

Siddiqi, Hammad; Quiggin, John C.

Article

The pricing kernel puzzle: A behavioral explanation

Cogent Economics & Finance

Provided in Cooperation with:

Taylor & Francis Group

Suggested Citation: Siddiqi, Hammad; Quiggin, John C. (2019) : The pricing kernel puzzle: A behavioral explanation, Cogent Economics & Finance, ISSN 2332-2039, Taylor & Francis, Abingdon, Vol. 7, Iss. 1, pp. 1-20,
<https://doi.org/10.1080/23322039.2019.1684609>

This Version is available at:

<https://hdl.handle.net/10419/270694>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



<https://creativecommons.org/licenses/by/4.0/>

The pricing kernel puzzle: A behavioral explanation

Hammad Siddiqi & John Quiggin |

To cite this article: Hammad Siddiqi & John Quiggin | (2019) The pricing kernel puzzle: A behavioral explanation, Cogent Economics & Finance, 7:1, 1684609, DOI: [10.1080/23322039.2019.1684609](https://doi.org/10.1080/23322039.2019.1684609)

To link to this article: <https://doi.org/10.1080/23322039.2019.1684609>



© 2019 The Author(s). This open access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.



Published online: 02 Nov 2019.



Submit your article to this journal [↗](#)



Article views: 1404



View related articles [↗](#)



View Crossmark data [↗](#)



Received: 07 July 2018
Accepted: 20 October 2019
First Published: 24 October 2019

*Corresponding author: Hammad Siddiqi, University of the Sunshine Coast, Sunshine Coast, Australia.
E-mail: hammadniu@yahoo.com

Reviewing editor:
David McMillan, University of Stirling,
Stirling, United Kingdom

Additional information is available at
the end of the article

FINANCIAL ECONOMICS | RESEARCH ARTICLE

The pricing kernel puzzle: A behavioral explanation

Hammad Siddiqi^{1*} and John Quiggin²

Abstract: We show that if sophisticated institutional managers and individual investors perceive tail-risks differently, then a new explanation for the pricing kernel puzzle emerges. We show, by example, that even a tiny difference in tail-risk perception by the two investor types can explain the pricing kernel puzzle.

Subjects: Finance; Corporate Finance; Investment & Securities

Keywords: Differential awareness; pricing kernel puzzle; rare disasters

Subjects: D00; D40; G12; G13

If investors are risk averse, a financial asset that pays well in bad times should be worth more, other things equal, than a financial asset that pays better in good times. The first asset helps in reducing consumption volatility, whereas the second asset makes consumption more volatile. If investors value reduction in consumption volatility, the first asset should be worth more than the second asset. In other words, the marginal value of a dollar in a bad state should be more than the marginal value of a dollar in a good state. This condition implies that the pricing kernel for financial assets is decreasing in aggregate consumption.

However, empirical analysis has yielded results inconsistent with this theoretical prediction (Ait-Sahalia & Lo, 2000; Jackwerth 2000; Rosenberg & Engle, 2002). The pricing kernel recovered from index options data (using S&P 500 index return as a proxy for aggregate wealth) is not everywhere decreasing as expected. Instead, the pricing kernel is increasing over some range of aggregate wealth. This empirical finding is known in the literature as the pricing kernel puzzle.

Three broad categories of explanations for the pricing kernel non-monotonicity have been discussed in the literature (Hens & Reichlin, 2012). These are: risk-seeking behavior; incomplete markets; and incorrect beliefs. Hens and Reichlin (2012) argue that there is no evidence of risk-seeking behavior in the aggregate, and that generating a non-monotonic pricing kernel with incomplete markets requires some extreme assumptions about wealth distribution. Shefrin (2005) explains the

ABOUT THE AUTHOR

Hammad Siddiqi is a financial economist interested in how investor interpretation of information matters for market behavior. Traditionally, information is viewed as a commodity or a currency which is simply transferred or transmitted. Hammad's research aims to change that. By drawing on insights developed in cognitive science, he looks at how information is adapted (rather than simply transmitted) into cognitive structures through processes involving *accommodation* and *assimilation*, and how this matters for financial markets in general and asset prices in particular.

PUBLIC INTEREST STATEMENT

This paper shows that an important puzzle in asset pricing is potentially resolved if perception of small probability events involving large losses is different across institutional and individual investors. Institutional investors generally are well-resourced, so they are expected to have a better understanding of events that have a small probability of large losses, when compared with individual investors. We show that such differences in awareness of risks provide a natural explanation for the puzzle.

puzzle with a model in which there are heterogeneous beliefs. Ziegler (2007) tests this explanation and finds that the degree of belief heterogeneity required is implausibly large. Polkovinchenko and Zhao (2013) also use a model with distorted beliefs to explain this puzzle.

Our approach is closest to the third category of explanations. We assume incomplete markets; however, rather than modelling heterogeneous beliefs over a known set of states, we examine the case where some investors may be unaware of some possible states. The analysis draws on the nascent literature on differential awareness (Grant & Quiggin, 2013; Halpern & Rego, 2006; Heifetz, Meier, & Schipper, 2006; Karni & Vierø, 2013).¹

It is increasingly being recognized that at least some investors tend to neglect tail-risks (see Gennaioli and Vishny (2012) and references therein). Tail-risks are risks of rare disasters or small probabilities of large losses. One expects relatively sophisticated institutional investors to show greater awareness of such tail-risks than unsophisticated individual traders. It is well-known that index-option trading is dominated by sophisticated institutional managers (Bollen and Whaley 2004). Due to the presence of these sophisticated traders, index-option prices are found to be less sentiment-prone than equity prices (Lemmon and Ni 2013).

In line with the above findings, we assume that sophisticated institutional managers (who dominate the index-option market) are better aware of tail-risks than individual investors (who dominate the equity market). We show that, in the presence of such differential awareness of tail-risks, there is a range of feasible index option prices (both call and put) that give rise to non-monotonic pricing kernels. We argue, and show by example, that small differences in tail-risks as perceived by the two investor types can give rise to non-monotonic pricing kernels consistent with the observed evidence.

If the presence of differential awareness causes a non-monotonic pricing kernel to emerge as argued in this article, then, the absence of differential awareness should lead to a monotonic pricing kernel. It is difficult to see how differential awareness can arise in any pair of option and equity markets in which the same investor types are dominant. Retail or individual investors are dominant in both individual stock option and equity markets (Lakonishok, Pearson, & Poteshman, 2007). Hence, we predict that a non-monotonic pricing kernel cannot arise in individual stock option markets. That is, it can only arise in index option markets where sophisticated institutional managers tend to be dominant. In accord with our prediction, Chadhuri and Schroder (2015) find that options written on individual stocks exhibit average returns consistent with a monotone pricing kernel.

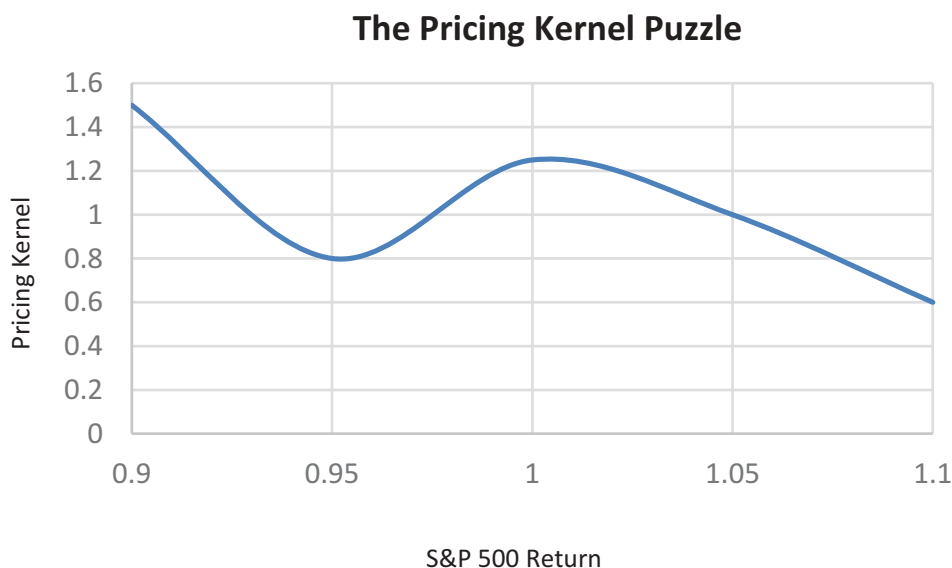
This paper is organized as follows. Section 2 summarizes the evidence on pricing kernel non-monotonicity, discusses key findings from the literature, and motivates the approach put forward in this article. Section 3 shows that a locally increasing pricing kernel similar to the empirical pricing kernel may arise if the option and equity investors are differentially aware. Section 4 shows that the pricing kernel so generated is consistent with the key-observed features of the empirical pricing kernel. Section 5 shows how the explanation put forward in this article relates to some of the existing explanations of the pricing kernel puzzle. Section 6 concludes.

