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Concentration Risk under Pillar 2: When are Credit Portfolios Infinitely Fine Grained?

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I. Introduction

There have been significant advances in analytical approaches to credit risk modeling since the first proposal of the new capital adequacy framework (Basel II) has been published in 1999 and it has been finalized in 2004/2006 by the *Basel Committee On Banking Supervision (BCBS)*.¹ In the supervisory capital rules for portfolio credit risk a closed form solution for the measures of risk like Value at Risk (VaR) and Expected Loss (EL) has been achieved in the Internal Ratings Based (IRB) Approach for quantifying credit risk in Pillar 1 of Basel II. Such a model avoids time consuming Monte-Carlo methods as described in *Marrison* (2002) and is widely used in credit portfolio models.² Furthermore, such analytical models like the IRB-approach³ also add benefit to the bank's credit risk management because the risk contribution of each exposure to the portfolio risk can be identified easily and additional approaches for risk-capital allocation as proposed by *Overbeck/Stahl* (2003) are not needed. However, several components of credit risk are not covered sufficiently by this analytical model. One of the most important "non-disclosures" seems to be (credit) concentration risk that is primarily considered in Pillar 2 of Basel II, since the *BCBS* might be aware of such shortcomings when using an analytical framework.⁴ Particularly, it is stated in Basel II

¹ See *Basel Committee On Banking Supervision* (1999, 2001, 2003, 2005a, 2006).

² Particularly, Monte-Carlo simulation is used in the commercial models of *CreditPortfolioView™*, see *Wilson* (1997a, b), and *CreditMetrics™*, see *Gupton/Finger/Bathia* (1997).

³ For the general scientific background of the IRB-model as an analytic credit risk model please refer to *Gordy* (2003) or *Finger* (2001).

⁴ One of the main tasks of Pillar 2 is to ascertain, that "risks [...] not fully captured by the Pillar 1 process (e.g. credit concentration risk)" should be captured

that dealing with concentration risk seems to be one of the most important future tasks. Since concentration risk is not covered in Pillar 1, its quantification and management are two of the important problems in credit risk management under Pillar 2.⁵

The BCBS mainly distinguishes two sorts of concentration risk that emerge from⁶

- (I) “significant exposures to an individual counterparty or group of related counterparties” and
- (II) “credit exposures to counterparties in the same economic sector or geographic region”.

The concentration risk (I) evolves from “single-name” concentrations. The BCBS distinguishes between two sorts of “single-name” concentrations.⁷ One type of concentration risk pertains an exposure to one firm or to a conglomerate of economic highly dependent firms⁸ that is extremely large compared to the rest of the exposures of the portfolio. In such a situation the default risk of the portfolio is mainly driven by the probability of default of this individual debtor. We call this type of risk “individual single-name” concentration risk. *Tasche/Theiler* (2004) and *Emmer/Tasche* (2005) integrated this type of risk in a IRB type model and concluded that consideration of an “individual single-name” concentration risk leads to spurious results when using VaR as a risk measure. Against this background we focus on the second type of “single-name” concentration that occurs if the bank holds a risk bucket (or sub-portfolio) containing a relatively small number of firms, each of them with large exposures. Such a risk bucket is hardly diversified because of the quite small number of debtors. Thus, a bank faces high losses if a large number of defaults appears, even if they are accidentally and not driven by default correlation of the firms. This type of concentration risk (I) can

adequately by the banks risk management using advanced methods, see *Basel Committee On Banking Supervision* (2005), paragraph 724.

⁵ For further issues on concentration risk see *Basel Committee On Banking Supervision* (2005), paragraphs 770–777, as well as *Basel Committee On Banking Supervision* (2000), principle 12 (paragraphs 65–68) including the Appendix to Concentrations, paragraphs 2–6.

⁶ See *Basel Committee On Banking Supervision* (2005), paragraph 773. For a general explanation of concentration risk under Basel II one may refer to Deutsche Bundesbank (2006) as well (only German language).

⁷ See *Basel Committee On Banking Supervision* (2005c, 2005d). Some of the literature referred there will be discussed later on.

⁸ Under Basel II such a conglomerate is called “connected group”, see *Basel Committee On Banking Supervision* (2005a), paragraph 423.

be denoted as “portfolio single-name” concentration risk. It is linked with concentration risk (II) that arises from “sector” concentration. “Sector” concentration risk occurs if defaultable claims belong to a single sector with high default correlation. Though such a concentrated credit portfolio might incorporate many debtors, these debtors might default at the same time due to sector concentration and resulting high default correlation. The major problem caused by this form of “sector” concentration risk is the possibility of the false conclusion that such a large risk bucket (or sub-portfolio) is diversified.

However, such concentration risks are not just a Pillar 2 problem. Indeed, the existence of “portfolio single-name” and “sector” concentration risk of Pillar 2 directly lead to misspecified assumptions of analytic VaR-models like the *Merton*-type model of *Vasicek* (1987, 1991, 2002), that builds the bottom of the Internal Ratings Based (IRB) model of Basel II for quantifying credit risk under Pillar 1 and has become one of the standards in analytical credit portfolio modeling.⁹ In the *Vasicek* model, portfolio credit risk is mainly quantified due to its default rate using the VaR as the risk measure. To achieve analytical tractability of the model, a so-called Asymptotic Single Risk Factor (ASRF) framework as explained in *Gordy* (2003) or *Bank/Lawrenz* (2003) is assumed. That is,

- (A) the portfolio is infinitely fine grained and thus it consists of a nearly infinite number of credits with small exposures, and
- (B) only one systematic risk factor influences the default risk of all loans in the portfolio.

Because of these two simplifications the measured VaR is portfolio-invariant, i. e. only the risk contribution of each defaultable claim to the systematic risk factor is of interest. Each individual claim does not cause any (further) diversification effect, since the portfolio already reached the highest possible degree of diversification. Unfortunately, both assumptions (A) and (B) are mutually exclusive in practice. Precisely, due to the limited factorization (assumption (B)) the model is only designed for small risk buckets, like rating grades as in *Gordy* (2000) or industry sectors as in *Rösch* (2003), rather than for whole credit portfolios. Hence, if one aims to meet assumption (B), the risk bucket under consideration is likely to consist of only a small number of loans. Resulting from this limitation, assumption (A) will become critical.

⁹ See *Merton* (1974). It is also known as the one factor two state approach of CreditMetricsTM, see *Finger* (1999). For the adoption in Basel II see *Finger* (2001) and additionally *Basel Committee On Banking Supervision* (2005b).

However, the violation of (A) or (B) do not have to lead to the fact, that the ASRF-framework can not be used at all for credit risk quantification. But one has to consider the consequences of the violation, i.e. the existence of concentration risk. Concretely, if assumption (A) is not met, we should account for concentration risk (I) especially in form of “portfolio single-name” concentration risk. However, if we exaggerate a risk bucket in order to meet assumption (A), assumption (B) is possibly violated and we should keep the existence of (“sector”) concentration risk (II) in mind. This issue is not only a problem that should be accounted for in credit risk management when dealing with analytical models, but it is also critical for supervisory capital measurement in banks.¹⁰ This raises the following question: Do the assumptions (A) and (B) of (the IRB-model under) Pillar 1 generally hold for our portfolio or do we have to quantify concentration risk for Pillar 2? Resulting from this, we identify two important tasks regarding risk concentrations:

- (i) In which cases are the assumptions of the ASRF framework of the *Vasicek* model critical concerning the credit portfolio size?
- (ii) In which cases are currently discussed adjustments for the VaR-measurement able to overcome the shortcomings of the *Vasicek* model?

The answers of both questions are available if we know the minimum number of loans in a risk bucket that is necessary in order to fulfill the granularity assumption (A) to achieve a required accuracy, say 5%, of the analytical determined VaR in comparison to the true VaR using the *Vasicek* framework.¹¹ Unfortunately, numerical analyses on that topic are scarce. Thus, firstly, we oppose the existing formulas for the VaR using the granularity adjustment assuming a coarse grained, an (infinitely) fine grained as well as a medium grained portfolio. Additionally, we extend the existent framework in order to account for small sized portfolios.¹²

¹⁰ Another solution to the problem of the violation of assumption (A) or (B) might be to cancel risk quantification under the IRB-approach and use internal models. However, this solution is not designated in Basel II.

¹¹ This question is also interesting when analysing the Basel II formula, because the designated add-on factor for the potential violation of assumption (A) was cancelled from the second consultative document to the third consultative document, see *Basel Committee On Banking Supervision* (2001, 2003). Thus, we only prove, under which conditions the assumption (A) of the *Vasicek* model is fulfilled. Of course, this model may suffer from other assumptions like the distributional assumption of standardised returns. However, since we only would like to address the topic of concentration risk, our focus should be reasonable. Additionally, the distributional assumptions seem not to have a deep impact on the measured VaR, see *Koyluoglu/Hickman* (1998a, b), *Gordy* (2000) or *Hamerle/Rösch* (2004, 2005a, b).

Secondly, we numerically infer the minimum number of loans in a portfolio using two definitions of accuracy in order to enhance the theoretical background with concrete facts on critical portfolio sizes.¹³ This could give an advice which sub-portfolios have significant risk concentrations and thus should be controlled on credit portfolio and not on individual credit level. Like in the *Vasicek* model, we focus on gross loss rates in homogeneous credit portfolios, i.e. each borrower has an identical probability of default as well as an identical credit exposure and the loss rate is equal to one.¹⁴ Furthermore we examine the granularity adjustment of an inhomogeneous portfolio based on a simulation as well. With our analysis we explain more about differences between simulated and analytically determined solutions to credit portfolio risk as well as between Basel II capital requirements for Pillar 2 with respect to Pillar 1.¹⁵

The rest of the paper is outlined as follows. In section II we briefly discuss the ongoing research and solutions in the field of measuring especially “single name” concentration risk in analytic credit portfolio models. In section III we describe the *Vasicek* model and derive the adjustment for small and medium sized risk buckets. The numerical analyses on homogeneous as well as on non-homogeneous risk buckets will be

¹² We motivate this procedure by the fact, that for market risk quantification of nonlinear exposures two factors of the *Taylor* series (first and second order) are common to achieve a higher accuracy, see e.g. *Crouhy/Galai/ Mark* (2001) or *Jorion* (2003). This might be appropriate for credit risk as well. Furthermore, the higher order derivatives of VaR given by *Wilde* (2003) make it possible to systematically derive such a formula, which was already mentioned by *Gordy* (2004), but neither derived nor tested so far.

¹³ BCBS already stated that in principle the effect of portfolio size on credit risk is well understood, but lacks practical analysis, see *Basel Committee On Banking Supervision* (2005c).

¹⁴ Precisely, we assume non-stochastic loss rates. This might be satisfied by the fact that the number of defaults in a portfolio is still of main interest and in the Foundation IRB approach the loss rate is fixed for banks anyway. However, we finally examine the granularity adjustment of an inhomogeneous portfolio as well. Our setup is comparable to the one of *Céspedes/Herrero/Kreinin/Rosen* (2005), who analyse the sector concentration effect assuming infinitely fine grained risk buckets.

¹⁵ Additionally, our article makes contribution to the ongoing research on analysing differences between Basel II capital requirements and banks internal “true” risk capital measurement approaches. Since the approximation of the regulatory capital requirements and the perceived risk capital of banks internal estimates for portfolio credit risk is often stated as the major benefit of Basel II, see e.g. *Hahn* (2005), p. 127, but often not observed in practice (see e.g. exemplary calculations for real world portfolios of SunGard Data Systems Inc.), this task might be of relevance in future.

taken out in section IV. Section V summarizes the results and points out some key issues on the use of the IRB-model of Basel II for credit risk management.

II. Literature on Portfolio Concentrations in the *Vasicek* Model

The principle of incorporating the effect of the portfolio size in the analytical *Vasicek* model discussed in literature is very simple. As a first step it is assumed, that the portfolio is infinitely fine grained and the VaR can be determined under the ASRF framework. However, an add-on factor is constructed, that accounts for the finite size of the portfolio and that converges to zero if the assumption (A) of infinite granularity is (nearly) met. A version of this so-called granularity adjustment was part of Basel II until the second consultative document,¹⁶ but because of some theoretical shortcomings of this model, a more convenient formula for the adjustment was presented by *Wilde* (2001).¹⁷ Precisely, this factor equals the first element different from zero that results from a *Taylor* series expansion of the VaR around the ASRF solution.¹⁸ However, a concrete number of loans that is required to meet a pre-defined accuracy interval for the VaR (including the granularity adjustment) is not discussed widely. *Gordy* (2003) comes to the conclusion that the granularity adjustment works fine for risk buckets of more than 200 loans considering low credit quality buckets and for more than 1000 loans for high credit quality buckets. However, he uses the CreditRisk⁺ framework from *Credit Suisse Financial Products* (1997) and not the *Vasicek* model that builds the basis of Basel II, and he does not analyse the effect of different correlation factors as they are assumed in Basel II. Additionally, solutions for further improvement of the granularity are postulated in the literature without examining the results.¹⁹

¹⁶ The effectiveness and the eligibility of the (cancellation of the) granularity add-on from the second to the third consultative document of Basel II is only discussed vaguely in the literature so far, see e.g. *Bank/Lawrenz* (2003), p. 543.

¹⁷ The main criticism of the formula in Basel II was that the granularity adjustment was derived via the CreditRisk⁺ methodology, whereas *Wilde* (2001) was able to derive a formula consistent with the *Vasicek* model.

¹⁸ For the derivation of the granularity adjustment in the *Vasicek* model see also *Pykhtin/Dev* (2002) as well as *Pykhtin* (2004). The derivation of the granularity adjustment by a *Taylor* series expansion is mainly motivated by *Gordy* (2004) and *Rau-Bredow* (2002/2004) and we come to that in section IV. Additionally, *Martin/Wilde* (2002) show that via the heat equation the same results can be achieved whereas the saddle point method agrees only in special cases, e.g. CreditRisk⁺ with one sector.

Finally, *Pykhtin* (2004) recently extended the analytical VaR derivation using a multi-factor adjustment in order to relax assumption (B). Due to a multi-factor layout of the model the observed risk buckets can be enlarged, so that the granularity-assumption (A) becomes less critical. Nonetheless, an additional adjustment would be needed and the analytical solution (and the parameter estimation) will become more complicated.²⁰ *Cespedes/Herrero/Kreinin/Rosen* (2005) presented a smart solution to reduce some of those troubles. They suggested to divide the portfolio under consideration into small risk buckets, so that each risk bucket belongs to one and only one sector.²¹ Consequently, even if the ASRF framework is used, “sector” concentration risk (II) is not a problem. However, each risk bucket has to fulfill assumption (A) of infinite granularity. Therefore, from the practitioners’ perspective it is interesting to know, which minimum size of the portfolio is needed to meet assumption (A) and how a granularity adjustment is able to improve the results.

III. Adjusting for Granularity in the *Vasicek* Model

1. Coarse and Fine Grained Risk Buckets

With reference to *Vasicek* (1987, 1991, 2002) and *Finger* (1999, 2001) we use a one-period one-factor model for determining the portfolio default rate of a homogeneous portfolio and its VaR.²² Precisely, we observe a risk bucket of J obligors at $t = 0$ with respect to $t = T$. Each obligor $j \in \{1, \dots, J\}$ holds an exposure of the amount $E_j = E$. The discrete time process of “normalized” returns²³ $\tilde{a}_{j,T}$ at $t = T$ of the assets of each obligor j is represented by the following one-factor model²⁴

$$(1) \quad \tilde{a}_{j,T} = \sqrt{\varrho} \cdot \tilde{x}_T + \sqrt{1 - \varrho} \cdot \tilde{\varepsilon}_{j,T}.$$

¹⁹ See footnote 12.

²⁰ Especially the data requirements for estimating asset correlations in this multi-factor model are a big challenge.

