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Strategy-Proof Allocation of Objects Revisited

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Department of Economics
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Strategy-Proof Allocation of Objects Revisited

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Strategy-Proof Allocation of Objects Revisited

Tommy Andersson and Lars-Gunnar Svensson^{*†}

September 03, 2018

Abstract

We consider an allocation problem with a finite number of objects, and agents that demand at most one of the objects. The study provides a characterization of a class of strategy-proof price mechanisms. A mechanism belongs to the class if and only if the price space is restricted in a special way and, given that restriction, the outcome prices are minimal. The domain of the mechanisms is the set of general preference profiles $(R_a)_{a \in A}$, i.e., where R_a is agent a 's rational, monotonic and continuous preference ordering over objects and prices.

Keywords Characterization, House-allocation, Strategy-proofness, Multi-object auction.

JEL Classification D44, D47, D63, D78, D82

1 Introduction

We consider an allocation problem with a finite number of indivisible objects, and a finite number of agents that demand at most one of the objects. Since an agent's willingness to pay for an object is private information, we analyze mechanisms (allocation rules) that use agents' reported information of preferences to determine an allocation consisting of an assignment of objects to agents in connection with a price vector telling how much an agent has

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to pay (or receive) for the object he is assigned. A mechanism is strategy-proof if for all preferences and all agents it is a dominant strategy to report preferences truthfully. It is well known that strategy-proofness is obtained if the mechanism always chooses an equilibrium allocation with the minimal equilibrium price vector (see e.g., Vickrey (1961), Leonard (1983), Demange and Gale (1986), Sun and Yang (2003) and Andersson and Svensson (2008)).

The present study shows that minimal prices are not only sufficient but also necessary for mechanisms, the range of which are equilibrium allocations, to be strategy-proof. We also show how the set of feasible prices has to be restricted in order to be consistent with strategy-proofness. The domain of the mechanisms is the set of general preference profiles $(R_a)_{a \in A}$, i.e., where R_a is agent a 's rational, monotonic and continuous preference ordering over objects and prices. The proofs of our results are fairly short.

There are papers in the literature that characterize the set of strategy-proof mechanisms under various conditions. A common result is the necessity of minimal prices for mechanisms to be consistent with strategy-proofness, while conditions on the price space are not in general analyzed. Miyake (1998) considers a multi-object model where the price space is defined as all prices above exogenously given lower bounds (the reservation prices). A similar model is considered in Morimoto and Serizawa (2014). Also in that case minimal prices follows from strategy-proofness. However, their result follows without assuming the outcome to be price equilibria. It is rather a consequence of strategy-proofness and some other assumptions, e.g., that the outcomes of the mechanisms are assumed to be efficient allocations and that the number of agents is strictly greater than the number of objects. Svensson (2009) shows in a multi-object model that not only prices have to be minimal but also that price vectors p necessarily have to satisfy a restriction $p \geq \underline{p}$. In that study the number of agents are the same as the number of objects. A more detailed comparison of the various studies is made at the end of Section 4.

The paper is organized as follows. Section 1 is an introduction and Section 2 gives a simple graphical illustration of the model and the problem analyzed, while Section 3 provides the formal model. The main results are derived in Section 4. A dynamic implementation of the results are suggested in Section 5, while Section 6 provides an informal discussion of the relationship between the concept of fairness and our main result.

2 An Introducing Example

In this section we give a graphical presentation of the model and the problem to be analyzed. Consider the situation in Figure 1. There are two agents, called 1 and 2 and there are two objects, also called 1 and 2, the prices of which are p_1 and p_2 . There is an outside option with price $p_0 = 0$. The BB line denotes a lower bound for feasible prices while the remaining lines denote agents' monotonic preferences (R_1, R_2) . The 45-degree lines denote price vectors where agents are indifferent between consuming object 1 and object 2 while at the horizontal and vertical lines the price vectors denote indifference between consuming an object and the outside option. Monotonicity means that an agent strictly prefers a lower price to a higher price on an object.

Given these presumptions, the area A denotes equilibrium price vectors, i.e., both agents can be assigned their most preferred object at an equilibrium price vector. Note that prices may be negative as well as positive. If the objects are e.g. houses prices may be non-negative while if the objects are jobs, the prices may be negative, i.e., $w = -p$ is a vector of wages. In that case the BB line is an upper contour line for a production possibility set.