1. Non-monotonicity of the empirical pricing kernel

A large literature on the non-monotonicity of the empirical pricing kernel can be traced back to the seminal work in Jackwerth (2000). By non-parametrically estimating the kernel price, Jackwerth (2000) finds that the shape of this function is not in accordance with economic theory. Jackwerth (2000) concludes that the likely reason is mispricing of options relative to the underlying index. More recently, pricing kernel non-monotonicity has been observed by Christoffersen, Jacobs, and Heston (2013) and Beare and Schmidt (2016).² Jackwerth (2004) and Cuesdeanu and Jackwerth (2016a) provide surveys of the literature.

A typical locally increasing pricing kernel is shown in Figure 1. According to economic theory, one expects the pricing kernel to decrease with aggregate wealth (S&P return is a proxy for aggregate

Figure 1.



wealth in Figure 1). The pricing kernel puzzle refers to the empirical finding that instead of being a decreasing function of wealth, the pricing kernel has an increasing portion, which often appears in the middle region as in Figure 1. Some studies have found a U-shaped pricing kernel as well (Bakshi, Madan, & Panayotov, 2010; Christoffersen et al., 2013).

Some key features of the empirically observed pricing kernel are listed below:

- (1) As noted in Hens and Reichlin (2012), a non-monotonic pricing kernel is typically shown with options data. This suggests that the root of the puzzle may lie in the relative price of index options with respect to the underlying index. Jackwerth (2000) argues that option mispricing relative to the underlying index seems to be the most likely cause of the pricing kernel non-monotonicity. Constantinides, Jackwerth, and Perrakis (2009) find evidence of frequent mispricing in options relative to the underlying index.
- (2) There is wide variation in the shape of the pricing kernel. During some months, the pricing kernel is monotonic, and during other months, it is non-monotonic. Using data on options on the S&P 500 index ranging from 1997 to 2013, Beare and Schmidt (2016) reject the null hypothesis of pricing kernel monotonicity 37% of the time. The shape of the pricing kernel appears to be period specific.
- (3) Barone-Adesi, Mancini, and Shefrin (2014) find that the times when the pricing kernel displays an upward sloping portion near the middle coincides with a greater fraction of investors in the Yale–Shiller survey assigning a very low chance to a left tail event. Polkovinchenko and Zhao (2013) also find evidence of underweighting of tail probabilities during 2005 and 2006. This is the period in which the Yale–Shiller crash confidence index also indicates a dramatic decline in the fear of a crash. Polkovinchenko and Zhao (2013) call this a period of ‘complacency in which asset prices were high and volatility low. Cuesdeanu (2016) also find that the pricing kernel tends to display an increasing portion in the middle during calm periods.

1.1. Equity traders vs. index option traders

Barone-Adesi et al. (2014) find that an increasing portion in the empirical pricing kernel tends to appear when equity investors are highly confident (as measured by the Yale Crash Confidence index) that a major crash will not happen in the immediate future. This evidence suggests that

there might be a connection between underweighting of left tail probabilities and the appearance of a locally increasing portion in the empirical pricing kernel.

Bollen and Whaley (2004) and Garleanu, Poteshman, and Poteshman (2009) point out that index option trading is dominated by institutional investors, who, in particular, buy put options as downside protection. Rubinstein (1994) argues that index option traders suffer from “crash-o-phobia” because they seem excessively worried about the disaster state. In fact, there is a long tradition in the literature of using far-out-of-the-money put options prices to infer the *ex-ante* probability of rare disasters.³ In sharp contrast with index option trading where disaster risk matters, perhaps excessively, stock option and equity markets include a large number of individual investors who trade equities and leveraged equities (call options) in anticipation of speculative capital gains.⁴ Hence, empirical evidence suggests that different types of investors with different concerns populate index option and equity markets, respectively. Lemmon and Ni (2013) present evidence that the relative absence of individual investors from index option market is behaviorally relevant as it makes the index option market less prone to sentiment.

1.2. Differential awareness

Various models of limited awareness have been developed in the literature. See Grant and Quiggin (2013), Heifetz et al. (2006), Halpern and Rego (2006), among many others.⁵ Karni and Vierø (2013) call the scenario in which an agent’s state space expands, “reverse Bayesianism”. As the state space expands, probability mass is shifted proportionally away from the non-null events in the prior state space to events created as a result of the expansion of the state space. We make use of “reverse Bayesianism” in this article when considering “growing awareness”. Quiggin (2016) defines “restriction” as a situation in which a decision-maker effectively assigns a probability of zero to some outcomes. We use this notion of “restriction” when considering differential awareness arising due to a shrinking state space.

2. The model

We start by discussing the motivation of our approach followed by formal derivation of key results. Section 3.1 discusses the motivation, provides intuition, summarizes key results, and discusses a novel prediction that arises from our approach. Section 3.2 presents formal results. Section 3.3 illustrates the key results with an example.

2.1. Motivation for the model

If investors are expected utility maximizers, then corresponding to given values of the equity index and the risk-free rate, there is a range of index option prices (call and put) within which the pricing kernel is monotonically decreasing. There are at least two ways of deriving the bounds implied by a decreasing pricing kernel: the expected utility comparisons under a zero-net-cost option strategy introduced by Perrakis and Ryan (1984) and the linear programming (LP) method pioneered by Ritchken (1985). The two approaches yield identical results. An empirical application of these bounds is Constantinides et al. (2009), who found that option prices appear to violate the bounds implied by decreasing pricing kernel quite frequently. In this article, we show that differential awareness of rare disasters between institutional investors and individual traders may cause the option prices to lie outside these bounds.

Our motivation is the observation that, whereas retail or individual investors play an important role in the equity market, the index option market is dominated by sophisticated institutional managers who purchase put options as protection against downside risk⁶ (Bollen & Whaley, 2004). What are the implications of the limited role of individual investors in the index option market? The ideal setting for studying the implications would be to find an asset that is identical to an index option but is mostly traded by individual investors.

Stock options in aggregate and index options are close to identical. The fact that less sophisticated individual investors trade stock options more actively than index options (Lakonishok et al.,

2007) creates an ideal setting for testing the implications of relative absence of individual investors from the index option market. Using a large dataset, Lemmon and Ni (2013) compare the behavior of aggregated stock options and index options. They find that the absence of individual investors makes the index option market less prone to sentiment when compared with stock options, leading to the conclusion that individual investors (who dominate the equity market and the stock option market) are more prone to trades based on sentiment than sophisticated institutional managers (who dominate the index option market).

The above findings are in line with evidence that sentiment influences stock prices. Baker and Wurgler (2007), (p. 130) write, “Now the question is no longer, as it was a few decades ago, whether investor sentiment affects stock prices, but rather how to measure investor sentiment and quantify its effects.”

Based on the empirical finding that individual investors are more prone to sentiment than sophisticated institutional managers, we hypothesize that they differ in their perception of tail-risks. We assume that the marginal index option trader (sophisticated institutional manager) is more conscious of tail risks than the marginal equity investor (retail or individual investors). We show that, if investors are expected utility maximizers, then any difference in how the tail-risks are perceived by the two types of investors gives rise to a range of option prices within which the pricing kernel is non-monotonic. In general, different tail-risk perceptions always split the feasible range of option prices into two parts. In one part, the pricing kernel is monotonic, and in the other part, the pricing kernel is non-monotonic. As expected, the range of option prices within which the pricing kernel is non-monotonic increases as the difference in the tail-risk perception increases.

In line with behavioral finance literature, we assume that the mere presence of differences in beliefs is not sufficient to lead to a counter-trade in which index option traders short-sell the equity index and cause the required correction in the equity price. This is not just because betting against individual investor sentiment is risky (Shleifer & Vishny, 1997), but also due to the fact that the presence of trading frictions such as transaction costs guarantees that the impact of different tail-risk beliefs can never be completely eliminated.

Tail risks are small probabilities of large losses. The 2008 financial crisis in the United States has provided direct evidence that investors underestimated the risk of a crisis. Coval, Jurek, and Stafford (2009) show that investors underestimated the probability that mortgage backed securities can default. Foote and Willen (2012) find that investors entirely neglected the possibility that home prices can decline by the magnitudes that actually materialized. We add to this literature by arguing that different types of investors may neglect tail-risks differently with the relatively sophisticated institutional investors showing greater awareness of tail-risks than individual investors.