²¹ At a stretch, the multiple (small) risk buckets are represented by a single systematic factor each. Since these systematic factors are not perfectly correlated, they present a solution for “sector” diversification, which is the opposite of the “sector” concentration. Thus, their view on the impact of sectors within a portfolio is slightly different from the one of the committee.

²² The following model outline is very similar to *Rösch* (2003).

²³ The returns are normalized by subtracting the expected return and dividing the resulting term by the standard deviation in order to get standard normally distributed variables.

²⁴ To keep track of the model, stochastic variables are marked with a tilde “~”.

in which $\tilde{x}_T \sim N(0,1)$ and $\tilde{\varepsilon}_{j,T} \sim N(0,1)$ are i.i.d. with $j \in \{1, \dots, J\}$, i.e. they are independent (and identical) normally distributed with mean zero and standard deviation one. Therefore, \tilde{x}_T serves as the common shared, systematic factor that represents the overall economic condition of all obligors. Besides this, the risk factors $\tilde{\varepsilon}_{j,T}$ are the idiosyncratic factors, that are independent from the systematic factor and account for the individual risk of each borrower. The asset correlation ϱ between all borrowers is assumed to be constant in the risk bucket and also expresses the fraction of risk to the common shared factor measured by the variance. Additionally, we assume that the obligor j defaults at $t = T$ when its “normalized” return falls short of an exogenously given default threshold

$$(2) \quad b_{j,T} = N^{-1}(PD_j),$$

in which $N^{-1}(\cdot)$ stands for the inverse cumulative standard normal distribution and PD_j defines the (unconditional) probability of default of obligor j . Due to homogeneity we set $PD_j = PD$ and thus $b_{j,T} = b_T$ for all $j \in \{1, \dots, J\}$. Conditional on a realisation of the systematic factor the probability of default of each obligor is²⁵

$$(3) \quad P(\tilde{a}_{j,T} < b_T | \tilde{x}_T) = E(I(\tilde{a}_{j,T} < b_T | \tilde{x}_T)) = N\left(\frac{N^{-1}(PD) - \sqrt{\varrho} \cdot \tilde{x}_T}{\sqrt{1 - \varrho}}\right) =: p(\tilde{x}_T)$$

in which $I(\cdot)$ represents the indicator function that is 1 in the event of default and 0 in case of survival of the obligor and $N(\cdot)$ stands for the cumulative standard normal distribution. Since conditional on a realization $\tilde{x}_T = x_T$ the individual default events are independent, the (conditional, still uncertain) number of defaults $\tilde{K}_T | x_T$ (and the gross loss rate) of the portfolio are binomial distributed with the probability $p(x_T)$, i.e.

$$(4) \quad \tilde{K}_T | x_T \sim B(J; p(x_T)).$$

With reference to *Vasicek* (1987), see also *Gordy/Heitfield* (2000), we are able to calculate the unconditional probability of having k_T defaults and we get

$$(5) \quad P\left(\tilde{D}_T = \frac{k_T}{J}\right) = \int_{-\infty}^{+\infty} \binom{J}{k_T} \cdot p(x_T)^{k_T} \cdot (1 - p(x_T))^{J - k_T} \cdot dN(x_T)$$

where \tilde{D}_T marks the (uncertain) portfolio gross loss rate.

²⁵ In the following “ P ” denotes the probability of an event and “ E ” stands for the expectation operator.

For risk quantification we use the VaR on confidence level z of the observed risk bucket, that is the z -quantile q_z of the loss variable, in which $z \in (0,1)$ is the target solvency probability. Precisely, like *Gordy* (2004), we define the VaR as the loss that is only exceeded with the probability of at most $1 - z$, i. e.

$$(6) \quad \text{VaR}_z(\tilde{D}_T) := q_z(\tilde{D}_T) := \inf(d_T : P(\tilde{D}_T \leq d_T) \geq z).$$

With respect to equation (5) we get

$$(7) \quad \text{VaR}_z^{(cg)}(\tilde{D}_T) = \inf\left(d_T : P(\tilde{D}_T \leq d_T) = \sum_{k_T=1}^{d_T \cdot J} P\left(\frac{k_T}{J}\right) \geq z\right)$$

for the VaR of the risk bucket. We call this the VaR of a coarse grained (homogeneous) bucket, since this formula is valid for any bucket size J . Thus, the granularity assumption (A) of section I is not considered in this situation. The result of expression (7) can only be derived numerically.

As a next step we apply the concept of an (infinitely) fine grained portfolio, i. e. we assume an infinite number of obligors in the risk bucket and the weight of each exposure shrinks to zero,²⁶ i. e.

$$(8) \quad \lim_{J \rightarrow \infty} \sum_{j=1}^J w_j^2 = 0 \quad \text{with} \quad w_j = E_j / \sum_{k=1}^J E_k \stackrel{E_j = E_k = E}{=} \frac{1}{J}.$$

For the VaR of the portfolio gross loss rate according to *Vasicek* (2002) or *Bluhm/Overbeck/Wagner* (2003) we receive

$$(9) \quad \begin{aligned} \lim_{J \rightarrow \infty} \text{VaR}_z^{(cg)}(\tilde{D}_T) &=: \text{VaR}_z^{(fg)}(\tilde{D}_T) = \text{VaR}_z^{(fg)}(E(\tilde{D}_T | \tilde{x}_T)) \\ &= N\left(\frac{N^{-1}(PD) - \sqrt{\varrho} \cdot q_{1-z}(\tilde{x}_T)}{\sqrt{1-\varrho}}\right), \end{aligned}$$

in which $q_{1-z}(\tilde{x}_T)$ stands for the $(1-z)$ -quantile of the systematic factor. This is the (well established) VaR-figure of an (infinitely) fine grained risk bucket and it is equal to the expected loss rate as defined in equation (3) conditional on $q_{1-z}(\tilde{x}_T)$. Obviously, the credit risk only relies on the systematic factor, since due to the infinite number of exposures the idiosyncratic risks associated with each individual obligor cancel each

²⁶ Here we used the assumption due to *Vasicek* (2002), p. 160, that can be derived from the assumption due to *Bluhm/Overbeck/Wagner* (2003), p. 87, by using *Kroneckers Lemma*.

other out and are diversified completely. However, in a real-world application assumption (8) surely not holds and a fraction of risk, that comes from the idiosyncratic factors, stays in the bucket.

2. Small and Medium Sized Risk Buckets

In this section we present two adjustments for the VaR formula (9) to take into account that in real world portfolios the idiosyncratic risk can not be diversified completely. The first formula was derived by *Wilde* (2001), the second is an extension and will be developed below. These adjustments can be derived as a *Taylor* series expansion of VaR around the ASRF solution.²⁷ Precisely, we subdivide the portfolio loss rate into a systematic and an idiosyncratic part, i. e.

$$(10) \quad \tilde{D}_T = E(\tilde{D}_T|\tilde{x}_T) + [\tilde{D}_T - E(\tilde{D}_T|\tilde{x}_T)] =: \tilde{Y} + \lambda\tilde{Z}.$$

Thus, the first term $E(\tilde{D}_T|\tilde{x}_T) =: \tilde{Y}$ describes the systematic part of the portfolio loss rate that can be expressed as the expected loss rate conditional on \tilde{x}_T (see also equation (3) and (9)). The second term $\tilde{D}_T - E(\tilde{D}_T|\tilde{x}_T) =: \lambda\tilde{Z}$ of equation (10) stands for the idiosyncratic part of the portfolio loss rate. Therefore, \tilde{Z} describes the general idiosyncratic component and λ decides on the fraction of the idiosyncratic risk that stays in the portfolio. Obviously, λ tends to zero if the number of obligors J converges to infinity, since this fraction (of the idiosyncratic risk) vanishes if the granularity assumption (A) from section I holds. However, for a granularity adjustment we claim that the portfolio is only “nearly” infinitely granular and thus λ is just close to but exceeds zero. In order to incorporate the idiosyncratic part of the portfolio loss rate into the VaR-formula we perform a *Taylor* series expansion around the systematic loss at $\lambda = 0$. We get

$$(11) \quad \begin{aligned} \text{VaR}_z(\tilde{D}_T) &= \text{VaR}_z(\tilde{Y} + \lambda\tilde{Z}) = \text{VaR}_z(\tilde{Y}) + \lambda \left[\frac{\partial \text{VaR}_z(\tilde{Y} + \lambda\tilde{Z})}{\partial \lambda} \right]_{\lambda=0} \\ &+ \frac{\lambda^2}{2!} \left[\frac{\partial^2 \text{VaR}_z(\tilde{Y} + \lambda\tilde{Z})}{\partial \lambda^2} \right]_{\lambda=0} + \frac{\lambda^3}{3!} \left[\frac{\partial^3 \text{VaR}_z(\tilde{Y} + \lambda\tilde{Z})}{\partial \lambda^3} \right]_{\lambda=0} + \dots \end{aligned}$$

Thus, the first term describes the systematic part of the VaR and all other terms add an additional fraction to the VaR due to the undiversified idio-

²⁷ The concept of this approach can be compared with the derivation of the Duration/Convexity in the context of bond management.

syncratic component. For the granularity adjustment it turns out, that only the terms of the order two and higher are non-zero.

To compute the elements of the *Taylor* series, we require the derivatives of VaR. With reference to *Wilde* (2003), the formula for the first five derivatives ($m = 1, 2, \dots, 5$) of VaR in this context is given as²⁸

$$(12) \quad \frac{\partial^m \text{VaR}_z(\tilde{Y} + \lambda \tilde{Z})}{\partial \lambda^m} = (-1)^m \frac{1}{f_Y} \left[-\frac{d^{m-1}}{dl^{m-1}} (\mu_m \cdot f_Y) + \alpha(m) \frac{d}{dx} \left(\frac{1}{f_Y} \frac{d}{dl} (\mu_2 \cdot f_Y) \cdot \frac{d^{m-3}}{dl^{m-3}} (\mu_{m-2} \cdot f_Y) \right) \right] \Bigg|_{l = \text{VaR}_z(\tilde{Y})},$$

with $\alpha(1) = \alpha(2) = 0$, $\alpha(3) = 1$, $\alpha(4) = 3$ and $\alpha(5) = 10$. Here f_Y is the density function of the systematic loss rate of the risk bucket and μ_m stands for the m^{th} (conditional) moment about the origin of the loss rate conditional on the systematic factor.

Concurrently, the first derivative of VaR equals zero,²⁹ so that the second derivative is the first relevant element underlying the granularity adjustment. With reference to *Wilde* (2001) and *Rau-Bredow* (2002) the *Taylor* series expansion up to this quadratic term leads to the following formula for the VaR including the granularity adjustment, that is³⁰

$$(13) \quad \text{VaR}_z^{(1.\text{Order Adj.})} = \text{VaR}_z^{(fg)} + \Delta l_1 \quad \text{with} \\ \Delta l_1 = -\frac{1}{2n(x)} \frac{\partial}{\partial x} \left(\frac{n(x) \cdot V[\tilde{D}_T | \tilde{x} = x]}{\frac{d}{dx} E[\tilde{D}_T | \tilde{x} = x]} \right) \Bigg|_{x=q_{1-z}(\tilde{x}_T)},$$

in which $n(x)$ describes the standard normal density function at x . Thus, the VaR figure of the infinitely fine grained portfolio due to equation (9) is adjusted by an additional term, that is the first term different from zero of the *Taylor* series expansion (11). We call this expression the ASRF solution with first order (granularity) adjustment. Under the condition of the *Vasicek* model, particularly the probability of default is assumed to be given by formula (3), we receive for the granularity add-on of a homogeneous portfolio³¹

²⁸ The first two derivatives were already presented by *Gourieroux/Laurent/Scaillet* (2000). *Wilde* (2003) presents a general formula for all derivatives of VaR. For our derivation the stated formula is sufficient.

²⁹ This is valid because the added risk of the portfolio is unsystematic; see *Martin/Wilde* (2002) for further explanations.

³⁰ “V” denotes the variance operator.

³¹ In Appendix A an analogous formula is stated for inhomogeneous portfolios.

$$(14) \quad \Delta l_1 = \frac{1}{2J} \left((1 - N(y)) \left[\frac{N(y)}{n(y)} \frac{q_{1-z}(\hat{x}_T) \cdot (1 - 2\rho) - N^{-1}(PD) \sqrt{\rho}}{\sqrt{\rho} \sqrt{1 - \rho}} - 1 \right] + N(y) \right) \Bigg|_{y = \frac{N^{-1}(PD) - \sqrt{\rho} q_{1-z}(\hat{x}_T)}{\sqrt{1 - \rho}}}$$

that is the formula presented by *Pykhtin/Dev* (2002) in the special case that we only model the gross loss rates. Obviously, the additional term is of order $O(1/J)^{32}$, that is in itself an asymptotic result, meaning that higher order terms are neglected.

Summing up both analytically derived formulas (9) and (13) for the VaR, the ASRF solution might only be exact if the term (14) of order $O(1/J)$ is close to zero, whereas the ASRF solution including the first order granularity adjustment might only be sufficient if the terms of order $O(1/J^2)$ vanish. For medium sized risk buckets this might be true, but if the number of credits in the portfolio is getting considerably small, an additional factor might be appropriate. Particularly, the mentioned granularity adjustment is linear in $1/J$ and this might not hold for small portfolios. Indeed, *Gordy* (2003) shows by simulation, that the portfolio loss seems to follow a concave function and therefore the adjustment (14) would slightly overshoot the theoretically optimal add-on for smaller portfolios.³³

An explanation of the described behaviour is that the first order adjustment only takes the conditional variance into account, whereas higher conditional moments are ignored, which result from the higher order terms (see the derivatives in equation (12)). With the intention to improve the adjustment for small portfolio sizes, now the $O(1/J^2)$ term will be derived and thus the error will be reduced to $O(1/J^3)$.³⁴ Having a closer look at the derivatives of VaR, the fourth and a part of the fifth element of the *Taylor* series can be identified to be relevant for the $O(1/J^2)$ terms.³⁵ Using the methodology of formula (11) this yields to the following term

$$(15) \quad \text{VaR}_z^{(1 + 2 \cdot \text{Order Adj.})} = \text{VaR}_z^{(fg)} + \Delta l_1 + \Delta l_2$$

with

³² The *Landau* symbol $O(\cdot)$ is defined as in *Billingsley* (1995), A18.

³³ *Gordy* (2003) observes the concavity of the granularity add-on for a high-quality portfolio (A-rated) up to a portfolio size of 1,000 debtors.

³⁴ See *Gordy* (2004), p. 112, footnote 5, for a similar suggestion.

