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Projection Estimators for Structural Impulse Responses*

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Abstract

In this paper we provide a general two-step framework for linear projection estimators of impulse responses in structural vector autoregressions (SVARs). This framework is particularly useful for situations when structural shocks are identified from information outside the VAR (e.g. narrative shocks). We provide asymptotic results for statistical inference and discuss situations when standard inference is valid without adjustment for generated regressors, autocorrelated errors or non-stationary variables. We illustrate how various popular SVAR models fit into our framework. Furthermore, we provide a local projection framework for invertible SVAR models that are estimated by instrumental variables (IV). This class of models results in a set of quadratic moment conditions used to obtain the asymptotic distribution of the estimator. Moreover, we analyse generalized least squares (GLS) versions of the projections to improve the efficiency of the projection estimators. We also compare the finite sample properties of various estimators in simulations. Two highlights of the Monte Carlo results are (i) for invertible VARs our two-step IV projection estimator is more efficient compared to existing projection estimators and (ii) using the GLS projection variant with residual augmentation leads to substantial efficiency gains relative to standard OLS/IV projection estimators.

I. Introduction

The analysis of dynamic effects in vector autoregressive (VAR) models by means of impulse responses has become a standard tool in empirical macroeconomics (cf. Kilian and Lütkepohl, 2017). Following Sims (1980) the dynamic effects of shocks are typically measured by the moving average (MA) coefficients derived from the finite-order VAR representation of the time series. In recent years it has become popular to estimate the effects of structural shocks by ‘local projections’ (e.g. Jordà, 2005; Jordà, Schularick, and Taylor, 2015; Ramey and Zubairy, 2018; Jordà, Schularick, and Taylor, 2020). This method is based on a *direct representation* of the time series vector shifted h periods

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ahead, whereas the traditional method traces out the impulse responses *iteratively* from the first up to the h th period.

The use of (local) projection estimators in structural VAR (SVAR) applications has been motivated by different arguments. First, local projection estimators are more robust against model misspecification. Second, in some applications there may not exist a suitable VAR framework (e.g. if shocks are determined outside the VAR models). Third, the VAR model may not be invertible with respect to the structural shocks (see e.g. Stock and Watson, 2018). Furthermore, asymptotic inference on structural impulse responses based on iterated VARs may be inaccurate as the quantity of interest is a highly nonlinear transformation of the VAR parameters. Inference for projection estimators, which are linear in the parameters, may be easier to implement and statistical inference is typically more accurate in small samples. Consequently, despite being potentially less efficient than iterated response estimators, local projections are nowadays a popular tool in empirical economics (for a review of the earlier literature see Ramey, 2016).

Our paper considers a general modelling framework for linear projection estimators of impulse responses in SVAR models and makes the following contributions: First, we adapt a two-step framework for shock identification and response estimation. This generalizes the VAR approach and can incorporate structural shocks that stem from a different information set (e.g. narrative or high-frequency shocks). Second, we provide an asymptotic framework for statistical inference on the projection estimator, which takes into account that the structural shocks are estimated in the first step. Third, we propose a new local projection estimator (2S-IV) for invertible SVARs identified by instrumental or proxy variables and compare it to existing methods. Although not robust against non-invertibility, our simulations indicate that in invertible settings this estimator provides substantial efficiency gains over existing projection estimators. Based on our approach, we derive a test for non-fundamental shocks that is similar to the test proposed by Plagborg-Møller and Wolf (2022). Fourth, we characterize applications where standard regression inference applies no matter of serial correlation, generated regressors or non-stationary time series. Fifth, we provide guidance on how to adapt projection estimators to various popular identification schemes. Sixth, we argue that the iterative VAR approach is asymptotically equivalent to a particular GLS version of the projection estimator. Finally, we compare the small sample properties of various estimators and of tests for non-fundamental shocks.

We stress that our paper does not offer any new reasons for preferring local projections over iterated VARs in empirical practice. Rather we start from the fact that many empirical economists apply this methodology in order to assess the effects of macroeconomic shocks. Therefore, we analyse popular strategies for identifying structural shocks such as recursive (Cholesky) schemes, proxy-VARs, narrative shocks, systems of simultaneous equations (the AB-model) and shocks identified by long-run restrictions. We show how all these popular variants of SVARs can be cast into a general framework consisting of two separate steps, the identification of shocks and the estimation of impulse responses. More formally the analysis can be characterized by two different steps (e.g. Ramey, 2016, Sec. 2.4):

$$\varepsilon_{j,t} = f(x_{j,t}, \beta_j), \quad (1)$$

$$y_{i,t+h} = \theta_{ij}^h \varepsilon_{j,t} + z'_{j,t} \pi_{ij}^h + e_{ij,t}^h, \quad (2)$$

where $\varepsilon_{j,t}$ denotes the j th structural shock ($j \in \{1, \dots, k\}$), $x_{j,t}$ is a vector of time series used to identify the shocks, $f(\cdot)$ is a (possibly nonlinear) function, β_j is a vector of structural parameters, $y_{i,t}$ denotes the target variable and $z_{j,t}$ is a vector of additional control variables, included to improve the efficiency of the estimator for θ_{ij}^h . The control variables should (i) have predictive power for $y_{i,t+h}$, (ii) be uncorrelated with the error $e_{ij,t}^h$, and (iii) be uncorrelated with the structural shock $\varepsilon_{j,t}$, because otherwise some part of the impulse response would be assigned to the control variables.

Equation (1) is used to identify the shocks based on a set of economic time series such that (i) the shock is unpredictable with respect to the relevant information set \mathcal{I}_t that typically includes the past of $x_{j,t}$ and $z_{j,t}$ and (ii) $\varepsilon_{j,t}$ is orthogonal to all other shocks. Equation (2) is called the *projection step*, where the parameter θ_{ij}^h measures the effect of the j th shock on the i th variable h steps ahead.

It is important to notice that the projection step (2) requires an estimate of the structural shock. If the shock is identified as a linear combination of the VAR residuals, it is assumed that the corresponding MA representation is invertible. Assume in contrast that the information sets employed in (1) and (2) are different. For example, the shock may be identified by using a high-frequency identification scheme (e.g. Wright, 2012; Matheson and Stavrev, 2014; Gertler and Karadi, 2015; Nakamura and Steinsson, 2018), whereas the projection step involves low-frequency variables only. It is well known (e.g. Leeper, Walker, and Yang, 2013) that if the shock is constructed from a different information set than the one used in the VAR for tracing out the impulse responses, then the moving average representation with respect to the structural shock may become non-invertible. In our framework this is not a serious problem as long as we correctly specify the shock in the identification step by using the suitable information set. For the projection step we do not require invertibility of the VAR representation. If the structural shock can be expressed as a linear combination of the VAR innovations, then the shock is called fundamental. Our framework gives rise to the simple test procedure of Plagborg-Møller and Wolf (2022) that allows us to indicate whether the shock is fundamental or not.

Our paper is related to previous work on projection estimators. Jordà (2005), for example, treats the structural shocks as given and focuses on the projection step only. For statistical inference, however, it is important to consider both estimation steps simultaneously. Our framework is also related to Plagborg-Møller and Wolf (2021), who consider a shock that is represented as a change in some observed variable $x_{j,t}$ when controlling for some other (potentially confounding) variables.¹ Montiel Olea and Plagborg-Møller (2021) propose lag augmentation for inference in the projection step (2) but again treat the structural parameters in the identification step as given. In contrast, we consider estimation and inference of the parameters in both the identification and projection step based on a method of moments framework.

Furthermore, we offer some theoretical and practical guidance for incorporating some popular structural VAR specifications in the framework characterized by (1) and (2). In

¹ Since their paper focuses on the identification of impulse responses rather than statistical inference in finite samples, their paper treats the structural parameters β_j in (1) as given. In equation (8) of Plagborg-Møller and Wolf (2021) the vector b is assumed to be known, whereas in equation (1) of their paper it is assumed that the shock can be represented by an observable variable that is not correlated with the error term. This is the case for recursive structural identification schemes.

particular we discuss how valid inference is achieved if the structural shock is estimated from the structural model (1). In many cases the estimator can be interpreted as a (nonlinear) method of moments estimator, where the asymptotic distribution follows from the asymptotic analysis of (G)MM estimators. We first discuss OLS projections, the case where equation (1) and (2) are both estimated by OLS. Then, we consider instrumental variable (IV) projections, that is, we explain how the framework can be employed when (1) is estimated by IV methods. Furthermore, we introduce different variants of GLS projection estimators that are all tailored to improve the efficiency and compare their asymptotic properties.