Our approach leads to a novel prediction. The core of our approach is the idea that pricing kernel is non-monotonic because different types of investors are dominant in equity and index option markets. This idea implies any pair of equity and option markets in which the same types of investors are dominant should only show a monotonic pricing kernel. Since individual investors are dominant in both individual stock and stock option markets, we predict that a non-monotonic pricing kernel cannot arise in such markets. Consistent with our approach, Chadhuri and Schroder (2015) find that options written on individual stocks exhibit average returns consistent with monotone pricing kernel.

2.2. Key results

Option trading is, in general, limited in volume relative to equity trading. Hence, the option pricing literature generally takes stock prices as given. Consistent with this literature, we assume that the marginal options trader takes equity prices as given. We setup the stochastic dynamic portfolio optimization problem of the representative options trader below.

Let t index time ($t = 0, 1, 2, 3, \dots, T$). The representative agent's environment is summarized by an $M \times 1$ state vector x_t . In particular, the first component of the state vector, x_{1t} , is the wealth of the representative investor. Other components include the parameters of the distribution of asset returns as perceived by the decisionmaker, the risk-free rate, as well as the price of the underlying index on which options are traded. There is a $K \times 1$ vector of controls y_t . It denotes how many dollars of wealth are allocated to each available asset. The agent consumes whatever is left after making allocation decisions. That is, with N available assets, $c_t = x_{1t} - \sum_{i=1}^{K=N} y_{it}$.

The above formulation is useful because it clarifies the channel through which differential awareness matters in dynamic portfolio allocation. To see how, consider the following case: Suppose, equity traders neglect a state in the left-tail of the probability distribution corresponding to a small probability of a large decline; however, option traders do not neglect that state. The equity price changes causing the distribution of equity returns as perceived by the option trader to change as well. Hence, the state vector x_t changes for the option trader whose portfolio optimization problem is modelled here. In principle, the option traders' and equity traders' awareness can be modelled as stochastic and inserted in the state transition equations.

The agent maximizes $\sum_{t=0}^T \beta^t u(x_t, y_t)$, where $u(x_t, y_t)$ is a time-invariant utility function and β is the subjective one-period discount factor.

From the Bellman principle:

$$V_t(x_t) = \max_{y_t} [u(x_t, y_t) + \beta E\{V_{t+1}(x_{t+1}) | x_t\}]$$

Assume that the agent's wealth does not influence the other components of the state vector, and that the agent has no control over her environment except for her wealth. The first-order conditions of the optimization problem are independent of the value function and are given by:

$$-\beta E_t \left[\frac{\frac{\partial u}{\partial x_{kt+1}}}{\frac{\partial u}{\partial y_{kt}}} R_{kt+1} | x_t \right] = 1,$$

where R_{kt+1} is the return from asset k .

Note that $c_t = x_{1t} - \sum_{i=1}^{K=N} y_{it}$. The first-order conditions expressed in terms of consumption are:

$$\beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} R_{kt+1} | x_t \right] = 1.$$

In particular, the agent trades in the equity index (S_t), a call option on the index (C_t), a put option on the index (P_t), and in the risk-free asset. Using $m = \frac{\beta u'(c_{t+1})}{u'(c_t)}$, denoting the net index return by z_{t+1} , and the gross risk-free return by R , the first-order conditions pertaining to the four assets are (time subscripts on E and m are suppressed for simplicity):

$$1 = E[m (1 + z_{t+1}) | x_t];$$

$$C_t = E[m C_{t+1} (S_t (1 + z_{t+1})) | x_t];$$

$$P_t = E[m P_{t+1} (S_t (1 + z_{t+1})) | x_t]; \text{ and}$$

$$1 = E[m | x_t] R.$$

As the above conditions specify an equilibrium, there cannot be any arbitrage opportunities from the perspective of options trader. In particular, put-call parity must be satisfied as it is a no-arbitrage restriction. For ease of exposition, we only discuss call options while noting that the corresponding argument for puts only requires an application of put-call parity.

From the last first-order condition, $E[m|x_t]R = 1$. It follows that $m = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{m}{E[m|x_t]R} = \frac{u'(c_{t+1})}{R E[u'(c_{t+1})]}$. The normalized marginal rate of substitution, also known as the stochastic discount factor or the pricing kernel, given by $\frac{u'(c_{t+1})}{R E[u'(c_{t+1})]}$ must be non-increasing in z_{t+1} if the trader has concave utility. This is because c_{t+1} , under typical conditions, is a non-decreasing function of z_{t+1} , and $u'(c_{t+1})$ falls as $u(c_{t+1})$ rises.

The monotonicity of the pricing kernel is a key concept in finance as it connects asset pricing to the fundamental economic notion of scarcity. It captures the notion that, given decreasing marginal utility of wealth, the valuation of payoffs must be lower in a good state of the world than in a bad state of the world. Observed violations of monotonicity therefore pose a challenge to economic and finance theory.

Empirical estimation of $m = \frac{\beta u'(c_{t+1})}{u'(c_t)}$ directly from consumption data is challenging as it requires assumptions about the form of the utility function and risk aversion. A non-parametric approach, pioneered in Jackwerth (2000), Ait-Sahalia and Lo (2000), and Rosenberg and Engle (2002), makes use of the fact that the pricing kernel is the ratio of risk-neutral to physical density discounted by the risk-free rate. To see this, note that the pricing kernel implies the following about the price of a call option that expires at $t + \Delta t$:

where $C_t = \int m(z_{t+\Delta t}) C_{t+\Delta t} (S_t(1 + z_{t+\Delta t})) \pi(z_{t+\Delta t}) dz$, is the probability density function of index returns. If the risk-neutral density is $q(z_{t+\Delta t})$, then the call price can also be expressed as:

$$C_t = e^{-r\Delta t} \int (z_{t+\Delta t}) C_{t+\Delta t} (S_t(1 + z_{t+\Delta t})) dz.$$

Comparing the above two equations, it follows that $m(z_{t+\Delta t}) = e^{-r\Delta t} \frac{q(z_{t+\Delta t})}{\pi(z_{t+\Delta t})}$. Hence, the pricing kernel is the ratio of risk-neutral to physical density discounted at the risk-free rate.

As discussed earlier, there are at least two equivalent ways of deriving the option pricing bounds implied by a decreasing pricing kernel. Given the underlying index value and the risk-free rate, lower and upper bounds may be placed on index option prices using linear programming (LP) methods (Ritchken, 1985). The idea is to use monotonicity of the pricing kernel as a constraint. In other words, the idea is to solve for the maximum and minimum option prices that are consistent with a decreasing pricing kernel. A brief sketch of the essential argument is provided below.

Suppose the distribution of index returns at $t + \Delta t$, given the current index value, is discrete. Specifically, assume that there are $n + 1$ states ordered from the lowest to the highest index return. The distribution of index returns is given by $(z_{j(t+\Delta t)}, \pi_{j(t+\Delta t)})$, where $\pi_{j(t+\Delta t)}$ is the probability associated with the j th state.

The following LP set-up can be used to find the option pricing bounds:

$$\max_{m_{j(t+\Delta t)}} \sum_{j=0}^{j=n} C_{t+\Delta t} (S_t(1 + z_{j(t+\Delta t)})) \pi_{j(t+\Delta t)} m_{j(t+\Delta t)}$$

$$\left(\min_{m_{j(t+\Delta t)}} \sum_{j=0}^{j=n} C_{t+\Delta t} (S_t(1 + z_{j(t+\Delta t)})) \pi_{j(t+\Delta t)} m_{j(t+\Delta t)} \right)$$

subject to(1)

$$1 = \sum_{j=0}^{j=n} \pi_{j(t+\Delta t)} (1 + z_{j(t+\Delta t)}) m_{j(t+\Delta t)}$$

$$1 = R \sum_{j=0}^{j=n} \pi_{j(t+\Delta t)} m_{j(t+\Delta t)}$$

$$m_{0t+\Delta t} \geq m_{2t+\Delta t} \geq \dots \geq m_{nt+\Delta t} \geq 0.$$

(1) uses a discrete version of the first order conditions and the requirement that the pricing kernel must be decreasing in index returns to derive bounds on the call option price. The bounds are presented in Appendix A.

2.2.1. Differential awareness

The main point of this article is that differential awareness can cause the option price to lie outside the bounds that follow from the linear programming set-up in (1). If the option price is outside these bounds, then the pricing kernel must be non-monotonic.

We focus on two ways of generating a non-monotonic pricing kernel given differential awareness between index option traders and equity index traders. Both involve left-tail events. Left-tail events are special because they correspond to small probabilities of large losses.