³⁵ See Appendix B for details.

$$\begin{aligned}
 \Delta l_2 = & \frac{1}{6n(x)} \frac{d}{dx} \left(\frac{1}{d\mu_1(x)/dx} \frac{d}{dx} \left[\frac{\eta_3(x) \cdot n(x)}{d\mu_1(x)/dx} \right] \right) \\
 (16) \quad & + \frac{1}{8n(x)} \frac{d}{dx} \left[\frac{1}{n(x)} \frac{1}{d\mu_1(x)/dx} \left(\frac{d}{dx} \left[\frac{\eta_2(x) \cdot n(x)}{d\mu_1(x)/dx} \right] \right)^2 \right] \Bigg|_{x = \text{VaR}_{1-z}(\tilde{x})}
 \end{aligned}$$

in which $\mu_1(x) = E(\tilde{D}_T | \tilde{x} = x)$ is the 1st (conditional) moment about the origin and $\eta_m(x) = \eta_m(\tilde{D}_T | \tilde{x} = x)$ is the m^{th} (conditional) moment about the mean. In the context of the *Vasicek* model and under consideration of homogeneity we receive for this second add-on factor³⁶

$$\begin{aligned}
 \Delta l_2 = & \frac{1}{6J^2 s^2 n_y^2} [(x^2 - 1 + s^2 + 3x s y + 2s^2 y^2) (N_y - 3N_y^2 + 2N_y^3) \\
 & + s n_y (2x + 3s y) (1 - 6N_y + 6N_y^2) - s^2 n_y (y - 6[N_y y - n_y] + 6N_y [N_y y - 2n_y])] \\
 (17) \quad & - \frac{1}{8J^2 s^3 n_y^3} [(-x - 3s y) ([N_y - N_y^2] [-x - s y] - s n_y [1 - 2N_y])^2 \\
 & + 2([N_y - N_y^2] [x + s y] + s n_y [1 - 2N_y]) \\
 & \cdot ([N_y - N_y^2] [1 - s^2] - s n_y [1 - 2N_y] [x + s y] + s^2 n_y [y + 2(n_y - N_y y)])],
 \end{aligned}$$

with $N_y = N(y)$, $n_y = n(y)$, $y = \frac{N^{-1}(PD) - \sqrt{Q} \cdot x}{\sqrt{1-Q}}$, $s = \frac{\sqrt{Q}}{\sqrt{1-Q}}$, and $x = q_{1-z}(\tilde{x}_T)$.

Thus, the additional term is of order $O(1/J^2)$ and equation (15) for the VaR only neglects terms of order $O(1/J^3)$. We will refer to this expression as the VaR under the ASRF solution with (first and) second order granularity adjustment. In terms of numbers of credits the error is reduced in the postulated way. Even if the formulas appear quite complex, both adjustments are easy to implement, fast to compute and we do not have to run Monte Carlo simulations and thereby avoid simulation noise.

IV. Numerical Analysis of Granularity

1. The Impact of the Approximations on the Portfolio Quantile

For a detailed analysis of the granularity assumption (A) as mentioned in section I, we firstly would like to discuss the general behaviour of the four procedures for risk quantification of homogeneous portfolios presented in section III.1 and section III.2, that are

³⁶ See Appendix C for the derivation of the more general inhomogeneous case.

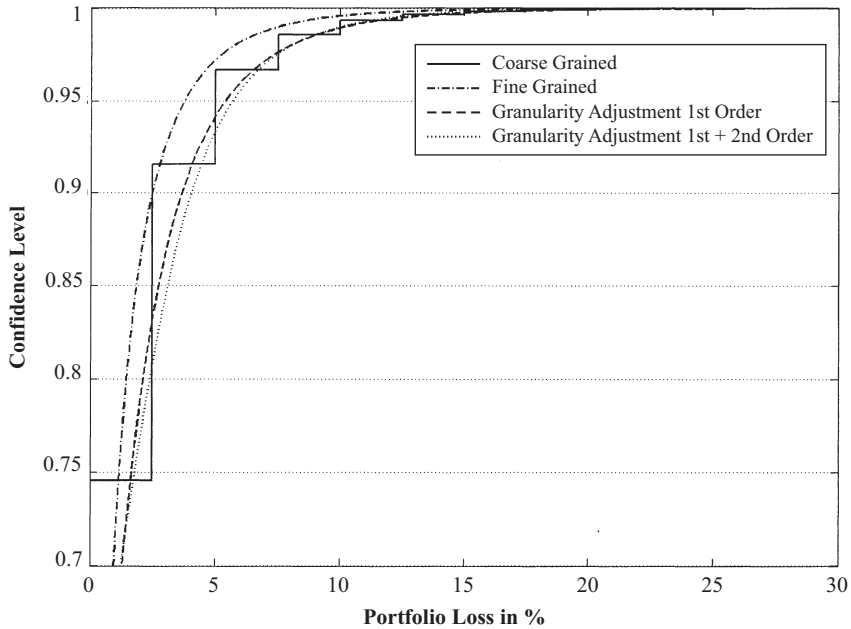


Figure 1: Distribution of Losses for a Wide Range of Probabilities

- (a) the numerically “exact” coarse grained solution (see equation (5))
- (b) the fine grained ASRF solution (see equation (9))
- (c) the ASRF solution with first order adjustment (see equations (13) and (14))
- (d) the ASRF solution with first and second order adjustments (see equations (14) to (17))

Therefore, we evaluate the portfolio loss distribution of a simple portfolio, that consists of 40 credits, each with a probability of default of $PD = 1\%$. We set the correlation parameter to $\rho = 20\%$.³⁷ Using these parameters, we calculate the loss distribution using the “exact” solution (a) as well as the approximations (b) to (d). The results are shown in Figure 1 for portfolio losses up to 30% (12 credits) and the corresponding

³⁷ The chosen portfolio exhibits high unsystematic risk and therefore serves as a good example in order to explain the differences of the four solutions. However, we evaluated several portfolios and the results do not differ widely. Additionally, we claim that the general statements can also be applied to heterogeneous portfolios as well.

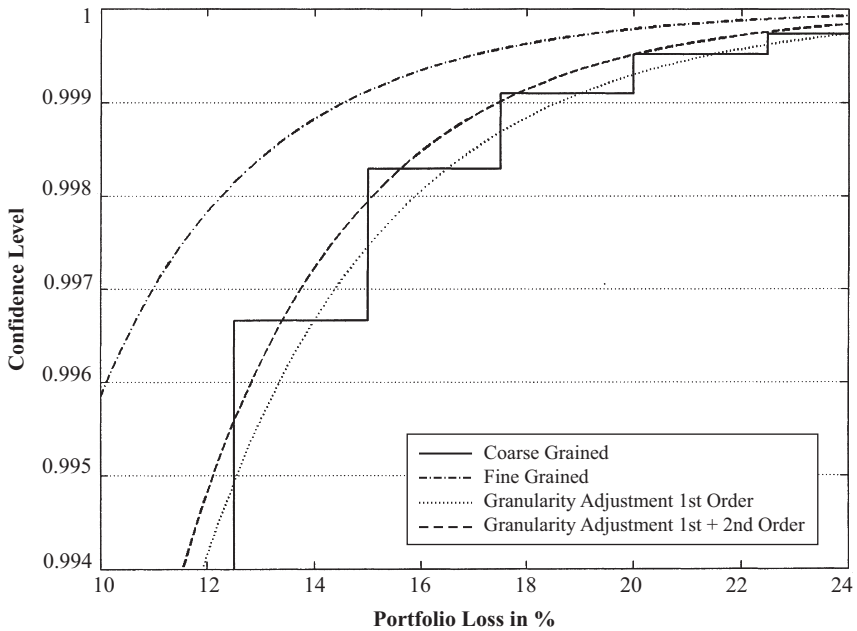


Figure 2: Distribution of Losses for High Confidence Levels

quantiles (of the loss distribution) starting at 0.7. See Figure 2 for the region of high quantiles from 0.994 on, that are of special interest in a VaR-framework for credit risk with high confidence levels.

It is obvious to see that the coarse grained solution (a) is not continuous, since the distribution of defaults is a discrete binomial mixture, whereas all other solutions (b) to (d) are “smooth” functions. This is caused by the fact, that these approximations for the loss distribution assume an infinitely granular portfolio, i. e. the loss distribution is monotonous increasing and differentiable (solution (b)), or at least are derived from such an idealized portfolio ((c) and (d)).

Firstly, we may examine the result for the VaR-figures at confidence levels 0.995 and 0.999. Using the exact, discrete solution (a) the VaR is 12.5% (or 5 credits) for the 0.995 quantile and 17.5% (or 7 credits) for the 0.999 quantile. Compared to this, the ASRF solution (b) exhibits significant lower loss rates at these confidence levels, that are 9.46% for the 0.995 quantile and 14.55% for the 0.999 quantile. Obviously, the ASRF solution underestimates the loss rate, since it does not take (additional) concentration risks into account.

If we add the first order adjustment (c), the VaR figures increase compared to the ASRF solution (b) with values 12.55% for the 0.995 quantile and 18.59% for the 0.999 quantile. Both values are good proxies for the “true” solution (a). Especially the VaR at 0.995 confidence level is nearly exact (12.55% compared to 12.5%). However, (c) seems to be a conservative measure, since the VaR is positively biased. Using the additional second order adjustment (d), the VaR lowers to 12.12% for the 0.995 quantile and 17.48% for the 0.999 quantile. In this case the VaR at 0.999 confidence level is nearly exact (17.48% compared to 17.5%). Nonetheless, (d) is likely to be a progressive approximation for the “true” solution (a), since the VaR is negatively biased.

Summing up the results from our experience (see also Figure 1 and Figure 2), using the ASRF solution (b) the portfolio distributions shift to lower loss rates for the VaR compared to the “exact” solution (a), since an infinitely high number of credits is presumed. Precisely, the idiosyncratic risk is diversified completely, resulting in a lower portfolio loss rate at high confidence levels. If one incorporates the first order granularity adjustment (c), this effect will be weakened and especially for the relevant high confidence levels the portfolio loss rate will increase compared to the ASRF solution (b). This means, that the first order granularity adjustment is usually positive.³⁸

However, if the second order granularity adjustment (d) is added, the portfolio loss distribution will shift backwards again (for high confidence levels). This can be addressed to the alternating sign of the *Taylor* series as can be seen in formula (12). Since the first order granularity adjustment is positive, the second order adjustment tends to be negative. Summing up, with the incorporation of the second order adjustment (d) the approximation of the discrete distribution of the coarse grained portfolio (a) is (in general) less conservative compared to the (only) use of the first order adjustment. However, a clear conclusion, that the application of second order adjustment (d) in order to approximate the discrete numerical derived distribution (a) for high confidence levels outperforms the only use of the first order adjustment (c), can not be stated.³⁹

To conclude, if we appraise the approximations for the coarse grained portfolio, we find both adjustments (c) and (d) to be a much better fit of the

³⁸ See *Rau-Bredow* (2005) for a counter-example for very unusual parameter values. This problem can be addressed to the use of VaR as a measure of risk which does not guarantee sub-additivity; see *Artzner/Delbaen/Eber/Heath* (1999).

³⁹ By contrast, we expected a significant enhancement by using the second order adjustment like mentioned in *Gordy* (2004), p. 112, footnote 5.

numerical solution in the (VaR relevant) tail region of the loss distribution than the ASRF solution, whereas the first order adjustment is more conservative and seems to give the better overall approximation in general.

2. Size of Fine Grained Risk Buckets

Reconsidering the assumptions of the ASRF framework (see section I), we found assumption (A) – the infinite granularity assumption – to be critical in a one factor model. Thus, we investigate in detail the critical numbers of credits in homogeneous portfolios that fulfill this condition.

Therefore, we firstly have to define a critical value for the derivation of the “true” VaR figure from solution (a) from the “idealized” VaR of the ASRF solution (b) to discriminate an infinite granular portfolio from a finite granular portfolio. We do that in two ways.

Firstly, one may argue, that the fine grained approximation (9) in order to calculate the VaR is only adequate, if its value does not exceed the “true” VaR from equation (7) of the coarse grained bucket minus a target tolerance β_T both using a confidence level of 0.999. Precisely, we define a critical number $I_{c,per}^{(fg)}$ of credits in the bucket, so that each portfolio with a higher number of credits than $I_{c,per}^{(fg)}$ will meet this specification. We use the expression⁴⁰

$$(18) \quad I_{c,per}^{(fg)} = \inf \left(J : \left| \frac{\text{VaR}_{0.999}^{(fg)}(\tilde{D}_T)}{\text{VaR}_{0.999}^{(cg)}(\tilde{D}_T = \tilde{K}_T/n)} - 1 \right| < \beta_T \text{ for all } n \in \mathbf{N}^{\geq J} \right)$$

with $\beta_T = 0.05$.

Here, we set the target tolerance β_T to 5%, meaning, that the “true” VaR specified by coarse grained risk buckets does not differ from the analytic VaR using the fine grained solution (9) by more than 5%, if the number of credits in the bucket reaches at least $I_{c,per}^{(fg)}$.

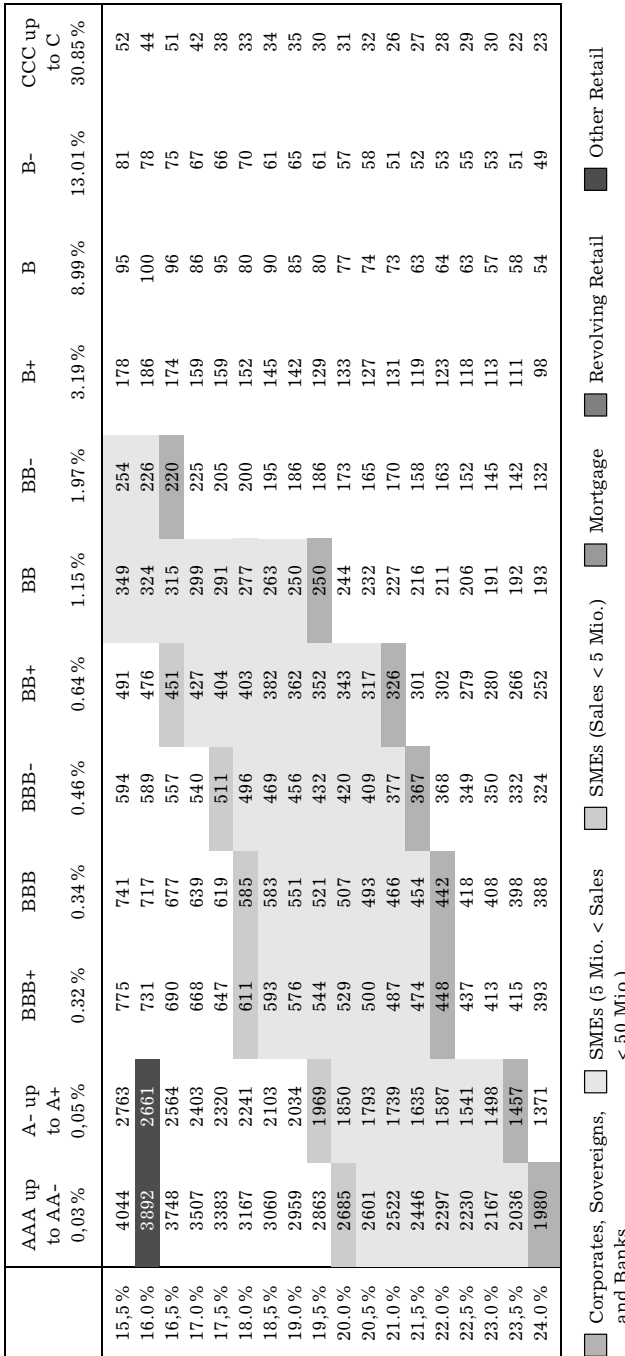
Secondly, the fine grained approximation (b) of the VaR (“idealized” VaR) may be sufficient as long as its result using a confidence level of 0.999 does not exceed the “true” VaR as defined by solution (a) of the coarse grained bucket using a confidence level of 0.995, i. e.

$$(19) \quad I_{c,abs}^{(fg)} = \sup \left(J : \text{VaR}_{0.999}^{(fg)}(\tilde{D}_T) < \text{VaR}_{0.995}^{(cg)} \left(\tilde{D}_T = \frac{\tilde{K}_T}{J} \right) \right).$$

⁴⁰ To address to the minimum number after which the target tolerance will permanently hold, we have to add the notation “for all $n > J$ ” because the function of the coarse grained VaR exhibit jumps dependent on the number of credits.