In the paper price mechanisms, i.e., functions that maps preference profiles on equilibrium allocations, are analyzed. The problem is to find out which conditions on the set of feasible price vectors and which equilibrium allocations are consistent with strategy-proofness. The existence of such mechanisms is well known from the literature but here we are looking for a full characterization of the class of strategy-proof mechanisms in this type of model.

3 The Formal Model

Let $A = \{1, 2, \dots, n\}$ be a finite set of *agents* and $H = \{1, 2, \dots, m\}$ a finite set of *indivisible objects*. There are various interpretations of the objects, e.g., as houses or jobs. In particular, negative prices may be interpreted as positive wages, so in that case the objects may be jobs. In this paper the objects will mainly be *called houses*, while consequences of the main result for the concept of fairness will be discussed shortly in Section 6 in the job interpretation of the model. There is also an outside option, called *null house*, denoted by 0, the copies of which are unlimited. No agent in A owns a house in H . There may be several owners but that will not affect the analysis.

An *assignment* is a mapping $\mu : A \rightarrow H \cup \{0\}$ such that $\mu_a = \mu_{a'}$, for $a, a' \in A$ and $a \neq a'$, only if $\mu_a = 0$. The set of assignments is denoted \mathcal{A} .

Prices on houses are real numbers and a price vector is $p \in \mathbb{R}^{m+1}$, where

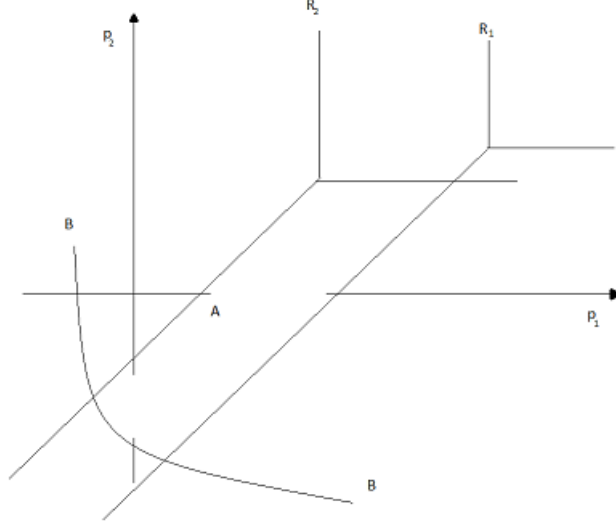


Figure 1:

$p = (p_0, p_1, \dots, p_n)$ and p_h denotes the price on house $h \in H \cup \{0\}$. A set $\mathbb{P} \subset \mathbb{R}^{m+1}$ is a set of *feasible* price vectors if (i) - (iii) below are satisfied:

- (i) For all $p \in \mathbb{P}$, $p_0 = 0$.
- (ii) \mathbb{P} is a *closed* subset of \mathbb{R}^{m+1} and *bounded below*, i.e., there is a lower bound $\underline{p} \in \mathbb{R}^{m+1}$ such that $\mathbb{P} \subset \{p \in \mathbb{R}^{m+1} : p \geq \underline{p}\}$.
- (iii) \mathbb{P} is *monotonic*: if $p \in \mathbb{P}$ and $p' \in \mathbb{R}^{m+1}$, $p' \geq p$ and $p'_0 = 0$, then $p' \in \mathbb{P}$.

Each agent $a \in A$ has *rational* preferences R_a on houses and prices, i.e., on "bundles" like $(h, p_h) \in (H \cup \{0\}) \times \mathbb{R}$. To simplify notation, let $(h, p) \equiv (h, p_h)$, i.e., by (h, p) we mean house h at price p_h in the price vector p .

All houses are (boundedly) *desirable* in the following meaning. To each agent $a \in A$ there is a "price vector" $q^a \in \overline{\mathbb{R}}^{m+1}$, where $\overline{\mathbb{R}}$ is the extended real line, with $q_0^a = 0$ and $q_h^a \in [-\infty, \infty)$ for all $h \in H$, such that for all $h, h' \in H \cup \{0\}$ with $q_h^a, q_{h'}^a \in \mathbb{R}$, $(h, q^a) I_a (h', q^a)$. This means that q^a is an indifference point for preferences R_a . Of course, $q_h^a = -\infty$ means that agent a cannot or will not consume house h at any price (or, in a job interpretation of the model, cannot or will not take job h at any wages).