The remainder of this paper is structured as follows: section II discusses OLS projections and their asymptotic properties and section III considers IV projections. Section IV suggests refinements leading to GLS projections. Section V presents Monte Carlo evidence on the relative performance of different projection estimators and invertibility tests. Section VI concludes.

II. OLS projections

In this section we analyse strategies, where the structural shock can be identified from OLS regressions. After briefly reviewing the iterated VAR framework, we compare our projection estimator to the original projection estimator of Jordà (2005) in the recursive (triangular) identification scheme ('Cholesky VAR'). We then illustrate how 'narrative' structural shocks obtained as residuals from OLS regressions fit into our framework and discuss conditions that allow us to adopt standard OLS inference to the response estimator.

The *iterated* approach is based on the VAR(p) model $y_t = A_1 y_{t-1} + \dots + A_p y_{t-p} + u_t$, where y_t is a k -dimensional vector of time series and u_t is a $k \times 1$ vector of white noise innovations with $\mathbb{E}(u_t) = 0$ and $\mathbb{E}(u_t u_t') = \Sigma$ (positive definite). The inclusion of further deterministic regressors like constants, trends or dummy variables is unproblematic and is therefore suppressed. If all roots of the lag polynomial $A(L) = I - A_1 L - \dots - A_p L^p$ are outside the unit circle, the MA representation of the VAR system is given by $y_t = u_t + \Phi_1 u_{t-1} + \Phi_2 u_{t-2} + \dots$, where the $k \times k$ matrices Φ_i result from the inverse $A(L)^{-1} = I + \Phi_1 L + \Phi_2 L^2 + \dots$ (e.g. Lütkepohl, 2005). In structural VARs, one often assumes $\varepsilon_t = \Gamma u_t$ and using the iterated approach obtains the responses to structural shocks ε_t from $\Phi_i \Gamma^{-1}$, $i = 0, 1, \dots$. In the case of unit roots or cointegration, the structural impulse responses from the *iterated* approach can still be obtained from $\Phi_i \Gamma^{-1}$ but the Φ_i can no longer be interpreted as MA coefficients (Lütkepohl and Reimers, 1992). The unrestricted estimator of VAR coefficients and any (nonlinear) functions (such as iterated impulse responses) at finite horizons are consistent. As shown by Phillips (1998), however, estimated impulse responses in an unrestricted VAR with unit roots may be inconsistent if the horizons increase with the sample size. Reduced rank regressions render consistent estimates of the impulse responses, provided the co-integrating rank is correctly specified or estimated by a consistent selection rule. In contrast, the projection estimators discussed below do not require the correct specification of the co-integrating rank.

Our analysis of projection estimators starts by comparing the estimator based on (1) and (2) to the original projection approach of Jordà (2005) in a recursive (Cholesky) VAR. As above, we use $\varepsilon_t = \Gamma u_t$, but assume that Γ is a lower triangular matrix and unit diagonal elements (because of a unit-effect normalization). To illustrate the difference of estimators, we use a bivariate VAR(1) and consider the response of the *first variable* $y_{1,t}$ to the *second shock* $\varepsilon_{2,t}$ at horizon h . For this DGP, it is easy to see (section A.1 in Data S1), that (1) and (2) result in

$$\begin{aligned} \varepsilon_{2,t} &= y_{2,t} - \beta_1 y_{1,t} - \beta_2 y_{1,t-1} - \beta_3 y_{2,t-1}, \\ y_{1,t+h} &= \theta_{12}^h \varepsilon_{2,t} + \pi_1^* \varepsilon_{1,t} + \pi_2^* y_{1,t-1} + \pi_3^* y_{2,t-1} + e_{12,t}^h. \end{aligned}$$

The shock $\varepsilon_{2,t}$ can be estimated as the residual from an OLS regression of $y_{2,t}$ on $y_{1,t}$, $y_{1,t-1}$, and $y_{2,t-1}$. The control variable are $y_{1,t-1}$, $y_{2,t-1}$ and $\varepsilon_{1,t}$. Since the shock $\varepsilon_{1,t}$ is a linear combination of $y_{1,t}$, $y_{1,t-1}$ and $y_{2,t-1}$, one may equivalently use the controls $z_{2,t} = (y_{1,t}, y_{1,t-1}, y_{2,t-1})'$. Furthermore, since $\varepsilon_{2,t}$ is a linear combination of $y_{2,t}$ and the vector $z_{2,t}$, we may estimate the response θ_{12}^h by running the regression

$$y_{1,t+h} = \theta_{12}^h y_{2,t} + \pi_1 y_{1,t} + \pi_2 y_{1,t-1} + \pi_3 y_{2,t-1} + e_{12,t}^h. \tag{3}$$

A generalization of this projection approach to higher-dimensional models with a triangular identification schemes is presented below.

The projection estimator based on (3) is similar but not identical to the estimator suggested by Jordà (2005), which is based on the direct h -step ahead forecast regression. We show in section A.1 of Data S1 that by using $y_t^* = (y_{1,t}^*, y_{2,t}^*)' = \Gamma y_t$, the projection equation in our bivariate recursive VAR(1) for $y_{1,t}$ from Jordà's approach can be written as

$$y_{1,t+h} = \theta_{11}^h y_{1,t}^* + \theta_{12}^h y_{2,t}^* + e_{1,t+h}. \tag{4}$$

Comparing (3) and (4) it turns out that the former approach decomposes $y_t^* = \Gamma A y_{t-1} + \varepsilon_t$ into the vector of structural shocks ε_t and a linear combination of the control variables in $z_{2,t}$. In other words, (4) results from (3) by imposing the restrictions $\pi_1 = \theta_{11}^h + \theta_{12}^h \gamma_{21}$ and $\pi_2 = \pi_3 = 0$, where γ_{21} is the (2, 1) element of Γ . Therefore, if Γ is known, the estimate of θ_{12}^h from (4) is more efficient than estimating the impulse response from (3). In practice the matrix Γ is unknown and has to be replaced by a consistent estimator. Then it is no longer clear whether Jordà's estimator is more efficient than using (3). The reason is that the estimation error $\widehat{\Gamma} - \Gamma$ increases the asymptotic variance when using $\widehat{y}_t^* = \widehat{\Gamma} y_t$ instead of y_t^* in (4), see also Kilian and Kim (2011). Indeed, simulation results (cf. section V and Montiel Olea and Plagborg-Møller, 2021) suggest that both approaches often have similar small sample properties and for some specifications the augmented regression (3) possesses superior sampling properties. Furthermore, as we discuss below, using (3) for estimating the impulse response facilitates statistical inference, as no corrections for generated regressors, autocorrelation or unit roots are required.

In the example above, the structural shocks ε_t can be represented as a linear combination of VAR innovations u_t . In such cases the shocks are labelled as *fundamental* or *internal*

(e.g. Kilian and Lütkepohl, 2017). If, on the other hand, the shocks involve variables that are not included in the VAR system, we refer to the shocks as *external* shocks. A typical example is the construction of ‘narrative’ shocks, which are obtained from a different information set than the VAR system. We analyse the case where the shock is determined outside the VAR system next and note that in most empirical applications the function $f(x_{j,t}, \beta)$ is linear. For example, the narrative shocks of Romer and Romer (2004) are obtained as the residuals from a linear regression of a set of external variables:

$$x_{0,t} = \beta'x_{1,t} + \varepsilon_t, \tag{5}$$

where we drop the index j for notational convenience. In this regression the vector $x_{1,t}$ is assumed to be uncorrelated with shock ε_t . For instance, in Romer and Romer (2004) $x_{0,t}$ is the change in the intended federal funds rate and the vector $x_{1,t}$ comprises the Greenbook forecast of real output growth, inflation, and unemployment. In a second step, Romer and Romer (2004) estimate an autoregressive distributed lag regression for the target variable y_t of the form

$$\alpha(L)y_t = \phi(L)\widehat{\varepsilon}_t + v_t, \tag{6}$$

with the lag polynomials $\alpha(L) = 1 - \alpha_1L - \alpha_2L^2 - \dots - \alpha_pL^p$ and $\phi(L) = \phi_0 + \phi_1L + \dots + \phi_qL^q$ and $\widehat{\varepsilon}_t$ denotes the residual from the OLS regression (5). The impulse response function is obtained from $\theta(L) = \phi(L)/\alpha(L)$ where the lag polynomials $\phi(L)$ and $\alpha(L)$ are estimated by least-squares using (6). As this method inverts the AR polynomial when computing the MA representation with respect to ε_t , this estimation procedure corresponds to the iterative approach in VAR models. A projection analog of this approach is

$$y_{t+h} = \theta^h\widehat{\varepsilon}_t + \pi_1y_{t-1} + \pi_2y_{t-2} + \dots + \pi_{p^*}y_{t-p^*} + \tilde{e}_{t+h}, \tag{7}$$

where p^* denotes the suitable lag length for approximating the infinite polynomial $\alpha(L)/\phi(L)$ in (6). It is important to note that standard regression inference on θ^h is invalid due to the generated regressor problem analysed in Pagan (1984). As the following proposition shows, this problem may easily be sidestepped by including the variables $x_{1,t}$ in the vector of control variables for the projection step such that $z_t = (x'_{1,t}, y_{t-1}, \dots, y_{t-p})'$.