Table 1
Critical Number of Credits from that ASRF Solution can be Stated to be Sufficient
for Measuring the True VaR (see Formula (18))

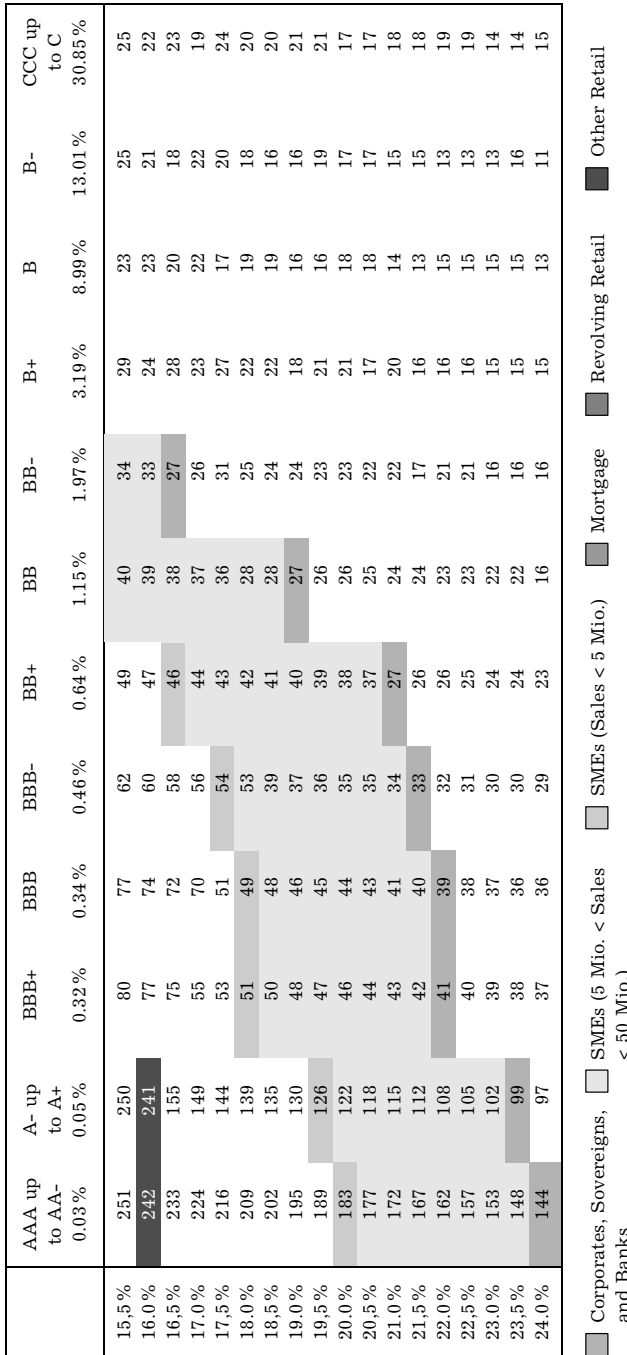
| | AAA up to AA- 0.03 % | A- up to A+ 0.05 % | BBB+ | BBB | BBB- | BB+ | BB | BB- | B+ | B | B- | CCC up to C 30.85 % |
|--------|----------------------------|--------------------------|--------|--------|--------|--------|--------|--------|--------|--------|---------|---------------------------|
| | | | 0.32 % | 0.34 % | 0.46 % | 0.64 % | 1.15 % | 1.97 % | 3.19 % | 8.99 % | 13.01 % | |
| 3.0 % | 35986 | 23985 | 5389 | 5184 | 4105 | 3176 | 2057 | 1390 | 988 | 478 | 370 | 205 |
| 3.5 % | 30501 | 20122 | 4627 | 4457 | 3544 | 2755 | 1801 | 1214 | 861 | 421 | 322 | 175 |
| 4.0 % | 26051 | 17272 | 4054 | 3851 | 3076 | 2402 | 1563 | 1077 | 760 | 350 | 295 | 161 |
| 4.5 % | 22372 | 14906 | 3569 | 3392 | 2719 | 2132 | 1398 | 958 | 690 | 350 | 271 | 145 |
| 5.0 % | 19669 | 13160 | 3153 | 3047 | 2412 | 1928 | 1273 | 866 | 628 | 320 | 255 | 128 |
| 5.5 % | 17723 | 11667 | 2840 | 2701 | 2180 | 1722 | 1145 | 784 | 564 | 289 | 229 | 125 |
| 6.0 % | 15715 | 10590 | 2611 | 2442 | 1977 | 1566 | 1032 | 711 | 515 | 264 | 205 | 116 |
| 6.5 % | 14276 | 9452 | 2366 | 2252 | 1828 | 1428 | 946 | 655 | 477 | 251 | 201 | 106 |
| 7.0 % | 12730 | 8637 | 2148 | 2045 | 1665 | 1327 | 869 | 615 | 457 | 226 | 185 | 101 |
| 7.5 % | 11633 | 7915 | 1990 | 1896 | 1547 | 1214 | 827 | 578 | 412 | 209 | 167 | 90 |
| 8.0 % | 10657 | 7272 | 1813 | 1761 | 1414 | 1133 | 762 | 527 | 389 | 206 | 160 | 87 |
| 8.5 % | 9785 | 6695 | 1720 | 1607 | 1318 | 1040 | 703 | 505 | 357 | 200 | 156 | 87 |
| 9.0 % | 9222 | 6176 | 1571 | 1498 | 1231 | 992 | 660 | 460 | 338 | 183 | 143 | 80 |
| 9.5 % | 8504 | 5707 | 1466 | 1427 | 1152 | 930 | 610 | 443 | 326 | 164 | 135 | 76 |
| 10.0 % | 7853 | 5281 | 1399 | 1334 | 1079 | 873 | 597 | 419 | 304 | 157 | 132 | 68 |
| 10.5 % | 7262 | 5015 | 1309 | 1249 | 1011 | 804 | 552 | 382 | 289 | 153 | 118 | 70 |
| 11.0 % | 6900 | 4655 | 1226 | 1170 | 949 | 756 | 532 | 376 | 285 | 144 | 120 | 65 |
| 11.5 % | 6398 | 4324 | 1149 | 1097 | 911 | 726 | 493 | 357 | 257 | 138 | 109 | 64 |
| 12.0 % | 6099 | 4127 | 1103 | 1053 | 838 | 684 | 466 | 332 | 254 | 135 | 107 | 58 |
| 12.5 % | 5669 | 3843 | 1036 | 989 | 806 | 645 | 450 | 315 | 242 | 127 | 103 | 60 |
| 13.0 % | 5419 | 3677 | 974 | 952 | 759 | 622 | 435 | 299 | 226 | 117 | 94 | 53 |
| 13.5 % | 5046 | 3430 | 915 | 896 | 732 | 587 | 395 | 284 | 211 | 117 | 98 | 55 |
| 14.0 % | 4701 | 3290 | 882 | 843 | 706 | 555 | 391 | 288 | 201 | 110 | 87 | 52 |
| 14.5 % | 4510 | 3073 | 851 | 794 | 666 | 536 | 362 | 263 | 200 | 101 | 91 | 50 |
| 15.0 % | 4331 | 2954 | 822 | 767 | 629 | 519 | 344 | 250 | 195 | 108 | 84 | 51 |



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Table 2
**Critical Number of Credits from that the Exact Solution on Confidence Level 0.995 Exceeds
 the Infinite Fine Granularity on Confidence Level 0.999 (see Formula (19))**

| | AAA up to AA- 0.03 % | A- up to A+ 0.05 % | BBB+ | BBB | BBB- | BB+ | BB | BB- | B+ | B | B- | CCC up to C 30.85 % |
|--------|----------------------------|--------------------------|--------|--------|--------|--------|--------|--------|--------|--------|---------|---------------------------|
| | | | 0.32 % | 0.34 % | 0.46 % | 0.64 % | 1.15 % | 1.97 % | 3.19 % | 8.99 % | 13.01 % | |
| 3.0 % | 5499 | 3885 | 997 | 1019 | 786 | 678 | 464 | 329 | 255 | 165 | 143 | 123 |
| 3.5 % | 4354 | 3126 | 836 | 793 | 665 | 542 | 380 | 274 | 217 | 138 | 122 | 110 |
| 4.0 % | 3428 | 2508 | 701 | 666 | 564 | 428 | 308 | 227 | 184 | 118 | 103 | 94 |
| 4.5 % | 3111 | 1998 | 588 | 558 | 434 | 364 | 266 | 200 | 155 | 100 | 93 | 79 |
| 5.0 % | 2436 | 1830 | 490 | 466 | 404 | 308 | 230 | 175 | 138 | 92 | 83 | 70 |
| 5.5 % | 2239 | 1445 | 406 | 386 | 339 | 288 | 198 | 154 | 123 | 77 | 71 | 65 |
| 6.0 % | 1724 | 1338 | 380 | 361 | 283 | 244 | 170 | 135 | 109 | 74 | 69 | 57 |
| 6.5 % | 1599 | 1037 | 312 | 297 | 266 | 204 | 161 | 117 | 97 | 68 | 58 | 56 |
| 7.0 % | 1489 | 968 | 294 | 280 | 220 | 193 | 138 | 112 | 85 | 62 | 57 | 50 |
| 7.5 % | 1114 | 906 | 238 | 264 | 208 | 183 | 131 | 97 | 82 | 57 | 50 | 46 |
| 8.0 % | 1044 | 681 | 225 | 214 | 197 | 152 | 111 | 93 | 72 | 52 | 46 | 42 |
| 8.5 % | 982 | 641 | 214 | 204 | 161 | 145 | 106 | 80 | 63 | 47 | 45 | 43 |
| 9.0 % | 925 | 605 | 203 | 194 | 153 | 119 | 102 | 77 | 61 | 46 | 39 | 41 |
| 9.5 % | 874 | 573 | 161 | 185 | 146 | 113 | 85 | 66 | 59 | 42 | 38 | 39 |
| 10.0 % | 621 | 543 | 154 | 147 | 140 | 109 | 82 | 64 | 51 | 38 | 37 | 38 |
| 10.5 % | 589 | 516 | 147 | 140 | 111 | 104 | 79 | 61 | 49 | 37 | 34 | 35 |
| 11.0 % | 559 | 368 | 141 | 134 | 107 | 100 | 76 | 52 | 48 | 36 | 31 | 30 |
| 11.5 % | 532 | 351 | 135 | 129 | 103 | 80 | 63 | 50 | 41 | 32 | 28 | 31 |
| 12.0 % | 507 | 335 | 130 | 124 | 99 | 77 | 61 | 49 | 40 | 32 | 30 | 28 |
| 12.5 % | 484 | 320 | 100 | 95 | 95 | 74 | 59 | 47 | 39 | 31 | 27 | 29 |
| 13.0 % | 463 | 306 | 96 | 92 | 91 | 72 | 57 | 46 | 38 | 28 | 29 | 26 |
| 13.5 % | 443 | 293 | 92 | 88 | 71 | 69 | 55 | 38 | 37 | 30 | 24 | 27 |
| 14.0 % | 425 | 281 | 89 | 85 | 68 | 67 | 44 | 37 | 31 | 27 | 26 | 24 |
| 14.5 % | 407 | 270 | 86 | 82 | 66 | 65 | 43 | 36 | 31 | 24 | 22 | 28 |
| 15.0 % | 261 | 260 | 83 | 79 | 64 | 50 | 42 | 35 | 30 | 21 | 23 | 21 |



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This definition of a critical number can be justified due to the development of the IRB-capital formula in Basel II: when the granularity adjustment (of Basel II) was cancelled, simultaneously the confidence level was increased from 0.995 to 0.999.⁴¹ Thus, the reduction of the capital requirement by neglecting granularity was roughly compensated by an increase of the target confidence level. The risk of portfolios with a high number of credits will therefore be overestimated, if we assume that the actual target confidence level is 0.995, whereas the risk for a low number of credits will be underestimated. Thus, a critical number $I_{c,abs}^{(fg)}$ of credits in the bucket exists, so that in each portfolio with a higher number of credits than $I_{c,abs}^{(fg)}$ the VaR can be stated to be overestimated.

The critical numbers $I_{c,per}^{(fg)}$ and $I_{c,abs}^{(fg)}$ for homogeneous portfolios with different parameterizations of ϱ and PD are reported in Table 1 and Table 2. We do not only report the critical numbers under Basel II conditions, but also a wide range of parameter settings that might be relevant, if banks internal data are used for estimating ϱ . Due to the supervisory formula, this parameter is a function of PD for Corporates, Sovereigns, and Banks as well as for Small and Medium Enterprises (SMEs) and (other) retail exposures and remains fixed for residential mortgage exposures and revolving retail exposures.⁴²

With definition type (18) the critical numbers $I_{c,per}^{(fg)}$ vary from 23 to 35,986 credits (see Table 1), dependent on the probability of default PD and the correlation factor ϱ . In buckets with small probabilities of default as well as low correlation factors the idiosyncratic risk is relatively high, so that the portfolio must be substantially bigger to meet the goal. This means that in the worst case a portfolio must consist of at least 35,986 creditors to meet the assumptions of the ASRF framework at an accuracy of 5%. The same tendency can also be found for the target tolerance specification (19). We get critical numbers $I_{c,abs}^{(fg)}$ ranging from 11 to 5,499 creditors (see Table 2), that are substantially lower compared to

⁴¹ These were the major changes of the IRB-formula from the second to the third consultative document, see *Basel Committee On Banking Supervision* (2001, 2003).

⁴² See *Basel Committee On Banking Supervision* (2004) paragraphs 272, 273, and 328 to 330. In both tables (rounded) parameters ϱ due to Basel II are marked. If one aims to measure ϱ from default series, one may refer to *Gordy/Heitfield* (2002), *Gordy* (2000) or *Düllmann/Trapp* (2004/2005). *Lopez* (2004) uses a KMV methodology. The results for estimating ϱ from portfolio data may differ from the correlations given in Basel II, see e.g. *Duellmann/Scheule* (2003) or *Dietsch/Petey* (2003), but overall the parameters given in Basel II are reasonable, see especially *Lopez* (2004).

the critical numbers of the target tolerance. Thus, the critical number $I_{c,abs}^{(Jg)}$ is less conservative. This is caused by the effect, that an increase of the confidence level for VaR calculations has a high impact especially on risk buckets with low default rates.

However, since for all those obligors still the ASRF assumptions (see section I) have to be valid, such big risk buckets may only be relevant for retail exposures in practice. Furthermore, it should be mentioned that these portfolio sizes are valid only for homogeneous portfolios. For heterogeneous portfolios these numbers can be considerably higher especially because the exposure weights differ between the obligors and thus concentration risk will occur.⁴³ Thus, an improvement of measuring the portfolio-VaR is indeed advisable. However, it has to be mentioned, that for portfolios with debtors incorporating low creditworthiness the ASRF solution is already sufficient for some hundred credits (or even less).

3. Probing First Order Granularity Adjustment

After auditing the adequacy of the ASRF solution (b) compared to the discrete, “true” solution (a) in context of a homogeneous risk bucket, we now investigate the accuracy of the first order granularity adjustment (solution (c)). Similar to section IV.2 we compare its accuracy with the discrete solution (a) but we additionally relate its result to the ASRF solution (b).

For the first (conservative) number $I_{c,per}^{(1,OrderAdj.)}$ we compare the analytically derived VaR including first order approximation (solution (c)) with the “true” VaR of the discrete, binomial solution (a) both on a 0.999 confidence level. Again, we aim to meet a target tolerance of β_T and we get

$$(20) \quad I_{c,per}^{(1,OrderAdj.)} = \inf \left(J : \left| \frac{\text{VaR}_{0.999}^{(1,OrderAdj.)}(\tilde{D}_T)}{\text{VaR}_{0.999}^{(cg)}(\tilde{D}_T = \tilde{K}_T/n)} - 1 \right| < \beta_T \text{ for all } n \in \mathbf{N}^{\geq J} \right)$$

with $\beta_T = 0.05$.

Thus, any analytically derived VaR of a risk bucket including more credits than $I_{c,per}^{(1,OrderAdj.)}$ does not differ from the “true” numerical derived VaR by more than 5%.

