{\sigma}\right) + \right. \\ &+ C_{5j} \mathfrak{J}\left(at - 1, w_j, z_j, \frac{u - \mu t}{\sigma}\right) + C_{6j} \mathfrak{J}\left(at - 1, 0, -\frac{\theta}{\sigma}, \frac{u - \mu_j t}{\sigma}\right) + \\ &\left. + \mathfrak{J}\left(at, 0, -\frac{\theta}{\sigma}, \frac{u - \mu_j t}{\sigma}\right) + C_{7j} \mathfrak{J}\left(at - 1, w_j, z_j, \frac{u - \mu_j t}{\sigma}\right) \right\}. \end{aligned} \quad (60)$$

Combining together (22), (23)–(27) and (60), we get (28). \square

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: I would like to thank the anonymous referees whose suggestions have improved the economic impact of the work.

Conflicts of Interest: The authors declare no conflict of interest.

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