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Dynamic Consistency and Ambiguous Communication*

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Abstract

We analyze dynamically consistent behavior for games with strategically ambiguous communication by introducing rectangularity. Gains from ambiguous communication persist if players behave dynamically consistently. We extend the setting of Beauchêne, Li, and Li (2019) and find a Perfect Bayesian Equilibrium which yields the same outcomes as in the original paper. In this Perfect Bayesian Equilibrium, the Sender can benefit from ambiguous persuasion even if the Receiver behaves dynamically consistently. Furthermore, we show how to apply rectangularity to settings with strategically ambiguous communication in cheap talk and mechanism design.

Keywords and phrases: Bayesian Persuasion, Ambiguity Aversion, Dynamic Consistency
JEL subject classification: C73, D81, D83

1 Introduction

The present paper studies the role of ambiguous communication and its strategic use for dynamically consistent and ambiguity averse players. It has been shown that one or even all players can benefit from ambiguous communication in mechanism design (Bose and Renou (2014)), cheap talk (Kellner and Le Quement (2018)), and persuasion (Beauchêne et al. (2019)).

However, it is well known that ambiguity averse preference may lead to dynamically inconsistent behavior. Roughly speaking, new information can lead to a change in the worst-case belief and, therefore, change the optimal strategy. Dynamically inconsistent behavior makes it impossible to use standard equilibrium concepts such as Perfect Bayesian Nash Equilibrium or Sequential Equilibrium. The most common approach in the literature on ambiguous communication to deal with dynamically inconsistent behavior is the

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consistent planning approach of Siniscalchi (2011). Consistent planning is based on backward induction and assumes that agents behave dynamically inconsistent but consider the preferences of future selves at the ex-ante stage.

We contribute to the literature on ambiguous communication by extending the results from Bose and Renou (2014), Beauchêne et al. (2019) and Kellner and Le Quement (2018) to dynamically consistent players. Thus, we show that the gain of ambiguous communication can be maintained if players behave dynamically consistently. Second, we define rectangularity and Perfect Bayesian equilibria in settings with strategic ambiguity.

Rectangularity is a condition on the belief set that leads to dynamically consistent behavior for a fixed information structure. Intuitively, rectangularity allows agents to take the information structure into account. Epstein and Schneider (2003) and Sarin and Wakker (1998) define rectangularity for decision theoretical settings with a fixed information structure. Pahlke (2021) generalizes the concept of rectangularity to multi-stage games with ambiguity about states but non-ambiguous strategies. She demonstrates that rectangular beliefs ensure the existence of a sequential equilibrium. However, ambiguous beliefs arise in these settings due to exogenous ambiguity about states or types. In the present paper, ambiguous beliefs arise due to ambiguous strategies. To our knowledge, Muraviev et al. (2017) is the only work analyzing rectangularity for strategic ambiguity. However, they do not define an equilibrium concept.

To analyze the role of rectangularity for ambiguous communication, we focus on the ambiguous persuasion setting of Beauchêne, Li, and Li (2019), henceforth BLL. However, the same methods and technics can be applied analogously for ambiguous communication in mechanism design and cheap talk. We discuss the settings of Bose and Renou (2014) and Kellner and Le Quement (2018) in more detail in Section 4.

BLL introduce ambiguity in the standard Bayesian persuasion setting of Kamenica and Gentzkow (2011) by allowing the Sender to choose a set of communication devices. Each communication device can generate a message that reveals information about an unknown (risky) state $\omega \in \Omega$. Sender and Receiver only observe one message without knowing which communication device generated the message. Therefore, ambiguity about the communication device induces ambiguity about the risky state ω . However, they claim that there is no gain of ambiguous persuasion compared to Bayesian persuasion if the players behave dynamically consistently.¹

First, we show that we can restrict the message set to messages that send recommended actions or synonyms of recommended actions without loss of generality. A synonym m' of a message m is a message that induces the same posterior belief or best response of the Receiver as the message m . This result generalizes Proposition 1 of Kamenica and Gentzkow (2011), which states that one can restrict the message set, without loss of generality, to straightforward messages to the ambiguous setting.

Then, we define beliefs over a more general state space of straightforward messages and states. The general state space allows for rectangular ambiguous beliefs. These beliefs take the dependence of the ambiguous signal and the ex-ante risky state into account and allow

¹See Proposition 5 of Beauchêne et al. (2019).

for a non-singleton ex-ante belief set. Given these beliefs, the optimal interim strategy of the Receiver in BLL is ex-ante optimal and, therefore, dynamically consistent. Hence, ambiguous persuasion can generate a higher value for the Sender even under dynamically consistent behavior.

Concurrent with this work, Cheng (2021) analyzes dynamic consistency for ambiguous persuasion. However, instead of rectangularity, he uses the updating rules of Hanany and Klibanoff (2007). Roughly speaking, these updating rules imply dynamically consistent behavior by assuming that players only update beliefs consistent with the ex-ante worst-case belief. Therefore, the ex-ante optimal choice of a player becomes interim optimal. Cheng (2021) shows that players using the updating rules of Hanany and Klibanoff (2007) can not gain from ambiguous persuasion. We discuss the relation between our work and Cheng (2021) in more detail in Section 5.

The paper is organized as follows: In Section 2, we formulate the ambiguous persuasion model and give an example that illustrates the gain of an ambiguous strategy and the dynamically inconsistent behavior. Further, Section 2.3 generalizes Proposition 1 of Kamenica and Gentzkow (2011) and defines rectangular beliefs. In Section 3 we define Perfect Bayesian Equilibria under rectangular beliefs and generalize the results of BLL. In Section 4 we discuss the models of Bose and Renou (2014) and Kellner and Le Que-ment (2018) and show how our analysis can be applied in their settings. Finally, Section 5 concludes and discusses related literature in more detail.

2 Model

The basic setting follows the model of BLL, which extends the standard Bayesian persuasion setting of Kamenica and Gentzkow (2011) by an ambiguous communication device.

2.1 Setting

As the standard Bayesian persuasion, an ambiguous persuasion game consists of a Sender (he) and a Receiver (she). The utility of both players depends on the state $\omega \in \Omega$ and action $a \in A$ chosen by the Receiver, where Ω and A are compact subsets of the Euclidean space. We denote with $u(a, \omega)$ and $v(a, \omega)$ the utility of Receiver and Sender, respectively. Further, Sender and Receiver have maxmin preferences á la Gilboa and Schmeidler (1989), i.e., they maximize their worst case expected utility.

Ex-ante, the state ω is unknown, and both players have the same prior state belief $p_0 \in \Delta\Omega$, i.e., ex-ante exists no ambiguity about the state.² The Sender tries to persuade the Receiver by choosing a signal that reveals information about the state. A signal consists of a finite set of signal realizations or messages M and a set of communication devices

²Our definition of belief differs from the one of BLL. To avoid confusion, we use the term state belief whenever we refer to beliefs in the sense of BLL.

$\Pi = \{\pi_k\}_{k \in K}$.³ Each communication device is a distribution over the set of messages M for each $\omega \in \Omega$, i.e., $\pi_k(\cdot|\omega) \in \Delta M$ for all $\omega \in \Omega$. As BLL, we assume that the π_k 's have common support for all $k \in K$.