The two ways of generating differential awareness involving left-tail events or rare disasters are as follows:

- (1) Growing Awareness: The options trader becomes aware that *the index value can fall to a value smaller than a fraction θ of the current value* while the equity trader remains unaware. Alternatively, we can keep the options trader unaware while the awareness of the equity trader grows to include the disaster state. In either case, the state space of one type of investor expands. Probabilities are adjusted as in Karni and Vierø (2013) by adding probability mass for the newly considered state, and proportionally reducing probabilities for the non-null events in the prior state space.
- (2) Shrinking Awareness: Both the equity investor and the options investor were initially aware that *the index value can fall to a value smaller than a fraction θ of the current value*. The equity investor is replaced by a more optimistic, less aware, investor, who disregards the disaster state. Such replacement of “hedge” investors by “speculative” investors is part of Minsky’s analysis of financial instability (Minsky, 1986). The less aware equity investor perceives a restricted or reduced state space as in Quiggin (2016) and the probability mass of eliminated states is shifted proportionally to other non-null events. Alternatively, we may assume that it is the options trader who disregards the disaster state, and the equity investor remains aware.

Growing awareness of the options trader, in which she becomes aware of sudden crashes or left-tail events, worsens her perception of the return distribution from the equity index. The marginal equity trader whose awareness has not increased perceives the same state space as before. Hence, equity prices do not change. Consequently, the equity becomes relatively less attractive to the options investor, so she shifts funds away from equity and into options. The lower and upper bounds on the option prices, derived as in (1), rise. Since the new option pricing upper bounds (both call and put) are higher, there is a non-zero interval of option prices (between the two upper bounds) that support a non-monotonic pricing kernel.

Shrinking awareness of the marginal equity trader, in which she ignores the states in the extreme left tail of the distribution, raises the equity price. Consequently, equity becomes less attractive to the more aware options trader. The marginal options investor shifts funds away from equity and into options, causing a rise in option prices. In particular, the upper bounds on option prices are higher with such differential awareness.

Both with “growing awareness” and “shrinking awareness” there must exist an interval of option prices in which the empirical pricing kernel will not be monotonically decreasing. Such an interval consists of prices between the upper bounds after and before changes in awareness.

Mathematically, the key to understanding the implications of an increase in the awareness of the marginal options trader is to focus on the following conditional expectation, which has been used in defining the option pricing bounds (see Appendix A):

$$\hat{z}_{jt+\Delta t} = \frac{\sum_{i=0}^j z_{it+\Delta t} \pi_{it+\Delta t}}{\sum_{i=0}^j \pi_{it+\Delta t}}.$$

Her state space expands to include a disaster state. Since this shift is unfavorable, $\hat{z}_{jt+\Delta t}$ values are lower for the expanded state space than for the initial state space.

The risk neutral distribution corresponding to the upper bound is (see Appendix A):

$$U_{0t} = \frac{R - 1 - \hat{z}_{0t+\Delta t}}{\hat{z}_{nt+\Delta t} - \hat{z}_{0t+\Delta t}} \pi_{0t+\Delta t} + \frac{\hat{z}_{nt+\Delta t} + 1 - R}{\hat{z}_{nt+\Delta t} - \hat{z}_{0t+\Delta t}}$$

$$U_{jt} = \frac{R - 1 - \hat{z}_{0t+\Delta t}}{\hat{z}_{nt+\Delta t} - \hat{z}_{0t+\Delta t}} \pi_{jt+\Delta t}, j = 1, \dots, n.$$

A decline in $\hat{z}_{jt+\Delta t}$ shifts the probability mass to the right, causing an increase in the option price upper bound.

Proposition 1 follows:

Proposition 1 (Growing Awareness): If the awareness of the marginal options (equity) investor grows and she becomes aware of the possibility of sudden large crashes, whereas the awareness of the marginal equity (options) investor remains unchanged, then there is a non-empty set of call and put option prices in which the empirical pricing kernel is non-monotonic and violates the upper (lower) bound constraint

The expansion of the option investor's state-space to include a disaster-state without a corresponding expansion in the state-space of the equity investor, implies that the option pricing bounds increase. In particular, the option pricing upper bound *after expansion* is larger than the option pricing upper bound *before expansion*. Consequently, there is an interval of prices (between the upper bounds before and after expansion) that give rise to a non-monotonic pricing kernel, if the disaster state does not materialize in-sample.

On the other hand, expansion of the equity investor's state-space to include a disaster-state without an equivalent expansion in the option investor's state-space, reduces option pricing bounds. In particular, the option pricing lower bound *after expansion* is less than the option pricing lower bound *before expansion*. Consequently, there is an interval of prices (between the lower bounds after and before expansion) that supports a non-monotonic pricing kernel.

Empirically, the observed violations of upper bounds are more prevalent (see Constantinides et al. (2009)) implying that it is often the marginal option trader that perceives an expanded state space. As discussed earlier, this is consistent with the finding that institutional investors seeking downside protection dominate in the index option market.

Next, consider the case of shrinking awareness. As before, the key to understanding what happens when a portion of the left tail is ignored is to focus on the following conditional expectation, which has been used in defining the option pricing bounds (see Appendix A):

$$\hat{z}_{jt+\Delta t} = \frac{\sum_{i=0}^j z_{it+\Delta t} \pi_{it+\Delta t}}{\sum_{i=0}^j \pi_{it+\Delta t}}.$$

If only one state at the left-end of the distribution is ignored, then the above conditional expectation can be written as:

$$\hat{z}_{jt+\Delta t}^T = \frac{\sum_{i=1}^j z_{it+\Delta t} \pi_{it+\Delta t}^T}{\sum_{i=1}^j \pi_{it+\Delta t}^T}$$

where the superscript T denotes truncation. That is, the summation is not over $i = 0, 1, 2, \dots, j$. Rather, it is over $i = 1, 2, \dots, j$. Also, $\pi_{it+\Delta t}^T = \frac{\pi_{it+\Delta t}}{1 - \pi_{0t+\Delta t}}$ for $i = 1, 2, \dots, j$. Hence, $\hat{z}_{jt+\Delta t}^T$ is larger than the corresponding value without truncation. That is, in states with negative stock returns, $\hat{z}_{jt+\Delta t}^T$ is less negative, and in states with positive returns, $\hat{z}_{jt+\Delta t}^T$ is more positive.

With truncation, the lower bound risk neutral distribution can be written (by using the expression from Appendix A) as:

$$L_{jt} = \frac{\hat{z}_{h+1,t+\Delta t}^T + 1 - R}{\hat{z}_{h+1,t+\Delta t}^T - \hat{z}_{h,t+\Delta t}^T} \left(\frac{\pi_{it+\Delta t}^T}{\sum_{k=1}^h \pi_{kt+\Delta t}^T} \right) + \frac{R - 1 - \hat{z}_{h,t+\Delta t}^T}{\hat{z}_{h+1,t+\Delta t}^T - \hat{z}_{h,t+\Delta t}^T} \left(\frac{\pi_{it+\Delta t}^T}{\sum_{k=1}^{h+1} \pi_{kt+\Delta t}^T} \right), j = 1, \dots, h$$

$$L_{h+1,t} = \frac{R - 1 - \hat{z}_{h,t+\Delta t}^T}{\hat{z}_{h+1,t+\Delta t}^T - \hat{z}_{h,t+\Delta t}^T} \left(\frac{\pi_{it+\Delta t}^T}{\sum_{k=1}^{h+1} \pi_{kt+\Delta t}^T} \right), L_{jt} = 0, j > h + 1$$

where h is a state index such that $\hat{z}_{h,t+\Delta t}^T \leq R - 1 \leq \hat{z}_{h+1,t+\Delta t}^T$.

Comparing the truncated distribution with the non-truncated distribution, the truncation shifts the probability mass to the left (across remaining states). Consequently, the option lower bound is lower with truncation. It follows that the option prices between the truncated lower bound and the non-truncated lower bound do not support a monotonic pricing kernel, if the physical probability distribution is correctly estimated. That is, within this range of prices, the pricing kernel must be non-monotonic. Proposition 2 formalizes this intuition.

Proposition 2 (Shrinking Awareness): If the marginal options (equity) investor displays restricted awareness and ignores some states in the extreme left tail of the stock return distribution, then there is a non-empty interval of call and put option prices in which the empirical pricing kernel is non-monotonic.

Proof.

See Appendix B.

The observed violations of upper bounds (Constantinides et al., 2009; Constantinides, Jackwerth, & Perrakis, 2011) can be rationalized either by the growing awareness of the marginal options trader, or by the shrinking awareness of the marginal equity trader.

What happens when the marginal equity investor displays a greater degree of restricted awareness? That is, what happens when she assigns a probability of zero to a bigger portion of the left-tail? This shifts a greater mass to the right in the risk-neutral distribution of the option upper bound. It follows that the upper bound is higher. Hence, the range of feasible option prices supporting a non-monotonic pricing kernel gets bigger as differential awareness rises. Proposition 3 formalizes this intuition.

Proposition 3 The range of feasible option prices supporting a non-monotonic pricing kernel increases if the marginal equity investor assigns a zero probability to a bigger portion of the left tail.

