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## Partially identified heteroskedastic SVARs

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Paper**

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**Emanuele Bacchiocchi, Andrea Bastianin, Toru Kitagawa,  
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## Summary

This paper presents new results on the identification of heteroskedastic structural vector autoregressive (HSVAR) models. Point identification of HSVAR models fails when some shifts in the variances of the structural shocks are suspected to be statistically indistinguishable from each other. This paper presents a new strategy that allows researchers to continue using HSVAR models in this empirically relevant case. We show that a combination of heteroskedasticity and zero restrictions can recover point identification in HSVAR models even in the absence of heterogeneous variance shifts. We derive the identified sets for impulse responses and show how to compute them. We perform inference on the impulse response functions, building on the robust Bayesian approach developed for set-identified SVARs. To illustrate our proposal, we present an empirical example based on the literature on the global crude oil market, where standard identification is expected to fail under heteroskedasticity.

**Keywords:** Heteroskedastic SVAR, point and set identification, robust Bayesian approach

**JEL classification:** C11, C32, C51, Q41

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# Partially identified heteroskedastic SVARs

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## Abstract

This paper presents new results on the identification of heteroskedastic structural vector autoregressive (HSVAR) models. Point identification of HSVAR models fails when some shifts in the variances of the structural shocks are suspected to be statistically indistinguishable from each other. This paper presents a new strategy that allows researchers to continue using HSVAR models in this empirically relevant case. We show that a combination of heteroskedasticity and zero restrictions can recover point identification in HSVAR models even in the absence of heterogeneous variance shifts. We derive the identified sets for impulse responses and show how to compute them. We perform inference on the impulse response functions, building on the robust Bayesian approach developed for set-identified SVARs. To illustrate our proposal, we present an empirical example based on the literature on the global crude oil market, where standard identification is expected to fail under heteroskedasticity.

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# I Introduction

In recent years, there has been growing interest in the use of external information to identify the parameters of econometric models. This external information, derived from features of the data or from proxies designed to capture specific structural shocks, is often combined with restrictions on the parameter space suggested by economic theory. A case in point is the literature that exploits heteroskedasticity to identify simultaneous equations systems (Rigobon, 2003) and structural vector autoregressive (SVARs) models (Lanne and Lütkepohl, 2008). When the data exhibit volatility clusters that can be attributed to shifts in the variance of structural innovations – while holding the parameters of the conditional mean constant – there are important gains in terms of identification of the structural parameters (see, for example, Kilian and Lütkepohl, 2017, Chapter 14, or the recent empirical work of Brunnermeier, Palia, Sastry, and Sims, 2021).

While heteroskedastic SVAR models have become a standard tool in macroeconometrics, it is important to recognize that two caveats apply when using them. First, identification through heteroskedasticity is a purely statistical identification strategy. In other words, shocks have a structural interpretation only if they give rise to impulse responses with a credible economic interpretation. Second, the size of the shifts in the variance of different shocks must be sufficiently heterogeneous for the identification to be valid.<sup>1</sup>

On the contrary, if some of the shifts in the variances are not distinct, heteroskedasticity cannot be used to identify structural shocks. In this case, the literature does not provide a specific solution for continuing to use HSVAR models, and other identification schemes - provided they are credible - must be considered to solve the identification problem. Interestingly, Carriero et al. (2023) propose a blended approach, where heteroskedasticity can be combined with sign restrictions, narrative restrictions, and external instruments.

This paper presents a new strategy that allows researchers to keep using HSVAR models even in the absence of information to identify the shocks of interest. Our approach starts where statistical tests would suggest stopping: namely, when some shifts in the variances of the structural shocks are suspected to be statistically indistinguishable from each other. Although

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<sup>1</sup>Lütkepohl et al. (2020) developed a formal test for identification by heteroskedasticity. Lewis (2022) proposes an alternative test of weak identification in the context of heteroskedastic SVARs (HSVARs). This test, based on the IV literature, can be applied in very specific specifications.

in a rather different setup, based on simultaneous equation models for cross-sectional data, Lütkepohl, Milunovich, and Yang (2020) also investigate the situation where (conditional) heteroskedasticity involves only a subset of equations. They propose tests for the heteroskedasticity rank and a way for estimating and doing classical inference on the parameters of the equations associated with conditional heteroskedasticity.

Instead, our idea is to combine the presence of heteroskedasticity with some zero and/or sign restrictions on parameters or functions of them. This strategy allows us to deal with HSVAR models that are not point, but only set identified. The paper makes three main contributions. The first concerns the development of analytical results on identification. We show that, apart from normalisation constraints, a combination of heteroskedasticity and zero restrictions allows point identification in HSVAR models even in the absence of heterogeneous variance shifts. Importantly, the number of zero restrictions in this case is much smaller than that generally used for point identification in traditional SVAR models.

Our second contribution, closely related to the first, is to extend the topological analysis of the identified set offered in Giacomini and Kitagawa (2021) for SVAR models to the HSVAR literature, and to derive analytical results for point identification and for set identification with convex identified sets. The mapping between the reduced and structural form parameters facilitates the extension of the literature on set identification, largely used in standard SVAR models, to the HSVAR framework.

Finally, another contribution concerns estimation and inference on the identified set. In this respect, we adapt the robust Bayesian approach of Giacomini and Kitagawa (2021) to our setup. In fact, this approach is perfectly suited to the peculiarities of HSVARs, where the identifying assumptions are violated due to heterogeneous variance shifts in the structural shocks. We provide a useful algorithm to implement our strategy for estimation and inference of the identified set. This provides applied economists and econometricians with a new tool for their empirical analyses when clusters of volatility provide a useful but insufficient source of information for the identification of structural shocks.

An empirical example about the identification of structural shocks driving the real price of crude oil illustrates our methodology. In this example, we show how to set or point identify structural shocks when the standard HSVAR approach fails.

The rest of the paper is organized as follows. The next section briefly surveys the literature; Section II introduces the econometric framework and provides some preliminary results on the identification of HSVARs. Section III is dedicated to the theory of identification in HSVARs. Section IV focuses on the inferential analysis of identified sets through a Robust Bayesian approach. Section V presents the empirical example and Section VI concludes. An appendix with the proofs and many other results completes the paper.

## I.1 Related literature

This paper is strongly related to the identification through heteroskedasticity literature both for simultaneous equations systems (Rigobon, 2003; Klein and Vella, 2010; Lewbel, 2012; Lütkepohl, Milunovich, and Yang, 2020) and SVARs (see e.g. Lanne and Lütkepohl, 2008; Bacchiocchi, 2017; Kilian and Lütkepohl, 2017; Lütkepohl and Netšunajev, 2017). However, to the best of our knowledge, the idea that heteroskedasticity can be helpful in identifying econometric models has been firstly proposed by Sentana and Fiorentini (2001) in a context of factor models, nesting SVAR models as well. Our contribution builds on the results of the statistical tests for heterogeneous variance shifts by Lütkepohl et al. (2020) and Lewis (2022), in the sense that our approach can be implemented when any statistical tests suggest a failure of the identifying information from heteroskedasticity to point identify the structural shocks.<sup>2</sup>

As previously mentioned, a paper that relates to our research is Carriero et al. (2023), where the authors combine different identification strategies, each of which can contribute resolving the critical issues of the others. Specifically, they propose different algorithms that combine sign restrictions, heteroskedasticity and external instruments. In our approach we focus on weak identification due to lack of (or, better, not enough) heterogeneity in the variances, where zero and sign restrictions are tools for recovering the identification issue. In their approach, instead, heteroskedasticity is mainly intended as a tool to reduce the identified set obtained by sign restrictions, narrative restrictions, and external instruments approaches.

While in our contribution the main assumption is that only the variances of the shocks

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<sup>2</sup>Different approaches exploiting heteroskedasticity for the identification of structural shocks are the recent contribution by Lewis (2021), that does not require volatility clusters, but simply needs for time-varying volatility of unknown form, as well as Lütkepohl and Schlaak (2021), who analyse identification through heteroskedasticity in the context of proxy SVAR models. See also Sims, 2020, for a recent contribution on heteroskedastic SVARs with misspecified regimes.

are subject to breaks, other authors find evidence of structural shifts among the structural parameters of the model, too (see, among others Sims and Zha, 2006; Inoue and Rossi, 2011; Boivin and Giannoni, 2006). This literature, that does not focus solely on SVAR models, allows for impulse responses to be different in the different regimes, while they are the same when only the variances do change.<sup>3</sup>

This paper contributes also to the literature on point and set identified SVARs. For point identification, we exploit the general criteria in Rubio-Ramírez et al. (2010), with subsequent modifications proposed by Bacchiocchi and Kitagawa (2021), for global identification on SVARs. As for set identification, we use sign restrictions to set identify the structural impulse response functions of interest (Rubio-Ramírez, Waggoner, and Zha, 2010; Uhlig, 2005).

Strictly connected to this last point is the literature on how to do inference on set identified models. As stated in the introduction, our approach builds on Giacomini and Kitagawa (2021), but other approaches have been proposed in the literature to pursue this purpose.<sup>4</sup>

## II SVARs and HSVARs: some definitions

### II.1 Econometric framework

Consider the following Structural Vector Autoregressive (SVAR) model

$$A_0 y_t = a + \sum_{i=1}^l A_i y_{t-i} + \varepsilon_t \quad (1)$$

where  $y_t$  is a  $n$ -dimensional vector of observable variables,  $\varepsilon_t$  is a vector of mutually orthogonal white noise processes, normally distributed with mean zero and time-varying covariance matrix. Specifically, let the covariance matrix of the structural shocks  $\varepsilon_t$  be as follows

$$E(\varepsilon_t \varepsilon_t') = \begin{cases} I_n & \text{if } 1 \leq t \leq T_B \\ \Lambda & \text{if } T_B < t \leq T \end{cases} \quad (2)$$

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<sup>3</sup>However, only few papers constructively use the presence of regime shifts to solve the identification issue (Magnusson and Mavroeidis, 2014; Bacchiocchi and Fanelli, 2015; Bacchiocchi and Kitagawa, 2023).

<sup>4</sup>Gafarov et al. (2018) and Granziera et al. (2018) provide results based on a frequentist setting, while, among others, Baumeister and Hamilton (2015) and Arias et al. (2018) adopt Bayesian inference.



where  $1 < T_B < T$  is the break date,  $I_n$  is the  $(n \times n)$  identity matrix, and  $\Lambda$  is a  $(n \times n)$  diagonal matrix made of strictly positive numbers.<sup>5</sup> The  $n \times 1$  vector  $a$  contains the intercepts and the  $n \times n$  matrices  $A_i$ , with  $i = 0, \dots, l$ , collect the structural parameters. The structural parameters can be indicated as  $\theta = (A_0, A_+, \Lambda) \in \Theta \subset \mathbb{R}^{(n+m)n+n}$ , with  $m = nl + 1$ , and where the  $n \times m$  matrix  $A_+ = (a, A_1, \dots, A_l)$ . We denote the open dense set of all structural parameters by  $\mathbb{P}^S \subset \mathbb{R}^{(n+m)n+n}$ . The model in Eq.s (1)-(2) is a standard SVAR model with structural shocks characterized by different volatility regimes. As shown in Eq. (2), structural innovations have unit variance before the break, and variance equal to the diagonal elements of  $\Lambda$ , denoted as  $\lambda_i$  after the break. Hereafter, this model is referred to as heteroskedastic SVAR (HSVAR).

The reduced-form VAR model can be written as

$$y_t = b + \sum_{i=1}^l B_i y_{t-i} + u_t \quad (3)$$

where  $b = A_0^{-1}a$ ,  $B_i = A_0^{-1}A_i$ . Furthermore, for both the regimes, the vector of error terms is defined as  $u_t = A_0^{-1}\varepsilon_t$ , with

$$E(u_t u_t') = \begin{cases} \Omega_1 = A_0^{-1} A_0^{-1'} & \text{if } 1 \leq t \leq T_B \\ \Omega_2 = A_0^{-1} \Lambda A_0^{-1'} & \text{if } T_B < t \leq T. \end{cases} \quad (4)$$

The VAR model, thus, presents different covariance matrices of the error terms  $\Omega_1$  and  $\Omega_2$ , and thus heteroskedasticity, as in, among others, Rigobon (2003), Lanne and Lütkepohl (2008), and Bacchiocchi and Fanelli (2015). The reduced-form parameters are denoted by  $\phi = (B, \Omega_1, \Omega_2) \in \Phi \subset \mathbb{R}^{n+n^2l} \times \Omega_n \times \Omega_n$ , where the  $n \times m$  matrix  $B = (b, B_1, \dots, B_l)$  and  $\Omega_n$  is the space of positive-semidefinite matrices of dimension  $n \times n$ . The set of all reduced-form parameters is denoted by  $\mathbb{P}^R \subset \mathbb{R}^{nm+n(n+1)}$ . The reduced form will be denoted HVAR.

Conditional on the restrictions of the domain  $\Phi$  such that all the roots of the characteristic polynomial lie outside the unit circle, there exists an equivalent VMA( $\infty$ ) representation for

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<sup>5</sup>We consider the initial conditions for the first regime,  $y_0, \dots, y_{1-l}$ , as given, while for the second regime they are fixed as the last  $l$  observations of the former, in order to guarantee the contiguity of the regimes on the whole sample.

the HVAR in Eq. (3), assuming the form<sup>6</sup>

$$y_t = c + \sum_{j=0}^{\infty} C_j A_0^{-1} \varepsilon_{t-j} \quad (5)$$

where  $C_j$  is the  $j$ -th coefficient matrix of  $\left(I_n - \sum_{i=1}^l B_i L^i\right)^{-1}$ . Based on the VMA representation, the long-run impulse response,  $IR^\infty$ , and the one at any  $h$ ,  $IR^h$ , are, respectively

$$IR^\infty = \lim_{h \rightarrow \infty} IR^h = \left(I_n - \sum_{j=1}^l B_j\right) A_0^{-1} \quad \text{and} \quad IR^h = C_h A_0^{-1}, \quad (6)$$

whose  $(i, j)$ -element represents the response of the  $i$ -th variable of  $y_{t+h}$  to a *unit* shock on the  $j$ -th element of  $\varepsilon_t$ , independently of the regime considered.

## II.2 Preliminary results on the identification of HSVARs

As is well known in the SVAR literature, without any restriction it is impossible to uniquely pin down the structural parameters based on the reduced form of the model. If, instead, we suppose the parameters of the conditional mean in the HVAR in Eq. (1) to remain stable across the two regimes, then Rigobon (2003), for a bivariate case, and Lanne and Lütkepohl (2008), for the general case, proved there is some gain in terms of identification. In this section we introduce some general theoretical results for the HVAR in Eq. (1). All the results will be formally presented in Appendix A.

Consider an  $n$ -variable HVAR model as in Eq.s (1)-(2). Following the parametrization and notations we have been using so far, we analyze identification of the  $n \times n$  matrix  $C$  that represents the inverse of structural coefficient matrix  $A_0$ , i.e.  $C \equiv A_0^{-1}$ , and  $\Lambda$ ,  $n \times n$  diagonal matrix with strictly positive elements. Given the reduced-form covariance matrix at regime 1

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<sup>6</sup>The HVAR in Eq.s (3)-(4) is characterized by the same parameters for the conditional mean over the two regimes, therefore breaks are confined to second moments parameters. As a consequence, the absence of unit roots is a characteristic of the model in the whole sample, and not within each regime. Furthermore, as shown in Eq. (5), the VMA representation is unique and not regime-specific. It follows that, if shocks have the same magnitude, impulse response functions in the two regimes are equivalent. If instead one considers a one standard deviation structural shock, impulse responses will be of different magnitude, but have exactly the same shape in different regimes.

and 2, denoted by  $\Omega_1$  and  $\Omega_2$ , respectively,  $C$  and  $\Lambda$  solve

$$\Omega_1 = CC' \quad \text{and} \quad \Omega_2 = C\Lambda C'. \quad (7)$$

The first important result on the identification of the HSVAR is about the set of solutions, in terms of  $(C, \Lambda)$ , of the system of equations in (7). In Theorem 4 in Appendix A we show that the solution is not unique, but any permutations and change of signs of the column vectors in  $C$ , as far as the same permutations are applied to the diagonal elements of  $\Lambda$ , remain observationally equivalent. In order to solve this indeterminacy, one possibility is to fix a specific ordering for the equations, that corresponds to fixing a specific ordering for the variances of the structural shocks in the second regime, i.e. the diagonal elements in  $\Lambda$ . Of course, this becomes problematic when some of the variances are equivalent.

In this direction, the second important result, reported in Theorem 5 in Appendix A, states how point identification is possible only once the solution for  $\Lambda$  is characterized by all distinct elements on the main diagonal. In this case, fixing a specific ordering of the structural shocks, as well as the standard sign normalization, leads the solution  $(C, \Lambda)$  to be unique, and thus point identified.