Preferences are further assumed to be strictly *monotonic*, i.e., for all houses $h \in H$ where $q_h^a \in \mathbb{R}$, $(h, p'_h) P_a (h, p_h)$ if $p'_h < p_h$, and *continuous*, i.e.,

for all $h \in H$, the sets $\{p_h \in \mathbb{R} : (h, p_h) R_a (h, p'_h)\}$ and $\{p_h \in \mathbb{R} : (h, p'_h) R_a (h, p)\}$ are closed for all $p'_h \in \mathbb{R}$.

A (preference) *profile* is a list $R = (R_a)_{a \in A}$ of agents' preferences. The set of profiles is $\mathcal{R} = \times_{a \in A} \mathcal{R}_a$, where agent a 's preferences are in the set \mathcal{R}_a of rational, monotonic, continuous and desirable preferences on $(H \cup \{0\}) \times \mathbb{R}$. A *state* is a pair $x = (\mu, p)$, where μ is an assignment and p a price vector. Here $x_a = (\mu_a, p)$ for $a \in A$. The set of unassigned houses in state x is denoted μ_0 , i.e., $\mu_0 = \{h \in H : \mu_a \neq h \text{ for all } a \in A\} \cup \{0\}$. Since there is an unlimited number of copies of the null house, also $0 \in \mu_0$. The set of states is denoted \mathcal{S} .

Definition 1 For a profile $R \in \mathcal{R}$, a state $x = (\mu, p)$ is a (price) equilibrium if: (i) $p \in \mathbb{P}$, (ii) for all $a \in A$, $x_a R_a (h, p)$ for all $h \in H \cup \{0\}$, and (iii) for all $h \in \mu_0$ and $\varepsilon > 0$, $(p_h - \varepsilon, p_{-h}) \notin \mathbb{P}$. For a given profile $R \in \mathcal{R}$, the set of equilibria is denoted \mathcal{E}_R and the set of corresponding price vectors Π_R .

The definition of equilibrium is the usual one; each agent is assigned his best alternative at the equilibrium price vector according to (ii). This is also an *individual rational* condition since $x_a R_a (0, p)$. Further, prices on unassigned houses are on the lower bound of the price space according to (iii).

The existence of equilibrium in this type of model is well known. See e.g. Alkan and Gale (1990) eller Kaneko and Yamamoto (1986).

Denote by \mathbb{P}^r sets of the type $\mathbb{P}^r = \{h \in H \cup \{0\} : p_h \geq r_h\}$, where $r \in \mathbb{R}^{m+1}$ and $r_0 = 0$. Clearly \mathbb{P}^r satisfies the conditions for a feasible set of price vectors and the vector r is a *lower bound* for \mathbb{P}^r . Note that for feasible sets \mathbb{P}^r of prices, the equilibrium condition (iii) means that $p_h = r_h$ if $h \in \mu_0$. If r is a vector of house owners' reservation prices and house h is not assigned to any agent at equilibrium, then its price is on its lower bound, i.e., $p_h = r_h$.

Definition 2 A mechanism is a mapping $f : \mathcal{R} \rightarrow \mathcal{S}$ of profiles to states. It is a price mechanism if for all $R \in \mathcal{R}$, the outcome $f(R) = (\mu, p)$ is an equilibrium state such that the number $|a \in A : \mu_a = 0|$ is minimal s.t. (i) and (ii) in the definition of an equilibrium.

The requirement of minimal number of assignments in a price mechanism is introduced to reduce the number of utility equivalent assignments. In particular, this is relevant when agents may be indifferent between the null house and a real house. In that case the assignment shall be the real house instead of the null house. Hence, the price mechanism prefers trade to no trade.

Definition 3 A mechanism f is manipulable at a profile $R \in \mathcal{R}$ by an agent $a \in A$ if there is a profile $(R'_a, R_{-a}) \in \mathcal{R}$ such that $x'_a P_a x_a$ for some $x' = f(R'_a, R_{-a})$ and some $x = f(R_a)$. A mechanism f is strategy-proof (SP) if no agent can manipulate at any profile.

4 The Class of Strategy-Proof Price Mechanisms

In this section necessary conditions on a strategy-proof price mechanism are investigated. First, we show that the set of feasible price vectors \mathbb{P} has to be restricted to a set $\mathbb{P}^r = \{p \in \mathbb{R}^{m+1} : p \geq r\}$, where $r \in \mathbb{R}^{m+1}$ and $r_0 = 0$. Clearly \mathbb{P}^r is closed, bounded below and monotonic and, hence, consistent with the definition of a set of feasible prices vectors. Second, we show that for a price mechanism $f(R) = (\mu, p)$ to be strategy-proof the outcome price vector p has to be chosen minimal in the set Π_R of equilibrium price vectors. Finally we conclude that minimal prices and the restricted domain for feasible price vectors also are sufficient for the price mechanism f to be strategy-proof.