The results for $I_{c,per}^{(1,OrderAdj.)}$ for homogeneous risk buckets with a specific (PD, ρ) -combination are reported in Table 3 (see p. 102). Obviously, the

⁴³ We will come to that in section IV.5.

Table 3
**Critical Number of Credits from that the First Order Adjustment can be Stated to be Sufficient
 for Measuring the True VaR (see Formula (20))**

| | AAA up to AA- 0.03 % | A- up to A+ 0.05 % | BBB+ | BBB | BBB- | BB+ | BB | BB- | B+ | B | B- | CCC up to C 30.85 % |
|--------|----------------------------|--------------------------|--------|--------|--------|--------|--------|--------|---------|----|----|---------------------------|
| | 0.32 % | 0.34 % | 0.46 % | 0.64 % | 1.15 % | 1.97 % | 3.19 % | 8.99 % | 13.01 % | | | |
| 3.0 % | 6100 | 4227 | 879 | 833 | 693 | 519 | 337 | 228 | 152 | 89 | 63 | 42 |
| 3.5 % | 5517 | 3491 | 810 | 768 | 590 | 443 | 291 | 199 | 133 | 67 | 54 | 32 |
| 4.0 % | 5027 | 3192 | 688 | 653 | 503 | 413 | 251 | 174 | 127 | 60 | 49 | 28 |
| 4.5 % | 4169 | 2936 | 641 | 609 | 470 | 355 | 237 | 165 | 112 | 54 | 38 | 24 |
| 5.0 % | 3846 | 2456 | 546 | 519 | 401 | 334 | 205 | 132 | 107 | 45 | 37 | 22 |
| 5.5 % | 3564 | 2283 | 513 | 488 | 378 | 287 | 195 | 138 | 94 | 51 | 35 | 20 |
| 6.0 % | 3317 | 2129 | 484 | 460 | 358 | 272 | 169 | 121 | 83 | 46 | 33 | 20 |
| 6.5 % | 3098 | 1993 | 413 | 435 | 339 | 258 | 177 | 105 | 80 | 34 | 28 | 18 |
| 7.0 % | 2902 | 1872 | 392 | 373 | 322 | 246 | 154 | 111 | 77 | 40 | 29 | 18 |
| 7.5 % | 2450 | 1762 | 373 | 354 | 277 | 235 | 133 | 97 | 61 | 29 | 27 | 13 |
| 8.0 % | 2309 | 1494 | 355 | 338 | 264 | 203 | 128 | 84 | 59 | 35 | 25 | 16 |
| 8.5 % | 2181 | 1414 | 338 | 322 | 253 | 215 | 136 | 81 | 57 | 31 | 21 | 16 |
| 9.0 % | 2065 | 1341 | 323 | 308 | 242 | 186 | 118 | 79 | 55 | 23 | 23 | 16 |
| 9.5 % | 1958 | 1274 | 309 | 295 | 232 | 179 | 114 | 76 | 54 | 30 | 19 | 14 |
| 10.0 % | 1861 | 1212 | 266 | 253 | 199 | 172 | 110 | 74 | 58 | 22 | 20 | 14 |
| 10.5 % | 1771 | 1156 | 255 | 231 | 214 | 148 | 106 | 64 | 51 | 19 | 15 | 11 |
| 11.0 % | 1689 | 1103 | 245 | 234 | 206 | 143 | 102 | 62 | 44 | 23 | 15 | 11 |
| 11.5 % | 1612 | 1055 | 263 | 225 | 178 | 154 | 89 | 60 | 43 | 21 | 17 | 11 |
| 12.0 % | 1541 | 1010 | 227 | 217 | 171 | 133 | 86 | 52 | 51 | 18 | 19 | 11 |
| 12.5 % | 1476 | 968 | 219 | 209 | 166 | 129 | 74 | 57 | 46 | 19 | 23 | 11 |
| 13.0 % | 1414 | 928 | 211 | 202 | 160 | 125 | 81 | 49 | 40 | 15 | 12 | 12 |
| 13.5 % | 1357 | 892 | 204 | 195 | 155 | 121 | 88 | 54 | 30 | 16 | 10 | 8 |
| 14.0 % | 1303 | 858 | 197 | 188 | 167 | 117 | 68 | 41 | 34 | 17 | 8 | 8 |
| 14.5 % | 1253 | 825 | 191 | 182 | 145 | 101 | 66 | 45 | 33 | 12 | 8 | 8 |
| 15.0 % | 1206 | 795 | 185 | 176 | 141 | 110 | 64 | 56 | 28 | 14 | 15 | 8 |

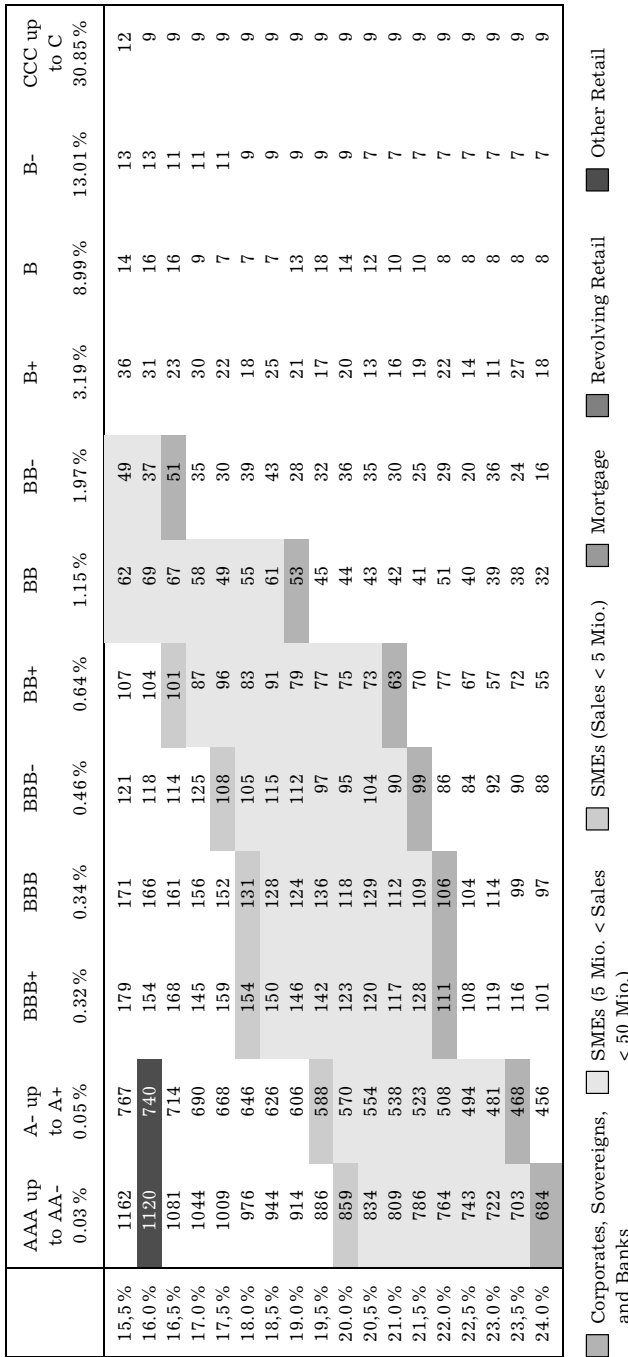
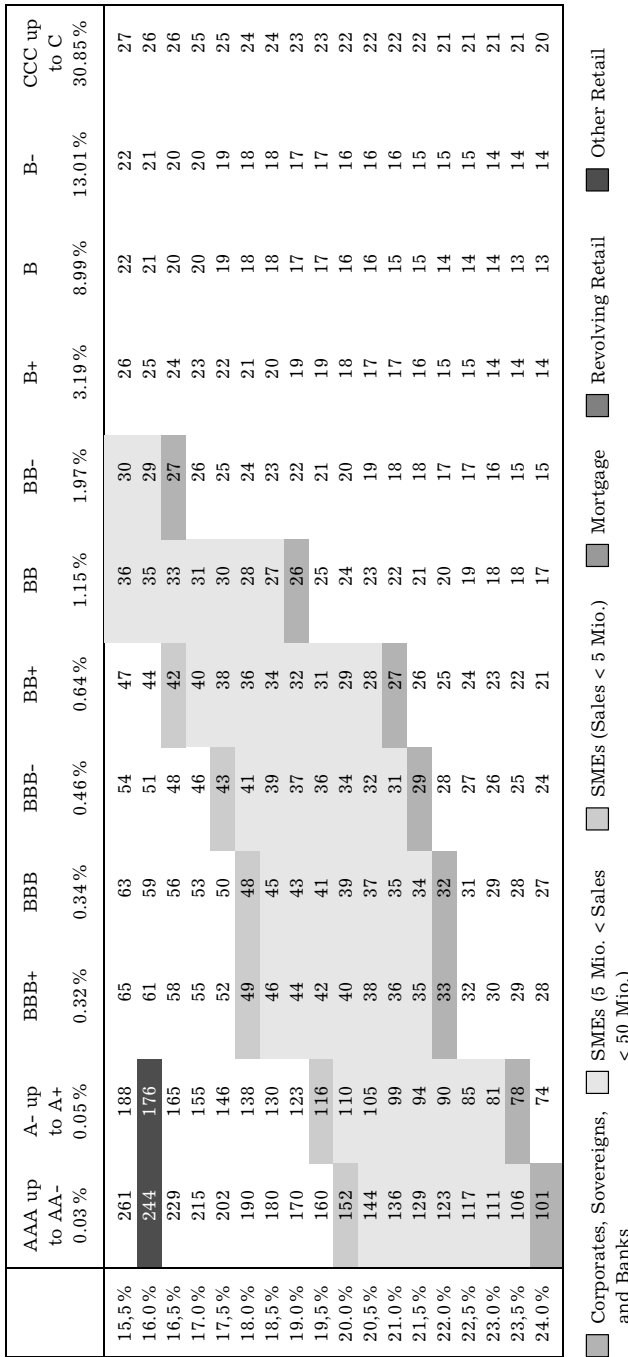


Table 4
**Critical Number of Credits from that the First Order Adjustment on Confidence Level 0.995 Exceeds
 the Infinite Fine Granularity on Confidence Level 0.999 (see Formula (21))**

| | AAA up to AA- 0.03 % | A- up to A+ 0.05 % | BBB+ | BBB | BBB- | BB+ | BB | BB- | B+ | B | B- | CCC up to C 30.85 % |
|--------|----------------------------|--------------------------|--------|--------|--------|--------|--------|--------|--------|--------|---------|---------------------------|
| | | | 0.32 % | 0.34 % | 0.46 % | 0.64 % | 1.15 % | 1.97 % | 3.19 % | 8.99 % | 13.01 % | |
| 3.0 % | 5170 | 3544 | 973 | 935 | 769 | 626 | 441 | 327 | 255 | 164 | 146 | 128 |
| 3.5 % | 4029 | 2773 | 774 | 744 | 615 | 501 | 356 | 265 | 209 | 136 | 122 | 109 |
| 4.0 % | 3231 | 2232 | 633 | 609 | 504 | 413 | 295 | 221 | 175 | 116 | 105 | 95 |
| 4.5 % | 2650 | 1836 | 528 | 508 | 422 | 347 | 249 | 188 | 150 | 101 | 91 | 85 |
| 5.0 % | 2213 | 1538 | 448 | 431 | 359 | 296 | 214 | 162 | 130 | 89 | 81 | 76 |
| 5.5 % | 1875 | 1307 | 385 | 371 | 310 | 256 | 186 | 142 | 114 | 79 | 72 | 69 |
| 6.0 % | 1609 | 1124 | 335 | 323 | 270 | 224 | 163 | 125 | 101 | 71 | 65 | 63 |
| 6.5 % | 1395 | 977 | 295 | 284 | 238 | 198 | 145 | 112 | 91 | 64 | 60 | 59 |
| 7.0 % | 1220 | 856 | 261 | 252 | 211 | 176 | 130 | 100 | 82 | 59 | 55 | 55 |
| 7.5 % | 1075 | 757 | 233 | 225 | 189 | 158 | 117 | 91 | 74 | 54 | 50 | 51 |
| 8.0 % | 955 | 673 | 209 | 202 | 170 | 142 | 106 | 83 | 68 | 50 | 47 | 48 |
| 8.5 % | 853 | 602 | 189 | 182 | 154 | 129 | 96 | 75 | 62 | 46 | 44 | 45 |
| 9.0 % | 766 | 542 | 171 | 165 | 140 | 117 | 88 | 69 | 58 | 43 | 41 | 43 |
| 9.5 % | 691 | 490 | 156 | 151 | 128 | 108 | 81 | 64 | 53 | 40 | 38 | 41 |
| 10.0 % | 626 | 445 | 143 | 138 | 117 | 99 | 75 | 59 | 50 | 38 | 36 | 39 |
| 10.5 % | 570 | 405 | 131 | 127 | 108 | 91 | 69 | 55 | 46 | 36 | 34 | 37 |
| 11.0 % | 521 | 371 | 121 | 117 | 100 | 84 | 64 | 51 | 43 | 34 | 32 | 36 |
| 11.5 % | 477 | 340 | 112 | 108 | 92 | 78 | 60 | 48 | 40 | 32 | 31 | 34 |
| 12.0 % | 439 | 313 | 104 | 100 | 86 | 73 | 56 | 45 | 38 | 30 | 29 | 33 |
| 12.5 % | 404 | 289 | 96 | 93 | 80 | 68 | 52 | 42 | 36 | 29 | 28 | 32 |
| 13.0 % | 374 | 268 | 90 | 87 | 74 | 63 | 49 | 40 | 34 | 27 | 27 | 31 |
| 13.5 % | 346 | 248 | 84 | 81 | 70 | 59 | 46 | 37 | 32 | 26 | 26 | 30 |
| 14.0 % | 322 | 231 | 78 | 76 | 65 | 56 | 43 | 35 | 30 | 25 | 24 | 29 |
| 14.5 % | 299 | 215 | 74 | 71 | 61 | 52 | 41 | 33 | 29 | 24 | 24 | 28 |
| 15.0 % | 279 | 201 | 69 | 67 | 58 | 49 | 39 | 32 | 27 | 23 | 23 | 28 |



critical number varies from 7 to 6,100 credits. Compared to the ASRF solution (see Table 1 in section IV.2), the critical values drop by 83.04% at a stretch. Precisely, we find that the number of credits that is necessary to ensure a good approximation of the “true” VaR is significantly lower with the adjustment (c) than without the adjustment (b). For example, a high quality retail portfolio (AAA) must consist of 5,027 compared to 26,051 credits if we neglect the first order adjustment. A medium quality corporate portfolio (BBB) must contain 106 compared to 442 credits. Thus, the minimum portfolio size should be small enough to hold for real world portfolios and we may come to the conclusion, that the first order adjustment works fine even with our conservative definition of a critical value.

Thus, we are able to use the ASRF formula with the first order granularity adjustment (c) as a (still progressive biased) proxy for the discrete numerical solution (a) and we are able to relate it to the ASRF formula (b). We do that by defining a critical value $I_{c,abs}^{(1, \text{Order Adj.})}$ of credits similar to the definition (19), but this time we proclaim, that the VaR of the ASRF solution without first order granularity adjustment (b) at confidence level of 0.999 should not exceed the VaR with first order granularity adjustment (c) at confidence level of 0.995. We write

$$(21) \quad I_{c,abs}^{(1, \text{Order Adj.})} = \sup \left(J : \text{VaR}_{0.999}^{(fg)}(\tilde{D}_T) < \text{VaR}_{0.995}^{(1, \text{Order Adj.})} \left(\tilde{D}_T = \frac{\tilde{K}_T}{J} \right) \right).$$

Consequently, the confidence level is increased by a buffer of 4 basis points, which should incorporate the idiosyncratic risk approximated by the first order granularity adjustment.