Proposition 1. Assume that $\varepsilon_{j,t}$ and $y_{i,t+h}$ can be represented as in (1) and (2), where $e^h_{ij,t}$ is a stationary and ergodic process obeying $\mathbb{E}(z'_{j,t}e^h_{ij,t}) = 0$ and $\mathbb{E}|e^h_{ij,t}|^{4+\delta} < \infty$ for some $\delta > 0$. The shock $\varepsilon_{j,t}$ is an i.i.d. sequence with $\mathbb{E}(z'_{j,t}\varepsilon_{j,t}) = 0$ and $\mathbb{E}(e^h_{ij,t}\varepsilon_{j,t}) = 0$. Let $\widehat{\varepsilon}_{j,t} = f(x_{j,t}, \widehat{\beta}_j)$, where $\widehat{\beta}_j - \beta_j = O_p(T^{-1/2})$. The derivative

$$g_{j,t}(\beta_j) = \frac{\partial f(x_{j,t}, \beta_j)}{\partial \beta_j},$$

exists with a constant M such that $\mathbb{E}(\|g_{j,t}(\beta_j)\|) < M < \infty$. If there exists a matrix C such that

$$g_{j,t}(\beta_j) = Cz_{j,t},$$

then (i) the OLS estimator $\widehat{\theta}_{ij}^h$ of θ_{ij}^h in the regression

$$y_{i,t+h} = \theta_{ij}^h \widehat{\varepsilon}_{j,t} + z'_{j,t} \pi_{ij}^h + \tilde{\varepsilon}_{ij,t}^h, \tag{8}$$

is consistent and asymptotically normally distributed. (ii) The usual OLS estimator of the standard error (SE) of $\widehat{\theta}_{ij}^h$ is consistent no matter of the serial correlation of $\tilde{\varepsilon}_{ij,t}^h$.

A proof is given in the Data S1 (section A.2). This proposition shows that augmenting the regression by suitable control variables, $z_{j,t}$, escapes the error-in-variables problem involved by working with estimated shocks. For the case that the estimated shock is a regression residual, our result (i) can be anticipated by the analysis of Pagan (1984) who studies a related situation (his model 4). Furthermore, the proposition shows that inference on the parameter θ_{ij}^h is not affected by the autocorrelation of $\tilde{\varepsilon}_{ij,t}^h$. For a structural identification where the structural parameters are known, Montiel Olea and Plagborg-Møller (2021) obtain a result similar to (ii) using a more general set of assumptions on the VAR innovations, whereas we consider *estimated* structural shocks that are not necessarily linear combinations of VAR innovations.

It is interesting to note that we do not require y_t and z_t to be stationary. Instead we only need to assume that the error $e_{ij,t}^h$ and the shock $\varepsilon_{j,t}$ is stationary. Accordingly, whenever z_t is non-stationary then there exists a co-integration relationship among the elements of the vector (y_t, z_t') . Provided that the impulse response coefficient is attached to the stationary variable $\varepsilon_{j,t}$, standard inference applies to the estimator $\widehat{\theta}_{ij}^h$, see Sims, Stock, and Watson (1990).

Example 1: Narrative shocks. Following Romer and Romer (2004) assume that the shock results from the OLS regression (5). The uncertainty about the shock is represented by the estimation error $x'_{1,t}(\widehat{\beta} - \beta)$. Accordingly, by augmenting the projection with the derivative $g_t = x_{1,t}$ solves the generated regressor problem and standard OLS inference applies to the estimator of $\widehat{\theta}_{ij}^h$.

Example 2: Cholesky VARs. We illustrate the usefulness of Proposition 1 in our earlier example of a triangular identification scheme but with an arbitrary number of time series. The j th structural shock results as

$$\varepsilon_{j,t} = \gamma'_j (y_t - A_1 y_{t-1} - \dots - A_p y_{t-p}) = \beta'_0 y_t + \beta'_1 y_{t-1} + \dots + \beta'_p y_{t-p},$$

where γ'_j is the j th row of Γ and β'_ℓ is the j th row of $-\Gamma A_\ell$ for $\ell = 1, \dots, p$ and $\beta_0 = \gamma_j$. Denote by $\beta_{0,r}$ the r th element of β_0 . For a triangular identification scheme we set $\beta_{0,r} = 0$ for $r > j$. Furthermore, the unit effect normalization implies that the diagonal elements of Γ are equal to unity which implies $\beta_{0,j} = 1$. Accordingly, the derivative with respect to the remaining parameters $\tilde{\beta} = (\beta_{0,1}, \dots, \beta_{0,j-1}, \beta'_1, \dots, \beta'_p)'$ is given by $g_{j,t}(\beta) = \partial f(x_{j,t}, \beta_j) / \partial \tilde{\beta}_j = z_{j,t}$, where $z_{j,t} = (y_{1,t}, \dots, y_{j-1,t}, y'_{t-1}, \dots, y'_{t-p})'$. Accordingly, the matrix C is the identity matrix, $\mathbb{E}(\varepsilon_{j,t} z_t) = 0$ and, therefore, Proposition 1 applies. Thus, the OLS estimator for θ_{ij}^h in the projection step results from $y_{i,t+h} = \theta_{ij}^h \widehat{\varepsilon}_{j,t} + z'_{j,t} \pi_{ij}^h + e_{ij,t}^h$ or

$$y_{i,t+h} = \theta_{ij}^h y_{j,t} + z'_{j,t} \tilde{\pi}_{ij}^h + e_{ij,t}^h, \tag{9}$$

where $\widehat{\varepsilon}_{j,t}$ is the residual from a regression of $y_{j,t}$ on $z_{j,t}$. Proposition 1 implies that inference on θ_{ij}^h is not affected by the autocorrelation of $e_{ij,t}^h$ and the estimation error in $\widehat{\varepsilon}_{j,t}$. Using the results of Sims *et al.* (1990) it follows that the OLS estimator $\widehat{\theta}_{ij}^h$ is asymptotically normal even if some control variables are non-stationary. This is due to the fact that the coefficient of interest θ_{ij}^h is attached to the stationary variable $\widehat{\varepsilon}_{j,t}$ in (8), whereas in Jordà's (2005) version of the projection estimator, the coefficient of interest may be attached to a non-stationary variable, see also Montiel Olea and Plagborg-Møller (2021). As mentioned above, Proposition 1 may also be used to find suitable control variables in the case of external (narrative) shocks. Furthermore, Proposition 1 may also be applicable for estimating the impulse response of some shocks in a proxy/IV SVAR, which is considered in the section III. A nonlinear example is discussed in section A.3 of Data S1 .

III. IV projections

The structural parameters in many empirical models used to analyse the effects of structural shocks can often be estimated by using (internal or external) instrumental variables. In other cases the structural model is estimated by maximum likelihood (ML). If the structural model is just-identified, the ML estimator can alternatively be obtained by applying a particular IV estimation scheme. Thus, IV methods have a wide range of applications within structural VAR studies. In this section, we therefore discuss how to apply projection methods in combination with an IV approach for estimating structural impulse responses. We first illustrate how IV projections fit into our general framework. In particular we consider a projection estimator that relies on the assumption of a fundamental shock and we derive a Hausman type test of the hypothesis that the shock can be represented as a linear combination of the VAR innovations. The details on statistical inference are presented at the end of this section.