Finally, the third important result concerns the representation of the system in (7) as an eigen-decomposition problem. To see this, let  $\Omega_{1,tr}$  be a lower triangular Cholesky decomposition of  $\Omega_1$ . Following Proposition A.1 of Uhlig (2005), the set of non-singular matrices solving Eq. (20) can be expressed as  $C = \Omega_{1,tr}Q$ ,  $Q \in \mathcal{O}(n)$ , where  $\mathcal{O}(n)$  is the set of  $n \times n$  orthogonal matrices. Plugging this representation of  $C$  into Eq. (7), leads to

$$\begin{aligned} C &= \Omega_{1,tr}Q \\ \Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'} &= Q\Lambda Q'. \end{aligned} \quad (8)$$

Symmetry of  $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'}$  and orthogonality of  $Q$  implies that solving Eq. (8) is precisely the eigen-decomposition problem. Identification of  $(C, \Lambda)$  can be therefore cast as uniqueness of the eigen-decomposition of  $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'}$  into the diagonal matrix of eigenvalues and the corresponding eigenvectors collected in  $Q$ . According to the previous result, an HSVAR can be point identified, up to permutations and sign changes, if the eigenvalues of  $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'}$  are

all distinct. A nice geometric interpretation for this result, for the simple bivariate case, is discussed in Appendix B.

### III Set-identification due to proportional volatility shifts

In this section we extend the preliminary results reported in Section II.2 where, conditional on the reduced-form parameters, the identification issue was addressed as an eigen-decomposition problem. Specifically, let  $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  be the eigenvalues of the eigen-decomposition problem in Eq. (8), where  $C = A_0^{-1}$  collects impact responses,  $Q$  is the orthogonal matrix containing the eigenvectors, and  $\Omega_i, i = \{1, 2\}$ , are the reduced-form covariance matrices with  $\Omega_{i,tr}$  the related Cholesky lower triangular matrices. Theorem 6 shows that a necessary condition for point identification of the structural parameters is the absence of multiplicity in eigenvalues in  $\Lambda$ . In the case of multiple eigenvalues, in fact, the identification of  $C$  fails. In particular, if two (or more) eigenvalues are equal, say  $\lambda_j = \lambda_{j+1}$ , then the corresponding columns of  $Q$  – i.e. the eigenvectors associated to  $\lambda_j$  and  $\lambda_{j+1}$  – denoted by  $q_j$  and  $q_{j+1}$ , are not unique. In fact they represent a basis for the two-dimensional vector space in  $\mathbb{R}^n$ , but any other couple of orthogonal unitary vector belonging to such a space could be an acceptable candidate to enter in the  $Q$  matrix. The matrix  $Q$ , thus, will not be a singleton in  $\mathcal{O}(n)$  anymore, but will be a set of admissible orthogonal matrices solving the eigen-decomposition problem in Eq. (8).

More suitable notations and formalization are thus necessary. We start by formalizing the eigen-decomposition problem, with the possibility of multiple eigenvalues.

**Definition 1** (Eigenspace of multiple eigenvalues). Let the eigen-decomposition problem in Eq. (8) be characterized by the following eigenvalues

$$\lambda_1 \neq \dots \neq \lambda_k$$

where the generic  $i$ -th distinct eigenvalue has algebraic multiplicity equal to  $m_i$ , i.e.  $g(\lambda_i) = m_i, i = 1, \dots, k$ , with  $\sum_{i=1}^k m_i = n$ . Let  $Q(\lambda_i)$  be the eigenspace associated to the  $i$ -th

eigenvalue  $\lambda_i$ , i.e.

$$Q(\lambda_i) = \left( \text{span} (q_1^i, \dots, q_{m_i}^i) \cap \mathcal{S}^{n-1} \right) \subset \mathbb{R}^n \quad (9)$$

where  $q_1^i, \dots, q_{m_i}^i$  are linearly independent (not unique) eigenvectors associated to  $\lambda_i$  with  $\mathcal{S}^{n-1}$  being the unit sphere in  $\mathbb{R}^n$ . Moreover, given the result in Lemma 3 in Appendix C,  $\dim(Q(\lambda_i)) = m_i$ .  $\square$

According to Definition 1, let  $Q_\lambda = Q(\lambda_1) \times \dots \times Q(\lambda_k)$ . It is possible to introduce the set of all admissible matrices  $Q$  as follows

$$\mathcal{Q}(\phi) = \left\{ (q_1, q_2, \dots, q_n) \in Q_\lambda \right\}. \quad (10)$$

As in the case of multiplicities  $\mathcal{Q}(\phi)$  is not a singleton in  $\mathcal{O}(n)$ , one could think of imposing restrictions, likewise it is traditionally done in SVARs. This will be the topic of the next section.

### III.1 Normalization, equality and sign restrictions

One of the characteristics of HSVARs is that the identification is obtained from a statistical point of view, without imposing restrictions on the parameters. However, we have seen in Section II.2 that normalization restrictions are important and play a relevant role. Moreover, we will see that in some cases, imposing equality or sign restrictions can be interesting to improve the results obtained through HSVARs, especially when some of the assumptions in Theorems 5 to 7 are no longer valid, as the presence of multiplicities. In this section we discuss normalization restrictions first, then we move to the equality restrictions before concluding with sign restrictions.

#### *Normalization restrictions*

The normalization issue has been largely debated in econometrics. Specifically for SVAR models, we refer to Waggoner and Zha (2003) and, more recently, to Hamilton, Waggoner, and Zha (2007). They show that a poor normalization rule can invalidate statistical inference on the parameters. In our setup, the first normalization restriction consists in imposing the

covariance matrix of the structural shocks to be the identity matrix in the first regime, i.e.  $E(\varepsilon_t \varepsilon_t') = I_n$ , and, as a consequence, the  $A$  matrix in the second regime, i.e.  $E(\varepsilon_t \varepsilon_t') = A$ , as already introduced in Eq. (2). However, as discussed in Section II.2, indeterminacy of the solutions also arises in terms of the sign of the columns of  $C$  and the particular ranking of the variaces in  $A$ . The normalization rule used in this paper is summarized here below.

**Definition 2.** (Normalization) A normalization rule can be characterized by a set  $N \subset \mathbb{P}^S$  such that for any structural parameter point  $\theta = (A_0, A_+, A) \in \mathbb{P}^S$ , there exists a unique permutation matrix  $P \in \mathcal{P}(n)$  and a unique diagonal matrix  $S \in \mathcal{D}(n)$ , with +1 and -1 along the diagonal, such that  $(PSA_0, PSA_+, PAP') \in N$ .  $\square$

For the sake of simplicity, concerning the ordering of the elements in the diagonal matrix  $A$ , we assume

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n. \quad (11)$$

It is important to stress that all the results developed in the paper can be rephrased in terms of different normalization rules coherent with Definition 2.

### *Equality restrictions*

The classical approach to address the identification issue in SVARs is to impose equality restrictions on the structural parameters or on particular linear and non-linear functions of them. Although not common in the literature of heteroskedastic SVARs, we do not preclude this possibility and allow for possible equality and sign restrictions. We first consider the former, while the latter will be presented in the next section.

Giacomini and Kitagawa (2021) and Arias et al. (2018), in the context of SVARs, stress that imposing constraints on the structural parameters, or on suitable functions of them, such as on the impulse responses, corresponds to restrict the columns of the orthogonal matrix  $Q \in \mathcal{Q}(\phi)$ . Here below, we show that it also happens in the context of HSVARs. In fact, as shown in Eq. (8), the structural-form parameters can be defined as the product of the orthogonal matrix  $Q$  and quantities coming from the reduced form. As these latter elements are unrestricted, imposing restrictions is equivalent to constrain the columns of  $Q$ . The set of

equality restrictions we consider, in compact notation, are as follows:

$$\mathbf{F}(\phi, Q) \equiv \left( (F_1(\phi)q_1)', (F_2(\phi)q_2)', \dots, (F_n(\phi)q_n)' \right)' = \mathbf{0} \quad (12)$$

where  $F_i(\phi)$ , of dimension  $f_i \times n$ , depends on the reduced-form parameters  $\phi = (B, \Omega_1, \Omega_2)$  only, while  $q_i$  is the  $i$ -th column of  $Q$ . The total number of restrictions characterizing the HSVAR is given by  $f = f_1 + \dots + f_n$ .

Focusing our attention on the  $i$ -th eigenspace as in the eigen-decomposition in Definition 1, let the set of restrictions on the vectors  $(q_1^i, \dots, q_{m_i}^i) \in Q(\lambda_i)$  be contained in the  $f^i \times n$  matrix  $\mathbf{F}^i(\phi, Q)$ , with  $f^i$  denoting the total number of restrictions on the vectors  $(q_1^i, \dots, q_{m_i}^i)$ . Moreover, let the  $f_j^i$  restrictions on the  $j$ -th vector  $q_j^i$  be defined as  $F_j^i(\phi)q_j^i = 0$ . This allows us to introduce the following definition:

**Definition 3** (Non redundant restrictions). Given reduced-form parameter  $\phi = (B, \Omega_1, \Omega_2)$ , let the HSVAR be characterized by the eigen-decomposition in Definition 1. Moreover, let the  $m_i$  vectors  $(q_1^i, \dots, q_{m_i}^i) \in Q(\lambda_i)$ ,  $i = 1, \dots, k$ , be characterized by zero restrictions of the form  $F_j^i(\phi)q_j^i = 0$ . Such identifying restrictions are *non redundant* if, for  $j = 1, \dots, m_i$ , the orthogonal vectors  $(q_1^i, \dots, q_{j-1}^i)$  are linearly independent of the row vectors of  $F_j^i(\phi)$ .  $\square$

Bacchiocchi and Kitagawa (2021) introduced first this definition of non redundant restrictions to complement the result in Theorem 7 in Rubio-Ramírez, Waggoner, and Zha (2010) on the identification of SVARs. In the same way, we will use it for developing conditions for point identification in our HSVARs.

### *Sign restrictions*

Uhlig (2005), among others, proposes sign restrictions to impulse responses in order to obtain identified sets rather than point identification. Giacomini and Kitagawa (2021) and Arias et al. (2018) combine sign and zero restrictions to tighten the impulse response identified sets.

As for the equality restrictions, sign restrictions can be seen as constraints on the columns of the  $Q$  matrix. Suppose to impose a set of  $s_{h,i}$  restrictions on the impulse responses to the  $i$ -th shock at the  $h$ -th horizon. We can write the sign restrictions as  $S_{h,i}(\phi)q_i \geq \mathbf{0}$ , where, given the definition of the impulse response provided in Eq. (6),  $S_{h,i} \equiv D_{h,i} C_h(B)\Omega_{1,tr}$  is a

$s_{h,i} \times n$  matrix with  $D_{h,i}$ , of dimension  $s_{h,1} \times n$ , a selection matrix made of 1 and  $-1$  elements indicating the restricted impulse responses. A compact notation for all the sign restrictions can be defined by

$$\mathbf{S}(\phi, Q) \geq \mathbf{0}. \quad (13)$$

*Admissible structural parameters and identified set*

Based on the parametrization of the model and the set of all possible restrictions considered above, it is now possible to formally define when a point in the parametric space can be indicated as admissible.

**Definition 4** (Admissible parameters). A structural parameter point  $(A_0, A_+, \Lambda)$  is said admissible if it satisfies the normalization restrictions of Definition 2, the equality restrictions in Eq. (12) and the sign restrictions in Eq. (13). Given the set of reduced-form parameters  $\phi \in \Phi$ , the set of admissible parameters can be defined as

$$\mathcal{A}^r(\phi) \equiv \left\{ (A_0, A_+, \Lambda) = (Q' \Omega_{1,tr}^{-1}, Q' \Omega_{1,tr}^{-1} B, \Lambda) \in N \mid Q \in \mathcal{Q}(\phi), \mathbf{F}(\phi, Q) = \mathbf{0}, \mathbf{S}(\phi, Q) \geq \mathbf{0} \right\}. \quad \square$$

At the same time, it is interesting to focus on the set of all the admissible matrices  $Q$ . We thus provide the following definition.

**Definition 5** (Admissible  $Q$  matrices). An orthogonal matrix  $Q$  is said admissible if, conditional on the reduced-form parameters, it satisfies the normalization restrictions of Definition 2, the equality restrictions in Eq. (12) and the sign restrictions in Eq. (13). The set of all admissible  $Q$  matrices is defined as

$$\mathcal{Q}(\phi | F, S) \equiv \left\{ Q \in \mathcal{Q}(\phi) \mid (A_0, A_+, \Lambda) \in \mathcal{A}^r(\phi) \right\}. \quad \square$$

Finally, given that the attention could not be limited to the structural parameters but on transformations of them, like impulse response functions, it is also important to define the so called *identified set*.

**Definition 6** (Identified set). Given the set of admissible  $Q$  matrices  $\mathcal{Q}(\phi | F, S)$  according to



Definition 5, an identified set is defined as

$$IS(\phi|F, S) \equiv \left\{ \eta(\phi, Q) \mid Q \in \mathcal{Q}(\phi|F, S) \right\}.$$

with  $\eta(\phi, Q)$  being the transformation of the structural parameters of interest, defined as

$$\eta(\phi, Q) = IR_{gj}^h = e'_g C_h(B) \Omega_{1,tr} Q e_j \equiv c'_{gh}(\phi) q_j$$

where  $IR_{gj}^h$  is the  $(g, j)$ -th element of  $IR^h$  and  $c'_{gh}(\phi)$  is the  $g$ -th row of  $C_h(B) \Omega_{1,tr}$ .  $\square$

## III.2 Point-identification in HSVARs with proportional volatility shifts

As discussed in the previous sections, point identification in HSVAR can be achieved only if the eigen-decomposition problem in Eq. (8) is characterized by  $n$  distinct eigenvalues. This feature does correspond to non proportional shifts in the variances of the structural shocks among the two regimes.

If this is not the case, or, practically speaking, we do not have credible evidence on structural breaks on the second moments of some of the variables in our HSVAR, point identification can still be reached by combining heteroskedasticity with zero restrictions. The next theorem formalizes this intuition.

**Theorem 1.** *Consider an HSVAR characterized by the eigenvalues and eigenspaces as in Definition 1 and by the admissible parameters as in Definition 4. The structural parameter  $(A_0, A_+, \Lambda) \in \mathcal{A}^r(\phi)$  is point identified if and only if, for each  $\lambda_i$ ,  $i = 1, \dots, k$ ,  $Q(\lambda_i) = (q_1^i, \dots, q_{m_i}^i)$ , the unit-length vector  $q_j^i$  is subject to  $f_j^i = m_i - j$  non-redundant zero restrictions, for  $j = 1, \dots, m_i$ .*

*Proof.* See the Appendix C.  $\square$

**Remark 1.** The previous theorem generalizes two important results in the literature of SVAR models. Firstly, when all the eigenvalues are distinct, then  $k = n$  and the algebraic multiplicity

$m_i = 1, i = 1, \dots, n$ . As a consequence,  $Q(\lambda_i) = (q^i)$ , and no zero restriction is needed for point identification, being  $f^i = 1 - 1 = 0$ , as originally introduced by Lanne and Lütkepohl (2008), and restated in the previous Theorems 5 and 6. Secondly, if one has no credible beliefs about structural breaks on the second moments of the observable variables, or the shift produces perfectly proportional covariance matrices in the two regimes, i.e.  $\Omega_2 = \lambda\Omega_1$  for any positive scalar  $\lambda$  (specifically,  $\lambda = 1$  in the case of no breaks), then  $\Omega = \Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'} = \Omega_{1,tr}^{-1}\sqrt{\lambda}\Omega_{1,tr}\sqrt{\lambda}\Omega_{1,tr}'\Omega_{1,tr}^{-1'} = \lambda I_n$ ; there is just an eigenvalue whose associated eigenspace is the whole  $\mathcal{S}^{n-1}$ , i.e the unit sphere in  $\mathbb{R}^n$ . The condition in Theorem 1 reduces to the identification condition for global identification in Rubio-Ramírez et al. (2010) (Theorem 7).

**Remark 2.** Identification in HSVARs is essentially a statistical issue, in the sense that, once the information contained in the data in terms of the two volatility regimes allows to point identify all the structural parameters, the path of the impulse responses allows the researcher to identify *a posteriori* the shocks of interest. In this respect, if the eigenvalues do not present multiplicity, the only task will be to see which shocks produce impulse responses coherent with the economic theory and label these shocks accordingly. The same happens if, even in the case of multiplicity, the shocks of interest are those corresponding to the eigenvalues with no multiplicity, whose eigenvectors (uniquely identified) will constitute the columns of  $Q$  one is interested in. A problem could arise when, in the case of multiplicity, none of the already identified impulse responses are consistent with what expected from economic theory for the shocks of interest. In this case, the results of Theorem 1 can be of extreme interest as including zero restrictions allows to point identify such shocks that, thus, will be identified based on economic restrictions rather than on statistical basis. Importantly, the number of restrictions is much less than what is required for traditional SVARs, as some of the columns of the  $Q$  matrix have been already identified by the heteroskedasticity. In our view, this can be an important added value of Theorem 1.

**Example 1** (Distinct eigenvalues). Consider an HSVAR with three variables, ( $n = 3$ ), and  $k = 3$  distinct eigenvalues,  $\lambda_1 \neq \lambda_2 \neq \lambda_3$ . In this case,  $m_i = 1$  for all  $i = 1, \dots, 3$ . Theorem 1 states that the HSVAR is point identified if and only if the unit vector  $q_j^i$  is subject to

$f_j^i = m_i - j$  zero restrictions for  $j = 1$  and  $i = 1, \dots, 3$ . It follows that no restrictions are needed because each  $\lambda_i$  is associated with a unique eigenvector  $q^i$ .