The critical numbers of credits $I_{c,abs}^{(1, \text{Order Adj.})}$ are shown in Table 4 (see p. 104). They contain a range from 14 to 5,170. It is interesting to note, that these critical values do not differ widely from the numbers $I_{c,abs}^{(1, \text{Order Adj.})}$, where we compared the VaR of the ASRF solution (b) with the “true” VaR using the numerical, time-consuming discrete formula. Precisely, the average percentage difference between the critical numbers of Table 2 and Table 4 is less than 10%. That means that the diversification behaviour of the coarse grained solution and the first order approximation is very similar, i.e. the first order adjustment is a good approximation of the idiosyncratic risk of coarse grained portfolios.

4. Probing Second Order Granularity Adjustment

Finally, we would like to test the approximation if the (first and) second order adjustment is added to the ASRF formula and we get the solution (d). Similar to section IV.2 and IV.3, we firstly examine the VaR according to this new formula (d) in comparison to the “exact” VaR from the coarse grained solution (a). Additionally, we analyse its performance with respect to the ASRF solution.

Again, we calculate a critical number $I_{c,per}^{(1.+2. Order Adj.)}$ of credits to test the approximation accuracy with reference to the coarse grained formula (a) according to the “percentage” accuracy with a target tolerance of 5% by

$$(22) \quad I_{c,per}^{(1.+2. Order Adj.)} = \inf \left(J : \left| \frac{\text{VaR}_{0.999}^{(1.+2. Order Adj.)}(\tilde{D}_T)}{\text{VaR}_{0.999}^{(cg)}(\tilde{D}_T = \tilde{K}_T/n)} - 1 \right| < \beta_T \text{ for all } n \in \mathbf{N}^{\geq J} \right)$$

with $\beta_T = 0.05$,

using the (first and) second order adjustment as an approximation of the coarse grained portfolio.

The results are presented in Table 5. Now, the critical number of credits ranges from 17 to 10,993. Compared to the ASRF solution (a), see Table 1 in section IV.2, the necessary number of credits to meet the requirements can be reduced to 33.5 percent on average. Thus, the second order adjustment is capable to detect idiosyncratic risk caused by a finite number of debtors to certain extend. However, if we compare the result with the ones of the only use of the first order adjustment (see Table 3 in section IV.3), the second order adjustment performs less. This might be due to the fact that the confidence level of 0.999 is very conservative and thus the more conservative first order adjustment (c) works better than the second order adjustment (d).

We are able to verify this result by analysing the second order adjustment (d) in comparison with the exact ASRF solution (a). Therefore we introduce a critical number $I_{c,abs}^{(1.+2. Order Adj.)}$ of credits, similar to the definition (22) in section IV.3. We get

$$(23) \quad I_{c,abs}^{(1.+2. Order Adj.)} = \sup \left(J : \text{VaR}_{0.999}^{(fg)}(\tilde{D}_T) < \text{VaR}_{0.995}^{(1.+2. Order Adj.)} \left(\tilde{D}_T = \frac{\tilde{K}_T}{J} \right) \right).$$

Table 5
Critical Number of Credits from that the First plus Second Order Adjustment can be Stated to be Sufficient for Measuring the True VaR (see Formula (22))

| | AAA up to AA- 0.03 % | A- up to A+ 0.05 % | BBB+ | BBB | BBB- | BB+ | BB | BB- | B+ | B | B- | CCC up to C 30.85 % |
|--------|-------------------------|-----------------------|--------|--------|--------|--------|--------|--------|--------|--------|---------|------------------------|
| | | | 0.32 % | 0.34 % | 0.46 % | 0.64 % | 1.15 % | 1.97 % | 3.19 % | 8.99 % | 13.01 % | |
| 3.0 % | 10993 | 7338 | 1796 | 1770 | 1417 | 1107 | 746 | 522 | 392 | 222 | 185 | 130 |
| 3.5 % | 9309 | 6251 | 1503 | 1427 | 1150 | 941 | 620 | 440 | 327 | 193 | 163 | 115 |
| 4.0 % | 7494 | 5077 | 1260 | 1252 | 1014 | 802 | 534 | 384 | 280 | 167 | 140 | 103 |
| 4.5 % | 6405 | 4367 | 1109 | 1054 | 858 | 683 | 460 | 323 | 255 | 148 | 120 | 90 |
| 5.0 % | 5864 | 3768 | 979 | 930 | 761 | 609 | 414 | 293 | 225 | 127 | 115 | 83 |
| 5.5 % | 5056 | 3256 | 866 | 824 | 677 | 544 | 373 | 266 | 199 | 118 | 103 | 78 |
| 6.0 % | 4362 | 3021 | 767 | 730 | 603 | 486 | 321 | 242 | 182 | 107 | 94 | 70 |
| 6.5 % | 4055 | 2622 | 680 | 647 | 537 | 485 | 304 | 210 | 167 | 100 | 86 | 64 |
| 7.0 % | 3509 | 2452 | 641 | 610 | 478 | 390 | 260 | 191 | 147 | 90 | 76 | 63 |
| 7.5 % | 3286 | 2132 | 570 | 542 | 453 | 349 | 248 | 183 | 141 | 84 | 74 | 60 |
| 8.0 % | 2844 | 2006 | 505 | 481 | 404 | 332 | 237 | 158 | 123 | 79 | 67 | 55 |
| 8.5 % | 2679 | 1892 | 480 | 457 | 385 | 297 | 214 | 160 | 119 | 71 | 63 | 51 |
| 9.0 % | 2529 | 1649 | 457 | 406 | 343 | 284 | 193 | 146 | 109 | 69 | 57 | 49 |
| 9.5 % | 2394 | 1563 | 406 | 387 | 328 | 254 | 174 | 133 | 105 | 67 | 58 | 51 |
| 10.0 % | 2077 | 1484 | 388 | 370 | 292 | 243 | 168 | 128 | 91 | 60 | 50 | 42 |
| 10.5 % | 1974 | 1412 | 344 | 354 | 280 | 234 | 161 | 116 | 88 | 56 | 49 | 43 |
| 11.0 % | 1879 | 1231 | 330 | 314 | 269 | 209 | 145 | 106 | 81 | 52 | 48 | 41 |
| 11.5 % | 1791 | 1175 | 316 | 302 | 239 | 201 | 140 | 109 | 88 | 51 | 45 | 38 |
| 12.0 % | 1710 | 1123 | 304 | 290 | 230 | 194 | 126 | 99 | 76 | 52 | 41 | 39 |
| 12.5 % | 1484 | 1075 | 269 | 257 | 222 | 173 | 131 | 96 | 74 | 51 | 42 | 37 |
| 13.0 % | 1421 | 1030 | 259 | 248 | 214 | 167 | 127 | 87 | 63 | 43 | 43 | 34 |
| 13.5 % | 1362 | 897 | 250 | 239 | 190 | 149 | 106 | 79 | 70 | 42 | 37 | 34 |
| 14.0 % | 1307 | 861 | 241 | 230 | 184 | 144 | 111 | 76 | 64 | 39 | 38 | 31 |
| 14.5 % | 1256 | 828 | 233 | 203 | 177 | 139 | 92 | 80 | 54 | 38 | 34 | 32 |
| 15.0 % | 1208 | 797 | 206 | 197 | 172 | 135 | 97 | 67 | 61 | 33 | 35 | 28 |

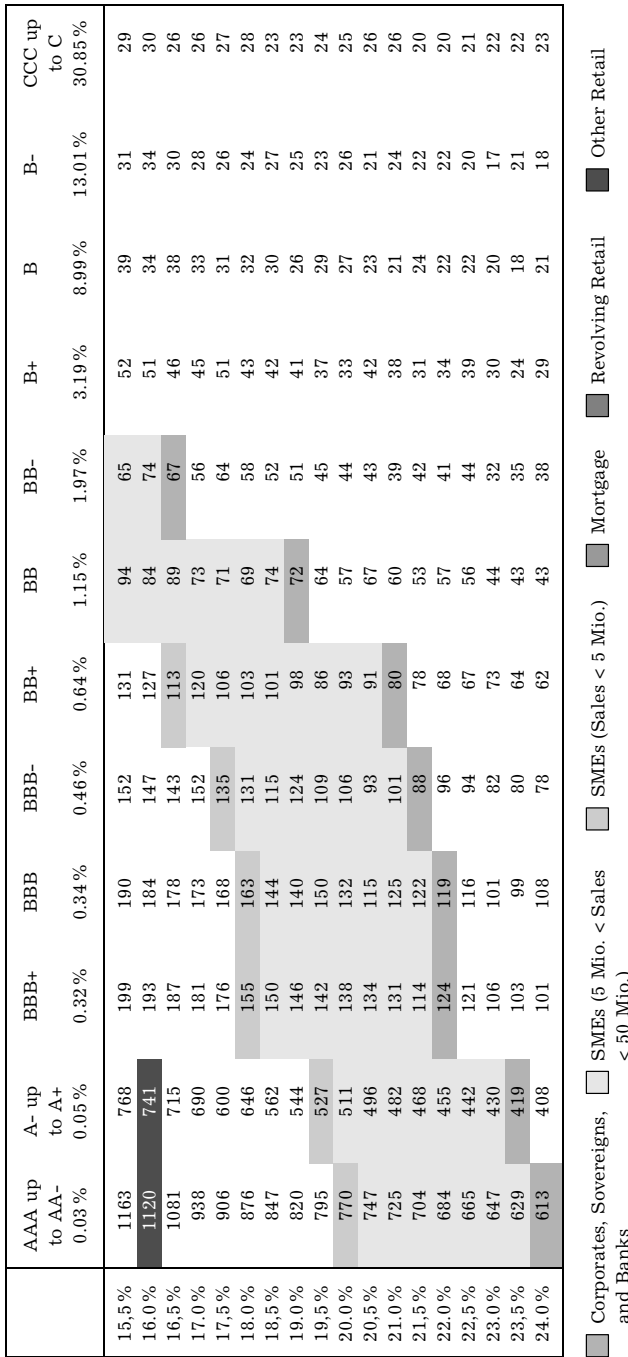
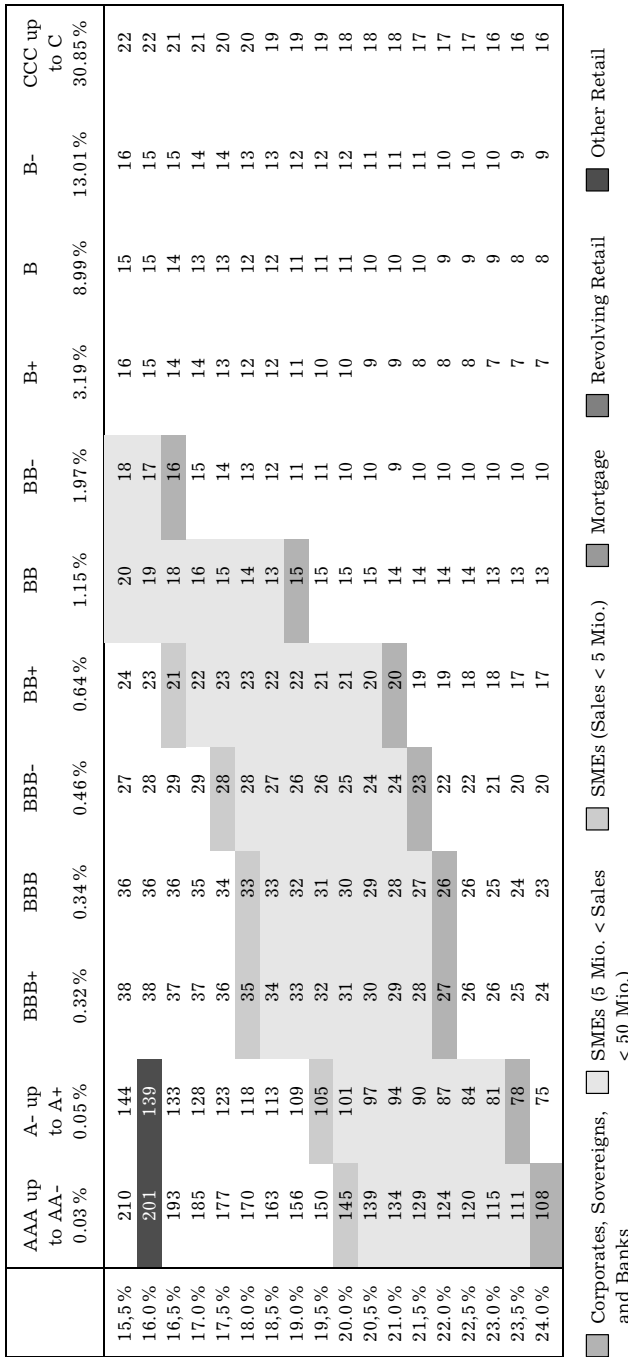


Table 6
Critical Number of Credits from that the First plus Second Order Adjustment on Confidence Level 0.995 Exceeds the Infinite Fine Granularity on Confidence Level 0.999 (see Formula (23))

| | AAA up to AA- 0.03% | A- up to A+ 0.05% | BBB+ | BBB | BBB- | BB+ | BB | BB- | B+ | B | B- | CCC up to C 30.85% |
|-------|------------------------|----------------------|-------|-------|-------|-------|-------|-------|-------|-------|--------|-----------------------|
| | | | 0.32% | 0.34% | 0.46% | 0.64% | 1.15% | 1.97% | 3.19% | 8.99% | 13.01% | |
| 3.0% | 4285 | 2942 | 810 | 778 | 640 | 521 | 367 | 272 | 214 | 140 | 125 | 114 |
| 3.5% | 3266 | 2254 | 633 | 609 | 503 | 411 | 292 | 218 | 173 | 115 | 104 | 97 |
| 4.0% | 2560 | 1776 | 508 | 489 | 406 | 333 | 238 | 180 | 143 | 97 | 89 | 84 |
| 4.5% | 2050 | 1429 | 417 | 401 | 334 | 275 | 198 | 151 | 121 | 83 | 77 | 75 |
| 5.0% | 1671 | 1170 | 347 | 335 | 279 | 231 | 168 | 128 | 103 | 73 | 68 | 67 |
| 5.5% | 1380 | 971 | 294 | 283 | 237 | 196 | 144 | 111 | 90 | 64 | 60 | 61 |
| 6.0% | 1153 | 815 | 251 | 242 | 203 | 169 | 124 | 96 | 79 | 57 | 54 | 56 |
| 6.5% | 973 | 691 | 216 | 209 | 176 | 147 | 109 | 85 | 70 | 52 | 49 | 51 |
| 7.0% | 827 | 590 | 188 | 182 | 153 | 128 | 96 | 75 | 62 | 47 | 44 | 48 |
| 7.5% | 708 | 507 | 164 | 159 | 135 | 113 | 85 | 67 | 56 | 43 | 41 | 44 |
| 8.0% | 610 | 439 | 145 | 140 | 119 | 100 | 76 | 60 | 50 | 39 | 38 | 42 |
| 8.5% | 527 | 382 | 128 | 124 | 106 | 89 | 68 | 54 | 46 | 36 | 35 | 39 |
| 9.0% | 458 | 333 | 114 | 110 | 94 | 80 | 61 | 49 | 42 | 33 | 32 | 37 |
| 9.5% | 399 | 292 | 102 | 98 | 84 | 72 | 55 | 45 | 38 | 31 | 30 | 35 |
| 10.0% | 349 | 257 | 91 | 88 | 76 | 65 | 50 | 41 | 35 | 29 | 28 | 33 |
| 10.5% | 306 | 226 | 82 | 79 | 68 | 59 | 46 | 37 | 32 | 27 | 27 | 32 |
| 11.0% | 268 | 200 | 74 | 72 | 62 | 53 | 42 | 34 | 30 | 25 | 25 | 31 |
| 11.5% | 264 | 177 | 67 | 65 | 56 | 48 | 38 | 32 | 28 | 24 | 24 | 29 |
| 12.0% | 271 | 156 | 60 | 59 | 51 | 44 | 35 | 29 | 26 | 22 | 22 | 28 |
| 12.5% | 266 | 173 | 55 | 53 | 46 | 40 | 32 | 27 | 24 | 21 | 21 | 27 |
| 13.0% | 257 | 172 | 50 | 48 | 42 | 37 | 30 | 25 | 22 | 20 | 20 | 26 |
| 13.5% | 248 | 167 | 45 | 44 | 39 | 34 | 27 | 23 | 21 | 19 | 19 | 25 |
| 14.0% | 238 | 162 | 41 | 40 | 36 | 31 | 25 | 22 | 20 | 18 | 18 | 24 |
| 14.5% | 229 | 156 | 38 | 37 | 33 | 29 | 24 | 20 | 18 | 17 | 18 | 24 |
| 15.0% | 219 | 150 | 34 | 34 | 30 | 26 | 22 | 19 | 17 | 16 | 17 | 23 |



So for each risk bucket with at least $I_{c,abs}^{(1.+2.\text{OrderAdj.})}$ number of credits the idiosyncratic risk, measured by the second order adjustment on a confidence level 0.995, is included in the confidence level premium of 4 basis points of the ASRF solution (on a confidence level 0.999).