To introduce alternative IV projection estimators, assume without loss of generality that we want to identify the first structural shock $\varepsilon_{1,t}$. As above, we drop the index $j (= 1)$ and write the structural equation for linear models as $x_{0,t} = \beta'x_{1,t} + \varepsilon_t$, where $x_{1,t}$ is a vector of exogenous variables not correlated with ε_t . Inserting the structural equation in the projection equation (2) yields

$$y_{t+h} = \theta^h x_{0,t} + \pi' z_t + e_{t+h}, \tag{10}$$

where $z_t = (x'_{1,t}, z_t^0)'$, and the vector z_t^0 collects any (exogenous) control variables not correlated with ε_t and e_{t+h} . Notice that $x_{0,t}$ may be correlated with other shocks that also enter e_{t+h} and, therefore, we cannot estimate (10) by OLS. Stock and Watson (2018) assume that there exists a valid instrumental variable w_t with the usual properties: $\mathbb{E}(w_t \varepsilon_t) \neq 0$, $\mathbb{E}(w_t e_{t+h}) = 0$, and w_t is uncorrelated with any other structural shock in the system. The relationship between the instrument (also called proxy variable) and the shock ε_t can be represented as $w_t = \phi \varepsilon_t + \eta_t$ which illustrates that the instrument is a measurement error contaminated (noisy version) of the underlying structural shock and thus w_t should not be used in the projection equation directly. To simplify the introduction of the IV approach, let us ignore the control variables in z_t . Since they are assumed to

be uncorrelated with ε_t and η_t they do not affect the probability limit of the IV estimator which is given by

$$\frac{\mathbb{E}(w_t y_{t+h})}{\mathbb{E}(w_t x_{0,t})} = \frac{\mathbb{E}(w_t x_{0,t} \theta^h)}{\mathbb{E}(w_t x_{0,t})} = \theta^h .$$

In other words, a simple IV estimator can be used to estimate the structural response coefficient θ^h . This approach can easily be extended to the case with additional control variables, see Stock and Watson (2018). For practical implementation, Stock and Watson (2018) suggest to estimate the projection equation

$$y_{t+h} = \theta^h x_{0,t} + \pi' z_t + e_{t+h}, \quad (11)$$

by an IV estimator using the proxy variable w_t as an instrument for $x_{0,t}$. They call this the LP-IV estimator if no controls z_t are present and LP-IV⁺ if additional control variables are used. It is important to notice that this approach does not provide estimates of the structural shock ε_t without further assumptions (like the invertibility of the VAR representation). Next, we therefore consider a local projection version of the SVAR-IV approach proposed by Stock and Watson (2012) and Mertens and Ravn (2013).

The 2S-IV approach

In this subsection, we outline a two-step estimation procedure that allows us to estimate the structural shocks and thereby provide a framework for statistical inference based on the representation (1) and (2). Earlier literature including Stock and Watson (2012) and Mertens and Ravn (2013) consider the so-called SVAR-IV method, which does not rely on projection methods, see also Stock and Watson (2018). In contrast, the SVAR-IV approach estimates the relevant column of the impact matrix by IV, which is in turn multiplied by the MA matrices obtained from inverting the VAR polynomial.

In this section we consider a local projection variant of the SVAR-IV approach, which is based on a two-stage estimation procedure (2S-IV) by using our framework in (1) and (2). To focus on the main issues we consider a simple bivariate structural model for the VAR innovations with

$$u_{1,t} = \alpha_1 \varepsilon_{2,t} + \varepsilon_{1,t}, \quad (12)$$

$$u_{2,t} = \alpha_2 \varepsilon_{1,t} + \varepsilon_{2,t} . \quad (13)$$

Note for this structural model we assume that the underlying VAR model is invertible as the structural shocks are assumed to be linear combinations of the VAR innovations. As before the external instrument is connected to the first structural shock. Let us first assume that $\varepsilon_{2,t}$ was observed. Inserting $\varepsilon_{1,t} = u_{1,t} - \alpha_1 \varepsilon_{2,t}$ in the projection equation (2) for the first shock yields

$$y_{i,t+h} = \theta_{i1}^h u_{1,t} + \pi_i' z_t + e_{i1,t}^h = \theta_{i1}^h y_{1,t} + \tilde{\pi}_i' z_t + e_{i1,t}^h, \quad (14)$$

where $z_t = (\varepsilon_{2,t}, y'_{t-1}, \dots, y'_{t-p})'$. Accordingly, estimating (14) by OLS yields a consistent estimator for θ_{i1}^h .

In empirical practice $\varepsilon_{2,t}$ is not observed and needs to be replaced by a suitable estimator. To this end we again insert $\varepsilon_{1,t} = u_{1,t} - \alpha_1 \varepsilon_{2,t}$ yielding

$$u_{2,t} = \alpha_2 u_{1,t} + \alpha_3 \varepsilon_{2,t}, \tag{15}$$

where $\alpha_3 = (1 - \alpha_2 \alpha_1)$. By estimating this equation using w_t as instrument for $u_{1,t}$ we obtain a consistent estimator for α_2 and the residual of this IV regression serves as an estimator for $\alpha_3 \varepsilon_{2,t}$. Note that the scaling factor α_3 does not matter for estimating θ_{i1}^h when ε_2 is replaced by the residual of (15) in the set of control variables. Furthermore, it is easy to see that this estimator is equivalent to an IV estimator of

$$y_{i,t+h} = \theta_{i1}^h y_{1,t} + \tilde{\phi}'_{i1} y_{t-1} + \dots + \tilde{\phi}'_{ip} y_{t-p} + e_{i1,t}^h \quad \text{for } i = 1, 2$$

by using $\widehat{\varepsilon}_{1,t}$ (the residual from a regression of $u_{1,t}$ on the residual of (15), see equation 12) as instrument for $y_{1,t}$.

The relationship between the LP-IV and 2S-IV estimators

For the 2S-IV approach we need to assume invertibility of the MA representation with respect to the structural shocks. This assumption is not required for the original LP-IV approach of Stock and Watson (2018). As mentioned in the previous subsection, the main difference between the LP-IV and the 2S-IV approach is that the former estimator employs the external instrument w_t , whereas the latter estimator uses $\widehat{\varepsilon}_{1,t}$ as instrument, which relies on the assumption that the shock $\varepsilon_{1,t}$ is fundamental. If this assumption is fulfilled, then the 2S-IV estimator is equivalent to the (Gaussian) ML estimator, see section B.2 in Data S1. Accordingly, for fundamental shocks the 2S-IV estimator is asymptotically more efficient than the LP-IV estimator, in particular if the parameter ϕ gets small and the instrument w_t is a poor proxy for $\varepsilon_{1,t}$.

These results suggest a test of the hypothesis that the shock $\varepsilon_{1,t}$ is fundamental by comparing the two estimators via the Hausman test. A related test was suggested by Stock and Watson (2018), which compares the LP-IV and the SVAR-IV estimators. Our simple alternative test procedure that avoids the possibility of a negative Hausman statistic is obtained by just augmenting the h -step ahead forecasting equation by the external instrument w_t

$$y_{i,t+h} = \psi_{ih} w_t + \varphi'_0 y_t + \varphi'_1 y_{t-1} + \dots + \varphi'_p y_{t-p} + \tilde{e}_{i,t+h} \tag{16}$$

and testing the hypothesis $\psi_{ih} = 0$ by the ordinary two-sided t -statistic. This test is similar to the one proposed by Plagborg-Møller and Wolf (2022), which checks for Granger causality of the external instrument to the target variables (see also Giannone and Reichlin, 2006; Forni and Gambetti, 2014). The main difference is that the Granger causality tests may be viewed as joint tests for all horizons $h = 1, 2, \dots$, whereas our test focuses on a particular horizon h . The notion behind this test is that whenever w_t contains

a relevant shock that cannot be represented by a linear combination of the innovations, then adding the external instrument to the direct forecasting regression should help to forecast $y_{i,t+h}$. Since $y_{1,t}$ and w_t are correlated, the 2S-IV projection is different from LP-IV estimator whenever $\psi_{ih} \neq 0$. In the section B.1 in Data S1, we derive the test statistic and argue that this simple test is related to the Hausman test.