Thus, the only difference to the standard Bayesian persuasion setting is that the Sender chooses a set of communication devices instead of one communication device. Which of the communication devices generates the observed message is ambiguous to both players. After observing a message m , the Receiver updates her prior state belief prior-by-prior using Bayes' rule. Since she does not know which communication device generated the message, she updates p_0 with respect to each communication device π_k . This leads to the following set of posterior state beliefs after observing the message $m \in M$

$$P_m = \left\{ p_m^{\pi_k}(\cdot) \in \Delta \Omega : p_m^{\pi_k}(\cdot) = \frac{p_0(\cdot)\pi_k(m|\cdot)}{\int_{\Omega} p_0(\omega)\pi_k(m|\omega) d\omega}, \pi_k \in \Pi \right\}. \quad (1)$$

Thus, after observing message m , the Receiver maximizes her interim worst-case expected utility

$$U(a, P_m) = \min_{p_m \in P_m} \mathbb{E}_{p_m}(u(a, \omega)). \quad (2)$$

As usual in the persuasion literature, we assume that the Receiver chooses the Sender preferred action if she has multiple maximizers. We denote with \hat{a}_m the (Sender preferred) best response of the Receiver after observing the message m .

The Sender chooses the signal (M, Π) that maximizes his ex-ante worst case expected utility

$$\sup_{(M, \Pi)} \min_{\pi \in \Pi} \mathbb{E}_{p_0} \left[\mathbb{E}_{\pi} [\nu(\hat{a}_m, \omega) | \omega] \right].$$

2.2 Dynamic Inconsistency

BLL analyze a consistent planning equilibrium, i.e., they assume that the Receiver maximizes her interim worst-case expected utility for each $m \in M$ and the Sender maximizes her ex-ante worst-case expected utility. Since the Sender only chooses an action at the ex-ante stage, he can never behave dynamically inconsistent. However, the interim best response of the Receiver is, in general, not ex-ante optimal. Intuitively, ex-ante the Receiver can hedge against ambiguity by playing a constant strategy. Thus, the Receiver may behave dynamically inconsistent in a consistent planning equilibrium.

The following example illustrates that ambiguity can lead to a higher expected payoff for the Sender. Furthermore, we show that the interim equilibrium strategy of the Receiver is not ex-ante ante optimal.

³Please note that we deviate from the model of BLL by defining Π as the set of communication devices. BLL define Π as the convex hull of the set of communication devices. Since Sender and Receiver have maxmin preferences, the minimization problems over $\{\pi_k\}$ or $\text{co}(\{\pi_k\})$ coincide.

Example 1. Assume that the Sender is a brand-name drug producer. The Receiver is a physician who can choose between prescribing the brand name drug ($a = a_B$) or a generic competitor of it ($a = a_G$). The Sender always prefers that the Receiver prescribes the brand-name drug. The Receiver's preferences depend on the state, which reflects the effectiveness of the generic drug. If the generic drug is effective ($\omega = \omega_e$), the Receiver prefers the generic drug; if not ($\omega = \omega_i$) she prefers the brand name drug. The payoffs of Sender and Receiver are given in the following table.

	ω_e	ω_i
a_B	(1, 2)	(1, 2)
a_G	(0, 3)	(0, -1)

Figure 1: Payoffs (S, R)

Sender and Receiver have a common ex-ante state belief $p_0 = \mathbb{P}(\omega = \omega_i) < \frac{1}{4}$.⁴ BLL show that the optimal Bayesian persuasion signal is such that the set of messages consists of two messages $M = \{i, e\}$ and the communication device is given by

$$\begin{aligned}\pi(e|\omega_e) &= \frac{1 - 4p_0}{1 - p_0} = 1 - \pi(i|\omega_e), \\ \pi(e|\omega_i) &= 0 = 1 - \pi(i|\omega_i).\end{aligned}$$

Then, the ex-ante expected payoff of the Sender given the optimal Bayesian persuasion is $\mathbb{P}(m = i) \cdot 1 + \mathbb{P}(m = e) \cdot 0 = 4p_0 < 1$.

Furthermore, BLL construct an ambiguous persuasion signal that leads to a higher expected payoff of the Sender. Let $M = \{e, i\}$ as before. The set of communication devices $\Pi = \{\pi, \pi'\}$ is given by a communication device that always reveals the true state and a communication device that does the opposite, i.e.,

$$\begin{aligned}\pi(i|\omega_i) &= 1 = 1 - \pi(e|\omega_i), & \pi(i|\omega_e) &= 0 = 1 - \pi(e|\omega_e), \\ \pi'(i|\omega_i) &= 0 = 1 - \pi'(e|\omega_i), & \pi'(i|\omega_e) &= 1 = 1 - \pi'(e|\omega_e).\end{aligned}$$

Given this ambiguous communication device the set of interim state beliefs are $P_m = \{(0, 1), (1, 0)\}$ for $m \in \{e, i\}$. Due to the maxmin preferences, the interim worst-case belief for both messages always gives probability one to the state ω_i . Therefore, the Receiver chooses the brand name drug with probability one. Then, the ex-ante expected payoff of the Sender is one, which is greater than the ex-ante expected payoff given the optimal Bayesian persuasion.

⁴Please note, that for simplicity we deviate from the Illustrating Example of BLL (page 317) by assuming $u_H = 3$, $u_L = -1$ and $c = 1$, which is consistent with the payoffs in Example 2 of BLL.

However, the ex-ante expected payoff of the Receiver is given by

$$\min_{\pi \in \Pi} \sum_{m \in \{e, i\}} (\pi(m|\omega_e) + \pi(m|\omega_i)) \mathbb{E}_{p_m^\pi} (u(a_m, \omega))$$

where a_m denotes her action after observing the messages m . If she chooses the brand name drug independently of the signal that she will observe, her ex-ante expected payoff equals

$$2 \cdot \mathbb{P}(\omega = \omega_e) + 2 \cdot \mathbb{P}(\omega = \omega_i) = 2.$$

Her expected payoff if she always choose the generic drug is

$$3 \cdot \mathbb{P}(\omega = \omega_e) + 1 \cdot \mathbb{P}(\omega = \omega_i) = 3 - 4p_0.$$

Since $p_0 < \frac{1}{4}$, the optimal interim strategy of always prescribing the brand name drug is not ex-ante optimal, and the Receiver behaves dynamically inconsistently.

2.3 Dynamically Consistent Beliefs

In our model, ambiguity arises due to the ambiguous communication device. Ambiguous interim beliefs only occur due to the combination of a risky state and an ambiguous signal. Consider the following two situations at the ex-ante stage:

- 1) The Receiver does not observe any message. All information about the state $\omega \in \Omega$ is represented by p_0 .
- 2) As in situation 1) the Receiver knows p_0 . Additionally, she knows that she will receive an ambiguous message before making her decision.

In the first situation, the Receiver knows that there will be no additional information. Hence, she chooses her optimal action, given the expected utility with respect to p_0 . In the second situation, ex-ante the Receiver has the same information about the state as in situation 1). However, she knows that she will receive some additional but ambiguous information before deciding. Furthermore, she knows that this ambiguous information influences her interim beliefs and, therefore, her best response. A rational player should consider this knowledge about a game's information structure when deciding at the ex-ante stage. Rectangularity takes the interplay of the prior state belief p_0 and the knowledge about the information structure into account and, therefore, ensures dynamically consistent behavior.