These critical numbers presented in Table 6 (see p. 110) range from 7 to 4,285. Obviously, these results are considerably higher than those of Table 4 and therefore the predefined target value of accuracy is reached with lower numbers of credits. Thus, the idiosyncratic risk is underestimated with the second order adjustment compared to the numerical “true” solution (a) (see the results in section IV.2) and is not measured with such a high accuracy as the first order adjustment does (see section IV.3). Concretely, this value is reduced by averaged 32.7 percent credits.

To conclude, the second order adjustment (d) converges faster to the asymptotic value of the ASRF solution (b), which confirms the findings of section IV.1. A possible reason is that the VaR measure using the first order approximation may be “corrected” into the direction of the ASRF solution by incorporating the second order adjustment. The possibility of this behaviour is given due to the alternating sign in the derivatives of VaR, see formula (12).⁴⁴ Thus, taking more derivatives into account could solve the problem, but would lead to even more uncomfortable equations.⁴⁵ Despite these theoretical questions, it can be stated that in homogeneous portfolios an excellent approximation of the true VaR can be achieved with the granularity adjustment.

5. Probing Granularity for Inhomogeneous Portfolios

The previous analyses showed that the granularity adjustment works fine for homogeneous portfolios. In this section we test if the approximation accuracy of the presented general formulas will hold for portfolios consisting of loans with different exposures and credit qualities. This means, that the credits in the portfolio vary in the exposure weight and in the probability of default, and we analyse if the gross loss rate for coarse grained portfolios could still be quantified satisfactory by the granularity adjustment.

⁴⁴ This is true not only for the first five derivatives but also for all following derivatives; see the general formula for all derivatives of VaR in *Wilde (2003)*.

⁴⁵ However, we also have to take into consideration that the Taylor series is potentially not convergent at all or does not converge to the correct value. For a further discussion see *Martin/Wilde (2002)* and *Wilde (2003)*.

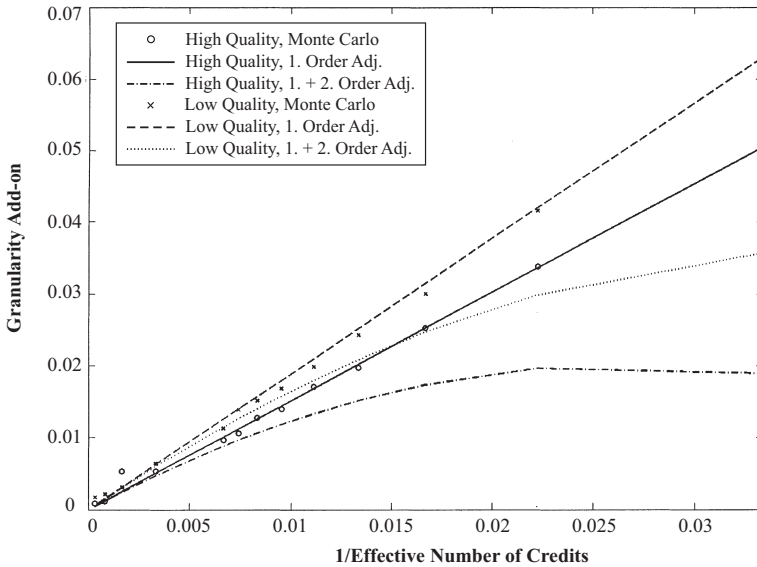


Figure 3: Granularity Add-on for Heterogeneous Portfolios Calculated Analytically with First Order (Solid Lines) and Second Order (Dotted Lines) Adjustments as well as with Monte Carlo Simulations (x and o) Using 3 Million Trials

Concretely, we examine high quality portfolios with probabilities of default ranging from 0.02% to 0.79% and lower quality portfolios with probabilities of default ranging from 0.2% to 7.9%. Additionally, we define a basic risk bucket consisting of 20 loans with exposures between 35 and 200 million €. ⁴⁶ In order to measure the portfolio size with respect to concentration risk we use the effective number of loans

$$(24) \quad J^* := 1 / \left(\sum_{j=1}^J w_j^2 \right)$$

rather than the number of loans J . ⁴⁷ Consequently, this effective number is more than 25% below the true number of credits.

⁴⁶ The used portfolio is based on *Overbeck* (2000), see also *Overbeck/Stahl* (2003), but reduced to 20 loans to achieve more test portfolios.

⁴⁷ The effective number J^* of credits is based on the *Herfindahl-Hirshman* index $H := 1/J^*$, that is preferably used as a measure of concentration in credit portfolios, see *Gordy* (2003) and *Basel Committee On Banking Supervision* (2001b), paragraphs 432 and 434.

A variation of portfolio size is reached by reproducing the basic risk bucket so that portfolios with 40, 60, ..., 400, 800, 1600 and 4000 loans result. Using an asset correlation $\rho = 20\%$ and confidence level of 0.999, we compute the granularity add-on with the presented first order and second order adjustment.⁴⁸ Because the exact value can not be determined analytically for heterogeneous portfolios, we compute the “true” VaR with Monte Carlo simulations using 3 million trials.⁴⁹ Finally, we compare this “true” VaR with the ASRF solution, so that we receive the granularity add-on.

The simulated results for granularity add-on for the high quality portfolios and low quality portfolios are presented in Figure 3 (see the circles and dots). Therefore, the add-on for the minimum size of 40 loans with $1/J \approx 0.035$ is 5.0% (6.2%) for the high (low) quality portfolio. This is equal to a relative correction of +112.5% (+30.5%) compared to a hypothetical infinitely fine grained portfolio. This shows again the relative high impact of idiosyncratic risks in small high quality portfolios. With shifting to bigger sized portfolios the effective number of credits shifts to zero and the granularity add-on decreases almost exactly linear in terms of $1/J^*$ – even for high quality portfolios. This result is contrary to *Gordy* (2003), who exhibits a concave characteristic of the granularity add-on. This might be due to the fact, that *Gordy* (2003) uses a CreditRisk⁺ framework, whereas we analysed the effect of the granularity with the CreditMetrics one-factor model that is consistent with the Basel II assumptions.

Thus, the granularity add-on in Figure 3 can be approximated with a linear function. Indeed, the (linear) first order adjustment is a very good approximation for heterogeneous portfolios of high as well as low quality. Just like in the previous sections, the second order adjustment leads to a reduction of the granularity add-on, thus it can be characterized as less conservative, but comparing the results we strongly recommend the first order adjustment.

V. Conclusion

Presently discussed analytical solutions for risk quantification of credit portfolio models especially rely on the assumptions of an infinite number

⁴⁸ For the concrete formulas see Appendices A and C.

⁴⁹ As in *Gordy* (2003) we firstly used 300,000 Monte Carlo trials for calculation of the 0.99 confidence level (leading to 3,000 hits in the tail). However, on a 0.999 confidence level the VaRs were not stable and thus we recommend 3 million trials (also with 3,000 hits in the tail) that seemed to be appropriate in our case.

of credits and of only one systematic factor. Thus, those analytical frameworks do not account for “single name” and “sector” concentration risks. This problem is discussed intensively by the financial authorities and it is especially considered in Pillar 2 of Basel II. Since one could get “sector” concentration risk under control by building small risk buckets, the “single name” concentration risk and the infinite granularity assumption, respectively, might be the most critical assumption. To cope with this problem, recently an add-on factor was developed, that adjusts the analytical solution for portfolios of finite size and therefore might serve as a simple solution for quantifying “single name” concentration risk under Pillar 2. In this article we briefly reviewed the general framework of this (first order) granularity adjustment for medium sized risk buckets. Furthermore, we derived an additional (second order) adjustment for small risk buckets, since an improvement due to the higher order term is expected in the literature. We implemented this adjustment on the *Vasicek* model that also builds the basis of the Basel II credit risk formula. We carried out a detailed numerical study. In this study we reviewed the accuracy of the infinite granularity assumption for credit portfolios with a finite number of credits, as well as the improvement of accuracy with so-called first and second order granularity adjustments. We received some critical values for the minimum numbers of credits for the analytical solutions compared to the numerical “exact” solutions under the risk measure Value at Risk (VaR). As far as we know, such a study was carried out for the first time. We came to the conclusion, that the critical number of credits for approving the assumption of infinite granularity is influenced by the probability of default, the asset correlation and of course the acquired accuracy of the analytical formula to great extent. The number of credits varies enormously, e. g. from 1,371 to 23,989 for a high-quality portfolio (A-rated) and from 23 to 205 for an extremely low-quality portfolio (CCC-rated). With the use of the first order granularity adjustment we could reduce these ranges drastically. The critical number of credits is in the bandwidth 456 to 4,227 (A-rated) and 9 to 42 (CCC-rated) and thus, the postulated accuracy should be obtained in many real-world portfolios. Additionally, the second order adjustment does not seem to work for a conservative risk measure like the VaR, since it reduces the add-on factor. To conclude, we think that in general the assumption of an infinitely fine grained portfolio seems to hold even for relatively small portfolios, especially if the first order granularity adjustment is incorporated.

Appendix A

With reference to *Emmer/Tasche (2005)*,⁵⁰ and to *Pykhtin/Dev (2002)* for the homogenous case, the first order granularity adjustment for inhomogeneous portfolios is

$$\begin{aligned}
 \Delta l_1 = & -\frac{1}{2} \left[q_{1-z}(\tilde{x}) \frac{\sum_{i=1}^J w_i^2 (N(y_i) - N^2(y_i))}{\sum_{i=1}^J w_i (\sqrt{\varrho}/\sqrt{1-\varrho}) n(y_i)} \right. \\
 (A1) \quad & + \frac{\sum_{i=1}^J w_i^2 \left(\frac{\sqrt{\varrho}}{\sqrt{1-\varrho}} n(y_i) - 2 \frac{\sqrt{\varrho}}{\sqrt{1-\varrho}} N(y_i) \cdot n(y_i) \right)}{\sum_{i=1}^J w_i (\sqrt{\varrho}/\sqrt{1-\varrho}) n(y_i)} \\
 & \left. + \frac{\sum_{i=1}^J [w_i^2 (N(y_i) - N^2(y_i))] \cdot \sum_{i=1}^J w_i \left[\frac{\varrho}{1-\varrho} y_i \cdot n(y_i) \right]}{\left(\sum_{i=1}^J w_i (\sqrt{\varrho}/\sqrt{1-\varrho}) n(y_i) \right)^2} \right]_{y_i = \frac{N^{-1}(PD_i) - \sqrt{\varrho} \cdot q_{1-z}(\tilde{x})}{\sqrt{1-\varrho}}}
 \end{aligned}$$

Appendix B

For any $m \in \mathbf{N}$ the $(m + 1)^{\text{th}}$ element of the *Taylor* series can be written as

$$(A2) \quad \frac{\lambda^m}{m!} \left[\frac{\partial^m \text{VaR}_\alpha(\tilde{\mathbf{Y}} + \lambda \tilde{\mathbf{Z}})}{\partial \lambda^m} \right]_{\lambda=0} = g \circ \left(\frac{\lambda^m}{m!} \sum_{p \prec m} \prod_{r=1}^m (\mu_r [\tilde{\mathbf{Z}} | \tilde{\mathbf{Y}} = l])^{e_{pr}} \right) \Big|_{l = \text{VaR}_z(\tilde{\mathbf{Y}})},$$

with the notation $p \prec m$ to indicate that p is a partition of m , e_i represents the frequency how often a number i appears in a partition p , and g is a function that is independent of the number of credits J . With μ_r as the r^{th} (conditional) moment about the origin and η_r as the r^{th} (conditional) moment about the mean it is possible to write

$$\begin{aligned}
 (A3) \quad & \lambda^m \prod_{r=1}^m (\mu_r [\tilde{\mathbf{Z}} | \tilde{\mathbf{Y}} = l])^{e_{pr}} = \prod_{r=1}^m (\mu_r [\lambda \tilde{\mathbf{Z}} | \tilde{\mathbf{Y}} = l])^{e_{pr}} \\
 & = \prod_{r=1}^m (\mu_r [(\tilde{D}_T | \tilde{\mathbf{Y}} = l) - E[\tilde{D}_T | \tilde{\mathbf{Y}} = l]])^{e_{pr}} = \prod_{r=1}^m (\eta_r [\tilde{D}_T | \tilde{\mathbf{Y}} = l])^{e_{pr}}
 \end{aligned}$$

⁵⁰ In addition to the formula for the granularity adjustment, the authors consider the contribution of single borrowers to entire portfolio risk via the partial derivative with respect to the exposure weight w_i , but this aspect is not the purpose of this article.

for each partition of m .⁵¹ Due to the limitation $\tilde{I}_i \in [-1,1] \forall i \in \{1, \dots, J\}$ there exists a finite constant η_r^* , so that under assumption of conditional independent defaults we have

$$(A4) \quad \eta_r[\tilde{D}_T|\bar{x} = x] = \eta_r \left[\sum_{i=1}^J w_i \cdot \tilde{I}_i | \tilde{Y} = l \right] = \sum_{i=1}^J w_i^r \cdot \eta_r[\tilde{I}_i | \tilde{Y} = l] = \eta_r^* \cdot \sum_{i=1}^J w_i^r.$$

Revisiting equations (A2) to (A4) it is straightforward to see that only for $m = 3$ and $m = 4$ there exist terms which are maximum of Order $O(1/J^2)$

$$(A5) \quad \begin{aligned} \sum_{p < 3} \prod_{r=1}^3 (\eta_r[\tilde{D}_T|\tilde{Y} = l])^{e_{pr}} &= \eta_3[\tilde{D}_T|\tilde{Y} = l] = \eta_3^* \cdot \sum_{i=1}^J w_i^3 \leq \eta_3^* \cdot \sum_{i=1}^J \left(\frac{b}{J \cdot a}\right)^3 \\ &= \eta_3^* \cdot \left(\frac{b}{a}\right)^3 \cdot \frac{1}{J^2} = O\left(\frac{1}{J^2}\right), \\ \sum_{p < 4} \prod_{r=1}^4 (\eta_r[\tilde{D}_T|\tilde{Y} = l])^{e_{pr}} &= \eta_4[\tilde{D}_T|\tilde{Y} = l] + (\eta_2[\tilde{D}_T|\tilde{Y} = l])^2 \\ &= \eta_4^* \cdot \sum_{i=1}^J w_i^4 + \left(\eta_2^* \cdot \sum_{i=1}^J w_i^2\right)^2 \\ &\leq \eta_4^* \cdot \sum_{i=1}^J \left(\frac{b}{J \cdot a}\right)^4 + \left(\eta_2^* \cdot \sum_{i=1}^J \left(\frac{b}{J \cdot a}\right)^2\right)^2 \\ &= \eta_4^* \cdot \left(\frac{b}{a}\right)^4 \cdot \frac{1}{J^3} + \left(\eta_2^* \cdot \left(\frac{b}{a}\right)^2 \cdot \frac{1}{J}\right)^2 = O\left(\frac{1}{J^3}\right) + O\left(\frac{1}{J^2}\right), \end{aligned}$$

with $a \leq E_i \leq b$ for some $0 < a \leq b$ and all i . All terms of higher derivatives of VaR are at least of Order $O(1/J^3)$.