Restrictions on the set of feasible price vectors

Theorem 1 Let $f : \mathcal{R} \rightarrow \mathcal{A} \times \mathbb{P}$ be a price mechanism. Then f is manipulable if $\mathbb{P} \neq \mathbb{P}^r$ for all $r \in \mathbb{R}^{m+1}$.

Proof. Suppose that f is strategy-proof and $\mathbb{P} \neq \mathbb{P}^r$ for all $r \in \mathbb{R}^{m+1}$, and let \underline{p} be a lower bound for \mathbb{P} . Then, since $\mathbb{P} \neq \mathbb{P}^{\underline{p}}$, there are two vectors $r', r'' \in \mathbb{P}$, $r' \neq r''$, both minimal in \mathbb{P} . To see this let r' be minimal in \mathbb{P} . There is such a vector since \mathbb{P} is closed and bounded below. Then consider a set $\mathbb{P}^{r'}$. Since $\mathbb{P}^{r'} \subset \mathbb{P}$ and $\mathbb{P}^{r'} \neq \mathbb{P}$, there is a vector $r'' \in \mathbb{P} - \mathbb{P}^{r'}$, and r'' can be chosen minimal in \mathbb{P} . Then r' as well as r'' are minimal in \mathbb{P} and $r' \neq r''$.

The vectors r' and r'' can be chosen so there are houses h_1 and h_2 such that $r'_{h_1} < r''_{h_1}$ and $r'_{h_2} > r''_{h_2}$. W.l.g., we assume that $h_1 = 1$ and $h_2 = 2$. Now define preference profiles $R' = (R'_a)_{a \in A}$ and $R'' = (R''_a)_{a \in A}$, according to:

- Preferences R'_a are quasi-linear and represented by utility functions $u'_{ah}(p)$,
- $u'_{10}(p) = 0$, $u'_{11}(p) = r'_1 - p_1$ and for $h > 1$, $u'_{1h}(p) = -\infty$,
- $u'_{20}(p) = 0$, $u'_{22}(p) = r'_2 - p_2$ and for $h = 1$ and $h > 2$, $u'_{2h}(p) = -\infty$,
- for $a > 2$: $u'_{a0}(p) = 0$ and $u'_{ah}(p) = -\infty$ for all $h \in H$.

Preferences R''_a are defined analogously where u' is replaced with u'' and r' replaced with r'' .

Clearly $x' = (\mu', p') \in \mathcal{E}_{R'}$ and $x'' = (\mu'', p'') \in \mathcal{E}_{R''}$ if $\mu' = \mu''$ and $\mu'_a = a$ for $a \leq 2$ and $\mu'_a = 0$ for $a > 2$. Moreover, $p'_h = r'_h$ for $h = 1, 2$ while for $h > 2$, p' must be chosen minimal in $\{p \in \mathbb{P} : p_h = r'_h \text{ for } h = 1, 2\}$. The price vector p'' is chosen analogously. Note that *all* equilibria in $\mathcal{E}_{R'}$ and $\mathcal{E}_{R''}$ must have this form. For instance, one cannot choose $p'_1 < r'_1$ because then $p' \notin \mathbb{P}$ by the equilibrium condition **Definition 1**(iii), and if $p'_1 > r'_1$ then $1 \in \mu'_0$ which is not consistent with equilibrium (**Definition 1**(iii)). Also $p'_1 = r'_1$ and $\mu'_1 = 0$ is not consistent with the maximal trade requirement of a price mechanism (**Definition 2**). Since all equilibria have this form we also have $f(R') = x'$ and $f(R'') = x''$.