Proof.

See Appendix C. ■

To bring out the intuition behind propositions 1 and 2 further, and to illustrate the implications for the shape of the pricing kernel, we show a worked-out example in the next sub-section.

2.3. Shape of the pricing kernel

In this section, we present a worked-out example that shows the implications of Proposition 1. Suppose there is a stock (interpreted as an equity index) with a current price of 50 and also a put option on the stock (interpreted as a put option on the index) with a strike price of 50. Suppose there are 11 states indexed by $j = -1, 0, 1, 2, \dots, 9$. The stock price in each state is shown in Table 1. The risk-free rate is 5%. Assume that based on historical time-series, only states $j = 0, 1, 2, \dots, 9$ have been realized in-sample with associated historical probabilities given in Table 1. That is, the state $j = -1$ has not materialized in-sample. Assume that initially both the option trader and the equity trader are only aware of states $j = 0, 1, 2, \dots, 9$. The put option price upper bound can be deduced by using the formula for the upper bound presented in Appendix A. The upper bound is 5.4.

Suppose that the option traders' awareness grows and she becomes aware of the state $j = -1$, whereas the perception of the equity trader does not change. The option trader perceives a 0.001 probability of the state $j = -1$ with a corresponding proportional redistribution of probability mass from states $j = 0, 1, 2, \dots, 9$. The new put option price upper bound becomes 5.48. As the state $j = -1$ has not materialized in-sample, the historical probabilities have not changed. It follows that put prices in the interval [5.40, 5.48] do not support a monotonic pricing kernel.

To recover the empirical pricing kernel, we follow the procedure in Rubinstein (1994). A typical non-parametric procedure for estimating the pricing kernel, such as that of Jackwerth (2000), uses some extension of Rubinstein (1994).

Extracting the pricing kernel from market data requires three steps. Step 1 is to estimate the risk-neutral density from option prices. Step 2 is to estimate the physical density from the historical distribution. Step 3 is to divide the risk neutral density by the historical density then discount it by the risk-free rate to recover the empirical pricing kernel.

In the example above, the historical density, and the risk-free rate are known, so it is only necessary to estimate the risk neutral density. Assume that the price of the put option is 5.44. Since this value lies in the interval [5.40, 5.48] the pricing kernel is non-monotonic. The shape of the kernel may be estimated using the method developed in Rubinstein (1994):

Step 1: Calculate the Black-Scholes implied volatility corresponding to the put option price of 5.44. We assume that the time to expiry is 1 year. The other parameters are as assumed earlier: $S = 50, K = 50, r = R - 1 = 5\%$. Given these parameter values, the implied volatility of 34% is obtained by inverting the Black-Scholes formula.

Step 2: Build an implied binomial tree with the following parameter values: *up factor* $= e^{\sqrt{\frac{T}{n}}\sigma}$, and *down factor* $= e^{-\sqrt{\frac{T}{n}}\sigma}$. Here T is 1 year, the number of binomial periods is, $n = 9$, as there are 10 states in the historical distribution, and $\sigma = 34\%$. So, *up factor* = 1.12 and *down factor* = 0.89

The risk-neutral probability of the up movement is, $p' = \frac{r - \text{downfactor}}{\text{upfactor} - \text{downfactor}} = 0.496$,

so the risk-neutral probability of the down movement is $1 - p' = 0.504$. As there are 10 states, we need a binomial tree with nine steps. The corresponding risk-neutral probabilities at step 9 in the binomial tree are calculated by applying the following formula: $C_j^9 (p')^j (1 - p')^{10-j}$ where C denotes the combination operator and j is the state index ranging from 0 to 9. This is our prior and is denoted by q' . The prior is shown in Table 1.

Step 3: The posterior risk neutral probabilities are calculated by minimizing the following quadratic function subject to the given constraints⁷:

Table 1.

State index	-1	0	1	2	3	4	5	6	7	8	9
Physical density:											
<i>Historical</i>	0.000	0.002	0.018	0.070	0.164	0.246	0.246	0.164	0.070	0.018	0.002
<i>Put-Trader</i>	0.001	0.002	0.018	0.0699	0.1638	0.2458	0.2458	0.1638	0.0699	0.018	0.002
Stock price	50										
R	1.05										
Payoffs:											
<i>Stock</i>	0	10	20	30	40	50	60	70	80	90	100
<i>Put</i>	50	40	30	20	10	0	0	0	0	0	0
Upper bound											
<i>Equal Awareness</i>	5.4										
<i>Diff. Awareness</i>	5.48										
Risk neutral											
<i>Prior</i>		0.002	0.017	0.068	0.160	0.244	0.248	0.168	0.073	0.019	0.002
<i>Posterior</i>		0.036	0.038	0.078	0.159	0.224	0.227	0.149	0.063	0.018	0.009
Pricing kernel		17.461	2.056	1.058	0.921	0.868	0.878	0.866	0.848	0.958	4.368

$$\sum_{j=0}^9 (q_j - q_j')^2$$

subject to

$$5.44 = e^{-0.05} \sum_{j=0}^9 P_j q_j$$

$$50 = e^{-0.05} \sum_{j=0}^9 S_j q_j$$

$$q_j > 0, \text{ and } \sum_{j=0}^9 q_j = 1$$

where P_j and S_j are state-wise Put and stock payoffs as shown in Table 1. The estimated posterior probabilities are shown in Table 1 next to “posterior” under Risk-Neutral.

Step 4: The posterior risk neutral probabilities are divided by the respective physical probabilities and discounted at the risk-free rate to arrive at the empirical pricing kernel, which is shown in Table 1 next to the heading Pricing Kernel.

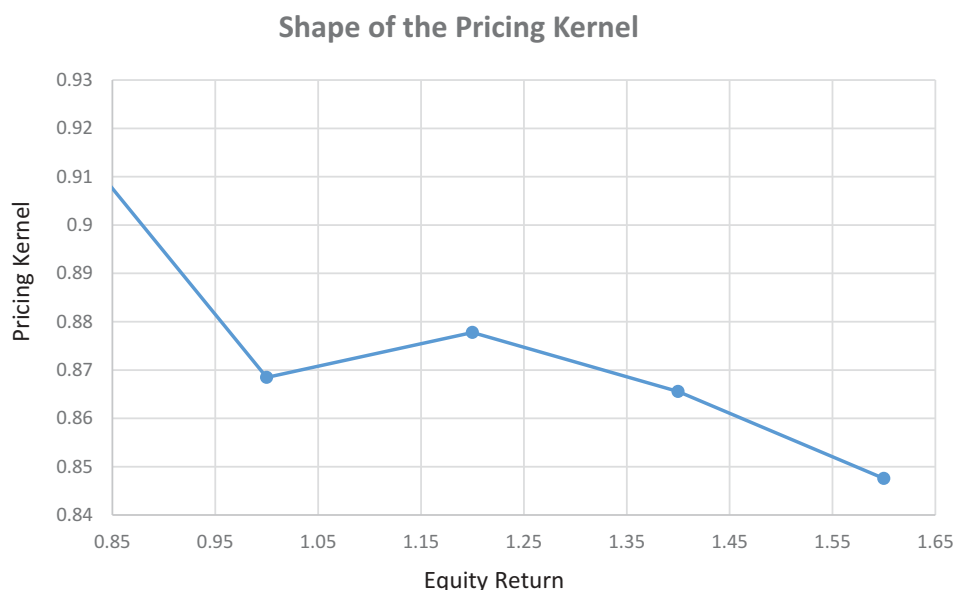
The estimated pricing kernel is plotted in Figure 2. As can be seen in Figure 2, the pricing kernel is non-monotonic and has both decreasing and increasing regions. This is generated with a very tiny difference in the state-spaces as perceived by the option and equity traders. Intriguingly, a tiny difference is all that we need to generate the observed shape of the empirical pricing kernel. In the next section, we show that the locally increasing pricing kernel generated by differential awareness is consistent with the key features of the observed empirical pricing kernel.

3. Key features of the empirical pricing kernel

In Section 2, we noted three features of the empirical pricing kernel.

- (1) A non-monotonic pricing kernel is typically shown with options data.
- (2) The shape of the pricing kernel appears to be period specific.

Figure 2.



(3) Non-monotonicity is associated with reduced fear of a crash.

The differential awareness approach is consistent with feature 1 noted above, which is that the root of the puzzle may lie in the relative price of the options in relation to the underlying index. This approach is also consistent with features 2 and 3 noted above. Our approach is consistent with feature 2 because differential awareness does not automatically imply that the pricing kernel must be monotonic or non-monotonic. Instead, irrespective of the severity of differential awareness, there is a range of feasible option prices in which the pricing kernel is monotonic and there is another range of feasible option prices in which the pricing kernel is non-monotonic. As shown by Propositions 1 and 2, differential awareness always splits the feasible range of option prices into two categories. In one category, the pricing kernel is monotonic, and in another it is non-monotonic.