Appendix C

In order to shorten the equation (16) we set $\mu_1 := \mu_1(x)$, $\eta_{2,3} := \eta_{2,3}(x)$, $n_x := n(x)$, and we get the following general form of the second order adjustment

$$(A6) \quad \begin{aligned} \Delta l_2 &= \left[\frac{1}{6 n_x} \frac{d}{dx} \left(\frac{1}{d\mu_1/dx} \frac{d}{dx} \left[\frac{\eta_3 n_x}{d\mu_1/dx} \right] \right) \right. \\ &\quad \left. + \frac{1}{8 n_x} \frac{d}{dx} \left(\frac{1}{n_x} \frac{1}{d\mu_1/dx} \left[\frac{d}{dx} \left(\frac{\eta_2 n_x}{d\mu_1/dx} \right) \right]^2 \right) \right]_{x=q_{1-z}(\bar{x})} \\ &=: [\Delta l_{2,1} + \Delta l_{2,2}]_{x=q_{1-z}(\bar{x})}. \end{aligned}$$

⁵¹ To illustrate that this will indeed hold for each partition, we demonstrate an example for $m = 5 : \lambda \sum_{p < 5} \prod_{r=1}^5 (\mu_r)^{e_{pr}} = \lambda (\mu_5 + \mu_4 \cdot \mu_1 + \mu_3 \cdot (\mu_1)^2 + \mu_3 \cdot \mu_2 + \mu_2 \cdot (\mu_1)^3 + (\mu_2)^2 \cdot \mu_1 + (\mu_1)^5)$.

First, the term $\Delta l_{2,1}$ will be examined

$$(A7) \quad \Delta l_{2,1} = \frac{1}{6} \left[\frac{d}{dx} \left(\frac{1}{d\mu_1/dx} \right) \underbrace{\left(\frac{1}{n_x} \frac{d}{dx} (\eta_3 n_x) \frac{1}{d\mu_1/dx} + \eta_3 \frac{d}{dx} \left(\frac{1}{d\mu_1/dx} \right) \right)}_I \right. \\ \left. + \frac{1}{d\mu_1/dx} \frac{1}{n_x} \frac{d}{dx} \left[\underbrace{\frac{d}{dx} (\eta_3 n_x) \frac{1}{d\mu_1/dx}}_{II} + \underbrace{\eta_3 n_x \frac{d}{dx} \left(\frac{1}{d\mu_1/dx} \right)}_{III} \right] \right].$$

During the derivation, there will be use of following expressions

$$(A8) \quad n_x = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \frac{dn_x}{dx} = -x \cdot n_x, \frac{d^2 n_x}{dx^2} = (x^2 - 1)n_x \quad \text{and} \\ \frac{d}{dx} \left(\frac{1}{d\mu_1/dx} \right) = -\frac{d^2 \mu_1/dx^2}{(d\mu_1/dx)^2}.$$

Then, (I) can be transformed into

$$(A9) \quad \frac{1}{n_x} \frac{d}{dx} (\eta_3 \cdot n_x) = \frac{d\eta_3}{dx} - \eta_3 \cdot x.$$

The derivative of (II) can be calculated as follows

$$(A10) \quad \frac{d}{dx} \left(\frac{d}{dx} (\eta_3 \cdot n_x) \frac{1}{d\mu_1/dx} \right) = \left(\frac{d^2 \eta_3}{dx^2} n_x + 2 \frac{d\eta_3}{dx} \frac{dn_x}{dx} + \eta_3 \frac{d^2 n_x}{dx^2} \right) \frac{1}{d\mu_1/dx} \\ - \left(\frac{d\eta_3}{dx} n_x + \eta_3 \frac{dn_x}{dx} \right) \frac{d^2 \mu_1/dx^2}{(d\mu_1/dx)^2}.$$

Taking the derivative of (III) results in

$$(A11) \quad \frac{d}{dx} \left(\eta_3 \cdot n_x \left(-\frac{d^2 \mu_1/dx^2}{(d\mu_1/dx)^2} \right) \right) = \left(-\frac{d\eta_3}{dx} n_x - \eta_3 \frac{dn_x}{dx} \right) \frac{d^2 \mu_1/dx^2}{(d\mu_1/dx)^2} \\ - \eta_3 \cdot n_x \left(\frac{(d\mu_1/dx)^2 (d^3 \mu_1/dx^3) - 2(d\mu_1/dx) (d^2 \mu_1/dx^2)^2}{(d\mu_1/dx)^4} \right).$$

Reconsidering the derivatives of the density function, equation (A7) is equivalent to

$$(A12) \quad \Delta l_{2,1} = \frac{1}{6(d\mu_1/dx)^2} \left[\eta_3 \left(x^2 - 1 - \frac{d^3 \mu_1/dx^3}{d\mu_1/dx} + \frac{3x(d^2 \mu_1/dx^2)}{d\mu_1/dx} + \frac{3(d^2 \mu_1/dx^2)^2}{(d\mu_1/dx)^2} \right) \right. \\ \left. + \frac{d\eta_3}{dx} \left(-2x - \frac{3(d^2 \mu_1/dx^2)}{d\mu_1/dx} \right) + \frac{d^2 \eta_3}{dx^2} \right].$$

Similarly, the second part of (A6) will be calculated

$$(A13) \quad \Delta l_{2,2} = \frac{1}{8 n_x} \frac{d}{dx} \left(\frac{n_x}{d\mu_1/dx} \underbrace{\left[\frac{1}{n_x} \frac{d}{dx} \left(\frac{\eta_2 n_x}{d\mu_1/dx} \right) \right]}_* \right).$$

For (*) we can use the derivation of the first order adjustment in *Wilde* (2001), so we get

$$(A14) \quad \begin{aligned} \Delta l_{2,2} &= \frac{1}{8 n_x} \frac{d}{dx} \left(\frac{n_x}{d\mu_1/dx} \left[\frac{-x \cdot \eta_2}{d\mu_1/dx} + \frac{d\eta_2/dx}{d\mu_1/dx} - \frac{\eta_2 \cdot d^2\mu_1/dx^2}{(d\mu_1/dx)^2} \right]^2 \right) \\ &= \frac{1}{8} \left[\underbrace{\frac{1}{n_x} \frac{d}{dx} \left(\frac{n_x}{(d\mu_1/dx)^3} \right)}_I \cdot \left(-x \cdot \eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2 \cdot d^2\mu_1/dx^2}{d\mu_1/dx} \right)^2 \right. \\ &\quad \left. + \frac{1}{(d\mu_1/dx)^3} \frac{d}{dx} \left(\underbrace{\left[-x \cdot \eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2 \cdot d^2\mu_1/dx^2}{d\mu_1/dx} \right]^2}_{II} \right) \right]. \end{aligned}$$

For term (I) we obtain

$$(A15) \quad \frac{1}{n_x} \frac{d}{dx} \left(\frac{n_x}{(d\mu_1/dx)^3} \right) = \frac{-x}{(d\mu_1/dx)^3} - 3 \frac{(d^2\mu_1/dx^2)}{(d\mu_1/dx)^4}$$

Calculating (II) leads to

$$(A16) \quad \begin{aligned} \frac{d}{dx} \left(\left[-x \cdot \eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2 \cdot d^2\mu_1/dx^2}{d\mu_1/dx} \right]^2 \right) &= 2 \left(-x \cdot \eta_2 + \frac{d\eta_2}{dx} - \frac{\eta_2 \cdot d^2\mu_1/dx^2}{d\mu_1/dx} \right) \\ &\cdot \left(-\eta_2 - x \frac{d\eta_2}{dx} + \frac{d^2\eta_2}{dx^2} - \frac{d\eta_2}{dx} \frac{d^2\mu_1/dx^2}{d\mu_1/dx} - \eta_2 \frac{d^3\mu_1/dx^3}{d\mu_1/dx} + \eta_2 \frac{d^2\mu_1}{dx^2} \frac{d^2\mu_1/dx^2}{(d\mu_1/dx)^2} \right). \end{aligned}$$

Therewith, we get for equation (A13)

$$(A17) \quad \begin{aligned} \Delta l_{2,2} &= \frac{1}{8(d\mu_1/dx)^3} \left[\left(-x - 3 \frac{d^2\mu_1/dx^2}{d\mu_1/dx} \right) \left(\eta_2 \left[-x - \frac{d^2\mu_1/dx^2}{d\mu_1/dx} \right] + \frac{d\eta_2}{dx} \right)^2 \right. \\ &\quad \left. + 2 \left(\eta_2 \left[x + \frac{d^2\mu_1/dx^2}{d\mu_1/dx} \right] - \frac{d\eta_2}{dx} \right) \left(\eta_2 \left[1 + \frac{d^3\mu_1/dx^3}{d\mu_1/dx} - \frac{(d^2\mu_1/dx^2)^2}{(d\mu_1/dx)^2} \right] \right. \right. \\ &\quad \left. \left. + \frac{d\eta_2}{dx} \left[x + \frac{d^2\mu_1/dx^2}{d\mu_1/dx} \right] - \frac{d^2\eta_2}{dx^2} \right) \right]. \end{aligned}$$

Thus, our primary equation (A6) can be expressed by the equations (A12) and (A17). Until this point, we only assumed the systematic factor to be normal distributed. For the contained conditional moments we get

$$(A18) \quad \begin{aligned} \mu_1 &= \sum_{i=1}^J w_i \cdot p_i(x), \quad \eta_2 = \sum_{i=1}^J w_i^2 (p_i(x) - p_i^2(x)) \quad \text{and} \\ \eta_3 &= \sum_{i=1}^J w_i^3 [p_i(x) - 3p_i^2(x) + 2p_i^3(x)]. \end{aligned}$$

Now, we perform the second order adjustment with respect to the probability of default

$$(A19) \quad p(x) = N(y), \quad \text{with} \quad y = \frac{N^{-1}(PD) - \sqrt{\varrho} \cdot x}{\sqrt{1 - \varrho}} \quad \text{and} \quad s = \frac{\sqrt{\varrho}}{\sqrt{1 - \varrho}},$$

of the *Vasicek* model. Having a closer look at (A17) and the conditional moments, we find that the following derivatives are needed

$$(A20) \quad \frac{d(p(x))}{dx} = -s \cdot n(y), \quad \frac{d^2(p(x))}{dx^2} = -s^2 \cdot y \cdot n(y), \quad \frac{d^3(p(x))}{dx^3} = -s^3 \cdot n(y) [y^2 - 1],$$

$$(A21) \quad \frac{d(p^2(x))}{dx} = -2s \cdot N(y) \cdot n(y), \quad \frac{d^2(p^2(x))}{dx^2} = 2s^2 \cdot n(y) [n(y) - N(y) \cdot y],$$

$$(A22) \quad \frac{d(p^3(x))}{dx} = -3s \cdot N^2(y) \cdot n(y), \quad \frac{d^2(p^3(x))}{dx^2} = 3s^2 \cdot N(y) \cdot n(y) [2n(y) - N(y) \cdot y].$$

Finally, we just have to use equations (A12), (A17)–(A22) in order to perform (A6). To simplify the illustration, we will reproduce the complete formula only for a homogeneous portfolio

$$(A23) \quad \begin{aligned} \Delta L_2 &= \frac{1}{6J^2 s^2 n_y^2} [(x^2 - 1 + s^2 + 3x s y + 2s^2 y^2) (N_y - 3N_y^2 + 2N_y^3) \\ &\quad + s n_y (2x + 3s y) (1 - 6N_y + 6N_y^2) - s^2 n_y (y - 6[N_y y - n_y] + 6N_y [N_y y - 2n_y])] \\ &\quad - \frac{1}{8J^2 s^3 n_y^3} [(-x - 3s y) ([N_y - N_y^2] [-x - s y] - s n_y [1 - 2N_y])^2 \\ &\quad + 2 ([N_y - N_y^2] [x + s y] + s n_y [1 - 2N_y]) \\ &\quad \cdot ([N_y - N_y^2] [1 - s^2] - s n_y [1 - 2N_y] [x + s y] + s^2 n_y [y + 2(n_y - N_y y)])], \end{aligned}$$

$$\text{with } N_y = N(y), \quad n_y = n(y), \quad y = \frac{N^{-1}(PD) - \sqrt{\varrho} \cdot x}{\sqrt{1 - \varrho}}, \quad s = \frac{\sqrt{\varrho}}{\sqrt{1 - \varrho}}, \quad x = q_{1-z}(\tilde{x}_T).$$

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Summary

Concentration Risk under Pillar 2: When are Credit Portfolios Infinitely Fine Grained?

The ongoing debate concerning credit concentration risk is mainly driven by the requirements on credit risk management due to Pillar 2 of Basel II since risks (e.g. concentration risk) that are not fully captured by Pillar 1 should be adequately considered in the banks' risk management. This instruction is indeed relevant since quantifying credit portfolio risk in Pillar 1 is based on an Asymptotic Single Risk Factor (ASRF) framework in which concentration risk is not covered. Against the background of the ASRF model, we determine the number of credits up to which concentration risk leads to a significant estimation error so that the assumption of an infinitely fine grained portfolio is inadequate. We conclude that the critical portfolio size varies from 22 up to 35,986 debtors, dependent on assets correlation and probability of default. Using a modified valuation function (granularity adjustment) it is possible to reduce the critical number of credits by averaged 83.04%. (JEL G21, G28)

Zusammenfassung

Konzentrationsrisiken unter Basel II: Wann sind Kreditportfolios unendlich granular?

Die Diskussion hinsichtlich Kredit-Konzentrationsrisiken wird hauptsächlich durch die Anforderungen an das Kreditrisikomanagement getrieben, wie diese in Säule 2 von Basel II formuliert sind. So fordert Säule 2, dass Risiken (z.B. Konzentrationsrisiken), die im Rahmen des Risikomanagements einer Bank nicht vollständig durch Säule 1 erfasst werden, adäquat zu berücksichtigen sind. Diese Forderung erscheint für Konzentrationsrisiken plausibel, da die Ermittlung des Kreditportfoliorisikos gemäß Säule 1 auf einem Asymptotischen Ein-Faktor-(engl.: Asymptotic Single Risk Factor, ASRF)Modell basiert, mit dem solche Risiken nicht abgedeckt werden. Vor dem Hintergrund des ASRF-Modells wird im vorliegenden Beitrag die kritische Größe von Kreditportfolios ermittelt, bis zu der Konzentrationsrisiken zu einem signifikanten Schätzfehler führen, sodass die Annahme unendlicher Granularität für entsprechend kleine Portfolios nicht adäquat ist. Unsere Untersuchung zeigt, dass die kritische Portfoliogröße – abhängig von der Kor-

relation zwischen den Kreditnehmerrenditen und der Kreditnehmerausfallwahrscheinlichkeit – von 22 bis 35.986 Krediten variiert. Durch Modifikation der Berechnungsfunktion (Granularitätsanpassung) kann die kritische Kreditanzahl um durchschnittlich 83,04 % gesenkt werden.