We show that defining beliefs over a more general state space allows the definition of non-singleton rectangular belief sets. Then, given a rectangular belief set, the Receiver behaves dynamically consistently, and the consistent planning equilibrium of BLL is an ex-ante and interim optimal and, therefore, a Perfect Bayesian Equilibrium.

2.3.1 Straightforward Messages

In the ambiguous persuasion setting, the set of messages M is part of the Sender's strategy. In the Bayesian persuasion setting, Kamenica and Gentzkow (2011) call a signal straightforward if $M \subseteq A$. They show that one can restrict to straightforward signals in a Bayesian persuasion setting without loss of generality. The next proposition generalizes this result to our ambiguous persuasion setting. It shows that the Sender chooses without loss of generality $M \subseteq A \cup \tilde{A}$ where \tilde{A} is a duplicated set of A such that there exists a bijection $b(\cdot)$ between A and \tilde{A} . Given this result, we can define rectangular ex-ante beliefs over $\Omega \times (A \cup \tilde{A})$.

Proposition 1. *Let $(M, \Pi) \in \operatorname{argsup} \min_{\pi \in \Pi} \mathbb{E}_{p_0} [\mathbb{E}_{\pi} [\nu(\hat{a}_m, \omega) | \omega]]$. Let \tilde{A} be such that there exist a bijection $b(\cdot) : A \rightarrow \tilde{A}$ between A and \tilde{A} . Then, there exist (M', Π') with $M' \subseteq A \cup \tilde{A}$ and $\Pi' = \{\pi'_1, \pi'_2\}$ such that (M', Π') generates the same value for the Sender as (M, Π) .*

The intuition of the result is as follows. Kamenica and Gentzkow (2011) show that for Bayesian persuasion it is, without loss of generality, that $M \subseteq A$. BLL show that ambiguous persuasion increases the value for the Sender compared to Bayesian persuasion only if the Sender uses a signal with synonyms. Synonyms are messages that copy the meaning of another message, i.e., they induce the same posterior state belief set or best response of the Receiver. Intuitively, the Sender uses synonyms to hedge himself against ambiguity.

Furthermore, they show that for any ambiguous signal, one can find an ambiguous signal which consists only of two communication devices and leads to the same value. Hence, we have to duplicate the message space to allow for synonyms, and duplication is enough to generate the same value as any ambiguous signal. Thus, without loss of generality, we can assume that $M \subseteq A \cup \tilde{A}$. The detailed proof can be found in Section A in the Appendix.

Due to the assumption that all π_k have common full support on M , a strategy of the Sender (M, Π) is completely characterized by Π . For the rest of the paper, we will use the term strategy of the Sender for such a Π . Furthermore, we denote with $\operatorname{supp}(\Pi) = \operatorname{supp}(\pi_k(\cdot | \omega)) \subset A \cup \tilde{A}$ the support of $\pi_k \in \Pi$ for all $k \in K$.

2.3.2 Rectangular Beliefs

Given the results from the previous section, we can define beliefs over the general state space $\Omega \times (A \cup \tilde{A})$. Defining beliefs over this general state space allows the Receiver to form a joint belief about the risky state $\omega \in \Omega$ and the message $m \in M$, i.e., the Receiver forms beliefs of the events "the state is ω , and I observe message m ." Then, the probability of this event depends on the risky state $\omega \in \Omega$ and the ambiguous communication device that generates the message.

Definition 1. *For a strategy Π of the Sender we define the **set of ex-ante beliefs** of the Receiver as*

$$\Phi_{\Pi}^0 = \left\{ \rho^k \in \Delta(\Omega \times (A \cup \tilde{A})) : \exists \pi_k \in \Pi \text{ s.t.} \right.$$

$$\rho^k(\omega, m) = \left\{ \begin{array}{ll} p_0(\omega)\pi_k(m|\omega) & \text{if } m \in \text{supp}(\Pi) \\ 0 & \text{otherwise} \end{array} \right\}.$$

Note that the strategy of the Sender generates the information structure of the persuasion games. Therefore, it has to influence the joint belief over states and messages, and Φ_{Π}^0 depends on Π .

At the interim stage the Receiver observes a message $m \in \text{supp}(\Pi)$. The information structure at the ex-ante stage ($t = 0$) and interim stage ($t = 1$) can be represented by the following partitions

$$\begin{aligned} \mathcal{F}_0 &= \Omega \times (A \cup b(A)), \\ \mathcal{F}_1 &= \left\{ \{\Omega \times m\}_{m \in A \cup b(A)} \right\}. \end{aligned}$$

Then, given an observation $\hat{m} \in \text{supp}(\Pi)$ the Receiver updates her ex-ante belief set prior-by-prior using Bayes' formula, i.e., she updates each prior belief in Φ_{Π}^0 with Bayes' formula

$$\rho^k|_{\hat{m}} = \rho^k((\omega, m)|\hat{m}) = \frac{p_0(\omega)\pi_k(m|\omega)}{\int_{\Omega} p_0(\omega')\pi_k(m|\omega') d\omega'}$$

if $m = \hat{m}$ and zero otherwise. Then, the set of updated beliefs given $\hat{m} \in \text{supp}(\Pi)$ is

$$\text{Bay}(\Phi_{\Pi}^0|\hat{m}) = \{\rho^k|_{\hat{m}} \text{ with } \rho^k \in \Phi_{\Pi}^0\}.$$

Remark 1. Note that $\rho^k((\omega, m)|\hat{m}) = 0$ for $\hat{m} \notin \text{supp}(\Pi)$. Furthermore, $\rho^k((\omega, \hat{m})|\hat{m}) = p_{\hat{m}}^{\pi^k}(\omega)$ as defined in Equation (1) for all ω . Therefore, Receivers maximization problem at the interim stage given our definition of beliefs coincides with the maximization problem of BLL of Equation (2).

To define rectangularity let us first look at the case without ambiguity, i.e., if $\Pi = \{\pi\}$ and $\Phi_{\Pi}^0 = \{\rho\}$ is singleton. After observing message m the updated belief is given by $\rho|_m$. Furthermore, the marginal beliefs of observing $m \in A \cup \tilde{A}$ under ρ is

$$\rho(\Omega, m) = \int_{\Omega} \rho(\omega, m) d\omega = \int_{\Omega} p_0(\omega)\pi(m|\omega) d\omega.$$

Then, the structure of Bayes' formula implies that multiplying the updated belief after observing message m with the marginal probability of observing m leads to the prior belief restricted to the events that the message is m . This holds for all messages m and therefore for all information sets of the partition defined above. Hence, integrating over all $m \in \text{supp}(\Pi)$ leads to the prior belief ρ

$$\rho(\omega, m) = \int_{\text{supp}(\Pi)} \rho(\Omega, m')\rho|_{m'}(\omega, m) dm'.$$

Now, we generalize these considerations to an ambiguous setting, i.e., Π is not a singleton. Rectangularity requires that any combination of marginal belief and updated belief is a prior belief that the agent considers possible. The Receiver knows which messages she could receive and, therefore, which updated beliefs potentially exist. Taking this knowledge into account, rectangularity requires that any combination of marginal and updated belief is an element of the ex-ante belief set.