Long-run restrictions

(Internal) instrumental variables are also useful for estimating more general SVARs. In section B.2 in Data S1 we discuss non-recursive AB-type models, while we focus on long-run restrictions here. The structural VAR with long-run restrictions is typically estimated by using a Cholesky factorization of the model’s long-run solution (e.g. Blanchard and Quah, 1989). In this section we follow Shapiro and Watson (1988) and Fry and Pagan (2005) in employing a simple IV estimator for the columns of Γ . Note that in just-identified models the IV estimator is equivalent to the ML estimator of the structural parameters in Γ . Thus, the IV estimator on the identification step together with a projection step provides a computationally convenient alternative to the ML estimator suggested by Blanchard and Quah (1989).

Let $y_t = (y_{1t}, y_{2t})'$ denote a bivariate vector of stationary time series with VAR(p) representation. The structural shocks $\varepsilon_t = \Gamma u_t$ have long-run effects given by $\sum_{h=0}^{\infty} \Theta_h = \sum_{h=0}^{\infty} \Phi_h \Gamma^{-1} = A(1)^{-1} \Gamma^{-1}$, where $A(1) = I_k - \sum_{i=1}^p A_i$. The identifying restriction is that $\varepsilon_{2,t}$ has a zero long-run effect on the first variable, that is, $A(1)^{-1} \Gamma^{-1}$ is lower-triangular. We re-write the VAR in error correction format $\Delta y_t = -A(1)y_{t-1} + \Delta \text{lags} + u_t$, where ‘ Δlags ’ represents a linear combination of the lagged differences $\Delta y_{t-1}, \dots, \Delta y_{t-p+1}$. Next we multiply the system by the matrix Γ yielding $\Gamma \Delta y_t = -\Gamma A(1)y_{t-1} + \Delta \text{lags} + \varepsilon_t$. From the identifying assumption, $A(1)^{-1} \Gamma^{-1}$ is lower triangular and so is its inverse $\Gamma A(1)$. By normalizing the diagonal elements of the matrix Γ to unity, the two equations of the system result as

$$\Delta y_{1,t} = -\gamma_{12} \Delta y_{2,t} + \psi_{11} y_{1,t-1} + \Delta \text{lags} + \varepsilon_{1,t}, \tag{17}$$

$$\Delta y_{2,t} = -\gamma_{21} \Delta y_{1,t} + \psi_{21} y_{1,t-1} + \psi_{22} y_{2,t-1} + \Delta \text{lags} + \varepsilon_{2,t}, \tag{18}$$

where ψ_{ij} denotes the (i, j) element of the matrix $-\Gamma A(1)$. Note in the system (17) and (18) the error $\varepsilon_{1,t}$ is the permanent shock and $\varepsilon_{2,t}$ is the transitory shock. Furthermore, the transitory (permanent) shock has a positive and unit impact effect on $y_{1,t}$ ($y_{2,t}$), which is easier to understand than the normalization of the shocks in the original Blanchard–Quah framework.

By imposing the restriction that both shocks are orthogonal, the system is just identified and can be estimated efficiently by IV methods. Equation (17) is estimated by using $y_{2,t-1}$ as an instrument for $\Delta y_{2,t}$. For the second equation (18) the residual of (17) (the estimated permanent shock) can be used as an instrument for $\Delta y_{1,t}$. This provides us with the estimated shocks of the identification step that can in turn be used in the projection step (2).²

²It should be noted that the instruments may be weak resulting in poor small sample properties of the estimator, compare Chevillon, Mavroeidis, and Zhan (2020).

In a recent paper Plagborg-Møller and Wolf (2021) propose an alternative way to estimate structural VARs with long-run restrictions. Consider the cumulative impulse responses up to horizon H :

$$\sum_{\ell=0}^H y_{1,t+\ell} = \beta'_H y_t + z'_t \pi^H + \tilde{e}_t^H, \tag{19}$$

where $y_{1,t}$ denotes the differenced variable (Δ gdp in study of Blanchard and Quah (1989)) and $z_t = (y'_{t-1}, \dots, y'_{t-p+1})'$. Let $\varepsilon_{1,t} = \gamma'_1 u_t$ denote the permanent shock, where γ'_1 is the first row of the matrix Γ . Plagborg-Møller and Wolf (2021) show that if H tends to infinity, then $\beta_H \rightarrow c\gamma_1$, where c is some non-zero factor. Accordingly, we can estimate the permanent shock as $\widehat{\varepsilon}_{1,t} = \widehat{\beta}'_H \widehat{u}_t$, where $\widehat{\beta}_H$ is the OLS estimator of β_H in (19) and H is sufficiently large to ensure that the transitory effect vanishes. The estimated shock can be used in the projection regression and the unit impact normalization can be imposed by dividing $\widehat{\theta}_{i1}^h$ by $\widehat{\theta}_{11}^0$.

There are several drawbacks with using this approach. First, by estimating (19) we lose H observations when estimating the structural parameters. Second, it is not clear whether $\widehat{\beta}_H$ converges to β_H . The problem is that if H gets large, then the error \tilde{e}_t^H approaches a non-stationary process and the regression may become spurious. Finally, statistical inference is complicated by the fact that the impulse response estimator is a ratio of estimated coefficients from two regressions that are not independent of each other. Note also that the denominator $\widehat{\theta}_{11}^0$ may be small resulting in poor small sample properties.

Statistical inference

In all applications discussed above, the projection estimator can be characterized by solving a set of moment equations derived from the structural equations in (1) and the projection step (2). Let $\varphi = (\beta', \theta_{ij}^h, \pi_{ij}^h)'$ denote the vector of all coefficients involved, where $\beta = (\beta'_1, \dots, \beta'_k)'$. The moment vector of the system (1) and (2) can be represented as

$$m(\varphi) = \begin{pmatrix} T^{-1} \sum_{t=1}^T m_{1,t}(\beta) \\ T^{-1} \sum_{t=1}^T m_{2,t}(\beta, \theta_{ij}^h, \pi_{ij}^h) \end{pmatrix}.$$

In our previous examples the moments in the identification step have the generic form

$$m_{1,t}(\beta) = w'_t(x_{0,t} - \beta'_j x_{1,t}),$$

where the vector of regressors $x_{1,t}$ and the vector of instruments w_t may contain estimated elements, typically structural shocks estimated from other structural equations. For example, estimating the transitory shock in the previous subsection employs the residual from the first structural equation (17) as an instrument. Accordingly, the parameter vector β may involve parameters from different structural equations which implies that the respective moments are a quadratic function of the structural parameters.

The moments of the projection step have the generic form $m_{2,t}(\beta, \theta_{ij}^h, \pi'_{ij}) = \tilde{w}'_{j,t}(y_{i,t+h} - \theta_{ij}^h \varepsilon_{j,t} - \pi'_{ij} z_{j,t})$, where $\varepsilon_{j,t} = (x_{0,t} - \beta'_j x_{1,t})$ and $\tilde{w}_{j,t} = (\varepsilon_{j,t}, z'_{j,t})'$. Again this moment condition is a quadratic function of the parameters, in general.

As all examples considered above result in a set of just-identified moment conditions we focus on the simple method of moments (MM) estimators,³ where the set of moment equations can be estimated sequentially. First, we solve $T^{-1} \sum m_{1,t}(\hat{\beta}) = 0$ (that is, we estimate the structural model (1)) and given $\hat{\beta}$ the second set of moments results as a linear function of θ_{ij}^h and π_{ij} . If the moment equations of the structural model have a unique solution, then also the projection step yields a unique solution no matter of the fact that the moments may be a quadratic function of the parameters.

If all variables y_t, z_t, w_t , and x_t are stationary, the asymptotic distribution of the MM estimator results from standard theory on (G)MM estimators (e.g. Hayashi, 2000, chapter 7). Since in general $m_t(\theta)$ is a quadratic function, the derivative required for computing the asymptotic covariance matrix is a linear function of the parameters.

IV. GLS projections

As the error of the projection is autocorrelated for $h > 1$, the projection estimators are not efficient in general. In order to analyse the asymptotic properties of different estimators let us first consider the univariate AR(1) model, $y_t = \alpha y_{t-1} + u_t$, which is used to estimate the response $\theta_2 = \alpha^2$ to the shock u_t , $h = 2$ steps ahead. The direct two-period ahead representation is

$$y_{t+2} = \theta_2 y_t + v_{t+2}, \quad t = 1, 2, \dots, T - 2, \tag{20}$$

where $v_{t+2} = (1 + \alpha L)u_{t+2}$. Ignoring the serial correlation of the error v_{t+2} , the projection estimator estimates θ_2 by OLS (labelled as ‘OLS-proj’). We provide the asymptotic distribution of this estimator in section C.1 of Data S1.