Now consider the profile $R = (R'_1, R'_{-1})$ and let $f(R) = x = (\mu, p)$. Equilibrium also now implies that $\mu_a = a$ for $a \leq 2$ and $\mu_a = 0$ for $a > 2$. Moreover, by strategy-proofness, $p_1 = p'_1 = r'_1$. But then $p = p'$ (not necessarily for $h > 2$). This means that $a = 2$ has R'_2 -utility $u'_{22}(p) = r'_2 - r'_2 = 0$, at state x while, with the same preferences (R'_2), the utility at state x'' becomes $u'_{22}(p) = r'_2 - r''_2 > 0$. Hence, agent 2 can manipulate at state x by change preferences from R'_2 to R''_2 , contradicting strategy-proofness. Note that $R'_a = R''_a$ for $a > 2$. **Q.E.D.**

Minimal prices

According to Theorem 1 the set of feasible price vectors has to be restricted to sets of the type \mathbb{P}^r , for some $r \in \mathbb{R}^{m+1}$ with $r_0 = 0$, in order to have a strategy-proof mechanism. Now we will show that strategy-proofness also requires the output prices to be minimal equilibrium prices. For a profile $R \in \mathcal{R}$ a price vector $p^m \in \Pi_R$ is *minimal* if for any $p \in \Pi_R$, $p \leq p^m$ only if $p = p^m$. The set Π_R of equilibrium prices is closed and bounded below. It is also a lattice¹ if $\mathbb{P} = \mathbb{P}^r$ so, in that case, there is a unique minimal price vector in Π_R . Further,

Definition 4 *A mechanism f is a minimal-price mechanism if f is a price mechanism and $f(R) = (\mu^m, p^m)$, where p^m is the minimal price vector in Π_R .*

Theorem 2 *A price mechanism $f : \mathcal{R} \rightarrow \mathcal{A} \times \mathbb{P}$ is strategy-proof if and only if the set of feasible price vectors is $\mathbb{P} = \mathbb{P}^r$ for some r and f a minimal-price mechanism.*

Proof. By Theorem 1, $\mathbb{P} = \mathbb{P}^r$ for some r is necessary for strategy-proofness. W.l.g., assume that $r_h = 0$ for all $h \in H \cup \{0\}$. Now suppose that f is

¹See e.g., Demange and Gale (1985).

strategy-proof but that there is a profile $R \in \mathcal{R}$ such that $f(R) = x^0 \equiv (\mu^0, p^0)$ and $p^0 \geq p^m$, $p^0 \neq p^m$, where p^m is minimal in Π_R . Then $p_h^0 > p_h^m$ for some $h \in H$, and $h \neq 0$ since $p_0^0 = p_0^m = 0$. W.l.g., assume that $p_1^0 > p_1^m$ and $\mu_1^0 = 1$. Then for $x^m = (\mu^m, p^m) \in \mathcal{E}_R$, $x_1^m P_1 x_1^0$, since $x_1^m R_1(\mu_1, p^m) P_1(\mu_1, p^0)$ by monotonicity. Also let $S = \{h \in H \cup \{0\} : (h, p^m) I_1 x_1^m\}$. This means that $0 \notin S$ and $\mu_1^m \neq 0$.

Let preferences $R_1 \in \mathcal{R}_1$ be represented by utility functions $u_{1h}(p)$, $h \in H \cup \{0\}$, with $u_{10}(p) = 0$. Then define preferences $R'_1 \in \mathcal{R}_1$ by utility functions $u'_{1h}(p) = u_{1h}(p) - \alpha_h$, $h \in H \cup \{0\}$, where the numbers α_h satisfy the following conditions:

1. $\alpha_0 = 0$, so $u'_{10}(p) = u_{10}(p) = 0$. Note that for all $p \in \mathbb{P}$, $p_0 = 0$.
2. For $h \in S$, let $u'_{1h}(p^m) = u_{1h}(p^m) - \alpha_1 > 0$.
3. For $h \in S$, $u'_{1h}(p^0) = u_{1h}(p^0) - \alpha_1 < 0$.
4. For $h \notin S \cup \{0\}$, $u'_{1h}(p^m) = -\infty$.

Note that $x_1^m P_1 x_1^0$, i.e., $u_{1h}(p^m) > u_{1h}(p^0)$ for all $h \in S$, since $p_1^0 > p_1^m$ and $\mu_1^0 = 1$ by assumption. This means that for $h \in S$, there is α_1 such that $u_{1h}(p^0) - \alpha_1 < 0 < u_{1h}(p^m) - \alpha_1$, i.e., conditions 1-3 are consistent.