Definition 2. *The pasting of an ex-ante belief $\bar{\rho} \in \Phi_{\Pi}^0$ and a collection of updated beliefs $(\rho|_{\hat{m}})_{\hat{m}} \in \times_{\hat{m} \in \text{supp}(\Pi)} \text{Bay}(\Phi_{\Pi}^0|\hat{m})$ is defined as⁵*

$$\begin{aligned} \bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\omega, m) &:= \int_{\text{supp}(\Pi)} \bar{\rho}(\Omega, \hat{m}) \rho(\omega, m|\hat{m}) \, d\hat{m} \\ &= \left(\int_{\Omega} p_0(\omega') \bar{\pi}(m|\omega') \, d\omega' \right) \frac{p_0(\omega) \pi(m|\omega)}{\int_{\Omega} p_0(\omega') \pi(m|\omega') \, d\omega'}. \end{aligned}$$

The set of ex-ante beliefs is called **rectangular** (or *stable under pasting*) if it contains all pastings of an ex-ante belief $\bar{\rho} \in \Phi_{\Pi}^0$ and interim beliefs $(\rho|_{\hat{m}})_{\hat{m}}$, i.e.,

$$\bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\cdot) \in \Phi_{\Pi}^0$$

for all $\bar{\rho} \in \Phi_{\Pi}^0$ and $(\rho|_{\hat{m}})_{\hat{m}} \in \times_{\hat{m} \in \text{supp}(\Pi)} \text{Bay}(\Phi_{\Pi}^0|\hat{m})$.

If Φ_{Π}^0 is not rectangular, one can always construct the smallest set, which is rectangular and contains Φ_{Π}^0 by backward induction. We call this set the rectangular hull and denote it with $\text{rect}(\Phi_{\Pi}^0)$. Simple calculations show that $\text{Bay}(\Phi_{\Pi}^0|\hat{m}) = \text{Bay}(\text{rect}(\Phi_{\Pi}^0)|\hat{m})$. The same holds for the set of marginal beliefs under Φ_{Π}^0 and $\text{rect}(\Phi_{\Pi}^0)$. For a more detailed explanation of the construction and the properties of the rectangular hull, please see Pahlke (2021) or Epstein and Schneider (2003).

So far, we focused on the beliefs of the Receiver. The Sender only chooses an action at the ex-ante stage. Therefore, the interim beliefs of the Sender do not influence the equilibria of the game. If the Sender does not know which communication device generated the message, his interim and ex-ante belief sets and the rectangular hull coincide with the Receiver's beliefs. However, even if $\Phi_{\Pi}^0 \subsetneq \text{rect}(\Phi_{\Pi}^0)$, the marginal beliefs of observing message m are the same for Φ_{Π}^0 and $\text{rect}(\Phi_{\Pi}^0)$. Thus, the ex-ante maximization problem of the Sender given $\text{rect}(\Phi_{\Pi}^0)$ is the same as given Φ_{Π}^0 .

Alternatively, we could define an information structure of the Sender that does not influence the ex-ante decision of the Sender but ensures that the ex-ante belief set of the Sender is rectangular for any Π . For example, the Sender could observe which communication device generated the observed message at the interim stage. If the Sender learns which communication device generated the message, Φ_{Π}^0 is rectangular for all Π . By definition, rectangularity depends on the information structure faced by a player. Therefore,

⁵Please note, that the pasting is always well defined due to the common support assumption. Furthermore, the second equality follows since $\rho(\omega, m|\hat{m}) = 0$ if $m \neq \hat{m}$.

assuming heterogeneous information structures for Sender and Receiver would induce heterogeneous rectangular hulls. However, heterogeneous rectangular beliefs only arise due to heterogeneous information structures. Pahlke (2021) discusses the relation between information structures and common rectangular beliefs in more detail. However, the present paper aims to find a belief formation process that ensures dynamically consistent behavior. Since the Sender can never behave dynamically inconsistent, we do not go into details.

3 Dynamic Consistency and Perfect Bayesian Equilibrium

Finally, we show that rectangularity implies dynamically consistent behavior of the Receiver and, therefore, the existence of a Perfect Bayesian equilibrium.

Definition 3. *A Perfect Bayesian equilibrium with rectangular beliefs consists of a strategy Π^* of the Sender, a strategy $(\hat{a}_m)_{m \in M}$ of the Receiver and a belief system Ψ for each player. Strategies and belief systems have to satisfy the following conditions:*

- *The belief systems of both players consist of an ex-ante belief set Ψ_i^0 and interim belief set Ψ_i^m for each message $m \in A \cup \tilde{A}$ such that*

$$\begin{aligned}\Psi_R^0 &= \text{rect}(\Phi_{\Pi^*}^0) \\ \Psi_S^0 &= \Phi_{\Pi^*}^0.\end{aligned}$$

Furthermore, the interim belief sets are derived by Bayes rule whenever possible, i.e., $\Psi_i^m = \text{Bay}(\Psi_i^0|m)$ for all $m \in \text{supp}(\Pi^)$.*

- *The equilibrium strategy of the Sender Π^* with $\text{supp}(\Pi^*) \subseteq A \cup \tilde{A}$ maximizes his ex-ante worst case expected utility*

$$\min_{\rho \in \Psi_S^0} \mathbb{E}_\rho [\nu(\hat{a}_m, \omega)].$$

- *The equilibrium strategy of the Receiver maximizes her interim worst case expected utility for all $m \in \text{supp}(\Pi^*)$*

$$\min_{\rho|m \in \Psi_R^m} \mathbb{E}_{\rho|m} (u(a_m, \omega))$$

and her ex-ante worst case expected utility given the ex-ante belief set Ψ_R^0

$$\min_{\rho \in \Psi_R^0} \mathbb{E}_\rho (u(a_m, \omega)).$$

The following proposition shows that we can generalize any consistent planning equilibrium of BLL to a Perfect Bayesian equilibrium using rectangularity.

Proposition 2. *Let (M, Π) be the optimal ex-ante choice of the Sender and $(\hat{a}_m)_{m \in M}$ the optimal interim choice of the Receiver as in BLL. Then, there exists (M^*, Π^*) , with $M^* \subseteq A \cup \tilde{A}$ and $|\Pi^*| = 2$ that generate the same value of the Sender as (M, Π) . Furthermore, Π^* , $(\hat{a}_m)_{m \in M^*}$ and $\Psi_R^0 = \text{rect}(\Phi_{\Pi^*}^0)$, $\Psi_S^0 = \Phi_{\Pi^*}^0$ and $(\Psi_i^m)_{m \in M^*} = (\text{Bay}(\Psi_i^0 | m))_{m \in M^*}$ are a Perfect Bayesian equilibrium with rectangular beliefs.*

Proof. First, due to Proposition 1, there exists (M^*, Π^*) , with $M^* \subseteq A \cup \tilde{A}$ and $|\Pi^*| = 2$ that generate the same value of the Sender as (M, Π) . The proof of Proposition 1 shows that the Receiver chooses the same action given M or M^* in the sense that any two messages $m, m' \in M$ that are not synonyms of each other but induce the same optimal strategy, i.e., $\hat{a}_m = \hat{a}_{m'}$, are replaced by the same message $\bar{m} \in M^*$. Therefore, even if the message sets M and M^* are different, the Receiver's played actions do not change and $(\hat{a}_m)_{m \in M^*}$ is induced by $(\hat{a}_m)_{m \in M}$.