To account for the serial correlation, we may adopt the GLS approach and transform the regression equation as $z_{t+2} = \theta_2 z_t + v_{t+2}$, where $z_t = (1 + \alpha L)^{-1} y_t = y_t - \alpha y_{t-1} + \alpha^2 y_{t-2} - \alpha^3 y_{t-3} + \dots$. In C.1 (online) we present the asymptotic distribution of this GLS estimator, and of the variant which estimates the covariance parameter α and the impulse response θ_2 jointly by ML. A common feature of the GLS variants is that they ignore the relationship between the covariance parameter α and the conditional mean parameter θ_2 . It turns out, however, that the main improvement in efficiency is achieved by imposing the relationship $\theta_2 = \alpha^2$. Note that the iterated estimator is equivalent to a (conditional) ML estimator that relies on this restriction. A natural approach for imposing the particular covariance structure is to estimate the parameters from the autoregression with $h = 1$ and including the one-step-ahead residual as an additional regressor $y_{t+2} = \theta_2 y_t + \alpha \hat{u}_{t+1} + \tilde{v}_{t+2}$, where $\tilde{v}_{t+2} = u_{t+2} + \alpha(\hat{\alpha} - \alpha)y_t$ and, therefore, if the estimation error is small, the error \tilde{v}_{t+2} is approximately white noise. A similar approach was suggested by Lusompa (2019). Unfortunately, since the residual \hat{u}_{t+1} is orthogonal to the regressor y_t the augmentation has a

³The MM estimator is simpler than the GMM estimator as it does not require estimating some weight matrix.

negligible effect and the resulting estimator is asymptotically equivalent to the *OLS-proj* estimator.⁴

Another possibility to eliminate the autocorrelation from the multistep projection error is to subtract \widehat{u}_{t+2} from the equation yielding

$$\widehat{y}_{t+2|t+1} = y_{t+2} - \widehat{u}_{t+2} = \theta_2 y_t + \tilde{v}_{t+1}, \tag{21}$$

where $\tilde{v}_{t+1} = \alpha u_{t+1} + (\widehat{\alpha} - \alpha)y_{t+1}$. Note that the error \tilde{v}_{t+1} is approximately white noise for large T so that this approach is also considered to be a GLS variant. The OLS regression of $\widehat{y}_{t+2|t+1}$ on y_t yields

$$\tilde{\theta}_{2|1} = \frac{\sum_{t=1}^{T-2} \widehat{\alpha} y_{t+1} y_t}{\sum_{t=1}^{T-2} y_t^2} = \widehat{\alpha} \left(\frac{\sum_{t=1}^{T-1} y_{t+1} y_t}{\sum_{t=1}^{T-1} y_t^2} + O_p(T^{-1}) \right) = \widehat{\alpha}^2 + O_p(T^{-1}).$$

The resulting estimator differs from the iterative estimator $\widehat{\alpha}^2$ by an asymptotically negligible term that is due to the fact that the direct regression (21) is based on $T - 2$ observations instead of $T - 1$ observations used for the iterated estimator $\widehat{\alpha}^2$. Since $\widehat{\alpha}^2$ is the ML estimator for θ_2 it turns out that this estimator is asymptotically efficient if the innovations are normally distributed.

In section C.2 in Data S1, we argue that for $h \geq 2$ the asymptotic equivalence holds whenever all residuals $\widehat{u}_{t+h}, \dots, \widehat{u}_{t+2}$ are employed to remove the autocorrelation. Reducing the lag length to $q < h - 2$ lags, $\widehat{u}_{t+h}, \dots, \widehat{u}_{t+h-q}$ may be useful in practice for larger VARs and h , as it limits the number of additional regressors. In section C.3 in Data S1, we present Monte Carlo results comparing various GLS variants for the univariate AR model studied above. The general conclusion is that only the residual-augmented GLS estimator (*res-aug*) yields a uniform improvement of the standard *OLS-proj* estimator, whereas all other variants perform substantially worse. Therefore, we focus on this GLS variant in the remainder of the paper.

In particular, the residual-augmented GLS approach can be extended to VAR systems for estimating impulse responses. Denoting the vector of innovations by $u_t = y_t - \mathbb{E}(y_t|y_{t-1}, \dots, y_{t-p})$, the *res-aug* GLS-type projection estimator is obtained from the regression

$$(y_{i,t+h} - u_{i,t+h}) = \theta_{ij}^h \varepsilon_{j,t} + z'_{j,t} \pi_{ij}^h + \gamma'_1 u_{t+h-1} + \dots + \gamma'_{h-2} u_{t+2} + v_{ij,t}^h, \tag{22}$$

where in practice the unknown innovations are replaced by their sample analogs (VAR residuals) and the parameters $\gamma_1, \dots, \gamma_{h-2}$ are additional coefficient vectors. The error term $v_{ij,t}^h$ is a linear combination of all shocks in period t that are not included as regressors and u_{t+1} . Accordingly, the error term has an MA(1) representation. To remove this autocorrelation, all shocks at time t can be included in the vector of control variables,

⁴More precisely, the OLS estimation implies $\sum_{t=1}^{T-1} \widehat{u}_{t+1} y_t = 0$, whereas the regression involves the observations $t = 1, \dots, T - 2$ only. It is not difficult to see that the difference between the Lusompa (2019) GLS estimator and the OLS estimator without \widehat{u}_{t+1} as additional regressor is $O_p(T^{-1})$. Since the estimation error is $O_p(T^{-1/2})$ it follows that both estimators are asymptotically equivalent and this estimator does not improve asymptotic efficiency of the projection estimator.

which makes the GLS approach more efficient but inference is more complicated as the set of moment conditions becomes larger.

Replacing the innovations $u_{t-\ell}$ by $\widehat{u}_{t-\ell}$ in (22) involves an additional estimation error that affects statistical inference. In order to derive the asymptotic distribution of the resulting estimator we characterize the estimator by the following (just-identified) set of moment conditions: $\mathbb{E}(m_{1t}(\theta_1)) = \mathbb{E}\left(\frac{\partial f(x_{j,t}, \beta_j)}{\partial \beta_j} \varepsilon_{j,t}\right) = 0$, $\mathbb{E}(m_{2t}(\theta_2)) = \mathbb{E}(Y_{t-1}^+ u_t') = 0$, $\mathbb{E}(m_{3t}(\theta_3)) = \mathbb{E}(Z_t v_{ij,t}^h) = 0$, where $Y_{t-1}^+ = (y'_{t-1}, \dots, y'_{t-p})'$ and $Z_t = (\varepsilon_{j,t}, z'_{j,t}, u_{i,t+h}, u'_{t+h-1}, \dots, u'_{t+2})'$. Compared to the set of moment conditions considered in section III we add the moment conditions $m_{2t}(\theta_2)$ for estimating the innovations of the VAR system that enter $v_{ij,t}^h$. In empirical practice it may be more convenient to apply a suitable (sieve) bootstrap method for statistical inference (cf. Bruns and Lütkepohl, 2022).

Note also that the three sets of moment conditions are recursive in the sense that the parameters of the previous moments may enter the subsequent moments but not vice versa. Hence we can solve the moment conditions by first solving $\sum_t m_{1,t}(\widehat{\theta}_1) = 0$ and $\sum_t m_{2,t}(\widehat{\theta}_2) = 0$ and inserting the resulting expressions $\widehat{\varepsilon}_{j,t}$ and \widehat{u}_t in $m_{3,t}(\widehat{\theta}_3 | \widehat{\theta}_1, \widehat{\theta}_2) = 0$. As in section III, the asymptotic covariance matrix follows from adapting standard MM results accordingly.

V. Monte Carlo evidence

We investigate the finite sample properties of the alternative estimators discussed in the previous sections by Monte Carlo simulations. We focus on multivariate settings and explore the relative performance of alternative structural impulse response estimators.

Simulation results for OLS and GLS projections

We have conducted a large number of Monte Carlo experiments from different recursive VARs as data generating processes (DGPs) and looked at the bias, standard deviation (SD), empirical coverage and average lengths of confidence intervals for the structural impulse responses θ_{ij}^h . To keep the paper short, we only briefly summarize the main findings here and refer to the detailed description of the Monte Carlo setup and results in the section D.1 in Data S1.