**Example 2** (Eigenvalue multiplicity). Consider a HSVAR with three variables ( $n = 3$ ) and  $k = 2$  distinct eigenvalues,  $\lambda_1 > \lambda_2$ ,  $\lambda_2 = \lambda_3$ . In this case, the first eigenvalue is distinct from the others, hence  $m_1 = 1$ , while the second eigenvalue has multiplicity  $m_2 = 2$ . Theorem 1 implies that, as far as the first unique eigenvalue  $\lambda_1$  is concerned, we do not need any restriction on  $q_1^1$  (i.e.  $f_1^1 = 1 - 1 = 0$ ). The second eigenvalue,  $\lambda_2$ , is associated with  $m_2 = 2$  linearly independent, not unique, eigenvectors (i.e.  $q_1^2$  and  $q_2^2$ ). Writing  $Q$  as  $[q_1^1 \ q_1^2 \ q_2^2]$  point identification is achieved with  $f_1^2 = 2 - 1 = 1$  zero restriction on  $q_1^2$  and  $f_2^2 = 2 - 2 = 0$  restriction on  $q_2^2$ .

### III.3 Set-identification in HSVARs with proportional volatility shifts

The results obtained in Section II.2 allow to point identify all the columns of  $Q \in \mathcal{O}(n)$  associated with eigenvalues without multiplicity. For all the other columns, they can be point identified according to the particular pattern of zero restrictions suggested by Theorem 1. Let  $\lambda_i$  be an eigenvalue with algebraic multiplicity  $g(\lambda_i) = m_i$ , in this section we consider restrictions that make the  $(q_1^i, \dots, q_{m_i}^i)$  columns of  $Q$  only set identified, being

$$f_j^i \leq m_i - j, \quad j = 1, \dots, m_i \quad (14)$$

with strict inequality for at least one  $j = \{1, \dots, m_i\}$ .

**Example 3** (Set identification of an HSVAR with multiple eigenvalues). Consider an HSVAR model with three variables  $y_t = (y_{1,t}, y_{2,t}, y_{3,t})'$ . Let us assume that there are  $k = 2$  distinct eigenvalues:  $\lambda_1 > \lambda_2$  and  $\lambda_2 = \lambda_3$ . Suppose that plotting the impulse responses to the first shock, we observe a pattern that is consistent with an economically meaningful structural shock. Given that the second eigenvalue,  $\lambda_2$ , has multiplicity  $m_2 = 2$ , we can write the matrix  $Q$  as  $[q_1^1 \ q_1^2 \ q_2^2]$ . Without any zero restriction,  $f_1^2 < 1$ , and the second and third column of  $Q$  are only set identified.

As in many empirical applications, suppose we are interested in one single shock, i.e. one column of  $Q$ . Moreover, according to Remark 2, it is crucial to understand whether the

point identified impulse responses obtained through the eigenvalues without multiplicity can be compatible with the shock of interest. If this is not the case, it is likely to suppose this latter to be associated with the eigenspace generated by  $\lambda_i$ , with multiplicity  $m_i$ . We first introduce a specific ordering of the shocks according to the identifying restrictions in Eq. (14), and then provide the conditions for the identified set to be convex.

**Definition 7** (Ordering of variables). The variables associated with the eigenvalue  $\lambda_i$ , with algebraic multiplicity  $m_i$ , are ordered according to the number of zero restrictions on  $(q_1^i, \dots, q_{m_i}^i)$ , and specifically, such that they follow the relation

$$f_1^i \geq f_2^i \geq \dots \geq f_{m_i}^i \geq 0. \quad (15)$$

In case of ties, the shock of interest, represented by the  $j^*$ -th column of  $(q_1^i, \dots, q_{m_i}^i)$ , is ordered first. In other words, let  $j^* = 1$  if no other column has a larger number of restrictions than  $q_{j^*}^i$ . If  $j^* \geq 2$ , then let the variables be ordered such that  $f_{j^*-1}^i > f_{j^*}^i$ .

The next theorems, based on Proposition 3 in Giacomini and Kitagawa (2021), provides sufficient conditions for the impulse response identified set  $IS(\phi | F, S)$  to be convex. Precisely, we first consider the case of zero restrictions only, and then extend to the case of zero and sign restrictions.

**Theorem 2** (Convexity of identified set under zero restrictions). *Consider an HSVAR characterized by the eigenvalues and eigenspaces as in Definition 1 and by the admissible parameters as in Definition 4. Let  $\lambda_i$  be an eigenvalue of algebraic multiplicity  $g(\lambda_i) = m_i$ , with associated eigenspace  $Q(\lambda_i)$  as in Eq. (9), containing  $q_{j^*}^i$ , the column of  $Q$  associated with the  $j^*$ -th structural shock (shock of interest). Moreover, let  $r = \eta(\phi, Q) = c'_{lh}(\phi)q_{j^*}^i \in IS(\phi | F, S)$  be the impulse responses to the shock of interest. Finally, let the variables be ordered as in Definition 7.*

*Then, the identified set for  $r$  is non empty and bounded for any  $l \in \{1, \dots, n\}$  and  $h = 1, 2, \dots, \phi$ -a.s. Moreover, a sufficient condition for the identified set to be convex is that any of the following exclusive conditions holds:*

1.  $j^* = 1$  and  $f_1^i < m_i - 1$ ;

2.  $j^* \geq 2$  and  $f_j^i < m_i - j$ , for  $j = 1, \dots, (j^* - 1)$ ;

3.  $j^* \geq 2$  and there exists  $1 \leq k < (j^* - 1)$  such that  $(q_1^i, \dots, q_k^i)$  is exactly identified as in Theorem 1 and  $f_j^i < m_i - j$ , for  $j = k + 1, \dots, j^*$ .

*Proof.* See Appendix C. □

The previous theorem just consider zero restrictions on the vectors of  $Q(\lambda_i)$ . The following one, instead, also allows for sign restrictions, although these last can be imposed on the vector  $q_{j^*}^i$  associated with the shock of interest.

**Theorem 3** (Convexity of identified set under zero and sign restrictions). *Consider an HSVAR as in Theorem 2, where, as before,  $q_{j^*}^i$  is the column vector corresponding to the shock of interest, and let  $q_{j^*}^i \in Q(\lambda_i)$ , the eigenspace associated with the eigenvalue  $\lambda_i$ , of algebraic multiplicity  $g(\lambda_i) = m_i$ . Moreover, let the sign restrictions be imposed on the shock of interest, only.*

1. Let the zero restrictions  $F^i(\phi, Q) = 0$  satisfy one of the conditions (1) and (2) of Theorem 2
2. If there exists a unit length vector  $q \in \mathbb{R}^n$  such that

$$F_{j^*}^i(\phi) q = 0 \quad \text{and} \quad \begin{pmatrix} S_{j^*}(\phi) \\ \sigma^{j^*'} \end{pmatrix} q > 0 \quad (16)$$

then the identified set is non empty and convex for every  $l \in \{1, \dots, n\}$  and  $h = 0, 1, 2, \dots$

2. Let the zero restrictions  $F^i(\phi, Q) = 0$  satisfy condition (3) of Theorem 2, and let  $(q_1^i(\phi), \dots, q_k^i(\phi))$  be the first  $k$  vectors that are exactly identified. If there exists a unit vector  $q \in \mathbb{R}^n$  such that

$$\left( F_{j^*}^i(\phi)', v_1^i, \dots, v_{(n-m_i)}^i, q_1^i(\phi), \dots, q_k^i(\phi) \right)' q = 0 \quad \text{and} \quad \begin{pmatrix} S_{j^*}(\phi) \\ \sigma^{j^*'} \end{pmatrix} q > 0 \quad (17)$$

where  $(v_1^i, \dots, v_{(n-m_i)}^i)$  is a basis for the space  $Q^\perp(\lambda_i)$ , then the identified set is non empty and convex for every  $l \in \{1, \dots, n\}$  and  $h = 0, 1, 2, \dots$

*Proof.* See Appendix C. □

Taken jointly, the results of Theorems 2 and 3 generalize Proposition 3 in Giacomini and Kitagawa (2021) to the case of a structural break on the variances of the shocks, with potential eigenvalue multiplicity.

On the other side, the two theorems provide important insights on the possibility to apply the standard identification-through-heteroskedasticity approach to the case in which some of the switches in the variances are the same. As we will see in the next section, these new results represent the foundations for developing an estimator for the bounds of the identified set and produce the related inference.

## **IV Inference in Set-identified HSVARs: a Robust Bayes Approach**

In this section we present a completely brand new approach to conduct inference on set-identified HSVARs, where the set identification comes from the fact that not all the shifts in the variances of the shocks are statistically different. Details on how to estimate the reduced-form parameters and on how to check for proportional variance shifts are reported in Appendix E and Appendix F, respectively. In this section, instead, we deal with all situations in which some of such variances are not significantly different each other, and we introduce our Robust Bayes approach to conduct inference on the identified set of interest.

### **IV.1 Inference on the identified set**

Now suppose some of the eigenvalues obtained by the eigen-decomposition in Eq. (8) present potential multiplicity. This evidence, for example, could be statistically checked by the Lütkepohl et al. (2020) or Lewis (2022) tests. Once this evidence is “statistically confirmed”, a natural way of proceeding is to impose such eigenvalues to be effectively equal. This choice, however, imposes implicitly restrictions on the covariance matrices of the reduced form. While ML estimator subject to constraints on the parameters is generally implementable, it is rather problematic in the specific case of imposing equality restrictions among the eigenvalues. In such particular case, in fact, firstly, imposing the restrictions makes the model no longer identified, and thus creating convergence problems of the algorithm maximizing the likelihood

function and, secondly, it is technically difficult to impose restrictions on some parameters that are observationally equivalent to permutations, as highlighted in Theorem 4 in the Appendix.

Our strategy, instead, is based on the following lemma, that, according to evidence of potential eigenvalue multiplicity, suggests imposing the multiplicities as the result of a minimization problem about the unrestricted and restricted covariance matrices of the HVAR.

**Lemma 1** (Similarities of positive-definite symmetric real matrices). *Let  $\Omega$  be a  $n \times n$  symmetric and positive definite real matrix characterized by the eigen-decomposition  $\Omega = Q\Lambda Q'$ , with the eigenvalues contained in the diagonal matrix  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ , and the associated eigenvectors contained in the  $n \times n$  orthogonal matrix  $Q$ . Moreover, let  $\tilde{\Omega} = Q\tilde{\Lambda}Q'$ , where the diagonal matrix  $\tilde{\Lambda}$  contains the first  $m$  elements fixed to a scalar  $\tilde{\lambda}$ , while the remaining  $n - m$  are the corresponding eigenvalues in  $\Lambda$ .*

*Then, according to the Frobenius metric,  $\min_{\tilde{\lambda}} \left\| \Omega - \tilde{\Omega} \right\|_F^2$  is reached when*

$$\tilde{\lambda} = \frac{1}{m} \sum_{h=1}^m \lambda_h. \quad (18)$$

*Proof.* See Appendix C. □

The previous lemma provides a theoretical ground for fixing the common eigenvalues, when they are not statistically distinct, such that the unrestricted and restricted reduced-form covariance matrices are as close as possible, according to a specific metric. The assumption considered in the previous lemma is that the matrix  $Q$ , containing the eigen-vectors, is common in the two matrices  $\Omega$  and  $\tilde{\Omega}$ . This assumption is completely reasonable for our problem in that it states that for the eigenvalues without multiplicity the eigenvectors are common in  $\Omega$  and  $\tilde{\Omega}$ . Those associated to the eigenvalue with multiplicity, say  $\lambda_i$ , being not identified, must simply lay on the sub-space  $Q(\lambda_i)$ , orthogonal to  $Q^\perp(\lambda_i)$ . In this respect, the eigenvectors  $(q_1^i, \dots, q_m^i)$  obtained through the eigen-decomposition of  $\Omega$  share this feature and can be used also as eigenvectors of  $\tilde{\Omega}$ . This explains the common  $Q$  matrix used in Lemma 1 both for  $\Omega$  and  $\tilde{\Omega}$ .

Let  $\tilde{\pi}_\phi$  be a probability measure on the space  $\Phi$  of reduced-form parameters. In order to obtain a prior distribution for  $\phi$  we need to restrict the support of  $\tilde{\pi}_\phi$  such that its elements satisfy the sign, normalization and equality restrictions, as well as the fact that they show

eigenvalue multiplicities as in Eq. (10). In this respect, we define the prior distribution for the reduced-form parameters as follows

$$\pi_\phi = \frac{\tilde{\pi}_\phi \mathbb{1}\{\mathcal{Q}(\phi|F, S) \neq \emptyset\}}{\tilde{\pi}_\phi \left( \{\mathcal{Q}(\phi|F, S) \neq \emptyset\} \right)}$$

that, by construction, assigns probability one to the distribution of data that admits eigenvalue multiplicity and is consistent with the identifying restrictions. As the structural parameters are a function of  $(\phi, Q) \in \Phi \times \mathcal{O}(n)$ , we define a joint prior for the two sets of parameters  $(\phi, Q)$  as  $\pi_{\phi, Q} = \pi_{Q|\phi} \pi_\phi$ , where  $\pi_{Q|\phi}$  is supported on  $\mathcal{Q}(\phi|F, S) \in \mathcal{O}(n)$ .

In the case of no multiplicity, sign and permutation normalizations allow to pin down just one admissible  $Q$ , and  $\pi_{Q|\phi}$  becomes a degenerate distribution centered on such  $Q$ . In the case of multiplicity and zero restrictions satisfying the pattern in Eq. (14), instead, the HSVAR will be only set identified and the prior  $\pi_{Q|\phi}$  has to be specified in order to obtain a posterior distribution for the structural parameters and impulse responses, as desired in standard Bayesian approach. Other than being a challenging task for applied economists to specify  $\pi_{Q|\phi}$ , it has been shown that the choice of such a prior, being never updated by the data, can have non-negligible impact on the posterior inference even asymptotically (Baumeister and Hamilton, 2015).

In order to fix this unpleasant issue, we use the robust Bayes inference proposed by Giacomini and Kitagawa (2021). This approach consists in fixing a single prior  $\pi_\phi$  for the reduced-form parameters, but a set of priors for  $\pi_{Q|\phi}$ . This strategy allows to obtain a class of posteriors for  $(\phi, Q)$  and, as a consequence, for the impulse response of interest  $r = r(\phi, Q)$ .

According to Giacomini and Kitagawa (2021), the results of this procedure can be summarized by reporting the posterior mean bounds interval, that can be seen as an estimator for the identified set, and an associated robustified credible region measuring the uncertainty related to the former. In particular, if we define  $\ell(\phi) = \inf\{r(\phi, Q) : Q \in \mathcal{Q}(\phi|F, S)\}$  and  $u(\phi) = \sup\{r(\phi, Q) : Q \in \mathcal{Q}(\phi|F, S)\}$ , the posterior mean bounds interval can be written as  $\left[ \int_\Phi \ell(\phi) d\phi_\phi; \int_\Phi u(\phi) d\phi_\phi \right]$ . The robustified credible region, instead, consists in an interval  $C_\alpha$  for which the posterior probability is greater than or equal to  $\alpha$  uniformly, i.e.  $\pi_{r|Y}(C_\alpha) \geq \alpha$ .



## IV.2 Computing posterior bounds

In this subsection we present an algorithm to be used in the case of two volatility regimes in the data with known break date. This last assumption is rather standard in the literature, being the break dates associated to well documented changes in the policy conduct or to financial crises.<sup>7</sup>

**Algorithm 1.** *Let  $y_{-l+1}, \dots, y_0, y_1, \dots, y_T$  be a sample of observations characterized by a break in the volatility occurred at time  $T_B$ , that is known or exogenously determined. Fix a normalization rule  $N$ .*

*(Step 1) Estimate the HSVAR model through the ML estimator as in Eq. (67),<sup>8</sup> obtain the estimated  $\hat{Q}$  and  $\hat{\Lambda}$  and check for eigenvalue multiplicity (e.g. Lütkepohl, Meitz, Netšunajev, and Saikkonen, 2021, or Lewis, 2021). If there is no multiplicity, or the shock of interest can be attributed to a particular  $q_{j^*}$  that comes out from an eigenvalue without multiplicity, then such shock is point identified (apart from sign) and the inference on the IRFs is standard. Then STOP.*

*If there are multiplicities and the shock of interest cannot be attributed to the already identified columns of  $Q$ , then consider equality and sign restrictions,  $\mathbf{F}(\phi, Q)$  and  $\mathbf{S}(\phi, Q)$ , respectively, to identify the shock of interest associated to  $q_{j^*}^i \in Q(\lambda_i)$ ; then move to Step 2.*

*(Step 2) Specify a prior for the reduced-form parameters  $\tilde{\pi}_\phi$  and estimate a Bayesian HVAR as suggested in Appendix E and obtain draws from the posterior distribution of  $\tilde{\pi}_{\phi|Y}$ , the parameters of the reduced form of the HVAR.*

*(Step 3) Take one draw  $\phi = (B, \Omega_1, \Omega_2)$  from the posterior distribution of  $\tilde{\pi}_{\phi|Y}$ . From this draw obtain the covariance matrices  $\Omega_1$  and  $\Omega_2$ . Solve the eigen-decomposition in Eq. (8) and collect the eigenvalues in the matrix  $\Lambda$  and the eigenvectors in the matrix  $Q$ .*

*(Step 4) Extract from  $Q$  the basis of the space  $Q(\lambda_i)$ , whose columns are associated with the possible multiple eigenvalues, and define the matrix  $\bar{Q}_{\lambda_i}$  containing the  $n - m_i$  ei-*

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<sup>7</sup>See, among many others, Lanne and Lütkepohl (2008), Boivin and Giannoni (2006), Angelini, Bacchiocchi, Caggiano, and Fanelli (2019), Rigobon (2003), Bacchiocchi (2017), Carriero, Marcellino, and Tornese (2023). Moreover, Rigobon (2003) also shows consistency of the estimated parameters in the case of break date misspecification.