Furthermore, the Sender never behaves dynamically inconsistent. We only have to show that the Receiver's interim best response of BLL is an interim and ex-ante best response given rectangular beliefs. Remember that $p_{\hat{m}}^{\pi^k}(\cdot) = \rho^k((\cdot, \hat{m}) | \hat{m})$ for all $\hat{m} \in \text{supp}(\Pi)$ and that the set of Bayesian updates given Φ_{Π}^0 or $\text{rect}(\Phi_{\Pi}^0)$ are the same. Therefore, the interim best response given the state beliefs of BLL is an interim best response given rectangular beliefs, as well. Furthermore, we can rewrite the ex-ante expected utility of the Receiver as

$$\min_{\rho \in \text{rect}(\Phi_{\Pi^*}^0)} \int_{\text{supp}(\Pi)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho | \hat{m}}(u(a_{\hat{m}}, \omega)) d\hat{m},$$

where $\rho |_{\hat{m}}$ is the Bayesian update of ρ given message \hat{m} . We first show the following relation of ex-ante and interim worst case expected utility. Let ρ^* denote the ex-ante worst case belief given rectangular beliefs. Then,

$$\begin{aligned} & \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho^* | \hat{m}}(u(a_{\hat{m}}, \omega)) d\hat{m} \\ &= \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho |_{\hat{m}} \in \text{Bay}(\text{rect}(\Phi_{\Pi^*}^0) | \hat{m})} \mathbb{E}_{\rho | \hat{m}}(u(a_{\hat{m}}, \omega)) d\hat{m}. \end{aligned} \quad (3)$$

To prove Equation 3 we first show that the left hand side is greater equal than the right hand side.

$$\begin{aligned} & \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \underbrace{\mathbb{E}_{\rho^* | \hat{m}}(u(a_{\hat{m}}, \omega))}_{\geq \min_{\rho |_{\hat{m}} \in \text{Bay}(\text{rect}(\Phi_{\Pi^*}^0) | \hat{m})} \mathbb{E}_{\rho | \hat{m}}(u(a_{\hat{m}}, \omega))} d\hat{m} \\ & \geq \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho |_{\hat{m}} \in \text{Bay}(\text{rect}(\Phi_{\Pi^*}^0) | \hat{m})} \mathbb{E}_{\rho | \hat{m}}(u(a_{\hat{m}}, \omega)) d\hat{m}. \end{aligned}$$

To prove the other direction, let $\rho' |_{\hat{m}}$ be the worst case belief given that she observed \hat{m} . Then, due to rectangularity, there exist $\bar{\rho} \in \text{rect}(\Phi_{\Pi^*}^0)$ such that $\rho^* \circ (\rho' |_{\hat{m}})_{\hat{m}} = \bar{\rho}$.

Furthermore rectangularity implies, that $\bar{\rho}(\cdot|\hat{m}) = \rho'(\cdot|\hat{m})$ and $\bar{\rho}(\Omega, \hat{m}) = \rho^*(\Omega, \hat{m})$ for all \hat{m} . Then,

$$\begin{aligned} \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho^*|\hat{m}}(u(a_{\hat{m}}, \omega)) \, d\hat{m} &\leq \int_{\text{supp}(\Pi^*)} \bar{\rho}(\Omega, \hat{m}) \mathbb{E}_{\bar{\rho}|\hat{m}}(u(a_{\hat{m}}, \omega)) \, d\hat{m} \\ &= \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \mathbb{E}_{\rho'|\hat{m}}(u(a_{\hat{m}}, \omega)) \, d\hat{m} \\ &= \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho|\hat{m} \in \text{Bay}(\text{rect}(\Phi_{\Pi^*}^0)|\hat{m})} \mathbb{E}_{\rho|\hat{m}}(u(a_{\hat{m}}, \omega)) \, d\hat{m}. \end{aligned}$$

Combining both directions proves Equation 3. Finally, we show that an interim best response of the Receiver is an ex-ante best response, as well. We denote with $\hat{a}_{\hat{m}}$ the (Sender preferred) interim best response of the Receiver given message \hat{m} , i.e.,

$$\min_{\rho|\hat{m} \in \text{Bay}(\Phi_{\Pi^*}^0|\hat{m})} \mathbb{E}_{\rho|\hat{m}}(u(\hat{a}_{\hat{m}}, \omega)) \geq \min_{\rho|\hat{m} \in \text{Bay}(\Phi_{\Pi^*}^0|\hat{m})} \mathbb{E}_{\rho|\hat{m}}(u(a_{\hat{m}}, \omega))$$

for any arbitrary $a_{\hat{m}} \in A$ and all $\hat{m} \in \text{supp}(\Pi^*)$. We have to show that $(\hat{a}_{\hat{m}})_{\hat{m} \in \text{supp}(\Pi^*)}$ is ex-ante optimal. Since $\rho(\Omega, \hat{m}) \geq 0$ for all $\hat{m} \in \text{supp}(\Pi^*)$ and $\rho(\Omega, \hat{m}) = 0$ for all $\hat{m} \notin \text{supp}(\Pi^*)$, Equation 3 implies

$$\begin{aligned} \min_{\rho \in \text{rect}(\Phi_{\Pi^*}^0)} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho|\hat{m}}(u(a_{\hat{m}}, \omega)) \, d\hat{m} \\ &= \min_{\rho \in \text{rect}(\Phi_{\Pi^*}^0)} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \min_{\rho'|\hat{m} \in \text{Bay}(\Phi_{\Pi^*}^0|\hat{m})} \mathbb{E}_{\rho'|\hat{m}}(u(a_{\hat{m}}, \omega)) \, d\hat{m} \\ &\leq \min_{\rho \in \text{rect}(\Phi_{\Pi^*}^0)} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \min_{\rho'|\hat{m} \in \text{Bay}(\Phi_{\Pi^*}^0|\hat{m})} \mathbb{E}_{\rho'|\hat{m}}(u(\hat{a}_{\hat{m}}, \omega)) \, d\hat{m} \\ &= \min_{\rho \in \text{rect}(\Phi_{\Pi^*}^0)} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho|\hat{m}}(u(\hat{a}_{\hat{m}}, \omega)) \, d\hat{m} \end{aligned}$$

for any arbitrary $(a_{\hat{m}})_{\hat{m} \in \text{supp}(\Pi)}$. Here the inequality follows from the interim optimality of $(\hat{a}_{\hat{m}})_{\hat{m} \in \text{supp}(\Pi^*)}$ and the last equality from Equation 3.

Hence, Receivers ex-ante best response equals the interim best response, and the interim equilibrium of Beauchêne, Li, and Li (2019) satisfies ex-ante optimality. \blacksquare

To illustrate the previous results, we come back to our Example from Section 2.2.