Proposition 3 implies that, as the severity of differential awareness increases, that is, as the equity trader becomes more “complacent” and drops as a greater portion of the left tail, the relative size of the set of option prices supporting a non-monotonic pricing kernel increases. This is consistent with Feature 3.

However, there are two crucial differences between our approach and existing explanations. The first difference is methodological. The approach in the literature so far has been to make bold assumptions in order to explicitly derive a pricing kernel which could be non-monotonic for certain parameter values. Perhaps, due to such bold assumptions, there is no consensus among economists as to which existing explanation is the correct one. In this article, we turn this approach on its head. Instead of explicitly deriving a pricing kernel, we ask the following question: Assuming risk-averse expected utility maximization, what is the range of option prices implied by a decreasing pricing kernel? A non-monotonic pricing kernel results if the option prices are outside these bounds.

The second key difference is that we propose a new mechanism (differential awareness) that may cause the option price to lie outside these bounds. Consistent with the findings in Barone-Adesi et al. (2014), when stock market confidence is high, the proposed mechanism increases the likelihood of finding the option price outside these bounds. A distinct advantage of this approach is the ability to see that a small distortion in beliefs is sufficient to generate a pricing kernel with an increasing region. This finding is highly relevant to the debate on the possible causes of pricing kernel non-monotonicity. As mentioned earlier, Zeigler (2007) finds that the degree of heterogeneity in beliefs required to generate a locally increasing pricing kernel is implausibly high. In contrast, our results suggest that a small difference between the beliefs of the marginal index options investor and those of the marginal equity investor is sufficient to generate a locally increasing pricing kernel.

4. Differential awareness and alternative approaches

An alternative approach to the pricing kernel puzzle relies on probability weighting. A probability weighting function, as in the rank-dependent utility model (Quiggin, 1982) transforms cumulative probabilities into decision weights. Polkovinchenko and Zhao (2013) argue that a rank dependent utility model with a probability weighting function that over-weights low probabilities and under-weights high probabilities provides an explanation for the puzzle.

The pricing kernel in a rank dependent model is proportional to the product of marginal utility and the derivative (density) of the weighting function. The derivative takes a value greater than 1 if the outcome probability is overweighted in the decision. It takes a value of less than 1 if the outcome probability is underweighted in the decision. What happens if we move from a low wealth state to a high wealth state? Two things: marginal utility falls if the utility is concave, and the value of the derivative of the weighting function changes. Commonly used weighting functions overweight low-probability extreme events and underweight intermediate events.

If the payoff distribution incorporates a low-probability high payoff event, the value of the derivative of the weighting function may rise. As the pricing kernel is proportional to the product of the marginal utility and the derivative of the weighting function, the pricing kernel does not necessarily fall as we move from a low wealth state to a high wealth state. This analysis is equivalent to that used to explain the purchase of lottery tickets by investors with concave utility functions (Quiggin, 1991).

The cumulative prospect theory model (Tversky & Kahneman, 1992) adds a reference point to the model above and incorporates an S-shaped utility function in which marginal utilities are rising below a reference point, and falling above it. This creates further possibilities for non-monotonicity of the pricing kernel (Hens & Reichlin, 2012).

In our approach, the marginal option investor perceives an expanded state space with a disaster state included, whereas the marginal equity investor does not. We find that no matter how small the difference in state space perception is, there exists a range of option prices in which a realistic non-monotonic pricing kernel arises.

In this context, it is important to recall that non-monotonic pricing kernels are typically only seen when options data is used. Our results support the argument in Jackwerth (2000) and in Constantinides et al. (2009) that the source of non-monotonicity is likely to be the relative mispricing of options when compared with underlying stocks. Differential awareness allows a minor belief difference (pertaining to only a portion in the left-tail) to generate a realistic non-monotonic pricing kernel.

Another explanation of the pricing kernel puzzle related to our approach is put forward in Shefrin (2005). In his model, there is a mix of optimistic and pessimistic investors. Optimistic investors overestimate expected stock returns and underestimate volatility, whereas pessimistic investors underestimate expected returns and overestimate volatility. Assuming a utility function with constant relative risk aversion, the probability belief of a representative agent is constructed. The resulting pricing kernel is proportional to the product of marginal utility and relative belief error. As we move from a low wealth state to a high wealth state, marginal utility falls; however, the relative belief error may rise, leading to a non-monotonic pricing kernel. Zeigler (2007) considers this explanation and concludes that the degree of heterogeneity required to generate the required non-monotonic pricing kernel is implausibly large. In contrast, our approach requires only a minor difference in state space perception across option and equity traders in order to generate a realistic non-monotonic pricing kernel.

The rank dependent utility and the heterogeneous belief explanations are similar in the sense that they both lead to a pricing kernel that is proportional to the product of marginal utility and another function. In the case of rank dependent utility, that function is the derivative of a probability weighting function, whereas in the heterogeneous belief case, it is the relative belief error of the representative investor. In both cases, it is the shape of this other function that gives rise to a non-monotonic pricing kernel. Unsurprisingly, large distortions in probability weighting functions or large errors in beliefs are required to get to the desired shape of the pricing kernel.

5. Conclusions

The pricing kernel is a bridge that connects finance to economic theory. It provides a connection between asset pricing and the basic economic concept of scarcity. It captures the intuition that the marginal value of a dollar in a good state is less than the marginal value of a dollar in a bad state. A large literature documents that the empirical pricing kernel is oddly shaped and inconsistent with finance and economic theory. In this article, we put forward a new explanation for the odd shape of the empirical pricing kernel, based on the notion of differential awareness. We show that if the marginal trader in options perceives an expanded state space with a disaster state included, whereas, the marginal equity investor perceives a state space without a disaster

state, then the resulting empirical pricing kernel may be locally increasing. Furthermore, we show that the oscillating shape of the empirical pricing kernel is easily generated with such differential awareness.

Funding

The authors received no direct funding for this research.

Author details

Hammad Siddiqi¹
E-mail: hammadniu@yahoo.com
E-mail: hsiddiqu@usc.edu.au
John Quiggin²
E-mail: j.quiggin@uq.edu.au

¹ University of the Sunshine Coast, Sunshine Coast, Australia.

² The University of Queensland, Brisbane, Australia.

Citation information

Cite this article as: The pricing kernel puzzle: A behavioral explanation, Hammad Siddiqi & John Quiggin, *Cogent Economics & Finance* (2019), 7: 1684609.

Notes

1. <http://faculty.econ.ucdavis.edu/faculty/schipper/unaw.htm> provides a bibliography.
2. Cuesdeanu and Jackwerth (2016b) confirm non-monotonicity with forward looking data and critically examine a few studies that did not find non-monotonicity such as Linn and Shumway (2015).
3. The earliest example is perhaps Bates (1991). For more recent examples, see Bollerslev and Todorov (2011); Bockus, Chernov, and Martin (2011); Seo and Wachter (2015); and Siriwardane (2015).
4. See Lakonishok et al. (2007) and Lemmon and Ni (2013).
5. For a bibliography of research on unawareness, see:
6. See Fournier and Jacobs (2017) for a discussion of institutional features of index option markets. For the related impact of finding costs and margin requirements, see Leippold and Su (2014).
7. Excel Solver is used to find a solution.