We first prove that p^m is minimal in $\Pi_{(R'_1, R_{-1})}$. Consider $u'_{1\mu_1^m}(p^m)$. Now for all $h \in S$,

$$u'_{1\mu_1^m}(p^m) = u'_{1h}(p^m),$$

while for $h \notin S$,

$$u'_{1\mu_1^m}(p^m) > 0 \gg u'_{1h}(p^m).$$

Since $x^m = (\mu^m, p^m) \in \mathcal{E}_R$, it now follows that $x^m \in \mathcal{E}_{(R'_1, R_{-1})}$ and, hence, that $p^m \in \Pi_{(R'_1, R_{-1})}$.

To prove that p^m also is minimal in $\Pi_{(R'_1, R_{-1})}$, assume there is $p' \in \Pi_{(R'_1, R_{-1})}$ such that $p' \leq p^m$ and $p' \neq p^m$. Let $x' = (\mu', p') \in \mathcal{E}_{(R'_1, R_{-1})}$ and let

$$H' = \{h \in H : p'_h < p_h^m\} \text{ and } H'' = H - H'.$$

Then $H' \neq \emptyset$ by assumption. Moreover, by the definition of equilibrium, $H' \cap \mu_0^m = \emptyset$ ($h \in \mu_0^m \implies p_h^m = 0$, and hence, $p'_h \geq p_h^m$) and by monotonicity, $\mu'_a \in H'$ if $\mu_a^m \in H'$. But then by monotonicity, for $a \in A$ with $\mu'_a \in H''$, $x_a^m P_a(h, p^m)$ for all $h \in H'$. Then, according to Alkan, Demange and Gale (1991), prices p_h^m , $h \in H'$, can be decreased such that there is an equilibrium $(\mu'', p'') \in \mathcal{E}_R$ with $p'' \leq p^m$ and $p'' \neq p^m$. This is a contradiction to p^m being the minimal price vector in Π_R . Hence, p^m is minimal also in $\Pi_{(R'_1, R_{-1})}$.

Now let $x^1 = (\mu^1, p^1) = f(R'_1, R_{-1})$ and, w.l.g., $\mu^1_1 = h_1$. $x^1 \in \mathcal{E}_{(R'_1, R_{-1})}$ so $x^1_1 R'_1(0, p^1)$, i.e., $u'_{1h_1}(p^1) \geq 0$. We first show that $h_1 = 0$.

If $h_1 \in S$ then $h_1 \neq 0$ and $u'_{1h_1}(p^0) < 0$ by point 3 above. Moreover, by equilibrium $u'_{1h_1}(p^1) \geq u'_{10}(p^1) = 0$, so $u'_{1h_1}(p^1) > u'_{1h'}(p^0)$, and $a = 1$ can manipulate at the state x^0 , contradicting strategy-proofness. On the other hand, if $h_1 \notin S$, then $u'_{10}(p^1) = 0$, and $u'_{1h_1}(p^1) = -\infty$ if $h_1 \notin S \cup \{0\}$ by point 4 above. Hence, $h_1 = 0$ is the only possible case.

Now we have $\mu^1_1 = 0$ and $\mu^m_1 \neq 0$. Let $H^1 = \{h \in H : p^1_h > p^m_h\}$. Then, $\mu^1_0 \cap H^1 = \emptyset$. Moreover, $\mu^1_a \in H^1$ implies that $\mu^m_a \in H^1$ by monotonicity. But then, since $\mu^1_0 \cap H^1 = \emptyset$ and $\mu^1_1 = 0$ and $\mu^m_1 \neq 0$, there are more agents assigned a house in H^1 at x^0 than at x^1 . This is not consistent with equilibrium. Hence, $p^0 = p^m$ must be the case.

Finally, a minimal-price mechanism is strategy-proof, see e.g., Leonard (1983), Demange and Gale (1985) or Andersson and Svensson (2008). **Q.E.D.**

Remark. Some of the results in Theorem 2 can be found in the literature. Svensson (2009, 2004) obtains the restriction on feasible prices (wages) as well as the necessity of minimal prices (wages) in a simplified version of the model in this paper; e.g., preferences are quasi-linear, no outside option and $n = m$, i.e., the same number of agents and objects. Miyake (1998) considers a model that is logically similar to ours with a domain for the mechanism somewhat smaller than ours. However, a main difference is that the set of feasible price vectors is exogenously given to be of type \mathbb{P}^r , $r_h \geq 0$. As in the present study, the characterization problem in Miyake assumes a price mechanism. That is not the case in Morimoto and Serizawa (2015) but they assume only that the range of the mechanism is efficient states. Given this weaker assumption, the minimal prices follows from the strategy-proof condition. Compared to the present study the domain for the mechanism is somewhat smaller and $n > m$, i.e., the number of agents are greater than the number of objects. The method of proofs in the mentioned studies is also different from the method used in the present study.