Example 2 (Example 1 cont.). *Remember that the optimal ambiguous communication device was given by $\Pi = \{\pi, \pi'\}$ with*

$$\begin{aligned} \pi(i|\omega_i) &= 1 = 1 - \pi(e|\omega_i), & \pi(i|\omega_e) &= 0 = 1 - \pi(e|\omega_e), \\ \pi'(i|\omega_i) &= 0 = 1 - \pi'(e|\omega_i), & \pi'(i|\omega_e) &= 1 = 1 - \pi'(e|\omega_e). \end{aligned}$$

Then, the set of ex-ante beliefs of the Receiver is $\Phi_{\Pi}^0 = \{\rho, \rho'\}$ with

$$\rho(\omega, m) = \begin{cases} p_0 & \text{if } m = i, \omega = \omega_i, \\ 1 - p_0 & \text{if } m = e, \omega = \omega_e, \\ 0 & \text{otherwise,} \end{cases} \quad \rho'(\omega, m) = \begin{cases} p_0 & \text{if } m = e, \omega = \omega_i, \\ 1 - p_0 & \text{if } m = i, \omega = \omega_e, \\ 0 & \text{otherwise.} \end{cases}$$

To construct the rectangular hull, we need to calculate all interim and marginal beliefs:

$$\begin{aligned} \rho(\omega, m|i) &= \begin{cases} 1 & \text{if } m = i, \omega = \omega_i, \\ 0 & \text{otherwise,} \end{cases} & \rho(\omega, m|e) &= \begin{cases} 1 & \text{if } m = e, \omega = \omega_e, \\ 0 & \text{otherwise,} \end{cases} \\ \rho'(\omega, m|i) &= \begin{cases} 1 & \text{if } m = i, \omega = \omega_e, \\ 0 & \text{otherwise,} \end{cases} & \rho'(\omega, m|e) &= \begin{cases} 1 & \text{if } m = e, \omega = \omega_i, \\ 0 & \text{otherwise,} \end{cases} \\ & \text{marg}(\rho(\cdot, i)) = p_0, & \text{marg}(\rho(\cdot, e)) &= 1 - p_0, \\ & \text{marg}(\rho'(\cdot, e)) = p_0, & \text{marg}(\rho'(\cdot, i)) &= 1 - p_0. \end{aligned}$$

By combining any marginal and interim belief we obtain the rectangular hull $\text{rect}(\Phi_{\Pi}^0) = \{\rho, \rho', \hat{\rho}, \bar{\rho}\}$ where ρ and ρ' are as before and

$$\bar{\rho}(\omega, m) = \begin{cases} 1 - p_0 & \text{if } m = i, \omega = \omega_i, \\ p_0 & \text{if } m = e, \omega = \omega_e, \\ 0 & \text{otherwise,} \end{cases} \quad \hat{\rho}(\omega, m) = \begin{cases} 1 - p_0 & \text{if } m = e, \omega = \omega_i, \\ p_0 & \text{if } m = i, \omega = \omega_e, \\ 0 & \text{otherwise.} \end{cases}$$

Given the rectangular hull, the worst-case belief of the Receiver if she plans to choose the generic drug after message m is $\mathbb{P}(\omega = \omega_i, m) = 1 - p_0 > \frac{3}{4}$. Therefore, always prescribing the brand name drug is ex-ante optimal, and the Receiver behaves dynamically consistently.

4 Further Models with Ambiguous Communication

A similar approach to define rectangularity can be used in various models with ambiguous communication, e.g., in cheap talk or mechanism design. The main task is to define an adequate general state space and generalize beliefs to the general state space. To illustrate the general applicability of our results, we discuss the settings of Bose and Renou (2014) and Kellner and Le Quement (2018) in more detail.

4.1 Ambiguous Mechanism Design

Bose and Renou (2014) analyze a mechanism design setting with ambiguous communication. In their setting, there is a finite set of players N , a finite set of payoff-relevant types Θ_i for each player $i \in N$ and a finite set of alternatives X . Types are privately known and there exists no exogenous ambiguity about types of opponents, i.e., ex-ante the distribu-

tion of types θ_{-i} is given by a singleton $p_i \in \Delta(\Theta_{-i})$ for player i . Players have maxmin preferences and update beliefs prior-by-prior.

Bose and Renou (2014) study the class of social choice functions $f : \times_{i \in N} \Theta_i \rightarrow X$ that is implementable by an ambiguous mechanism. An ambiguous mechanism consists of two steps: The second step, called the allocation mechanism is a usual static mechanism specifying a finite set of messages M_i for each player and an allocation rule $g : \times_{i \in N} M_i \rightarrow X$. The first step adds an ambiguous communication device before the allocation mechanism is played. An ambiguous communication device consists of a finite set of messages that player i can send to the communication device $\hat{\Omega}_i$, a finite set of messages that player i can receive from the communication device Ω_i , and a set of probability systems Λ . The set of probability systems Λ corresponds to the set of communication devices Π in our setting. Hence, each $\lambda \in \Lambda$ specifies the probability that a profile of messages ω is received by the players given that they send the profile $\hat{\omega}$ to the communication device, i.e., $\lambda : \hat{\Omega} \rightarrow \Delta(\Omega)$, where $\hat{\Omega} = \times_{i \in N} \hat{\Omega}_i$ and $\Omega = \times_{i \in N} \Omega_i$.

They define a consistent planning equilibrium, i.e., players may behave dynamically inconsistent. However, similar to ambiguous persuasion, all these results can be extended to dynamically consistent players, if players have rectangular beliefs. Here, the general state space is given by $\Theta_{-i} \times \hat{\Omega} \times \Omega$.⁶ Given an ambiguous communication device, the set of ex-ante beliefs of a type θ_i is

$$\Phi_{\Lambda}^0 = \left\{ \phi \in \Delta(\Theta_{-i} \times \hat{\Omega} \times \Omega) : \exists \lambda \in \Lambda \text{ s.t. } \phi(\theta_{-i}, \hat{\omega}, \omega) = \lambda(\hat{\omega})[\omega] p_i[\theta_{-i}] \right\}.$$

Now, we can define rectangularity analogously to Definition 2 and extend all results from Bose and Renou (2014) to dynamically consistent players. Thus, if a social choice function is implementable by an ambiguous mechanism of Bose and Renou (2014) for dynamically inconsistent players, then the same social choice function can be implemented by the same ambiguous mechanism for dynamically consistent players with rectangular beliefs.

4.2 Ambiguous Cheap Talk

Kellner and Le Quement (2018) study a cheap talk setting with ambiguous communication. They prove that an ambiguous strategy of the Sender can lead to a pareto improvement compared to the standard non-ambiguous cheap talk. Their setting is based on the standard non-ambiguous cheap talk setting of Crawford and Sobel (1982). The game consists of two players, a Sender and a Receiver. The Sender has private information about a risky payoff-relevant state $\omega \in \Omega = [0, 1]$ and an ambiguous payoff-irrelevant state $\theta \in \Theta$. An Ellsbergian communication strategy is a standard communication strategy $q_{\theta}(\cdot | \omega) \in \Delta(M)$ for each $\theta \in \Theta$, where M is a finite message space. A strategy of the Receiver is a mapping $M \rightarrow \Delta(\mathbb{R})$. The Receiver's interim belief set is derived by updating the prior state belief

⁶Note, that $\hat{\Omega}$ and Ω are specified by the mechanism and not part of the strategy of the players. Therefore, an analog to Proposition 1 is not needed here.

p on Ω with respect to each communication strategy $q_\theta(\cdot|\omega)$.

As in the ambiguous persuasion setting, the equilibrium strategy of the Receiver is not ex-ante optimal. However, similarly to the procedure described above, defining beliefs and rectangularity over the general state space $\Omega \times \Theta$ leads to a Perfect Bayesian equilibrium with the same strategies as in the interim equilibrium of Kellner and Le Quement (2018). Thus, ambiguous cheap talk can lead to a pareto improvement compared to the standard non-ambiguous cheap talk even if players behave dynamically consistently.