We find (in line with the literature) that in small samples and at larger horizons h , Jordà's (2005) projection estimator tends to be less efficient than those from iterated VARs. Second, the two-step estimator and Jordà's projection estimator perform quite similarly in terms of bias and SDs. Again, the main advantage of the two-step estimator is that SEs and confidence intervals can be computed without any correction for generated regressors and serial correlation. Third, the OLS projections lead to much wider confidence intervals at larger horizons h . Fourth, the GLS refinement (in the form of residual-augmentation) leads to substantial improvements bringing GLS projections on par with iterated VAR results. Finally, using the more parsimonious GLS with less than $h - 2$ terms in residual augmentation is typically not associated with a large loss of efficiency.

Results for IV projections

To investigate the properties of IV projection estimators, we focus on a model where we identify the first structural shock ε_{1t} by a single external instrument w_t . To keep the exposition concise, we only discuss the basic Monte Carlo setup here and refer to section D.2 in Data S1 for more details. To mimic an external instrument setup, we simulate data for the instrument w_t from $w_t = \phi\varepsilon_{1t} + \sigma_\eta\eta_t$, where ϕ, σ_η are scalar-valued and $\eta_t \stackrel{iid}{\sim} \mathcal{N}(0, 1)$. This setup has a measurement interpretation (see e.g. Braun and Brüggemann, 2022). We set $\phi = 0.5$ and the choice of σ_η determines the correlation ρ_{w,ε_1} between w_t and the ε_{1t} (IV strength). We use $\rho_{w,\varepsilon_1} = 0.5$ in our baseline setup.

In the simulations, we first focus on y_t obtained from an invertible VAR(1) with medium persistence and study the properties of estimators of θ_{21}^h , the response of the second variable to the first structural shock in the system. Table 1 and Figure 1 report results for $T = 200$ and the following five alternative response estimators: SVAR-IV, LP-IV and LP-IV⁺ are the estimators described in Stock and Watson (2018). 2S-IV denotes the two-step IV projection estimator introduced in section III and 2S-IV-GLS is the corresponding GLS version by adding future reduced form residuals to the projection equation. Intervals for impulse responses are obtained by using HAC SEs for LP-IV and LP-IV⁺ and a parametric bootstrap (with $B = 499$ bootstrap replications) for SVAR-IV, 2S-IV and 2S-IV-GLS following appendix A.2 of Stock and Watson (2018).

We observe from Table 1 that LP-IV has a comparably large bias at very low and high horizons. Adding control variables as in LP-IV⁺ decreases the bias somewhat at shorter horizons. Nevertheless, we observe the familiar pattern that for large h the projection based methods lead to larger biases than SVAR-IV, which is based on an iterated estimator for the impulse responses. Similar to the OLS projections considered above, we again find that the two-step GLS projection estimator (2S-IV-GLS) performs very much like the SVAR-IV method. The differences between the two groups is also evident from the standard deviations: All projection methods without GLS refinement lead to substantially higher SDs compared to SVAR-IV and 2S-IV-GLS at all horizons. Interestingly, within the group of projection estimators without GLS refinement, we find that the two-step IV method often performs somewhat better in terms of bias and SD. Note again, that the

TABLE 1
Bias and SD of impulse response estimators for θ_{21}^h

| | | Bias | | | | | SD | | | | |
|-----------|-----|--------|--------------------|--------|-----------|---------|-------|--------------------|-------|-----------|---------|
| | h | LP-IV | LP-IV ⁺ | 2S-IV | 2S-IV-GLS | SVAR-IV | LP-IV | LP-IV ⁺ | 2S-IV | 2S-IV-GLS | SVAR-IV |
| $T = 200$ | 0 | -0.029 | -0.003 | -0.003 | -0.003 | -0.003 | 0.208 | 0.146 | 0.141 | 0.141 | 0.143 |
| | 1 | -0.021 | -0.007 | -0.006 | -0.006 | -0.005 | 0.187 | 0.175 | 0.102 | 0.102 | 0.096 |
| | 2 | -0.024 | -0.017 | -0.014 | -0.016 | -0.017 | 0.195 | 0.197 | 0.102 | 0.093 | 0.088 |
| | 3 | -0.017 | -0.015 | -0.017 | -0.017 | -0.021 | 0.213 | 0.218 | 0.106 | 0.086 | 0.082 |
| | 4 | -0.015 | -0.014 | -0.017 | -0.015 | -0.019 | 0.219 | 0.227 | 0.107 | 0.075 | 0.072 |
| | 5 | -0.019 | -0.018 | -0.016 | -0.012 | -0.015 | 0.223 | 0.231 | 0.113 | 0.061 | 0.059 |
| | 6 | -0.021 | -0.020 | -0.018 | -0.009 | -0.010 | 0.215 | 0.226 | 0.108 | 0.048 | 0.046 |

Notes: Results show bias and SD of different structural impulse response estimators using the IV approach. DGP: 2-variable VAR(1) with medium persistence. Population correlation of instrument and structural shock is $\rho = 0.5$. Estimated models use lag length $p = 1$. Results are based on $M = 1,000$ Monte Carlo replications.

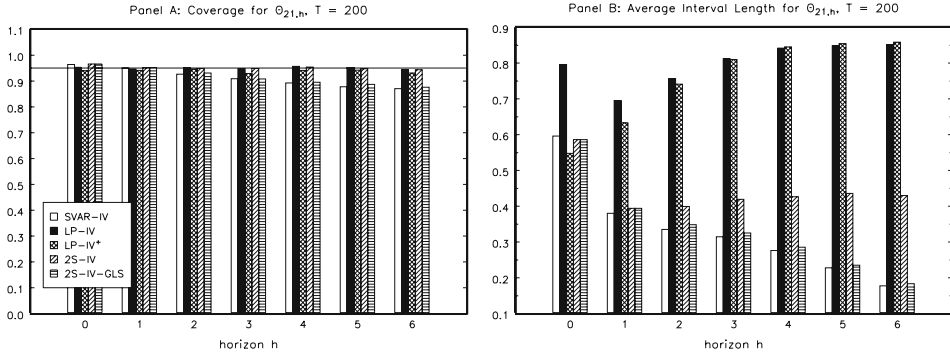


Figure 1. Coverage and average length of confidence intervals impulse response estimators of θ_{21}^h identified by IV. DGP: Two-variable VAR(1) with medium persistence. Population correlation of instrument and structural shock is $\rho = 0.5$. Estimated models use lag length $p = 1$. Results are based on $M = 1,000$ Monte Carlo replications

DGPs used to produce the results in Table 1 and Figure 1 are invertible and thus all estimators are consistent. In this setting, using our 2S-IV estimator leads to substantial efficiency gains relative to LP-IV and LP-IV+.

Panel A of Figure 1 shows the response coefficient interval coverage. At larger horizons h , the SVAR-IV method and 2S-IV-GLS show a somewhat lower coverage compared to the other projection methods. Panel B shows that this is mostly due to the much wider confidence intervals (especially for LP-IV and LP-IV+). The length of intervals produced by 2S-IV is much smaller than those from LP-IV and LP-IV+. It seems that within the group of projection estimators without GLS refinement, 2S-IV with bootstrap intervals has the best trade-off of coverage and interval length. In section D.2 of Data S1, we report additional results for $T = 500$ and also discuss simulations obtained from varying the persistence of the VAR and the correlation ρ_{w,ε_1} . For parameter combinations that imply a reasonable instrument strength, the conclusions above do not change.