5 Dynamic Implementation

The minimal-price mechanism is a direct mechanism where agents are asked to report their entire preference relations. In Andersson and Svensson (2018) a dynamic mechanism is constructed and analyzed where only partial preferences are required to reach the minimal-price equilibrium. A simplified version of that mechanism can also be applied here and a short description of the mechanism is as follows. W.l.g. we assume that feasible prices are the

vectors in $\mathbb{P} = \{p \in \mathbb{R}_+^{m+1} : p_0 = 0\}$.

The price space is partitioned by a grid where each box is of type $S_\alpha \subset \mathbb{P}$ satisfying²:

$$\begin{aligned} \delta\mathbb{N}^m &= \{\alpha \in \mathbb{R}_+^m : \alpha_j = \delta k_j \text{ for some } k_j \in \mathbb{N}\}, \\ \text{for each } \alpha &\in \delta\mathbb{N}^m, S_\alpha = (0, 0, \dots, 0) \times (\times_{h \in H} [\alpha_h, \alpha_h + \delta)). \end{aligned}$$

The vector α is the minimal corner in a box and $\delta > 0$ defines the size of the boxes. The outcome of the dynamic mechanism is a finite increasing sequence $(p^t)_{t=1}^T$ of price vectors, called an *English Price Sequence*, and a corresponding finite increasing sequence of boxes, where no price is increased more than δ . In each box only prices on over-demanded houses are raised. Agents' demands are required only in the various boxes in the sequence, and the sequence of boxes ends when the unique minimal-price equilibrium is achieved. The sequences are formally defined in the following way.

Let $a \in A$ and $p \in \mathbb{P}$, and denote by $d_a^p \subset H \cup \{0\}$ the (reported) *demand set* at the price vector p , i.e.,

$$d_a^p = \{h \in H \cup \{0\} : (h, p) R_a (h', p) \text{ for all } h' \in H \cup \{0\}\}.$$

A set $H' \subset H$ of houses is *over-demanded* if $|a \in A : d_a^p \subset H'| > |H'|$ ³ and a *minimal over-demanded set* if there is no over-demanded set $H'' \subset H'$, $H'' \neq H'$.

In each box S_α , $x = (\mu, p)$ is a *temporary state* if $p \in S_\alpha$ and $\mu_a \in d_a^p$. Given S_α and a temporary state $x = (\mu, p)$, $p \in S_\alpha$, a *price regime* $\Pi^x \subset S_\alpha$ is defined according to:

$p' \in \Pi^x$ iff there is a temporary state $x' = (\mu', p')$, with $p' \in S_\alpha$, such that $p'_h = p_h$ if h is not in a minimal over-demanded set at p and for such an h , $\mu'_a = \mu_a = h$.

Let $x = (\mu, p)$ be a temporary state and let $\xi(x) = \sup \Pi^x$. Note that $\xi(x)$ need not be a singleton. Now we can recursively define a price sequence in the following way.

Definition 5 *Given the partition $\{S_\alpha : \alpha \in \delta\mathbb{N}^m\}$ of the set of feasible price vectors, a sequence $(p^t)_{t=1}^T$ of price vectors constitutes an English Price Sequence (EPS) if there is a sequence $(x^t)_{t=1}^T$ of supporting temporary states, with $x^t = (\mu^t, p^t)$, such that $p^{t+1} \in \xi(x^t)$. The starting point is p^1 with $p_h^1 = 0$ for all $h \in H \cup \{0\}$. The EPS terminates at step T if $p^T \neq p^{T-1}$ and $p^{T+1} = p^T$.*

² $\mathbb{N} = \{0, 1, 2, \dots\}$

³ $|S|$ denotes the number of elements in a set S .

Note that there can be several price changes in one and the same box, so the corresponding sequence of boxes can have fewer steps than T . It can be proved that $T < \infty$ and that the end-point p^T is the unique minimal-price vector, while the sequence $(p^t)_{t=1}^T$ is not necessarily unique. Let $(S_{\alpha^j})_{j=1}^{T'}$, $T' \leq T$, be the sequence of boxes containing a price vector from the EPS $(p^t)_{t=1}^T$. Then the measure of the set $\cup_j S_{\alpha^j} \rightarrow 0$ as $\delta \rightarrow 0$, so the part of the price space where agents recursively report their demand can be arbitrarily small by choosing δ small. For proofs of the results in this paragraph, see Andersson and Svensson (2018). The EPS can be seen as the outcome of an auction rule defined as follows.