5 Discussion

We study dynamically consistent behavior in an ambiguous persuasion setting. First, we show that restricting the message set to straightforward messages and synonyms is without loss of generality. Given this result, we can define beliefs over the more general state space $\Omega \times A \cup \tilde{A}$. This state space allows for the dependence of the risky state and ambiguous signals. Therefore, the Receiver can consider the ambiguous information structure at the ex-ante stage. Then, rectangular beliefs ensure dynamically consistent behavior in ambiguous persuasion and the existence of a Perfect Bayesian equilibrium. Thus, ambiguity induces new equilibria in persuasion settings, even if the players behave dynamically consistently. To conclude, we discuss some related issues and literature.

Cheng (2021) Concurrent with our work, Cheng (2021) shows that the Sender cannot gain from ambiguous persuasion if the Receiver can commit to his ex-ante optimal choice. The same results can be achieved without commitment if the Receiver uses the updating rule of Hanany and Klibanoff (2007). Even if rectangularity and Hanany and Klibanoff (2007) both lead to dynamically consistent behavior, they may induce different equilibria. The updating rules of Hanany and Klibanoff (2007) restrict the interim belief set to beliefs that maintain the ex-ante optimal choice interim optimal. In contrast, rectangularity enlarges the ex-ante belief set such that the interim optimal choice becomes ex-ante optimal.

Our example can illustrate the difference between the approaches. Using the updating rules of Hanany and Klibanoff (2007), the ex-ante belief set of the Receiver is Φ_{Π}^0 . Then, ex-ante, she would prefer always to choose the generic drug. After updating Φ_{Π}^0 with the updating rules of Hanany and Klibanoff (2007), always choosing the generic drug is still interim optimal. Hence, given the updating rules of Hanany and Klibanoff (2007), the dynamically consistent Receiver would always choose the generic drug, and the Sender cannot benefit from ambiguous persuasion. Given rectangularity, the Receiver's ex-ante belief set is given by the rectangular hull $\text{rect}(\Phi_{\Pi}^0)$ and always choosing the brand name drug is ex-ante and interim optimal. Hence, a dynamically consistent Receiver with rectangular beliefs will always choose the brand name drug, and the Sender can gain from ambiguous persuasion.

Even if both approaches imply dynamically consistent behavior, the interpretation is different. A Receiver using the updating rules of Hanany and Klibanoff (2007) commits to her ex-optimal choice and ignores any information that would change her worst-case

belief. On the other hand, a Receiver with rectangular beliefs considers that she will receive ambiguous information before deciding. Thus, the results of Cheng (2021) do not contradict our result. Further, if $\Phi_{\Pi}^0 = \text{rect}(\Phi_{\Pi}^0)$ both approaches induce the same equilibria.

Value of Information Our work is related to the literature on the negative value of ambiguous information. In our example, the Receiver would prefer to ignore the ambiguous information. Her ex-ante expected utility given p_0 is higher than her ex-ante expected utility given the rectangular hull. Thus, the Receiver can have a negative value of information.

Ambiguous information induces two effects. On the one hand, an ambiguous communication device generates ambiguous beliefs and, therefore, decreases the worst-case expected utility of the Receiver. On the other hand, the communication device still reveals information about the state. The Receiver’s value of information is negative if the first effect dominates the second effect. We discuss the (negative) value of information for ambiguous persuasion in greater detail in Section B in the Appendix.

Li (2020) studies the relation of ambiguity aversion and an aversion of (partial) information. He shows that an ambiguity averse decision maker (DM) with maxmin preferences is always (weakly) averse to partial information. However, Li (2020) assumes that the DM’s set of acts is the same with and without ambiguous information. In contrast, our setting implies that given p_0 , the DM can only choose from constant acts. Given an ambiguous communication device, the DM can choose any act which is measurable with respect to the information partition induced by the communication device. These are precisely the two effects we describe above. On the one hand, an ambiguous information device induces ambiguity, which decreases the utility of an ambiguity averse Receiver. On the other hand, anticipating this information at the ex-ante stage allows the Receiver to choose an action for each message that could occur with positive probability. Li (2020) focuses only on the first effect. Therefore, his result about partial information aversion of maxmin preferences does not imply that the value of information of the Receiver is always negative.⁷

References

- Dorian Beauchêne, Jian Li, and Ming Li. Ambiguous persuasion. *Journal of Economic Theory*, 179:312–365, 2019.
- Subir Bose and Ludovic Renou. Mechanism design with ambiguous communication devices. *Econometrica*, 82(5):1853–1872, 2014.
- Xiaoyu Cheng. Ambiguous persuasion: An ex-ante perspective. mimeo, 2021.

⁷The same consideration apply to the cheap talk setting of Kellner and Le Quement (2018). Therefore, the result of Li (2020) does not contradict the pareto improvement result of Kellner and Le Quement (2018).

- Vincent P Crawford and Joel Sobel. Strategic information transmission. *Econometrica: Journal of the Econometric Society*, pages 1431–1451, 1982.
- Larry G. Epstein and Martin Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31, 2003.
- Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.
- Eran Hanany and Peter Klibanoff. Updating preferences with multiple priors. *Theoretical Economics*, 2:261–298, 2007.
- Brian Hill. Dynamic consistency and ambiguity: A reappraisal. *Games and Economic Behaviour*, 120:289–310, 2020.
- Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.
- Christian Kellner and Mark T Le Quement. Endogenous ambiguity in cheap talk. *Journal of Economic Theory*, 173:1–17, 2018.
- Jian Li. Preferences for partial information and ambiguity. *Theoretical Economics*, 15(3):1059–1094, 2020.
- Igor Muraviev, Frank Riedel, and Linda Sass. Kuhn’s theorem for extensive form ellsberg games. *Journal of Mathematical Economics*, 28:26–41, 2017.
- Marieke Pahlke. Dynamic consistency in incomplete information games with multiple priors. mimeo, 2021.
- Rakesh Sarin and Peter P Wakker. Dynamic choice and nonexpected utility. *Journal of Risk and Uncertainty*, 17(2):87–120, 1998.
- Marciano Siniscalchi. Dynamic choice under ambiguity. *Theoretical Economics*, 6(3):379–421, 2011.

A Proofs

Proof of Proposition 1. Corollary 1 of BLL shows that there exists π_1 and π_2 such that $(M, \{\pi_1, \pi_2\})$ generates the same value as (M, Π) . Hence, we have to show that (M', Π') generates the same value as $(M, \{\pi_1, \pi_2\})$. We first look at the case where the Sender does not use synonyms.

i) Sender does not use synonyms.

Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there does not exist $m, m' \in M$ with $m \neq m'$ such that $\hat{a}_m = \hat{a}_{m'}$. Remember, that p_m^π denotes the posterior state belief of the Receiver given the message m and the communication device π . Furthermore, \hat{a}_m denotes Receivers best response given message $m \in M$ and the communication devices $\{\pi_1, \pi_2\}$. Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there exists at most one $m \in M$ for each $a \in A$ such that $a = \hat{a}_m$. We define $\bar{\pi}_i(\cdot|\omega) \in \Delta M'$ with $M' \subset A$ such that

$$\bar{\pi}_i(a|\omega) = \begin{cases} \pi_i(m|\omega) & \text{if } \exists m \in M \text{ with } a = \hat{a}_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the posterior state belief $p_m^{\bar{\pi}_i}$ equals the posterior state belief $p_a^{\bar{\pi}_i}$ if $a = \hat{a}_m$. Therefore, $(M, \{\pi_1, \pi_2\})$ and $(M', \{\bar{\pi}_1, \bar{\pi}_2\})$ generate the same set of posterior state beliefs and the same best response of the Receiver. Since the best response does not change, the value of the Sender is the same for both signals.

ii) Sender uses synonyms.