References

- Ait-Sahalia, Y., & Lo, A. (2000). Nonparametric risk management and implied risk aversion. *Journal of Econometrics*, 94, 9–51. doi:10.1016/S0304-4076(99)00016-0
- Backus, D., Chernov, M., & Martin, I. (2011). Disasters implied by equity index options. *Journal of Finance*, 66(6), 1969–2012. doi:10.1111/j.1540-6261.2011.01697.x
- Baker, M., & Wurgler, J. (2007). Investor sentiment in the stock market. *Journal of Economic Perspectives*, 21(2), 129–151. doi:10.1257/jep.21.2.129
- Bakshi, G., Madan, D., & Panayotov, G. (2010). Returns of claims on the upside and the viability of U-shaped pricing kernels. *Journal of Financial Economics*, 97(1), 130–154. doi:10.1016/j.jfineco.2010.03.009
- Barone-Adesi, G., Mancini, L., & Shefrin, H. (2014). Sentiment, risk aversion, and time preference. Swiss Finance Institute Research Paper No.12-21.
- Bates, D. (1991). The crash of '87: Was it expected? The evidence from options markets. *Journal of Finance*, 46(3), 1009–1044. doi:10.1111/j.1540-6261.1991.tb03775.x
- Beare, B., & Schmidt, L. (2016). An empirical test of pricing kernel monotonicity. *Journal of Applied Econometrics*, 31(2), 338–356. doi:10.1002/jae.2422
- Berndt, A. (2015). A credit spread puzzle for reduced-form models. *Review of Asset Pricing Studies*, 5(1), 48–91. doi:10.1093/rapstu/rav002
- Bollen, N. P. B., & Whaley, R. E. (2004). Does net buying pressure affect the shape of the implied volatility functions?. *Journal of Finance*, 59(2), 711–753. doi:10.1111/j.1540-6261.2004.00647.x
- Bollerslev, T., & Todorov, V. (2011). Estimation of jump tails. *Econometrica*, 79(Issue6), 1727–1783.
- Chadhuri, R., & Schroder, M. (2015). Monotonicity of the stochastic discount factor and expected option returns. *Review of Financial Studies*, 28(5), 1462–1505. doi:10.1093/rfs/hhv011
- Chen, Z., Lookman, A. A., Schürhoff, N., & Seppi, D. J. (2014). Rating-based investment practices and bond market segmentation. *Review of Asset Pricing Studies*, 4(2), 162–205. doi:10.1093/rapstu/rau005
- Christoffersen, P., Jacobs, K., & Heston, S. (2013). Capturing option anomalies with a variance dependent pricing kernel. *Review of Financial Studies*, 26, 1963–2006. doi:10.1093/rfs/hht033
- Constantinides, G., Jackwerth, J., & Perrakis, S. (2009). Mispricing of S&P 500 index options. *Review of Financial Studies*, 22(3), 1247–1277. doi:10.1093/rfs/hhn009
- Constantinides, G., Jackwerth, J., & Perrakis, S. (2011). Are options on index futures profitable for risk averse investors? Empirical evidence. *Journal of Finance*, 66(4), 1407–1437. doi:10.1111/j.1540-6261.2011.01665.x
- Constantinides, G., & Perrakis, S. (2002). Stochastic dominance bounds on derivatives prices in a multiperiod economy with proportional transaction costs. *Journal of Economic Dynamics and Control*, 26, 1323–1352. doi:10.1016/S0165-1889(01)00047-1
- Constantinides, G. M., & Perrakis, S. (2007). Stochastic dominance bounds on american option prices in markets with frictions. *Review of Finance*, 11, 71–115. doi:10.1093/rof/rfl001
- Coval, J., Jurek, J., & Stafford, E. (2009). Symposium: Early stages of the credit crunch: The economics of structured finance. *Journal of Economic Perspectives*, 23(1), 3–25. doi:10.1257/jep.23.1.3
- Cuesdeanu, H. (2016). 'Empirical pricing kernels: A tale of two tails and volatility. Retrieved from <https://ssrn.com/abstract=2870372>
- Cuesdeanu, H., & Jackwerth, J. C. (2016a). *The pricing kernel puzzle: Survey and outlook*. University of Konstanz: Working paper.
- Cuesdeanu, H., & Jackwerth, J. C. (2016b). The pricing kernel in forward looking data. Working paper, University of Konstanz.
- Foote, G., & Willen. (2012). Why did so many people make so many ex post bad decisions? The causes of the foreclosure crisis. Federal Reserve Bank of Boston Public Policy Discussion Paper 12–2. doi:10.1094/PDIS-11-11-0999-PDN.
- Fournier, & Jacobs (2017). Fournier, Mathieu and Jacobs, Kris, A Tractable framework for option pricing with dynamic market maker inventory and wealth. Rotman School of Management Working Paper No. 2334842.
- Garleanu, P., Potoshman, L. H., & Potoshman, A. M. (2009). Demand based option pricing. *Review of Financial Studies*, 22, 4259–4299. doi:10.1093/rfs/hhp005
- Gennaioli, S., & Vishny, A. (2012). Neglected risks, financial innovation, and financial fragility. *Journal of Financial Economics*, 104, 452–468. doi:10.1016/j.jfineco.2011.05.005
- Grant, S., & Quiggin, J. (2013). Inductive reasoning about unawareness. *Economic Theory*, 54(3), 717–55. doi:10.1007/s00199-012-0734-y

- Halpern, J., & Rego, L. (2006), 'Extensive games with possibly unaware players', *Proceedings of the Fifth International Joint Conference on Autonomous Agents and Multiagent Systems*, Hakodate, Japan, May 8–12
- Heifetz, A., Meier, M., & Schipper, B. (2006). Interactive unawareness. *Journal of Economic Theory*, 130(1), 78–94. doi:10.1016/j.jet.2005.02.007
- Hens, T., & Reichlin, C. (2012). Three solutions to the pricing kernel puzzle. *Review of Finance*, 17(3), 1065–1098. doi:10.1093/rof/rfs008
- Jackwerth. (2004). Option-implied risk-neutral distributions and risk-aversion. Research Foundation of AIMR. Charlottesville, USA.
- Jackwerth, J. C. (2000). Recovering risk aversion from option prices and realized returns. *Review of Financial Studies*, 13(2), 433–51. doi:10.1093/rfs/13.2.433
- Karni, E., & Vierø, M. (2013). Reverse bayesianism: A choice-based theory of growing awareness. *American Economic Review*, 103(7), 2790–2810. doi:10.1257/aer.103.7.2790
- Lakonishok, L., Pearson, N., & Poteshman, A. (2007). Option market activity. *Review of Financial Studies*, 20(3), 813–857. doi:10.1093/rfs/hhl025
- Leippold, & Su (2014). Collateral smile. Swiss Finance Institute Research Paper No. 11–51. Retrieved from <https://ssrn.com/abstract=1956449>
- Lemmon, M., & Ni, S. X. (2013). Differences in trading and pricing between stock and index options. *Management Science*, 60, 1985–2001. doi:10.1287/mnsc.2013.1841
- Levy, H. (1985). Upper and lower bounds of put and call option value: stochastic dominance approach. *The Journal of Finance*, 40, 1197–1217. doi:10.1111/j.1540-6261.1985.tb02372.x
- Linn, S., & Shumway, (2015), Pricing kernel monotonicity and conditional information. Retrieved from <https://ssrn.com/abstract=2383527>
- Minsky, H. (1986). *Stabilizing an unstable economy*. Yale University Press: New Haven, CT.
- Oancea, & Perrakis, (2010). Jump diffusion option valuation without a representative investor: A stochastic dominance approach. Working Paper, Concordia University.
- Perrakis, S. (1986). Option bounds in discrete time: Extensions and the pricing of the american put. *Journal of Business*, 59, 119–141. doi:10.1086/296319
- Perrakis, S. (1988). Preference-free option prices when the stock returns can go up, go down or stay the same. In F. J. Fabozzi (Ed.), *Advances in futures and options research* (pp.209–235). Greenwich, Conn: JAI Press.
- Perrakis, S., & Ryan, P. J. (1984). Option pricing bounds in discrete time. *Journal of Finance*, 39(2), 519–525. doi:10.1111/j.1540-6261.1984.tb02324.x
- Palkovitchenko, V., & Zhao, F. (2013). Probability weighting functions implied in option prices. *Journal of Financial Economics*, 107(3), 580–609. doi:10.1016/j.jfineco.2012.09.008
- Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior and Organization*, 3, 323–343. doi:10.1016/0167-2681(82)90008-7
- Quiggin, J. (1991). On the optimal design of lotteries. *Economica*, 58(1), 1–16. doi:10.2307/2554972
- Quiggin, J. (2016). The value of information and the value of awareness. *Theory and Decision*, 80(2), 167–85. doi:10.1007/s11238-015-9496-x
- Ritchken, P. (1985). On option pricing bounds. *Journal of Finance*, 40(4), 338–356. doi:10.1111/j.1540-6261.1985.tb02373.x
- Ritchken, P., & Kuo, S. (1988). Option bounds with finite revision opportunities. *The Journal of Finance*, 4(3), 301–308. doi:10.1111/j.1540-6261.1988.tb03940.x
- Rosenberg, J., & Engle, R. (2002). Empirical pricing kernel. *Journal of Financial Economics*, 64(3), 341–372. doi:10.1016/S0304-405X(02)00128-9
- Rubinstein, M. (1994). Implied binomial trees. *Journal of Finance*, 49(3), 1771–1818. doi:10.1111/j.1540-6261.1994.tb00079.x
- Seo, B., & Wachter, J. (2015). Option prices in a model with stochastic disaster risk. working paper, University of Pennsylvania.
- Shefrin, H. (2005). *A behavioral approach to asset pricing*. London: Academic Press.
- Shleifer, A., & Vishny, R. W. (1997). The limits of arbitrage. *Journal of Finance*, 52(1), 35–55. doi:10.1111/j.1540-6261.1997.tb03807.x
- Siriwardane, E. (2015). The probability of rare disasters: Estimation and implications.' Harvard Business School, working paper, no. 16–061.
- Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4), 297–323. doi:10.1007/BF00122574
- Zeigler, A. (2007). Why does implied risk aversion smile?. *Review of Financial Studies*, 20(3), 859–904. doi:10.1093/rfs/hhl023

Appendix A Option Pricing Bounds

Suppose the distribution of stock returns at $t + \Delta t$, given current stock price, $P(z_{t+\Delta t}|S_t)$, is discrete. Specifically, assume that there are $n + 1$ states indexed from the lowest to the highest stock return by $j = 0, 1, 2, \dots, n$. That is, the distribution is given by $(z_{j(t+\Delta t)}, \pi_{j(t+\Delta t)})$, where $\pi_{j(t+\Delta t)}$ is the probability associated with the j th state.