The Iterative English Auction Rule. Initialize the price vector to p^1 . For each Step $t := 1, \dots, T$:

1. Each agent $a \in A$ reports his demand set $d_a^{p^t}$ at prices p^t .
2. Calculate a supporting temporary state $x^t = (\mu^t, p^t)$.
3. Define a small price regime Π^{x^t} and calculate $p^{t+1} \in \xi(x^t)$.
4. If $p^{t+1} = p^t$, stop. Otherwise, set $t := t + 1$ and continue.

If the reported demand sets d_a^p in the Iterative Auction Rule are consistent with rational preferences $R \in \mathcal{R}$, i.e., a reported demand set d_a^p can be derived from some preference ordering $R_a \in \mathcal{R}_a$, then bidding truthfully is an ex post Nash equilibrium.

6 Fair Wages

Any definition of fairness is faced with two fundamental problems. It is the uniqueness problem and it is the implementation problem. For instance, if fairness requires an allocation to be envy-free there are in general many allocations satisfying this condition.

If we consider the job interpretation of our model, Theorem 2 may be useful in analyzing those two problems. Let H be a set of various jobs, and negative prices be wages, e.g., $w_j = -p_j$, $j \in H$. Feasible wages are $\mathbb{W} \subset \mathbb{R}^{m+1}$, where $w \in \mathbb{W}$ if and only if $w = -p$ for some $p \in \mathbb{P}^r$. Suppose now that necessary for fairness is a state $x = (\mu, w)$ with no envy, i.e., $x_a R_a x_{a'}$ for all agents $a, a' \in A$. Since the calculation of such a state requires agents' private information, and that information can be obtained by using a strategy-proof mechanism, Theorem 2 shows that the degree of freedom in choosing a fairness criterion is reduced to the choice of a feasible set \mathbb{W}

of wages of the type, $\mathbb{W}^r = -\mathbb{P}^r \subset -\mathbb{P}$, where $-\mathbb{P}$ is the exogenously given set of production possibilities. Hence, the choice for the mechanism designer is one choice of a feasible wage structure r which determines \mathbb{W}^r , and then there is only one mechanism that solves the uniqueness problem and the implementation problem, and that is the maximum-wage mechanism.

Here the vector r may be interpreted as a vector of reservation wages of the employers. Such a choice of mechanism results in a non-manipulable rule for the employees, while the employers may manipulate by their choice of r .

References

- Alkan, A., Demange, G., Gale, D. (1991) Fair allocation of indivisible objects and criteria of justice, *Econometrica* 59:1023-1039.
- Andersson, T. and Svensson, L.-G. (2018) Sequential Rules for House Allocation with Price Restrictions, *Games and Economic Behavior* 127:42-59.
- Andersson, T. and Svensson, L.-G. (2008) Non-manipulable assignment of individuals to positions revisited, *Mathematical Social Sciences* 56:350-354.
- Demange, G. and Gale, D. (1985) The strategy structure of two-sided matching markets. *Econometrica* 53:873-888.
- Leonard, H. (1983) Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy* 91:461-479.
- Miyake, M. (1998) On the incentive properties of multi-item auctions. *International Journal of Game Theory* 27:1-19.
- Morimoto, S. and Serizawa, S. (2015) Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule. *Theoretical Economics* 10(2):445-487.
- Mukherjee C. (2014) Fair and group strategy-proof good allocation with money. *Social Choice and Welfare* 42:289-311.
- Sakai, T. (2008) Second price auctions on general preference domains: two characterizations. *Economic Theory* 37:347-356.
- Sakai, T. (2013) An equity characterization of second price auctions when preferences may not be quasi-linear. *Review of Economic Design* 17:17-26.

- Sun, N. and Yang, Z. (2003) A general strategy proof fair allocation mechanism. *Economic Letters* 81:939-954.
- Svensson, L.-G. (2004) Strategy-proof and fair wages. WP 2004:8, Department of Economics Lund University.
- Svensson, L.-G. (2009) Coalitional strategy-proofness and fairness. *Economic Theory* 40:227-245.
- Vickrey, W. (1961) Counterspeculation, auction, and competitive sealed tenders. *Journal of Finance* 16:8-37.