If $(M, \{\pi_1, \pi_2\})$ uses synonyms we can split M in M_1 and M_2 such that there exist a bijection between M_1 and M_2 and $M_1 \cup M_2 = M$. Then $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$ with

$$\hat{\pi}_i(m|\omega) = \frac{\pi_i(m|\omega)}{\sum_{m \in M_1} \pi_i(m|\omega)}$$

defines a signal that does not use synonyms. Thus, as in Case i), there exists $(M'_1, \{\bar{\pi}_1, \bar{\pi}_2\})$ with $M'_1 \subset A$ that generates the same value as $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$. Similar one can define the restriction of π_i to M_2 and find $(M'_2, \{\tilde{\pi}_1, \tilde{\pi}_1\})$ with $M'_2 \subset \tilde{A}$, that generates the same value as M_2 and the restriction of π_i to M_2 . Then, $(M', \{\pi'_1, \pi'_2\})$ with $M' = M'_1 \cup M'_2$ and

$$\pi'_i(a|\omega) = \begin{cases} \bar{\pi}_i(a|\omega) \sum_{m \in M_1} \pi_i(m|\omega) & \text{if } a \in A, \\ \tilde{\pi}_i(a|\omega) \sum_{m \in M_2} \pi_i(m|\omega) & \text{if } a \in \tilde{A}, \end{cases}$$

generates the same value as $(M, \{\pi_1, \pi_2\})$. ■

B Value of Information

Our example shows that the Receiver is better off by making her decision based on p_0 . Therefore, she would prefer getting no additional information than getting ambiguous information. This result is consistent with the recent literature on the (negative) value

of information under ambiguity, e.g., Li (2020) or Hill (2020). However, BLL show in their subsections 6.3 and 6.4 that the Receiver may benefit from listening to an ambiguous device.

We denote with $U^0(a)$ the ex ante expected utility of action a of the Receiver without any additional information, i.e.,

$$U^0(a) = \int_{\Omega} u(a, \omega) p_0(\omega) d\omega.$$

Definition 4. A communication device Π has a positive value of information for the Receiver if

$$\max_{(a_m)_{m \in \text{supp } \Pi} \in A^{|\text{supp } \Pi|}} \min_{\rho \in \text{rect}(\Phi_{\Pi}^0)} \mathbb{E}_{\rho}(u(a_m, \omega)) \geq \max_{a \in A} U^0(a).$$

Ambiguous information induces two effects. On the one hand an ambiguous communication device generates ambiguous beliefs and therefore, decreases the worst case expected utility of the Receiver. On the other hand, the communication device still reveals information about the state. This information allows the Receiver to choose an action which better fits the state and increases her expected utility. Then, the value of information is positive if the second effect exceed the negative effect of ambiguity and ambiguity aversion.

BLL say that a communication device satisfies a participation constrain if

$$\max_{(a_m)_{m \in \text{supp } \Pi} \in A^{|\text{supp } \Pi|}} \min_{\pi \in \Pi} \int_{\Omega} \int_M \pi(m|\omega) u(a_m, \omega) dm p_0(\omega) d\omega \geq \max_{a \in A} U^0(a).$$

They call this condition a participation constraint, since it ensures, that the Receiver is willing to pay attention to the communication device. If the participation constraint is not satisfied, the Receiver would be better off ignoring the communication device, ex ante. Since $\Phi_{\Pi}^0 \subseteq \text{rect}(\Phi_{\Pi}^0)$, it follows

$$\begin{aligned} & \max_{(a_m)_{m \in \text{supp } \Pi} \in A^{|\text{supp } \Pi|}} \min_{\pi \in \Pi} \int_{\Omega} \int_M \pi(m|\omega) u(a_m, \omega) dm p_0(\omega) d\omega \\ &= \max_{(a_m)_{m \in \text{supp } \Pi} \in A^{|\text{supp } \Pi|}} \min_{\rho \in \Phi_{\Pi}^0} \mathbb{E}_{\rho}(u(a_m, \omega)) \\ &\geq \max_{(a_m)_{m \in \text{supp } \Pi} \in A^{|\text{supp } \Pi|}} \min_{\rho \in \text{rect}(\Phi_{\Pi}^0)} \mathbb{E}_{\rho}(u(a_m, \omega)). \end{aligned}$$

Hence, any communication device with a positive value of information satisfies the participation constrain of BLL.

BLL characterize a condition that guarantees that the Receiver benefits from listening to a communication device (see BLL Proposition 8). We now translate this condition to our setting. We denote with a_0 the default actions, i.e., the action that maximizes $U^0(a)$.

Definition 5. Let \hat{a}_m denote the interim optimal action of the Receiver given the belief set $\text{Bay}(\text{rect}(\Phi_{\Pi}^0)|m)$. A message m is value-increasing (to the Receiver) if $\mathbb{E}_{\rho|m}(u(\hat{a}, \omega)) \geq U^0(a_0)$ for all $\rho|m \in \text{Bay}(\text{rect}(\Phi_{\Pi}^0)|m)$.

BLL show, that a communication device Π satisfies the participation constraint if Π uses only value-increasing messages. The next proposition proves a stronger result and very intuitive result: A communication device that increases the worst case expected utility of the Receiver for any message has a positive value of information.

Proposition 3. If Π only uses value-increasing messages, then Π has a positive value of information for the Receiver.

Proof. Since $\mathbb{E}_{\rho|m}(u(\hat{a}, \omega)) \geq U^0(a_0)$ for all $\rho|m \in \text{Bay}(\text{rect}(\Phi_{\Pi}^0)|m)$ it follows that

$$\min_{\rho|m \in \text{Bay}(\text{rect}(\Phi_{\Pi}^0)|m)} \mathbb{E}_{\rho|m}(u(\hat{a}, \omega)) \geq U^0(a_0). \quad (\text{B.1})$$

Then, rectangularity and Equation B.1 imply

$$\begin{aligned} & \max_{(a_m)_m \in \text{supp } \Pi \in A^{|\text{supp } \Pi|}} \min_{\rho \in \text{rect}(\Phi_{\Pi}^0)} \mathbb{E}_{\rho}(u(a_m, \omega)) \\ &= \min_{\rho \in \text{rect}(\Phi_{\Pi^*}^0)} \int_{\text{supp}(\Pi^*)} \rho(\Omega, m) \min_{\rho'|m \in \text{Bay}(\Phi_{\Pi^*}^0|m)} \mathbb{E}_{\rho'|m}(u(\hat{a}_m, \omega)) \, dm \\ &\geq \min_{\rho \in \text{rect}(\Phi_{\Pi^*}^0)} \int_{\text{supp}(\Pi^*)} \rho(\Omega, m) U^0(a_0) \, dm \\ &= U^0(a_0). \end{aligned}$$

■