The following linear programming set-up can be used to find the option pricing bounds:

$$\max_{m_{j(t+\Delta t)}} \sum_{j=0}^{j=n} C_{t+\Delta t}(S_t(1 + z_{j(t+\Delta t)})) \pi_{j(t+\Delta t)} m_{j(t+\Delta t)}$$

$$\left(\min_{m_{j(t+\Delta t)}} \sum_{j=0}^{j=n} C_{t+\Delta t}(S_t(1 + z_{j(t+\Delta t)})) \pi_{j(t+\Delta t)} m_{j(t+\Delta t)} \right)$$

subject to (A1)

$$1 = \sum_{j=0}^{j=n} \pi_{j(t+\Delta t)} (1 + z_{j(t+\Delta t)}) m_{j(t+\Delta t)}$$

$$1 = R \sum_{j=0}^{j=n} \pi_{j(t+\Delta t)} m_{j(t+\Delta t)}$$

$$m_{0t+\Delta t} \geq m_{2t+\Delta t} \geq \dots \geq m_{nt+\Delta t} \geq 0.$$

Instead of expressing the solution of (A1) in terms of physical probabilities, as done in Ritchken (1985), it is more useful to express the solution by using the associated risk neutral probabilities. Oancea and Perrakis (2010) provide the following solution. The solution is expressed in terms of the following conditional expectation:

$$\hat{z}_{jt+\Delta t} = \frac{\sum_{i=0}^j z_{it+\Delta t} \pi_{it+\Delta t}}{\sum_{i=0}^j \pi_{it+\Delta t}}$$

Call Upper Bound

$$\bar{C}_t = \frac{1}{R} E^{U_t} [C_{t+\Delta t} (S_t (1 + z_{t+\Delta t}))]$$

The superscript U_t indicates that the above expectation is taken at time t with respect to the following risk neutral distribution:

$$U_{0t} = \frac{R - 1 - \hat{z}_{0t+\Delta t}}{\hat{z}_{nt+\Delta t} - \hat{z}_{0t+\Delta t}} \pi_{0t+\Delta t} + \frac{\hat{z}_{nt+\Delta t} + 1 - R}{\hat{z}_{nt+\Delta t} - \hat{z}_{0t+\Delta t}}$$

$$U_{jt} = \frac{R - 1 - \hat{z}_{0t+\Delta t}}{\hat{z}_{nt+\Delta t} - \hat{z}_{0t+\Delta t}} \pi_{jt+\Delta t}, j = 1, \dots, n$$

Call Lower Bound

$$L_t = \frac{1}{R} E^{L_t} [C_{t+\Delta t} (S_t (1 + z_{t+\Delta t}))]$$

The superscript L_t indicates that the above expectation is taken at time t with respect to the following risk neutral distribution:

$$L_{jt} = \frac{\hat{z}_{h+1,t+\Delta t} + 1 - R}{\hat{z}_{h+1,t+\Delta t} - \hat{z}_{ht+\Delta t}} \left(\frac{\pi_{jt+\Delta t}}{\sum_{k=0}^h \pi_{kt+\Delta t}} \right) + \frac{R - 1 - \hat{z}_{ht+\Delta t}}{\hat{z}_{h+1,t+\Delta t} - \hat{z}_{ht+\Delta t}} \left(\frac{\pi_{jt+\Delta t}}{\sum_{k=0}^{h+1} \pi_{kt+\Delta t}} \right), j = 0, \dots, h$$

$$L_{h+1,t} = \frac{R - 1 - \hat{z}_{ht+\Delta t}}{\hat{z}_{h+1,t+\Delta t} - \hat{z}_{ht+\Delta t}} \left(\frac{\pi_{h+1,t+\Delta t}}{\sum_{k=0}^{h+1} \pi_{kt+\Delta t}} \right), L_{jt} = 0, j > h + 1, \tag{A2}$$

where h is a state index such that $\hat{z}_{ht+\Delta t} \leq R - 1 \leq \hat{z}_{h+1,t+\Delta t}$.

Appendix B

There are two cases: 1) Truncation does not change the location of state h . Recall, this is the state index at which the conditional stock return expectation, \hat{z}_h , is just below $R - 1$, whereas at $h + 1$, the conditional expectation, \hat{z}_{h+1} , is just above $R - 1$. 2) Truncation shifts h to the left. We consider both cases in turn.

Case 1

From (A2), the risk neutral probability of the lower bound is given by:

$$L_{jt} = \frac{\hat{z}_{h+1,t+\Delta t} + 1 - R}{\hat{z}_{h+1,t+\Delta t} - \hat{z}_{ht+\Delta t}} \left(\frac{\pi_{jt+\Delta t}}{\sum_{k=0}^h \pi_{kt+\Delta t}} \right) + \frac{R - 1 - \hat{z}_{ht+\Delta t}}{\hat{z}_{h+1,t+\Delta t} - \hat{z}_{ht+\Delta t}} \left(\frac{\pi_{jt+\Delta t}}{\sum_{k=0}^{h+1} \pi_{kt+\Delta t}} \right), \quad j = 0, \dots, h$$

$$L_{h+1,t} = \frac{R - 1 - \hat{z}_{ht+\Delta t}}{\hat{z}_{h+1,t+\Delta t} - \hat{z}_{ht+\Delta t}} \left(\frac{\pi_{h+1,t+\Delta t}}{\sum_{k=0}^{h+1} \pi_{kt+\Delta t}} \right), \quad L_{jt} = 0, j > h + 1$$

where h is a state index such that $\hat{z}_{ht+\Delta t} \leq R - 1 \leq \hat{z}_{h+1,t+\Delta t}$.

That is, the risk neutral probability has the form:

$$L_{jt} = Z \times w_{j,h} + (1 - Z) \times w_{j,h+1}$$

$$L_{h+1,t} = (1 - Z) \times w_{h+1,h+1}.$$

With truncation, both \hat{z}_h and \hat{z}_{h+1} rise, however \hat{z}_h rises more than \hat{z}_{h+1} . Since h does not change in case 1, Z must rise and $1 - Z$ must fall. With truncation, both w_{jh} and $w_{j,h+1}$ rise but the increase in w_{jh} is greater. As probabilities must sum to one, it follows that the probability $L_{h+1,t}$ falls and probabilities L_{jt} rise. For the lower bound, all L_{jt} must be equal. It follows that the probability mass shifts from $L_{h+1,t}$ and is distributed uniformly across probabilities L_{jt} . Consequently, the truncated lower bound must be lower.

Case 2

The proof follows as in case 1. In this case, Z must fall and $1 - Z$ must rise. However, the rise in w_{jh} outweighs the fall in Z . As before, the probability mass shifts to the left uniformly, implying that the truncated lower bound must be lower.

Appendix C

Proof of Proposition 3

Follows from the proof of proposition 1 by realizing that both \hat{z}_0 and \hat{z}_n fall more if a bigger portion is truncated. Hence, a relatively bigger mass is shifted to the right causing a greater increase in the upper bound.



© 2019 The Author(s). This open access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.

You are free to:

Share — copy and redistribute the material in any medium or format.

Adapt — remix, transform, and build upon the material for any purpose, even commercially.

The licensor cannot revoke these freedoms as long as you follow the license terms.

Under the following terms:

Attribution — You must give appropriate credit, provide a link to the license, and indicate if changes were made.

You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use.

No additional restrictions

You may not apply legal terms or technological measures that legally restrict others from doing anything the license permits.



***Cogent Economics & Finance* (ISSN: 2332-2039) is published by Cogent OA, part of Taylor & Francis Group.**

Publishing with Cogent OA ensures:

- Immediate, universal access to your article on publication
- High visibility and discoverability via the Cogent OA website as well as Taylor & Francis Online
- Download and citation statistics for your article
- Rapid online publication
- Input from, and dialog with, expert editors and editorial boards
- Retention of full copyright of your article
- Guaranteed legacy preservation of your article
- Discounts and waivers for authors in developing regions

Submit your manuscript to a Cogent OA journal at www.CogentOA.com