<sup>8</sup>Or through the feasible GLS as in Eq. (66).

genvectors orthogonal to  $Q(\lambda_i)$ . If the zero restrictions meet the rank condition in Theorem 1, then  $\mathcal{Q}(\phi|F, S)$  is non-empty and the point identified columns of  $Q(\lambda_i)$ , for the draw of  $\phi$ , can be easily determined through Algorithm 1 in RWZ; then move to Step 5. If, instead, the zero restrictions do not meet the rank condition in Theorem 1, then the model is only set identified and, given the draw of  $\phi$ , check whether  $\mathcal{Q}(\phi|F, S)$  is empty or not by following the sub-routines below:

(Step 4.1) Let  $z_1 \sim N(0, I_n)$  be a draw of an  $n$ -variate standard normal random variable. Let  $\tilde{q}_1^1 = M_1 z_1$  be the  $n \times 1$  residual vector in the linear projection of  $z_1$  onto an  $n \times f_1^i$  regressor matrix  $F_1^i(\phi)'$ . For  $k = 2, \dots, m_i$ , run the following procedure sequentially: draw  $z_k \sim N(0, I_n)$  and compute  $\tilde{q}_k^i = M_k z_k$ , where  $M_k z_k$  is the residual vector in the linear projection of  $z_k$  onto the  $n \times (f_k^i + n - m_i + k - 1)$  matrix  $(F_k^i(\phi)', \bar{Q}_{\lambda_i}, \tilde{q}_1^i, \dots, \tilde{q}_{k-1}^i)$ . The vectors  $\tilde{q}_1^i, \dots, \tilde{q}_{m_i}^i$  are mutually orthogonal, orthogonal to  $\bar{Q}_{\lambda_i}$ , and satisfy the equality restrictions.

(Step 4.2) Given  $\tilde{q}_1^i, \dots, \tilde{q}_{m_i}^i$  obtained in the previous step, define

$$Q_{\lambda_i} = \left[ \pm \frac{\tilde{q}_1^i}{\|\tilde{q}_1^i\|}, \dots, \pm \frac{\tilde{q}_{m_i}^i}{\|\tilde{q}_{m_i}^i\|} \right],$$

where  $\|\cdot\|$  is the Euclidean metric in  $\mathbb{R}$ , then arrange the sign of each column of  $Q_{\lambda_i}$  according to the sign normalization as defined by  $S \in \mathcal{D}(n)$ . Based on the obtained  $Q_{\lambda_i}$  with appropriate sign normalization, form the  $Q$  matrix by collecting the columns in  $\bar{Q}_{\lambda_i}$  and  $Q_{\lambda_i}$  according to the correct ordering determined by the permutation matrix  $P \in \mathcal{P}(n)$ .

(Step 4.3) Check whether  $Q$  obtained in (Step 4.2) is such that

$(A_0, A_+) = (PSQ' \Sigma_{1, tr}^{-1}, PSQ' \Sigma_{1, tr}^{-1} B)$ , for appropriate  $S \in \mathcal{D}(n)$  and  $P \in \mathcal{P}(n)$ , satisfies the sign restrictions  $S(\phi, Q) \geq 0$ . If so, retain this  $Q$  and proceed to (Step 5). Otherwise, repeat (Step 4.1) and (Step 4.2) a maximum of  $L$  times (e.g.  $L = 3000$ ) or until  $Q$  is obtained satisfying  $S(\phi, Q) \geq 0$ . If none of the  $L$  draws of  $Q$  satisfies  $S(\phi, Q) \geq 0$ , approximate  $\mathcal{Q}(\phi|F, S)$  as being empty and return to (Step 3) with the following draw of  $\phi$ .

(Step 5) Given  $\phi$  and  $Q = (\bar{Q}_{\lambda_i}, Q_{\lambda_i})$ , with the correct ordering determined by  $P \in \mathcal{P}(n)$ , and

correct sign normalization determined by  $S \in \mathcal{D}(n)$ , obtained in (Step 4), compute the lower and upper bounds of  $IS(\phi|F, S)$  by solving the following constrained nonlinear optimization problem:

$$\begin{aligned} \ell(\phi) &= \arg \min_{Q_{\lambda_i}} c'_{gh}(\phi) q_{j^*}^i, \\ \text{s.t.} \quad Q'Q &= I_n, \quad \mathbf{F}(\phi, Q) = 0 \\ (PSQ' \Sigma_{1,tr}^{-1}, PSQ' \Sigma_{1,tr}^{-1} B) &\in N, \quad \text{and} \quad \mathbf{S}(\phi, Q) \geq 0 \end{aligned}$$

and  $u(\phi) = \arg \max_{Q_{\lambda_i}} c'_{gh}(\phi) q_{j^*}^i$  under the same set of constraints. If the zero restrictions meet the rank condition in Theorem 1, then  $Q = (\bar{Q}_{\lambda_i}, Q_{\lambda_i})$  is a singleton and  $\ell(\phi) = u(\phi)$ .

(Step 6) Repeat (Step 3) - (Step 5)  $M$  times to obtain  $[\ell(\phi_m), u(\phi_m)]$ ,  $m = 1, \dots, M$ . Approximate the set of posterior means by the sample averages of  $(\ell(\phi_m), m = 1, \dots, M)$  and  $(u(\phi_m), m = 1, \dots, M)$ .

(Step 7) To obtain an approximation of the smallest robust credible region with credibility  $\alpha \in (0, 1)$ , define  $d(\eta, \phi) = \max\{|\eta - \ell(\phi)|, |\eta - u(\phi)|\}$ , and let  $\tilde{z}_\alpha(\eta)$  be the sample  $\alpha$ -th quantile of  $(d(\eta, \phi_m) : m = 1, \dots, M)$ . An approximated smallest robust credible region for  $\eta$  is an interval centered at  $\arg \min_\eta \tilde{z}_\alpha(\eta)$  with radius  $\min_\eta \tilde{z}_\alpha(\eta)$ .

Some remarks about the algorithm are in order. The first one is about the prior for the two covariance matrices  $\Omega_1$  and  $\Omega_2$ . As the aim of the analysis is to highlight the possible eigenvalue multiplicity, it would be preferable to use diffuse priors, like diagonal matrices with equal values on the main diagonal, that from one side are non-informative and on the other side consider all the eigenvalues to be equal and let the likelihood function to play the relevant role in this respect.

Second, the way the draws from the posterior distribution are obtained depends on the theoretical results of Appendix E. Using independent priors for  $\Omega_1$ ,  $\Omega_2$  and  $\phi_B$  allows to develop a Gibbs sampler that is rather simple and permits to explore the joint posterior distribution in a very convenient way. Step 3 of our algorithm is based on this approach for generating the draws  $\phi$  from the distribution  $\tilde{\pi}_{\phi|Y}$ . Any alternative way, however, can be performed without altering the other steps of the algorithm.

Third, checking for the emptiness of the identified set in the case of sign restrictions is performed in Step 4 by using linear projections starting from normal draws, as in Giacomini and Kitagawa (2021) and many other contributions in the Bayesian literature. As an alternative, we could use the QR decomposition as proposed by Arias et al. (2018).

Fourth, for each of the draws consistent with the zero and sign restrictions, we consider it as if there were eigenvalue multiplicities, regardless of whether it is effectively so from a statistical point of view, for the draw  $\phi$ . In this respect, the eigenvectors associated to the potential multiple eigenvalues act as a basis for the space of the not identified columns of  $Q$ . From one side, this way of proceeding is extremely conservative as it completely ignores the amount of information contained in all those draws where all the eigenvalues are substantially distinguished. From the other side, however, it avoids the consequences of a pre-testing step to be applied to each draw to statistically check for eigenvalue multiplicity. Our inference, thus, is robust to eigenvalue multiplicity in the sense that we apply the robust Bayesian approach to the set identified columns of  $Q$  associated to the suspected multiple eigenvalues.

Fifth, the constrained nonlinear optimization problem in Step 5 is less demanding than the one in Algorithm 1 by Giacomini and Kitagawa (2021), as the argument is not the entire matrix  $Q$  but just a subset of its columns. Even if the HVAR model is relatively large, the number of eigenvectors generating the subspace  $Q(\lambda_i)$  is in general relatively small and we do not expect concerns about the convergence properties of the numerical optimization step. On the contrary, we could replace Step 5 by a new algorithm in the spirit of Algorithm 2 in Giacomini and Kitagawa (2021), where the constrained nonlinear optimization problem is substituted by iterating many times Step 4.1-Step 4.3 and approximate the interval  $[\ell(\phi_m), u(\phi_m)]$  with the minimum and maximum values obtained in such iterations. If the number of iterations goes to infinity, such alternative bounds still provide a consistent estimator of the identified set.

Sixth, the algorithm works even in the case the zero restrictions allow to point identify the matrix  $Q$ . In this case, the set  $\mathcal{Q}(\phi|F, S)$  is always non-empty, and the constrained nonlinear optimization problem simply returns  $\ell(\phi) = u(\phi)$ . The inference, then, becomes standard.

From a theoretical point of view, Giacomini and Kitagawa (2021) discuss the importance of convexity, continuity and differentiability of the identified set  $IS(\phi|F, S)$  for the posterior means to have a valid frequentist interpretation. Obviously, the same has to be verified in our

setup. If this is the case, they show that the set of posterior means is a consistent estimator of the true identified set and the robust credible region is an asymptotically valid confidence set for the true identified set.

Concerning convexity, we have already proved it in the previous Theorems 2 and 3. About continuity and differentiability, since we use the same set of zero and sign restrictions as in Giacomini and Kitagawa (2021), extending their Proposition 4 and Proposition 5 to our setup is straightforward. Definitely, appropriate choice of zero and sign restrictions associated with mild regularity conditions on the coefficient matrices of such restrictions guarantee our results on HSVARs to have a valid frequentist interpretation, as for traditional SVARs.

## V Empirical Application

We apply our methodology to the SVAR model for the global crude oil market of Kilian (2009) that includes three variables: the percent change in global crude oil production ( $\Delta prod_t$ ), an index of global real economic activity ( $reat_t$ ) and the logarithm of the real price of crude oil ( $rpo_t$ ). Data are monthly and the sample period runs from January 1973 through December 2007.<sup>9</sup> While Kilian (2009) identifies three structural shocks that drive the real price of crude oil using a simple recursive scheme, Lütkepohl et al. (2020), Lütkepohl and Netšunajev (2014) and Lütkepohl (2013) show that the same structural innovations can also be recovered by exploiting the existence of distinct volatility regimes.<sup>10</sup> We set the lag order of the VAR equal to 12 and follow Lütkepohl et al. (2020) that distinguish between two volatility regimes with – an exogenously determined – change point in October 1987.<sup>11</sup>

Table 2 in Appendix G reports the estimated  $\lambda$ s and the results of the test by Lütkepohl et al. (2020) for eigenvalue multiplicity. From the test it clearly emerges that  $H_0 : \lambda_2 = \lambda_3$  cannot be rejected by the data, implying that standard identification through heteroskedasticity fails. In fact, only one structural shock can be statistically identified relying on changes in volatility. If the identified shock cannot be given an interpretation that is consistent with

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<sup>9</sup>We rely on exactly the same dataset as Kilian (2009). See Appendix G for details and additional results.

<sup>10</sup>Identification through heteroskedasticity has found other applications in studies of the crude oil market. See e.g. Bruns and Lütkepohl (2023) and Känzig (2021).

<sup>11</sup>Note that Lütkepohl et al. (2020), Lütkepohl and Netšunajev (2014) and Lütkepohl (2013) rely on a VAR of order 3, while Kilian (2009) stresses that a VAR of order 24 is necessary to capture long price cycles of crude oil and hence for accurately estimating the impulse responses in global oil market models. The selected lag order is thus a compromise between these two approaches.

Table 1: Sign Restrictions on impact responses ( $C \equiv A_0^{-1}$ ) in model  $\mathcal{M}_2$

	Oil supply disruption	Positive aggregate demand shock	Positive oil-specific demand shock
$\Delta prod_t$	(-)	+	*
$rea_t$	-	(+)	*
$rpo_t$	+	+	(+)

Notes: “\*” denotes that the sign of the impact response is unrestricted. Signs along the main diagonal are in brackets to highlight that these are not actual sign restrictions, but sign normalizations placed on  $C \equiv A_0^{-1}$ .

economic theory, or if one wishes to identify other structural shocks, additional sign or exclusion restrictions are needed. In this case, Theorem 1 and its implementation in Algorithm 1 become extremely useful.

## V.1 Point and set identification of oil supply and demand shocks

We write the relationship between structural innovations,  $\varepsilon_t$  and reduced-form errors,  $u_t = C\varepsilon_t$  as follows:

$$\begin{pmatrix} u_t^{\Delta prod} \\ u_t^{rea} \\ u_t^{rpo} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_t^{oil\ supply\ shock} \\ \varepsilon_t^{aggregate\ demand\ shock} \\ \varepsilon_t^{oil-specific\ demand\ shock} \end{pmatrix} \quad (19)$$

where  $C$  corresponds to the impact response matrix ( $A_0^{-1}$ ). We consider four SVARs based on different identifying restrictions:

- $\mathcal{M}_0$ , a recursively identified SVAR model with  $c_{12} = c_{13} = c_{23} = 0$ ;
- $\mathcal{M}_1$ , a standard HSVAR model identified exploiting changes in volatility and assuming distinct eigenvalues;
- $\mathcal{M}_2$ , an HSVAR model that imposes eigenvalue multiplicity and exploits static and dynamic sign restrictions;
- $\mathcal{M}_3$ , an HSVAR model that allows for eigenvalue multiplicity and imposes one exclusion restriction ( $c_{21} = 0$ ).

Model  $\mathcal{M}_0$  is the recursively identified SVAR model of Kilian (2009) used as a benchmark against which we compare results from HSVAR models. Three exclusion restrictions –  $c_{12} = c_{13} = c_{23} = 0$  – allow to point identify an oil supply shock and two demand shocks (i.e. aggregate and oil-specific demand shocks). Oil supply shocks represent innovations to the current

physical availability of crude oil. Aggregate demand shocks capture unexpected changes of the demand for all industrial commodities driven by fluctuations in the global business cycle, while oil-specific demand shocks represent shifts in the precautionary demand for crude oil triggered by concerns about the future availability of supplies.

The HSVAR model  $\mathcal{M}_1$  exploits changes in volatility for identifying structural shocks, without imposing additional exclusion or sign restrictions. We assume the eigenvalues to be all distinct, although results in Table 2(b) in the Appendix highlight the existence of eigenvalue multiplicity.

For this reason, in model  $\mathcal{M}_2$  we impose the constraint  $\lambda_2 = \lambda_3$  in Step 3 of Algorithm 1. With two distinct eigenvalues, we can point identify only one structural shock that, as will be shown in Section V.2, yields impulse responses consistent with those associated with an oil-specific demand shock. The remaining structural shocks are set identified combining static and dynamic sign restrictions. See Table 1 in Appendix G. We postulate that a negative oil supply shock increases the real price of crude oil and depresses global real economic activity on impact. A positive aggregate demand shock is expected to raise oil price and production on impact. Notice that we also place sign normalizations on the main diagonal of  $C \equiv A_0^{-1}$ . Furthermore, we constrain the sign of the response of real crude oil price to oil supply disruptions to be positive for twelve months, starting from the impact response. These additional restrictions rule out models with the real price of crude oil decreasing below its starting level after a negative oil supply shock.

While specification  $\mathcal{M}_2$  allows to point identify a single structural shock, because of eigenvalue multiplicity the HSVAR model remains set identified. In this case, Theorem 1 shows that point identification of the HSVAR model can be achieved with a single exclusion restriction. In model  $\mathcal{M}_3$ , we add one exclusion restriction on the impact response matrix:  $c_{21} = 0$ . This restriction implies that the first shock does not affect real economic activity within the same month. Compared to the recursively identified model,  $\mathcal{M}_0$ , when a volatility shift is exploited the identification scheme is less demanding. In fact in model  $\mathcal{M}_3$  point identification is achieved – at least in statistical sense – combining the shift in the volatility of one of the structural shocks with a single zero restriction.

## V.2 Impulse response analysis

All specifications are estimated with Bayesian techniques drawing from the posterior of reduced-form parameters until we obtain 1000 realizations of the non-empty identified set. We estimate the whole set of structural impulse response functions over an horizon of 24 months. Notice that we report the implied response of world crude oil production obtained by cumulating that for  $\Delta prod_t$ . We focus on shocks that are expected to raise the real price of crude oil. Therefore, in the case of supply shocks we plot the responses to a negative shock representing a disruption of crude oil supply.