In the second part of our simulation exercise, we analyse the role of invertibility on the estimation performance and investigate the finite sample properties of the Hausman tests for invertibility. To this end, we generate data y_t from a possibly non-invertible vector MA (VMA) representation $y_t = \Theta_0\varepsilon_t + \Theta_1\varepsilon_{t-1}$, where

$$\Theta_0 = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix} \text{ and } \Theta_1 = \begin{pmatrix} r & 0 \\ 0.5 & 0.3 \end{pmatrix}. \tag{23}$$

The crucial parameter for the invertibility of the process is r , which is identical to the largest root of the process whenever $r > 0.3$. If $r \geq 1$, then the VMA process is not invertible as the first shock is not fundamental. Within this setup, we report the precision (as measured by the RMSE) of the alternative estimators of θ_{21}^h for $h = 1$ in invertible and non-invertible settings by varying r . Moreover, we look at rejection frequencies of the Hausman test discussed in section III and compare it against the invertibility test of Stock and Watson (2018). Table 2 shows the results for $T = 200$ and $T = 500$ with IV/shock correlations $\rho_{w,\varepsilon_1} = 0.5$ and $\rho_{w,\varepsilon_1} = 0.7$. We find from columns (1) to (4) that for invertible models, 2S-IV and SVAR-IV have smaller RMSE errors throughout. Comparing

TABLE 2
RMSEs of estimators for θ_{21}^1 and Hausman test rejection frequencies

| r | $T = 200, \rho_{w,\varepsilon_1} = 0.5$ | | | | | | $T = 200, \rho_{w,\varepsilon_1} = 0.7$ | | | | | |
|-----|---|-----------------|--------------|----------------|----------------|-----------|---|-----------------|--------------|----------------|----------------|-----------|
| | (1) LP-IV | (2) LP - IV+ | (3) 2S-IV | (4) SVAR-IV | (5) Hausman | (6) SW | (1) LP-IV | (2) LP - IV+ | (3) 2S-IV | (4) SVAR-IV | (5) Hausman | (6) SW |
| 0.5 | 0.170 | 0.178 | 0.096 | 0.095 | 0.052 | 0.026 | 0.119 | 0.125 | 0.089 | 0.089 | 0.055 | 0.032 |
| 0.8 | 0.179 | 0.178 | 0.096 | 0.097 | 0.050 | 0.024 | 0.121 | 0.126 | 0.090 | 0.091 | 0.056 | 0.032 |
| 1 | 0.184 | 0.179 | 0.116 | 0.120 | 0.068 | 0.037 | 0.123 | 0.127 | 0.108 | 0.113 | 0.101 | 0.072 |
| 1.2 | 0.200 | 0.188 | 0.176 | 0.180 | 0.168 | 0.090 | 0.126 | 0.127 | 0.173 | 0.177 | 0.334 | 0.233 |
| 1.5 | 0.234 | 0.205 | 0.250 | 0.252 | 0.305 | 0.154 | 0.134 | 0.133 | 0.246 | 0.247 | 0.588 | 0.444 |
| 2.0 | 0.915 | 0.256 | 0.303 | 0.294 | 0.412 | 0.117 | 0.165 | 0.145 | 0.290 | 0.293 | 0.709 | 0.532 |

| r | $T = 500, \rho_{w,\varepsilon_1} = 0.5$ | | | | | | $T = 500, \rho_{w,\varepsilon_1} = 0.7$ | | | | | |
|-----|---|-----------------|--------------|----------------|----------------|-----------|---|-----------------|--------------|----------------|----------------|-----------|
| | (1) LP-IV | (2) LP - IV+ | (3) 2S-IV | (4) SVAR-IV | (5) Hausman | (6) SW | (1) LP-IV | (2) LP - IV+ | (3) 2S-IV | (4) SVAR-IV | (5) Hausman | (6) SW |
| 0.5 | 0.105 | 0.105 | 0.059 | 0.059 | 0.048 | 0.042 | 0.072 | 0.075 | 0.054 | 0.054 | 0.047 | 0.040 |
| 0.8 | 0.108 | 0.107 | 0.058 | 0.059 | 0.050 | 0.044 | 0.074 | 0.076 | 0.056 | 0.056 | 0.054 | 0.053 |
| 1 | 0.111 | 0.110 | 0.084 | 0.091 | 0.108 | 0.112 | 0.074 | 0.076 | 0.084 | 0.091 | 0.205 | 0.222 |
| 1.2 | 0.118 | 0.114 | 0.159 | 0.163 | 0.351 | 0.329 | 0.077 | 0.078 | 0.158 | 0.162 | 0.687 | 0.671 |
| 1.5 | 0.122 | 0.117 | 0.240 | 0.241 | 0.649 | 0.609 | 0.083 | 0.081 | 0.239 | 0.239 | 0.944 | 0.933 |
| 2.0 | 0.149 | 0.134 | 0.287 | 0.288 | 0.795 | 0.749 | 0.091 | 0.086 | 0.287 | 0.288 | 0.981 | 0.976 |

Notes: Columns (1)–(4) show root mean squared errors (RMSE) of different structural impulse response estimators for θ_{21}^1 using the IV approach. Columns (5) and (6) show the rejection frequencies of Hausman tests ($h = 1$) based on (16) and the one from Stock and Watson (2018), respectively. DGP: two-variable VMA with parameters as in (23). ρ_{w,ε_1} is the population correlation of instrument and structural shock. Lag length in VAR models and for controls $p = 6$. Results are based on $M = 5,000$ Monte Carlo replications.

LP-IV and 2S-IV for invertible models, we find again a relative advantage for our 2S-IV especially for lower IV/shock correlation ($\rho_{w,\varepsilon_1} = 0.5$). Note that the relative efficiency gain of 2S-IV over LP-IV⁺ depends on instrument strength and is particularly large for lower IV/shock correlations. Interestingly, even for non-invertible models (especially for $r = 1$), the inconsistent estimators 2S-IV and SVAR-IV produce often smaller or at least very comparable RMSEs indicating a favourable bias-variance trade-off for the latter methods. Only with larger r , that is, an ‘increasing degree of non-invertibility’, LP-IV and LP-IV⁺ show superior behaviour illustrating that these estimators are still consistent. Overall, these simulation results provide useful insights on the issue of robustness versus efficiency. Columns (5) and (6) show the rejection frequencies of the Hausman tests. We find that both tests have power against the alternative and hence are able to detect situations where 2S-IV and SVAR-IV are inconsistent. For large T and with strongly correlated IVs, there is not much to choose between the two tests. In contrast, in smaller samples and with instruments that are not very strongly correlated with the structural shock, the test suggested in section III tends to be more powerful. Thus, in empirical work using our test may be a good and simple alternative.

Finally, we refer readers interested in the effect of neglecting the generated regressor problem in the context of IV projections to the simulation results in the section D.2.3 of Data S1.

The main findings from the simulations are (i) that standard projection estimators of structural responses at long response horizons are less efficient than those from iterated VARs, (ii) using GLS projections (in the form of residual-augmentation) leads to substantial improvements, bringing GLS projections on par with iterated VAR results, (iii) using the more parsimonious GLS with less than $h - 2$ terms in residual augmentation is typically not associated with a large loss of efficiency, (iv) for invertible VARs our two-step IV projection estimator (2S-IV) is more efficient than existing projection estimators, (v) the simple Hausman type test of III has good power properties and can be used to detect non-invertibility and (vi) ignoring the generated regressor problem may understate the true uncertainty around response estimates substantially.

VI. Conclusion

In this paper we consider projection estimators for impulse responses in a structural VAR framework. In the identification step the structural shock is estimated within an adequately identified structural model. In the projection step, the impulse response is estimated from a regression of the future variable of interest $y_{i,t+h}$ on the estimated shock $\hat{\varepsilon}_{j,t}$ and a vector of control variables $z_{j,t}$. The control variables are included for improving the efficiency but they may also be used to eliminate the estimation error from the first estimation step. We provide conditions that ensure the asymptotic negligibility of the estimation error from the identification step. Furthermore, standard OLS inference applies even if the projection residuals are autocorrelated. Another advantage is that inference is valid no matter whether the variables are stationary or non-stationary (integrated).

In many empirical applications the structural model is estimated by using instrumental variables. We show how the projection approach can be adapted for estimating popular SVAR models like the proxy-VAR, the AB-model and the SVAR model with long-run

restrictions. In particular we propose a projection estimator for invertible proxy-VARs which allows us to test the hypothesis that the structural shock is fundamental. The asymptotic distribution of all projection estimators can be derived from the method-of-moment representation. Finally, we point out that the OLS and IV projection methods are inefficient as the projection residuals are correlated up to $h - 1$ lags. In order to improve the efficiency we propose a GLS projection that removes the serial correlation from the projection equation. We show that GLS projections are closely related (but not identical) to the iterative method of estimating impulse responses from the MA representation of a finite order VAR.

Using Monte Carlo simulations we compare the small sample properties of various estimators for impulse responses. While different variants of OLS projections perform quite similar, our two-step IV projection estimator VARs (2S-IV) turns out to be more efficient than the LP-IV estimator of Stock and Watson (2018) whenever the invertibility assumption is fulfilled. We also find that our simple Hausman test for non-invertible shocks tends to be slightly more powerful than the test based on the SVAR-IV estimator. Applying the GLS refinement we observe substantial efficiency gains relative to OLS (or IV) projections.

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Supporting Information

Additional Supporting Information may be found in the online version of this article:

Data S1. Supporting information