Impulse responses for  $\mathcal{M}_0$  and  $\mathcal{M}_1$ , as not informative for the implementation of our methodology, are reported in Appendix G to save space (Figures G.8 and G.9, respectively). Not surprisingly, the impulse responses for  $\mathcal{M}_0$  are totally in line with the original ones in Kilian (2009).<sup>12</sup> Model  $\mathcal{M}_1$ , instead, builds on the standard identification through heteroskedasticity approach assuming implicitly that all eigenvalues are distinct. However, as the test by Lütkepohl et al. (2020) shows, changes in volatility alone here might not convey enough information to point identify all the structural shocks. We overcome this issue by imposing restrictions in  $\mathcal{M}_2$  and  $\mathcal{M}_3$ . Impulse responses from HSVAR models  $\mathcal{M}_2$ - $\mathcal{M}_3$  are displayed in Figures 1-2.<sup>13</sup>

Static and dynamic sign restrictions in model  $\mathcal{M}_2$  allow to set identify supply and demand shocks that are expected to drive the real price of crude oil. Since the only distinct eigenvalue is associated with the oil-specific demand shock, sign restrictions are imposed on the remaining columns of  $Q$ . The impulse responses appearing in the first two columns of Figure 1 specifically refer to those columns of  $Q$  and, as such, we also present the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% obtained with Algorithm 1.<sup>14</sup>

Imposing sign restrictions we recover oil supply and aggregated demand shock whose effects on the endogenous variables of the VAR are consistent with expectations from economic theory and previous analyses (see e.g. Kilian and Murphy, 2012). The shape of impulse responses in

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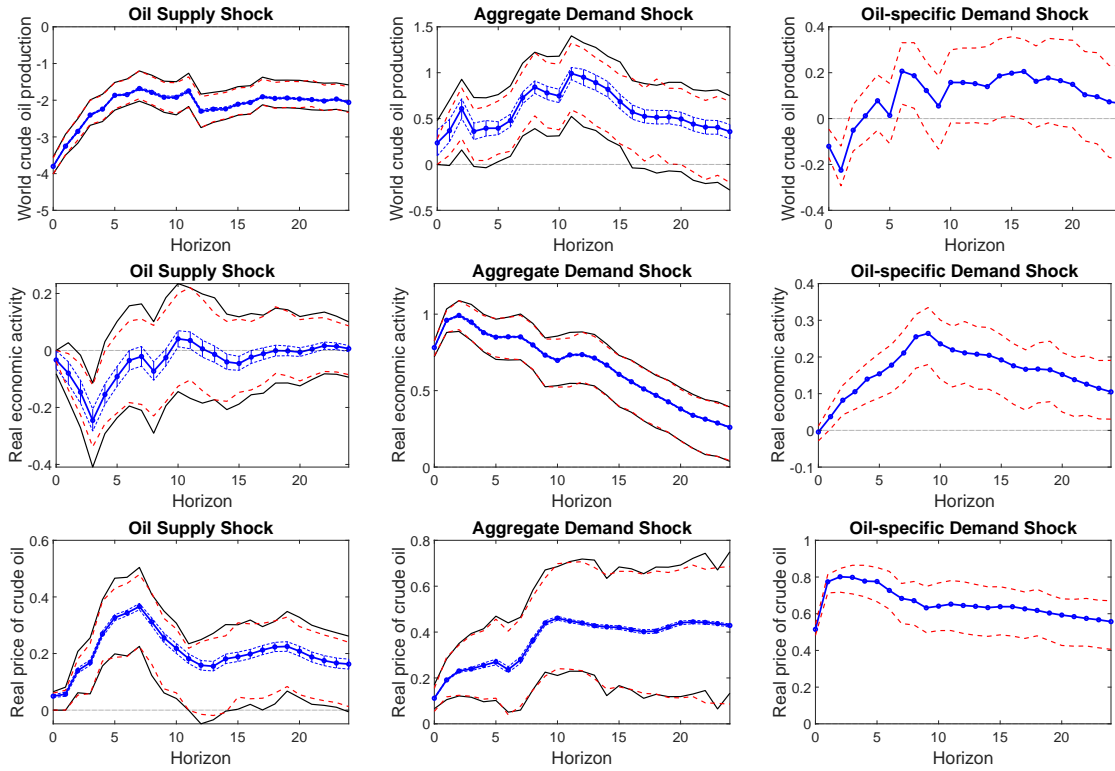
<sup>12</sup>As for the estimation, we rely on a noninformative improper Jeffreys' prior that allows to draw reduced-form parameters from a normal-inverse-Wishart posterior.

<sup>13</sup>Since HSVAR models normalize structural residuals to have identity covariance matrix in the first regime, the scaling of these figures is not the same as that of Figure G.8.

<sup>14</sup>Our implementation is based on the constrained nonlinear optimization problem highlighted in Step 5 of Algorithm 1. We follow Giacomini and Kitagawa (2021) that mitigate possible convergence problems using five different starting values for the optimization problem in Step 5.

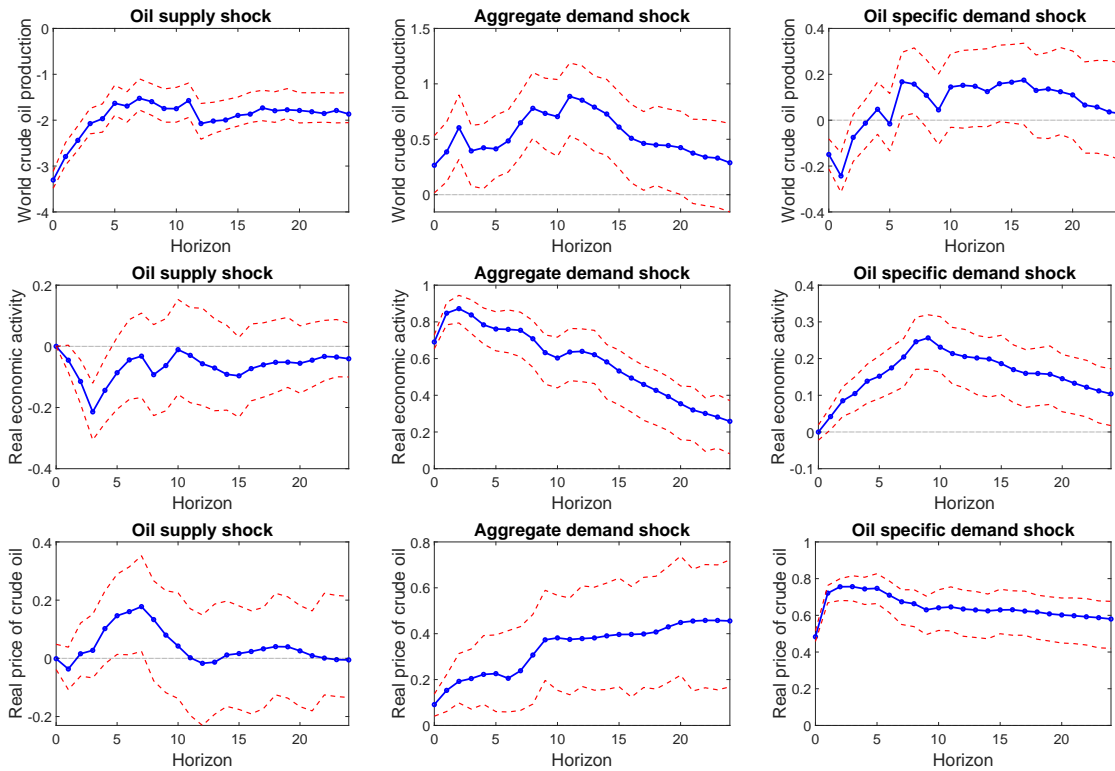


Figure 1: Impulse response functions  $\mathcal{M}_2$



Notes: the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Plots in first and second columns of the figure also report the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% (solid black curves). Identification via heteroskedasticity with multiple eigenvalues (i.e. only one shock is point identified), static and dynamic sign restrictions.

Figure 2: Impulse response functions  $\mathcal{M}_3$



Notes: the blue line with dots represents the posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Identification through heteroskedasticity exploiting the fact that one eigenvalue is distinct from the others; moreover, we impose  $c_{21} = 0$ .

the first two columns of Figure 1 are in line with those implied by the recursively identified model,  $\mathcal{M}_0$ . Also notice that the combination of volatility changes and sign restrictions delivers reasonable impulse responses even in the presence of multiple eigenvalues and using less sign restrictions than what is usually done in the literature.<sup>15</sup> In fact, we leave one of the columns of  $Q$  completely unrestricted and exploit changes in volatility to point identify the corresponding shock. Contrary to the standard HSVAR,  $\mathcal{M}_1$ , oil supply shocks implied by  $\mathcal{M}_2$  are associated with impulse responses that reasonably summarize the expected effects of such shocks on real economic activity and the real price of crude oil. Here we observe a positive response of the price of crude oil with a peak after 6 months.

The width of the HPD and of the robust credible regions for both oil supply and aggregate demand shocks are similar. We can thus draw essentially the same conclusions using any of them.<sup>16</sup> Giacomini and Kitagawa (2021) propose a measure of the informativeness of the choice of an unrevisable prior for  $Q$  that compares the width of such regions. The fact that in our case such measure is generally small (at any horizon), indicates that the fraction of the credible region tightened by choosing a particular unrevisable prior is very modest.<sup>17</sup>

Model  $\mathcal{M}_3$  illustrates that our methodology allows to point identify HSVAR models in the presence of multiple eigenvalues and that this can be achieved with less zero restrictions than in the case of recursive identification. In  $\mathcal{M}_3$  we impose that the first shock does not affect real economic activity within the same month. Interestingly, this restriction is consistent with the evidence in Figures 1 and G.8 where we see that the impact response of real economic activity to an oil supply shock is close to zero.

Results for  $\mathcal{M}_3$  are reported in Figure 2. Here, the zero restriction has the effect of tightening the width of the HPD regions, when compared to the standard HSVAR model  $\mathcal{M}_1$ . Focusing

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<sup>15</sup>In Appendix E we show results based on the implementation of Algorithm 1 while performing the Lütkepohl et al. (2020) test for heteroskedasticity on each draw of  $\phi$  in the context of set identified impulse response functions. Figure G.11 shows how performing the test for identification through heteroskedasticity does not affect the results.

<sup>16</sup>Results based on an alternative implementation of Algorithm 1 are almost identical. In such implementation, we follow Giacomini and Kitagawa (2021) and substitute Step 5 with 10000 iterations of Step 4.1-Step 4.3. The interval  $[\ell(\phi_m), u(\phi_m)]$  is then approximated by the minimum and maximum values over such iterations. This also confirms the convergence of the numerical algorithm in Step 5. See Appendix G.

<sup>17</sup>The informativeness of the prior with credibility  $\alpha$  is defined as  $\{1 - [\text{width highest posterior density}(\alpha)/\text{width robust credible region}(\alpha)]\}$ . Such fraction is in the range 0.05-0.52 for the impact response to an oil supply shock (i.e. the largest fraction is that associated with the response of real economic activity) and in the range 0.03-0.37 for the impact response to an aggregate demand shock (i.e. the largest fraction is that associated with the response of world crude oil production). At horizon 12 such intervals become 0.03-0.21 and 0.06-0.22.

on the response of real price of oil to an oil disruption, we see that in this case the response is slightly more significant, as the HPD region does not always contain the zero. One difference with the benchmark,  $\mathcal{M}_0$ , concerns the response of world crude oil production to an aggregate demand shock that here is positive, with highest posterior density region that does not contain the zero up to horizon 20.

## VI Conclusion

This paper deals with SVAR models with structural breaks on the second moments of the structural shocks, offering some new contributions. We first study the identification theory and propose a set of results for easily checking whether the model is globally identified. Second, we study the consequences on the impulse response functions of HSVARs that do not satisfy such identifying conditions. We deal with a SVAR model with heteroskedasticity, where non distinct changes in the variance shifts raise an identification issue. We solve the identification problem by imposing equality and sign restrictions and provide a methodology that helps giving a structural economic interpretation to the set or point identified shocks, by requiring fewer restrictions to be imposed. A way to do inference on the model both in case of point and set-identification is also proposed, as well as an empirical application of our approach to the global crude oil market model. Some issues remain to be addressed by future research, such as extending the model to more than two volatility regimes and analysing the consequences of having an unknown break date.

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# Appendix A Preliminary results on the identification of HSVARs

Some of the theoretical results we present in this section are not completely new in the literature, although they have been derived independently from other authors. However, we have decided to report them as prerequisites for a better understanding of the main results provided in the paper. Detailed references will be reported accordingly.

Consider an  $n$ -variable SVAR model with two regimes in structural shock variances, but maintaining the homogeneity of the structural coefficients. We normalize the covariance matrix of the structural shocks to  $n \times n$  identity matrix in the first regime. Following the parametrization and notations we have been using so far, we analyze identification of the  $n \times n$  matrix  $C$  that represents the inverse of structural coefficient matrix  $A_0$ , i.e.  $C \equiv A_0^{-1}$ , and  $\Lambda$ ,  $n \times n$  diagonal matrix with strictly positive elements. Given the reduced-form covariance matrix at regime 1 and 2, denoted by  $\Omega_1$  and  $\Omega_2$ , respectively,  $C$  and  $\Lambda$  solve

$$\Omega_1 = CC', \tag{20}$$

$$\Omega_2 = C\Lambda C' \tag{21}$$

The next theorem characterizes the set of  $(C, \Lambda)$  solving this equation system. To state it, we define  $\mathcal{P}(n)$  as the set of  $n \times n$  permutation matrices, such that pre-multiplying  $P \in \mathcal{P}(n)$  to any matrix  $M$  performs a row-permutation of  $M$ , and post-multiplying it performs a column-permutation. Moreover, in the case of a diagonal matrix  $D$  of size  $n$ ,  $P'DP$  performs a permutation of the diagonal elements of  $D$ . Let  $\mathcal{D}(n)$  be the set of  $n \times n$  diagonal matrices whose diagonal entries are either  $+1$  or  $-1$ . That is, if the  $i$ -th diagonal entry of  $S \in \mathcal{D}(n)$  is  $-1$ , pre-multiplying (post-multiplying)  $S$  to any matrix  $M$  flips the sign of the  $i$ -th row (resp. column) vector of  $M$ .

**Theorem 4** (Sign normalization and column-permutation). *Assume  $\Omega_1$  and  $\Omega_2$  are non-singular. Suppose  $(C^*, \Lambda^*)$  is a solution of the equation system (20)-(21). Then, the set of*

solutions solving (20)-(21) is represented as

$$\{(C, \Lambda) = (C^*SP', P\Lambda^*P') : P \in \mathcal{P}(n), S \in \mathcal{D}(n)\}. \quad (22)$$

*Proof.* See the appendix. □

This theorem clarifies the fundamental indeterminacy of the solutions in the equation system (20)-(21). Specifically, the representation of the solutions in Eq. (22) shows that  $(C, \Lambda)$  remains observationally equivalent with respect to any permutation and change of signs of the column vectors in  $C$  as far as the same permutation is applied to the diagonal elements of  $\Lambda$ . The observational equivalence with respect to  $S \in \mathcal{D}(n)$  corresponds to the indeterminacy of the signs of structural shocks common in any SVAR modelling (see Lanne, Lütkepohl, and Maciejowska, 2010, for an equivalent result on HSVARs). We often control such sign indeterminacy by imposing the sign normalization restrictions that pin down  $S$ , e.g., restricting the diagonal elements of  $A_0 = C^{-1}$  to be non-negative. The observational equivalence with respect to the permutations corresponds to the indeterminacy of the structural parameters with respect to the reordering of the structural equations. Rigobon (2003) noted this indeterminacy of the ordering of the structural equations in bivariate HSVAR models and argued that sign restrictions placed on the off-diagonal elements of  $A_0 = C^{-1}$  resolve such indeterminacy.

Theorem 4 implies that with sign normalization restrictions imposed, point-identification of  $(C, \Lambda)$  requires an assumption that pins down the ordering of the equations (i.e., permutation matrix  $P$ ). One way to constrain the ordering of the equations is to exploit available knowledge on the ratios of the structural shock variances of regime 1 to regime 2. In particular, assuming a complete ordering of the structural shocks according to their variance ratios can fix the order of the structural equations based on the diagonal entries of the true  $\Lambda$ . Hence, if a solution of  $\Lambda$  is such that all of its diagonal elements are distinct, a complete ordering of such elements reduces the set of solutions in Eq. (22) to a singleton. The following theorem hence follows as a corollary of Theorem 4.

**Theorem 5** (Point identification). *In addition to the assumptions of Theorem 4, assume that a solution of  $\Lambda$  has the diagonal terms all distinct. Then, with sign normalization restrictions*



and complete ordering of the structural shocks according to the variance ratios imposed,  $(C, \Lambda)$  is point-identified.

*Proof.* The theorem, apart from the potential indeterminacy due to the column permutation, corresponds to Proposition 1 in Lanne, Lütkepohl, and Maciejowska (2010), and can be proved in exactly the same way. However, according to Theorem 4 here before, fixing the ordering of the shocks is also necessary in order to have point identified  $(C, \Lambda)$ .  $\square$

If a solution of  $\Lambda$  has some of the diagonal elements identical, then invariance of  $\Lambda$  with respect to a permutation that permutes only these elements fails to uniquely pin down  $C$  within the set of solutions in Eq. (22). Partial identification of  $C$  matrix in this case is to be considered below.

The identification result of Theorem 5 is not constructive and it does not provide an explicit analytical expression of  $(C, \Lambda)$  as a function of  $(\Omega_1, \Omega_2)$ . A more constructive identification result for  $(C, \Lambda)$  can be obtained by representing the equation systems (20) and (21) as a certain eigen-decomposition problem, as already presented in Eq. (8). This perspective yields the following succinct analytical characterization of the solutions of the equation system.

**Theorem 6** (Identification and eigen-decomposition). *The set of solutions solving system (20)-(21) can be represented by  $(\Omega_{1,tr}Q, \Lambda)$ , where  $\Lambda$  is a diagonal matrix of eigenvalues of  $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'}$  and  $Q$  is an orthogonal matrix of the corresponding eigenvectors. Hence, if the eigenvalues of  $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'}$  are all distinct,  $(C, \Lambda)$  is identified up to permutations and sign changes of the structural equations.*

*Proof.* The result immediately follows from Theorem A9.9, and the related proof, in Muirhead (1982).  $\square$

The claim of this theorem simplifies computation of an estimator of  $(C, \Lambda)$ ; the maximum likelihood estimator for  $(C, \Lambda)$  can be computed by performing an eigen-decomposition on the maximum likelihood estimator of  $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1'}$  subject to the sign normalization. If a complete ordering assumption on  $\Lambda$  is available (e.g., the diagonal elements of  $\Lambda$  is decreasing), we can obtain a point-estimator for  $(C, \Lambda)$  by ordering the eigenvalues accordingly through the

decomposition. If the ordering assumption is not available, then permutations of the diagonal elements in  $\Lambda$  and the corresponding eigenvectors in  $Q$  span the identified set of  $(C, \Lambda)$ . The idea of treating identification and estimation of HSVARs as an eigen-decomposition issue has been also pursued by Lütkepohl, Meitz, Netšunajev, and Saikkonen (2021), that developed their test for identification via heteroskedasticity as a test on equivalent eigenvalues.

An alternative way to see and address the identification problem of  $(C, \Lambda)$  is to look at system (20)-(21) in a slightly different way. In fact, given that  $\Lambda$  is made of positive elements, it is possible to rewrite Eq. (21) as  $\Omega_2 = C\Lambda^{1/2}\Lambda^{1/2}C'$ . The quantity  $C\Lambda^{1/2}$  could not be unique because of the presence of an orthogonal matrix  $Q_2$  such that  $\Omega_2 = C\Lambda^{1/2}Q_2Q_2'\Lambda^{1/2}C'$ . Using the result in Proposition A.1 of Uhlig (2005) for the decomposition of  $\Omega_1$  and  $\Omega_2$  yields the following system

$$\begin{aligned} C &= \Omega_{1,tr}Q_1 \\ \Omega_{2,tr} &= C\Lambda^{1/2}Q_2 \end{aligned}$$

and plugging the definition of  $C$  into the second equation we obtain

$$\Omega_{1,tr}^{-1}\Omega_{2,tr} = Q_1\Lambda^{1/2}Q_2. \tag{23}$$

The next theorem discusses the identification issue of the structural parameters  $(C, \Lambda)$  in terms of the uniqueness of  $Q_1$  and  $Q_2$ .

**Theorem 7** (Identification and Single Value Decomposition). *The set of solutions of system (20)-(21) can be represented by  $(\Omega_{1,tr}Q_1, \Lambda)$ , where  $\Lambda$  is a diagonal matrix made of positive elements and  $Q_1$  is an orthogonal matrix solving the Single Value Decomposition of  $\Omega_{1,tr}^{-1}\Omega_{2,tr} = Q_1\Lambda^{1/2}Q_2$ . If the entries in  $\Lambda$  are all distinct, then  $Q_1$  and  $Q_2$  are unique apart from simultaneous sign changes and permutation of their corresponding columns. Hence,  $(C, \Lambda)$  is identified up to permutations and sign changes of the structural equations.*

*Proof.* See the Appendix C. □

# Appendix B   Geometry of identification in bivariate SVARs and HSVARs

This appendix is dedicated to the identification issue in bivariate SVAR models. We first introduce the notion of set identification in standard SVARs as in Giacomini and Kitagawa (2021), and derive point identification as a particular case. Second, we move to the core of the paper and extend the set and point identification notions to SVARs characterized by structural breaks, that, as shown in Bacchiocchi and Fanelli (2015), is more general than the separate analysis of each single regime.

Consider the following bivariate model, where, for simplicity, the dynamics is omitted, as not directly involved in the identification issue:

$$\begin{pmatrix} 1 & -\beta \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \quad (24)$$

or, more compactly,

$$AY_t = \epsilon_t. \quad (25)$$

In order to ease the explanation, we introduce a theoretical foundation to the model and interpret the first equation as a demand equation while the latter as a supply equation. The vector  $Y_t = (p_t, q_t)'$  collects the two observable variables and  $\epsilon_t = (\varepsilon_t, \eta_t)'$  the two structural shocks. Furthermore, let the structural shocks be characterized by null expected values and by the following covariance matrix:

$$\Sigma \equiv \text{Cov} \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix}.$$

The  $A$  matrix, containing the parameters of the simultaneous relationships among the observable variables,  $\alpha$  and  $\beta$ , can also be rescaled by dividing for the standard deviations of the

shocks and obtain

$$A_0 \equiv \Sigma^{-1/2} A = \begin{pmatrix} 1/\sigma_\varepsilon & -\beta/\sigma_\varepsilon \\ -\alpha/\sigma_\eta & 1/\sigma_\eta \end{pmatrix}.$$

Actually, from the observation of  $p_t$  and  $q_t$ , the amount of information is contained in the estimable covariance matrix

$$\Omega \equiv \text{Cov} \begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} \omega_p^2 & \omega_{pq} \\ \omega_{pq} & \omega_q^2 \end{pmatrix}$$

that is connected to the structural parameters through the non-linear system of equations

$$\begin{aligned} \Omega &= A^{-1} \Sigma A^{-1'} \\ &= A_0^{-1} A_0^{-1'} \end{aligned} \quad (26)$$

for which, when the solution with respect to the structural parameters is unique, the identification problem is clearly solved. As is well known, however, without imposing any restriction on the structural parameters, the solution cannot be unique as the three (estimable) empirical moments contained in  $\Omega$  are not sufficient to consistently estimate the four structural parameters in  $A_0$ . In fact, the amount of information contained in  $\Omega$  is the same as the one contained in the three elements of its lower triangular Cholesky factorization, that can be given by

$$\Omega_{1,tr} = \begin{pmatrix} \omega_p & 0 \\ \omega_{pq}/\omega_p & (\omega_q^2 - \omega_{pq}^2/\omega_p^2)^{1/2} \end{pmatrix}$$

whose inverse is given by

$$\Omega_{1,tr}^{-1} = \begin{pmatrix} \frac{1}{\omega_p} & 0 \\ -\frac{\omega_{pq}}{\omega_p^2} \left( \omega_q^2 - \frac{\omega_{pq}^2}{\omega_p^2} \right)^{-1/2} & \left( \omega_q^2 - \frac{\omega_{pq}^2}{\omega_p^2} \right)^{-1/2} \end{pmatrix} = (\omega_1, \omega_2) \quad (27)$$

where the two  $(2 \times 1)$  vectors  $\omega_1$  and  $\omega_2$  are the two columns of  $\Omega_{1,tr}^{-1}$ .

According to Uhlig (2005), the identification issue can be seen in terms of the non-uniqueness of an orthogonal matrix  $Q \in \mathcal{O}(2)$ , where  $\mathcal{O}(2)$  is the set of  $(2 \times 2)$  orthonormal

matrices, such that  $A_0 = Q' \Omega_{1,tr}^{-1}$  for which  $\Omega = (A_0' A_0)^{-1}$ . In other words, a unique  $Q$  guarantees a unique  $A_0$  through a suitable rotation of  $\Omega_{1,tr}^{-1}$ , containing all the information coming from the reduced form parameters (or, better, the information coming from the data).

Denoting with  $Q \equiv (q_1, q_2)$ , with  $q_1$  and  $q_2$  being the columns of  $Q$ , the  $A_0$  matrix can be given by

$$A_0 = Q' \Omega_{1,tr}^{-1} = \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} (\omega_1, \omega_2). \quad (28)$$

Moreover, we consider the following assumptions:

**Assumption 1.** (Sign normalization) Coherently with the sign normalization presented in Section III.1, among all the possible rotations of  $\Omega_{1,tr}^{-1}$  through the orthogonal matrix  $Q \in \mathcal{O}(2)$ , we consider only those guaranteeing the elements on the main diagonal of  $A_0 = Q' \Omega_{1,tr}^{-1}$  to be positive. In different words, we consider only  $q_1$  and  $q_2$  such that  $q_1' \omega_1 > 0$  and  $q_2' \omega_2 > 0$ .

**Assumption 2.** (Sign restriction on  $\alpha$  and  $\beta$ ) Coherently with the demand and supply curves in Eq. (24), we assume  $\alpha \geq 0$  and  $\beta \leq 0$ .

The first assumption, that is standard in the SVAR literature, asserts that, as the product of  $A_0^{-1} A_0^{-1'}$  is invariant to sign changes on the columns of  $A_0^{-1}$ , we select only  $A_0 = Q' \Omega_{1,tr}^{-1}$  such that the elements on the main diagonal are strictly positive. Equivalently, we assume the shock to have a positive on impact effect on the corresponding observable variable. The second assumption, instead, refers to the economic interpretation of the two equations of the bivariate model in Eq. (24) as a demand and a supply equation, respectively.

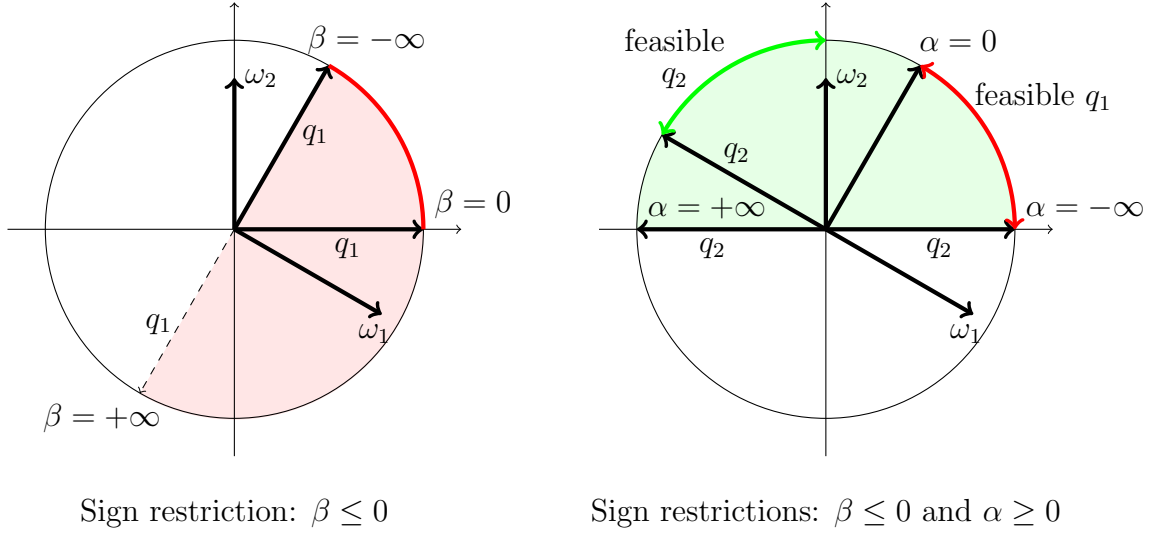
## B.1 Set identification in bivariate SVARs

Given the bivariate SVAR model discussed before, the following proposition provides the identification set for the two structural parameters  $\alpha$  and  $\beta$ , according to the two potential cases of  $\omega_{pq} \geq 0$  or  $\omega_{pq} < 0$ .

**Theorem 8** (Set identification in bivariate SVARs). *Given the bivariate model in Eq. (24), under Assumption 1, then:*

- (Case I): if  $\omega_{pq} \geq 0$ , then  $\alpha \in \left(-\infty; \frac{\omega_q^2}{\omega_{pq}}\right]$  and  $\beta \in (-\infty; \infty)$ ;
- (Case II): if  $\omega_{pq} < 0$ , then  $\alpha \in (-\infty; \infty)$  and  $\beta \in (-\infty; \infty)$ ;

Figure B.3: Identification of  $\alpha$  and  $\beta$ . Case I:  $\omega_{pq} \geq 0$



*Notes:* Set identification of the parameter  $\beta$  (left panel) and joint set identification of  $\alpha$  and  $\beta$  (right panel). The identified set, under the sign restriction consistent with a demand curve, i.e.  $\beta < 0$ , is represented by the red arc in both panels. In the right panel, the set identification of  $\alpha$  under the further sign restriction consistent with a supply curve, i.e.  $\alpha \geq 0$ , is represented by the green arc. In both cases, the standard assumption of positive diagonal terms on  $A_0$  is considered ( $\sigma_\varepsilon > 0$  and  $\sigma_\eta > 0$ ): in light red for the first equation and in light green for the second equation.

under Assumption 1 and Assumption 2, then:

(Case I): if  $\omega_{pq} \geq 0$ , then  $\alpha \in \left[ \frac{\omega_{pq}}{\omega_p^2}; \frac{\omega_q^2}{\omega_{pq}} \right]$  and  $\beta \in (-\infty; 0]$ ;

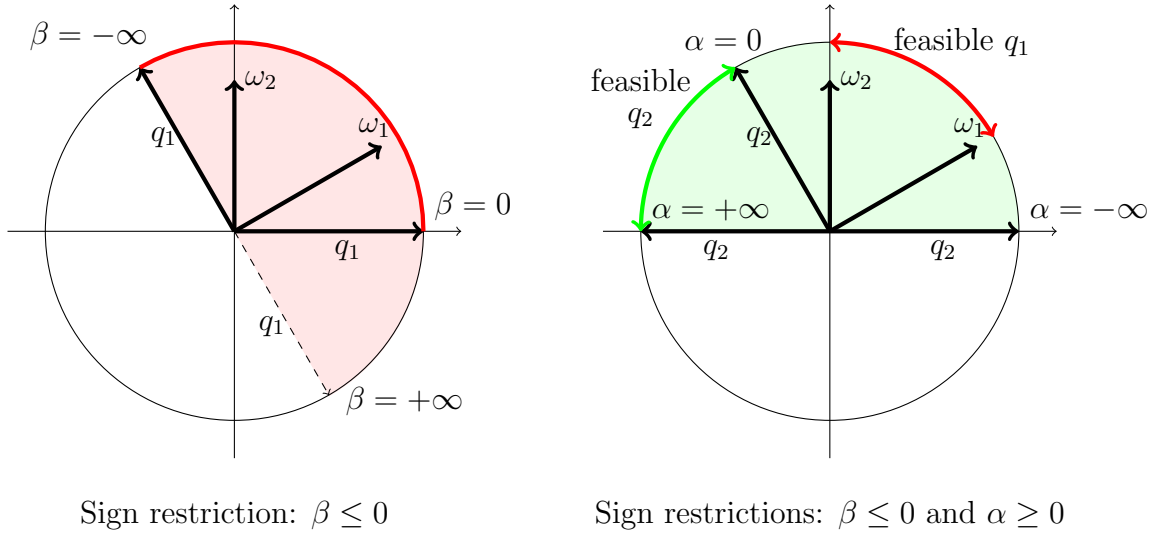
(Case II): if  $\omega_{pq} < 0$ , then  $\alpha \in [0; \infty)$  and  $\beta \in \left[ \frac{\omega_p^2}{\omega_{pq}}; \frac{\omega_{pq}}{\omega_q^2} \right]$ .

*Proof.* See the Appendix D. □

As is well known, unless we impose at least one equality restriction on one of the structural parameters, the bivariate model cannot be point identified. More specifically, according to the recent contribution by Rubio-Ramírez, Waggoner, and Zha (2010), if either  $\alpha = 0$  or  $\beta = 0$  (homogeneous restrictions) are imposed, the model will be globally identified, otherwise, if any other non-homogeneous restriction is imposed, the model will be simply locally identified, see Bacchiocchi and Kitagawa (2020). If no point restriction is imposed, the structural model remains unidentified, but focusing on the sign restrictions coherent with the theoretical interpretation of the model, together with the sign and magnitude of the elements in the reduced form covariance matrix among the observable variables, it is possible to obtain an identified set for  $\alpha$  and  $\beta$ . This is what Theorem 8 reports.

Although the formal proof is confined in the appendix, the intuition of the results can be obtained from the graphical representations reported in Figure B.3 and Figure B.4, depending

Figure B.4: Identification of  $\alpha$  and  $\beta$ . Case II:  $\omega_{pq} < 0$



*Notes:* Set identification of the parameter  $\beta$  (left panel) and joint set identification of  $\alpha$  and  $\beta$  (right panel). The identified set, under the sign restriction consistent with a demand curve, i.e.  $\beta < 0$ , is represented by the red arc in both panels. In the right panel, the set identification of  $\alpha$  under the further sign restriction consistent with a supply curve, i.e.  $\alpha \geq 0$ , is represented by the green arc. In both cases, the standard assumption of positive diagonal terms on  $A_0$  is considered ( $\sigma_\varepsilon > 0$  and  $\sigma_\eta > 0$ ): in light red for the first equation and in light green for the second equation.

on the two potential values of  $\omega_{pq} \geq 0$  (Case I) and  $\omega_{pq} < 0$  (Case II), respectively, under the sign normalization restriction discussed in Assumption 1 and sign restrictions of Assumption 2.

In Figure B.3, left panel, we report the  $\omega_1$  and  $\omega_2$  vectors when the estimated  $\omega_{pq} \geq 0$  (Case I), as well as all the possible  $q_1$  vectors generating strictly negative values for the  $\beta$  parameter, as in an hypothetical demand curve. In the right panel, we also include the further restriction of positive values for the parameter  $\alpha$ , consistent with a supply curve. The two sign restrictions, jointly, given the orthogonality condition regarding  $q_1$  and  $q_2$ , implicitly impose a set restriction for  $\alpha$ , in terms of the *feasible*  $q_2$  vectors highlighted with the green arc in the right panel of Figure B.3. The width of the set, as discussed in Proposition 8, depends on the estimable elements on the covariance matrix of the observable variables  $\Omega$ , or equivalently, on the inverse of its Cholesky decomposition  $\Omega_{1,br}^{-1}$ .

Similarly, in Figure B.4, we discuss the set identification of  $\alpha$  and  $\beta$  when  $\omega_{pq} < 0$  (Case II). In this latter case, a joint analysis of the sign restrictions on  $\alpha$  and  $\beta$  (as well as the sign normalization), provides an identified set for  $\beta$ , leaving instead  $\alpha$  to be unrestricted (though positive). The identified set for  $\beta$ , in the right panel, is highlighted by the red arrow indicating all feasible values for the  $q_1$  vector.

**Corollary 1** (Point identification and OLS estimation of  $\alpha$  and  $\beta$ ). *Given the bivariate model in Eq. (24), under Assumption 1:*

(Case I): if  $\omega_{pq} \geq 0$  and we restrict  $\beta = 0$ , then  $\alpha = \frac{\omega_{pq}}{\omega_p^2}$

(Case II): if  $\omega_{pq} < 0$  and we restrict  $\alpha = 0$ , then  $\beta = \frac{\omega_{pq}}{\omega_q^2}$ .

*Proof.* See the Appendix D. □

The previous Corollary 1 simply restates a standard result in econometrics. In fact, if we introduce a zero restriction on one of the two structural parameters, then the identified set, depending on the observed  $\omega_{pq}$ , reduces to a single point (point identification of the other parameter), that can be consistently obtained through the OLS estimator, as stated in the corollary.

## B.2 Point and set identification in bivariate HSVARs

Consider the model in Eq.s (24)-(25), but with a clear evidence of a shift in the variances of the observable variables. According to the HSVAR model introduced in Section II, such shift is simply due to a structural change involving the variances of the structural shocks, leaving unaffected the structural relationships among the variables, captured by the two parameters  $\alpha$  and  $\beta$ , i.e.

$$\Sigma_i \equiv \begin{pmatrix} \sigma_{\varepsilon,i}^2 & 0 \\ 0 & \sigma_{\eta,i}^2 \end{pmatrix}, \quad i = \{1, 2\},$$

where  $i = 1$  denotes the first regime (before the break),  $i = 2$  indicates the second regime (after the break), while the  $A$  matrix remains as defined in Eq. (25). Similarly as before, we standardize with respect to the standard deviations of the structural shocks in the first regime, and define

$$A_0 \equiv \Sigma_1^{-1/2} A = \begin{pmatrix} 1/\sigma_{\varepsilon,1} & -\beta/\sigma_{\varepsilon,1} \\ -\alpha/\sigma_{\eta,1} & 1/\sigma_{\eta,1} \end{pmatrix} \quad \text{and} \quad \Lambda \equiv \Sigma_1^{-1} \Sigma_2 = \begin{pmatrix} \sigma_{\varepsilon,2}^2/\sigma_{\varepsilon,1}^2 & 0 \\ 0 & \sigma_{\eta,2}^2/\sigma_{\eta,1}^2 \end{pmatrix},$$

and, coherently with the definitions in Sections II and II.2, let  $C \equiv A_0^{-1}$ .



From a different perspective, we consider that the change involves only the second moments of the distribution of the observable variables in  $Y_t = (p_t, q_t)'$ , showing the two covariance matrices

$$\Omega_i \equiv \begin{pmatrix} \omega_{p,i}^2 & \omega_{pq,i} \\ \omega_{pq,i} & \omega_{q,i}^2 \end{pmatrix}, \quad i = \{1, 2\}, \quad (29)$$

that are connected to the structural parameters through the non-linear system of equations (20)-(21). When the solution of the system with respect to the structural parameters is unique, the identification problem is clearly solved. As before, we define the lower triangular Cholesky factorization of  $\Omega_i$  as  $\Omega_{tr,i}$ ,  $i = \{1, 2\}$ , whose inverses are given by

$$\Omega_{i,tr}^{-1} = \begin{pmatrix} \frac{1}{\omega_{p,i}} & 0 \\ -\frac{\omega_{pq,i}}{\omega_{p,i}^2} \gamma_i & \gamma_i \end{pmatrix} = (\omega_{1,i}, \omega_{2,i}), \quad i = \{1, 2\}, \quad (30)$$

where the two  $(2 \times 1)$  vectors  $\omega_{1,i}$  and  $\omega_{2,i}$  are the two columns of  $\Omega_{tr,i}^{-1}$ , and where

$$\gamma_i = \left( \omega_{q,i}^2 - \frac{\omega_{pq,i}^2}{\omega_{p,i}^2} \right)^{-1/2}, \quad i = \{1, 2\}. \quad (31)$$

Based on the connections between the reduced-form and the structural-form parameters highlighted in Eq.s (20)-(21), the identification issue can be addressed by studying the solutions of the system of equations reported in Eq. (8). Theorem 6 shows that it can be addressed as an eigen-decomposition problem that, under the condition of distinct eigenvalues, proves the structural parameters contained in  $(C, \Lambda)$  (or, equivalently in  $A, \Sigma_1$  and  $\Sigma_2$ ) to be point identified, up to permutations and sign changes of the structural equations. However, as a matter of comparison, we first report the Rigobon's condition for identification in bivariate HSVARs.

**Theorem 9** (Rigobon (2003) condition for point identification in bivariate HSVAR). *Given the HSVAR model described in Eq. (24) with the two covariance matrices reported in Eq. (29), under Assumption 1, a necessary and sufficient condition for the uniqueness of the structural parameters  $(C, \Lambda)$  is that*

$$\Omega_1 \neq a\Omega_2 \quad (32)$$

for any scalar  $a > 0$ .

*Proof.* See the proof of Theorem 1 in Rigobon (2003) and the proof of our Theorem 10 in the Appendix D.  $\square$

The following theorem, instead, restates the Rigobon's condition (Proposition 1, page 780, or our Theorem 9) as the solution of the eigen-decomposition problem based on the observed covariance matrices in Eq. (29). Moreover, it extends the results to the case of lack of identification due to equivalent eigenvalues.

**Theorem 10** (Point and set identification in bivariate HSVAR). *Given the HSVAR model described in Eq. (24) with the two covariance matrices reported in Eq. (29), the structural parameters  $(C, \Lambda)$  are obtained through the eigen-decomposition problem discussed in Theorem 6. In particular, the two variances of the structural shocks contained in  $\Lambda$  are given by the two eigenvalues of  $\Omega_{i,tr}^{-1} \Omega_2 \Omega_{i,tr}^{-1'}$ , i.e.*

$$\lambda_{1,2} = \frac{\omega_{p,1}^2 \omega_{q,2}^2 + \omega_{p,2}^2 \omega_{q,1}^2 - 2\omega_{pq,1} \omega_{pq,2} \pm \Delta}{2(\omega_{p,1}^2 \omega_{q,1}^2 - \omega_{pq,1}^2)} \quad (33)$$

with

$$\Delta = \left[ (\omega_{p,1}^2 \omega_{q,2}^2 - \omega_{p,2}^2 \omega_{q,1}^2)^2 + 4(\omega_{p,1}^2 \omega_{pq,2} - \omega_{p,2}^2 \omega_{pq,1})(\omega_{q,1}^2 \omega_{pq,2} - \omega_{q,2}^2 \omega_{pq,1}) \right]^{1/2}.$$

The associated unit eigenvectors  $q_1$  and  $q_2$  form the columns of the orthogonal matrix  $Q = (q_1, q_2)$  such that  $C = \Omega_{1,tr} Q$ . Under Assumption 1, the necessary and sufficient condition for the uniqueness of the structural parameters  $(C, \Lambda)$  is that  $\Delta \neq 0$ . If, instead,  $\Delta = 0$ , then Rigobon's condition fails and the HSVAR will only be set identified according to the results of the previous Theorem 8.

*Proof.* See the Appendix D.  $\square$

Theorem 10 provides analytical formula to calculate the structural parameters as a function of the eigenvalues and eigenvectors of observable matrices, i.e. the covariance matrices of the reduced form in the two regimes,  $\Omega_1$  and  $\Omega_2$ . Furthermore, in proving the theorem, in the

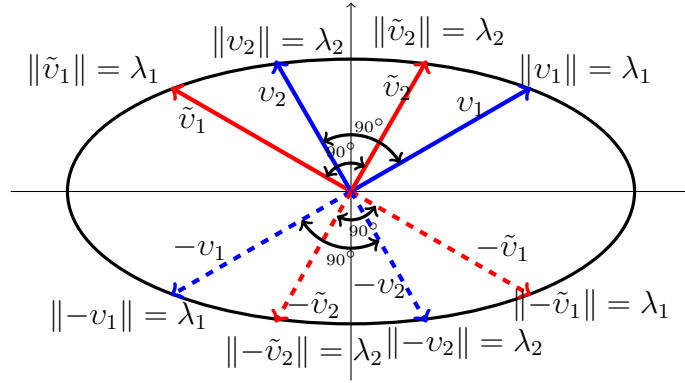
Appendix, we also show that the necessary and sufficient condition in Eq. (32), as expected, is equivalent to say that the two eigenvalues in Eq. (33) must be distinct (as postulated in Theorem 6) or, put differently, the shift in the variances of the structural shocks must be different.

However, when the quantity  $\Delta = 0$ , from Eq. (33) we have that the two eigenvalues  $\lambda_1$  and  $\lambda_2$  coincide and, according to Theorem 6, there will be infinite (not parallel) eigenvectors  $q_1$  and  $q_2$  and, as a consequence, the orthogonal  $Q = (q_1, q_2)$  is not unique. The model, thus, is not point identified though we are in the presence of a structural break with distinct covariance matrices  $\Omega_1$  and  $\Omega_2$ . Put differently, the information coming from the two different covariance matrices is not sufficient for point-identifying the structural parameters of the bivariate model.

We now move to the geometric interpretation of this result. Starting from Eq. (8), we easily obtain that  $Q' \Omega_{1,tr}^{-1} \Omega_{2,tr} \Omega_{2,tr}' \Omega_{1,tr}^{-1} Q = \Lambda$ . Fixing the quantity  $\Upsilon \equiv \Omega_{2,tr}' \Omega_{1,tr}^{-1} Q$ , then we have that  $\Upsilon' \Upsilon = \Lambda$ , or, equivalently,  $\Upsilon \Lambda^{-1} \Upsilon' = I_n$ , with  $I_n$  the  $(n \times n)$  identity matrix. Interestingly, the columns of  $\Upsilon$ , obtained as a linear transformation of the columns of  $Q$ , maintain the orthogonality condition, although their length is no longer unity, but given by the elements on the main diagonal of  $\Lambda$ . Coming back to the bivariate case, it is easy to remark that the equation  $\Upsilon \Lambda^{-1} \Upsilon' = I_n$  is the representation of two orthogonal vectors, of length  $\|v_1\| = \lambda_1$  and  $\|v_2\| = \lambda_2$ , belonging to an ellipse of equation  $\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} = 1$ , as shown in Figure B.5.

Once we know  $\lambda_1$  and  $\lambda_2$ , being the two eigenvalues of the eigen-decomposition highlighted in Theorem 6, there will be just two pairs of orthogonal vectors (other than their opposite),  $(v_1, v_2)$  and  $(\tilde{v}_1, \tilde{v}_2)$ , having  $\lambda_1$  and  $\lambda_2$  as their length, shown in blue and in red, respectively, in Figure B.5. Starting from these four pairs of vectors, using the definition of  $\Upsilon$ , it is possible to obtain four values of  $Q$  simply by linearly transforming the columns of  $\Upsilon$  by the known quantities  $\Omega_{1,tr}$  and  $\Omega_{2,tr}$ , i.e.  $Q = \Omega_{1,tr}' \Omega_{2,tr}^{-1} \Upsilon$ . Based on Assumption 1 (sign normalization), just two of the four  $Q$  matrices will be retained (upper-half or lower-half of the ellipse). Fixing a specific ordering of the eigenvalues, or, equivalently, fixing the permutation matrix  $P \in \mathcal{P}(n)$ , helps reducing to one single admissible  $Q$ , making the HSVAR point identified. The problem arises when  $\lambda_1 = \lambda_2$ . The ellipse will collapse into a circle and infinite orthogonal vectors will be potentially admissible. In this case, of course, the HSVAR will be no longer identified.

Figure B.5: Identification of a bivariate HSVAR



*Notes:* Representation of the ellipse of equation  $\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} = 1$ , where  $\lambda_1$  and  $\lambda_2$  are the eigenvalues of  $\Omega_{1,tr}^{-1}\Omega_2(\Omega_{1,tr}^{-1})'$ . In blue the pairs of orthogonal vectors  $(v_1, v_2)$  and  $(-v_1, -v_2)$ . In red the other two pairs of orthogonal vectors  $(\tilde{v}_1, \tilde{v}_2)$  and  $(-\tilde{v}_1, -\tilde{v}_2)$ . The vectors  $v_1, \tilde{v}_1, -v_1, -\tilde{v}_1$  have length exactly equal to  $\lambda_1$ , and  $v_2, \tilde{v}_2, -v_2, -\tilde{v}_2$  have length exactly equal to  $\lambda_2$ . The following orthogonality conditions hold:  $(v_1 \perp v_2)$ ,  $(\tilde{v}_1 \perp \tilde{v}_2)$ ,  $(-v_1 \perp -v_2)$ ,  $(-\tilde{v}_1 \perp -\tilde{v}_2)$ .

## Appendix C Proofs

We first introduce some notation that will be used in the following proofs. For any reduced-form parameter  $\phi \in \Phi$ , let  $\lambda_i$  be an eigenvalue of the eigenproblem as in Definition 1 with algebraic multiplicity  $g(\lambda_i) = m_i$ , with associated eigenspace  $Q(\lambda_i)$  as in Eq. (9), containing  $q_{j^*}^i$ , the column of  $Q$  associated with the  $j^*$ -th structural shock (shock of interest). We have defined the zero restrictions on the vectors  $(q_1^i, \dots, q_{m_i}^i) \in Q(\lambda_i)$  in terms of the matrix  $F_j^i(\phi)$ , with  $\phi$ -a.s. full row rank equal to  $f_j^i$ . Let  $Q^\perp(\lambda_i)$ , instead, be the linear space in  $\mathbb{R}^n$ , of dimension  $(n - m_i)$ , whose elements are orthogonal to  $Q(\lambda_i)$ . A basis for this linear space is given by  $(v_1^i, \dots, v_{(n-m_i)}^i)$ . We define  $\mathcal{F}_j^{i\perp}(\phi)$  the linear subspace of  $\mathbb{R}^n$  that is orthogonal to the row vectors of  $F_j^i(\phi)$  and to  $Q^\perp(\lambda_i)$ . We let  $\mathcal{H}_j(\phi)$  be the half-space in  $\mathbb{R}^n$  defined by the sign-normalization constraint  $\{x \in \mathbb{R}^n \mid (\sigma^j)'x \geq 0\}$ , with  $\sigma^j$  being the  $j$ -th column of  $\Omega_{1,tr}^{-1}$ . As before,  $\mathcal{S}^{n-1}$  indicates the unit sphere in  $\mathbb{R}^n$ . Finally, given  $k$  linearly independent vectors in  $\mathbb{R}^n$ ,  $V = (v_1, \dots, v_k) \in \mathbb{R}^{n \times k}$ , let  $\mathcal{P}(V)$  be the linear subspace in  $\mathbb{R}^n$ , of dimension  $(n - k)$  that is orthogonal to the column vectors of  $V$ .

**Lemma 2** (Diagonalization of symmetric matrices). *Let  $\Omega$  be a symmetric matrix in  $\mathbb{R}^{n \times n}$ , then it is diagonalizable, i.e. there exists an orthogonal matrix  $Q \in \mathcal{O}(n)$ , made of the (unit) eigenvectors of  $\Omega$ , such that  $\Omega Q = QD$ , or equivalently,  $Q'\Omega Q = D$ , where  $D$  is diagonal. Moreover, the matrix  $D$  contains the (real) eigenvalues of  $\Omega$ , corresponding to the eigenvectors in  $Q$ .*

*Proof of Lemma 2.* See Magnus and Neudecker (2007), Chapter 1, Theorem 13 (page 17).  $\square$

**Lemma 3.** *In real symmetric matrices the algebraic multiplicity does correspond to the geometric multiplicity.*

*Proof of Lemma 3.* Let  $A$  be an  $n \times n$  symmetric matrix whose  $i$ -th eigenvalue is represented by  $\lambda_i$ , with algebraic multiplicity equal to  $1 < m_i \leq n$ . Then, there exists some unit-length eigenvector  $p_{i1}$ . Let  $B = (p_{i1} \ C)$  be an orthogonal matrix. Then we have

$$B'AB = \begin{pmatrix} \lambda_i & 0 \\ 0 & C'AC \end{pmatrix}.$$

As the algebraic multiplicity  $m_i$  is greater than one, from the characteristic polynomial we have that  $|C'AC - \lambda_i I_{n-1}| = 0$ , that implies there will be some non-null vector  $q$  such that  $(C'AC - \lambda_i I_{n-1})q = 0$ . Let  $p_{i2} = Cq$ . It is easy to show that  $p_{i2}$  is an eigenvector of  $A$ . In fact, from the previous relation  $(C'AC - \lambda_i I_{n-1})q = 0$  we have  $ACq = \lambda_i Cq$ , that implies  $Ap_{i2} = \lambda_i p_{i2}$ . Moreover, by construction,  $p_{i1}$  will be orthogonal to  $p_{i2}$ . It will be possible, thus, to define a new  $B$  of the form  $B = (p_{i1} \ p_{i2} \ C)$  such that

$$B'AB = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & C'AC \end{pmatrix}.$$

and proceed as before for all the algebraic multiplicity of  $\lambda_i$ . The matrix  $E = (p_{i1}, \dots, p_{im_i})$  will be a basis for the eigenspace of  $A$  associated with  $\lambda_i$ , and the dimension of such space will be clearly  $m_i$ , being all the columns of  $E$  orthogonal.  $\square$

*Proof of Theorem 4.* Let  $(C, A)$  be a solution of the equation system different from  $(C^*, A^*)$  with non singular  $C$ . Then, there exists a  $n \times n$  matrix  $A$  such that  $C^* = CA$  holds. Note that  $A$  has to be an orthogonal matrix  $AA' = I$  as otherwise  $\Omega_1 = C^*C'^* = CC'$  violates. In order for (21) to hold for both  $(C^*, A^*)$  and  $(C, A)$ ,

$$\Omega_2 = C^*A^*C'^* = CAA^*A'C' \tag{34}$$

must hold and, hence,  $\Lambda = A\Lambda^*A'$  holds. We therefore investigate the conditions on orthogonal matrix  $A$  such that  $A\Lambda^*A'$  yields a diagonal matrix with non negative entries.

Let  $(\lambda_1^*, \dots, \lambda_n^*)$  be the diagonal elements of  $\Lambda^*$  and  $a_{ij}$  be  $(i, j)$ -element of  $A$ . Note that non-singularity of  $\Omega_2$  implies  $\lambda_k^* > 0$  for all  $k = 1, \dots, n$ . Noting that the  $(i, j)$ -element of  $A\Lambda^*A'$  can be expressed as  $\sum_{k=1}^n \lambda_k^* a_{ik} a_{jk}$ ,  $A$  has to satisfy

$$\begin{cases} \sum_{k=1}^n \lambda_k^* a_{ik} a_{jk} = 0, & \forall i \neq j \\ \sum_{k=1}^n \lambda_k^* a_{ik} a_{jk} \geq 0, & \forall i = j. \end{cases}$$

The second set of conditions does not at all constrain  $A$ , while the first set of conditions constrains  $A$  to those such that every row vector in  $A$  has only one nonzero element and none of the row vectors in  $A$  shares the column-index for the non-zero entry. Combined with orthogonality of  $A$ , feasible  $A$ 's can be therefore represented by  $A = PS$ .  $\square$

*Proof of Theorem 7.* The proof of the theorem is trivial and completely based on the proof of the Single Value Decomposition for square matrices, see among many others Magnus and Neudecker (2007), pages 19-20. The first point to remark is that, if we call  $\Omega_{tr} = \Omega_{1,tr}^{-1} \Omega_{2,tr}$ , then  $\Lambda^{1/2}$  contains the positive square root of the eigenvalues of  $\Omega = \Omega_{1,tr}^{-1} \Omega_{2,tr} \Omega'_{2,tr} \Omega_{1,tr}^{-1} = \Omega_{1,tr}^{-1} \Omega_{2,tr} \Omega_{1,tr}^{-1}$ , as described in Theorem 6. However, for symmetric and non-singular real matrices like  $\Omega$ , the number of identical eigenvalues (real and different from zero) corresponds to the number of degenerate singular values in  $\Omega_{tr}$ . As a consequence, if all the elements in  $\Lambda^{1/2}$  are distinct, then all the singular values are non-degenerate, and the singular value decomposition is unique ( $Q$  and  $Q_2$  are unique), up to multiplication of a specific column of  $Q$  and  $Q_2$  by -1, or changing the ordering of the elements in  $\Lambda^{1/2}$  (or  $\Lambda$ ).  $\square$

*Proof of Theorem 1.* The proof of the theorem takes inspiration from Rubio-Ramírez, Waggoner, and Zha (2010) (proof of their Theorem 7). When the two covariance matrices are perfectly proportional, or even equal, then the condition in the theorem collapses to the well known condition in Rubio-Ramírez, Waggoner, and Zha (2010) and Bacchiocchi and Kitagawa

(2021), and the proof is thus immediate. On the other side, if all eigenvalues are distinct, i.e.  $k = n$ , then the results of Theorem 6 apply. Similar results apply for all eigenvalues with algebraic multiplicity equal to one, i.e.  $m_i = 1$ . According to Lemma 3, the geometric multiplicity is equal to the algebraic one, and thus, if  $m_i = 1$  the eigenspace associated to such eigenvalues will generate spaces of dimension one, each. Imposing unit length and sign normalization allows to uniquely identify such vectors. Moreover, given Lemma 2, such vectors will be mutually orthogonal. They will constitute the columns of  $Q$  associated with the eigenvalues of multiplicity one.

We will now turn to the case when  $\lambda$  has multiplicity greater than 1. Let  $\lambda_i$  be characterized by algebraic multiplicity  $m_i \leq 2$ . Given Lemma 3, the  $m_i$  associated eigenvectors, although not unique, represent an orthonormal basis for the subspace  $Q(\lambda_i)$ , of dimension  $m_i$  in  $\mathbb{R}^n$ . For the condition in Theorem 1 to be sufficient, we need to prove that imposing such particular pattern of zero restrictions allows to uniquely pin down orthonormal vectors lying in  $Q(\lambda_i)$ . Let  $V(\lambda_i) = (v_1^i, \dots, v_{m_i}^i)$  be a basis for the eigenspace associated to  $\lambda_i$ . The identified vectors  $(q_1^i, \dots, q_{m_i}^i)$  must satisfy the following conditions:

- they must be a linear combination of the orthonormal basis identified through the eigenproblem;
- they must be orthogonal each other;
- they must satisfy the zero and normalization restrictions;
- they must have unit length.

We can think of writing a system of equations. The number of unknowns is  $m_i$  for each vector  $q_j^i$ ,  $j = 1, \dots, m_i$ . Let the ordering of the vectors be fixed according to the number of restrictions, from the more the less constrained. We start from the first and most constrained vector

$$q_1^i = v_1^i x_1 + \dots + v_{m_i}^i x_{m_i}$$

that is subjected to the  $m_i - 1$  zero restrictions  $F_1^i(\phi)q_1^i = 0$ . Substituting the definition of  $q_i$

according to the previous relation, we simply obtain:

$$\left[ F_1^i(\phi)v_1^i \quad \cdots \quad F_1^i(\phi)v_{m_i}^i \right] \begin{pmatrix} x_1 \\ \vdots \\ x_{m_i} \end{pmatrix} = 0. \quad (35)$$

As  $\text{rank}(F_1^i(\phi)) = m_i - 1$ ,  $\phi$ -almost surely (a.s.), then the matrix  $\left[ F_1^i(\phi)v_1^i \quad \cdots \quad F_1^i(\phi)v_{m_i}^i \right]$  projecting  $m_i$  orthogonal vectors in  $\mathbb{R}^n$  onto an  $m_i - 1$  dimensional space, will generate  $m_i - 1$  linearly independent vectors in  $\mathbb{R}^{m_i}$ . As a consequence, there will be just a uni-dimensional space in  $\mathbb{R}^{m_i}$  that is orthogonal to  $F_1^i(\phi)$ . Let  $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{m_i})'$  be a unit vector representing a basis for this vector space, then

$$q_i = v_1^i \alpha \tilde{x}_1 + \cdots + v_{m_i}^i \alpha \tilde{x}_{m_i}.$$

However,  $q_i$  must have unit length, thus

$$\begin{aligned} q_1' q_1 = 1 &\implies \alpha^2 (v_1^i \tilde{x}_1 + \cdots + v_{m_i}^i \tilde{x}_{m_i})' (v_1^i \tilde{x}_1 + \cdots + v_{m_i}^i \tilde{x}_{m_i}) = 1 \\ &\implies \alpha^2 (\tilde{x}_1^2 + \cdots + \tilde{x}_{m_i}^2) = 1 \\ &\implies \alpha^2 = 1 \implies \alpha = \pm 1, \end{aligned}$$

indicating that there will be two opposite vectors candidates for  $q_1^i$ , one of which, however, is ruled out by the normality sign restrictions. The following step consists in determining  $q_2^i \in Q(\lambda_i)$ , orthogonal to  $q_1^i$  and satisfying the  $(m_i - 2)$  restrictions  $F_2^i(\phi)q_2^i = 0$ . We can think of a system of equations of the form

$$\begin{cases} F_2^i(\phi)q_2^i = 0 \\ q_1^i q_2^i = 0 \end{cases}$$

where  $q_1^i$  is known from the previous step. Substituting for the definition of  $q_2^i$  in terms of the basis of the vector space it belongs, i.e.  $V(\lambda_i) = (v_1^i, \dots, v_{m_i}^i)$ , with simple algebra, the



system can also be written as

$$\left[ \begin{pmatrix} F_2^i(\phi) \\ q_1^i \end{pmatrix} v_1^i \cdots + \begin{pmatrix} F_2^i(\phi) \\ q_1^i \end{pmatrix} v_{m_i}^i \right] \begin{pmatrix} x_1 \\ \vdots \\ x_{m_i} \end{pmatrix} = 0.$$

According to the assumed non-redundancy of the restrictions, as in Definition 3, the quantity  $\begin{pmatrix} F_2^i(\phi) \\ q_1^i \end{pmatrix}$  has full row rank  $m_i - 1$ , we are exactly in the same situation as in Eq. (35).

We can thus proceed as before and obtain two potential opposite unit-length vectors  $q_2^i$ , one of which, however, is ruled out by the sign normalization restrictions. This strategy allows to prove the point identification of all the  $(q_1^i, \dots, q_{m_i}^i)$  vectors associated with the  $i$ -th multiple eigenvalue  $\lambda_i$ . The *sufficient* direction of the condition is thus proved.

The *necessary* part of the condition can be proved as follows. Let the parameter  $(A_0, A_+, A) \in \mathcal{A}^r(\phi)$  be point identified. As a consequence, the set of admissible orthogonal matrices  $\mathcal{Q}(\phi|F, S)$  will be a singleton, say  $Q$ . If the  $i$ -th column of  $Q$  is the eigenvector associated to an eigenvalue with no multiplicity ( $m_i = 1$ ), then it is unique and no zero restriction is needed, thus  $f_j^i = m_i - j = 1 - 1 = 0$  as predicted by the theorem. For those columns of  $Q$  associated with an eigenvalue with multiplicity  $m_i > 1$  the condition can be directly proved by using Lemma 4 in Bacchiocchi and Kitagawa (2021), that extends Lemma 9 in Rubio-Ramírez, Waggoner, and Zha (2010) to the case of non-redundant restrictions.  $\square$

*Proof of Theorem 2.* Let  $j^*$  the shock of interest, that is associated with  $q_{j^*}^i \in Q(\lambda_i)$ , the eigenspace related to  $\lambda_i$ , with multiplicity  $m_i$ . According to Definition 1, the space  $Q(\lambda_i)$  is orthogonal to the linear space generated by all the other eigenvectors of the eigenproblem, that we denote by  $Q^\perp(\lambda_i)$ . The dimension of  $Q^\perp(\lambda_i)$  is  $(n - m_i)$  and let the vectors  $(v_1^i, \dots, v_{(n-m_i)}^i)$  be a possible basis. Moreover, let the vectors  $(q_1^i, \dots, q_{m_i}^i) \in Q(\lambda_i)$  be defined in the following

recursive way:

$$\begin{aligned}
q_1^i &\in V_1^i(\phi) \equiv \mathcal{F}_1^{i\perp}(\phi) \cap \mathcal{H}_1(\phi) \cap \mathcal{S}^{n-1} \\
q_2^i &\in V_2^i(\phi, q_1^i) \equiv \mathcal{F}_2^{i\perp}(\phi) \cap \mathcal{H}_2(\phi) \cap \mathcal{P}(q_1^i) \cap \mathcal{S}^{n-1} \\
q_3^i &\in V_3^i(\phi, Q_{1:2}^i) \equiv \mathcal{F}_3^{i\perp}(\phi) \cap \mathcal{H}_3(\phi) \cap \mathcal{P}(Q_{1:2}^i) \cap \mathcal{S}^{n-1} \\
&\vdots \\
q_{m_i}^i &\in V_{m_i}^i(\phi, Q_{1:m_i-1}^i) \equiv \mathcal{F}_{m_i}^{i\perp}(\phi) \cap \mathcal{H}_{m_i}(\phi) \cap \mathcal{P}(Q_{1:m_i-1}^i) \cap \mathcal{S}^{n-1}
\end{aligned} \tag{36}$$

where the generic  $\mathcal{F}_j^{i\perp}(\phi)$  is the linear subspace of  $\mathbb{R}^n$  that is orthogonal to the row vectors of  $F_j^i(\phi)$  and to  $Q^\perp(\lambda_i)$ . The dimension of  $\mathcal{F}_j^{i\perp}(\phi)$  is  $\dim(\mathcal{F}_j^{i\perp}(\phi)) = n - (n - m_i) - f_j^i = m_i - f_j^i$ .

If  $j^* = 1$ , then we know that  $f_1^i \leq m_i - 1$ , and  $\mathcal{F}_1^{i\perp}(\phi) \cap \mathcal{H}_1(\phi)$  is the half-space of the linear subspace of  $\mathbb{R}^2$  with dimension  $m_i - f_1^i \geq 1$ . As a consequence,  $V_1^i(\phi, q_1^i)$  is non empty for every  $\phi \in \Phi$ . Similarly, if  $j^* = 2, \dots, m_i$ ,  $\mathcal{F}_{j^*}^{i\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:j^*}^i)$  is the half-space of the linear subspace of  $\mathbb{R}^n$  of dimension at least  $m_i - f_{j^*}^i - (j^* - 1) \geq 1$ , being  $f_{j^*}^i \leq m_i - j^*$ . For  $j^* = 1, \dots, m_i$ , thus,  $V_{j^*}^i(\phi, Q_{1:j^*-1}^i)$  is non empty and, as a consequence,  $Q(\phi, F)$  will be non empty, too. Non emptiness of the impulse responses is a direct consequence. Concerning the boundedness, it immediately follows from the fact that  $|r_{lj^*}^h| \leq \|c_{lh}(\phi)\| \leq \infty$  for any  $l \in \{1, \dots, n\}$ ,  $j^* \in \{1, \dots, m_i\}$  and  $h = 0, 1, 2, \dots$ , where  $\|c_{lh}(\phi)\| \leq \infty$  is guaranteed by the invertibility of the VAR characteristic polynomial. The first part of the proof is thus complete. We can move to prove the convexity of the identified set.

Let  $j^* = 1$  and  $f_1^i \leq m_i - 1$  (condition 1). Being  $V_1^i(\phi)$  the intersection of an half-space of dimension at least 2 and an unit sphere it is path connected for all values of the reduced-form parameters  $\phi$ . Hence, the identified set  $r_{l1}^h = c_{lh}(\phi)q_1^i$  will be an interval, being the impulse response a continuous function with a path connected domain always an interval. Concerning condition (2) of the Theorem, we can prove the result by applying Lemma A.1 in Giacomini and Kitagawa (2014). According to the definition of  $\mathcal{F}_{j^*}^{i\perp}(\phi)$ , that collects not simply vectors orthogonal to the row vectors of  $F_{j^*}^i(\phi)$ , but also orthogonal to all other vectors belonging to  $Q^\perp(\lambda_i)$ , Lemma A.1 in Giacomini and Kitagawa (2014) allows to simplify the set of admissible  $q_{j^*}^i$ . In fact, if we define  $\mathcal{E}_{j^*}^i(\phi)$  the set of admissible  $q_{j^*}^i$ , then using Lemma A.1 we derive that  $\mathcal{E}_{j^*}^i(\phi) = \mathcal{F}_{j^*}^{i\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{S}^{n-1}$ . Hence, being  $\mathcal{E}_{j^*}^i(\phi)$  the intersection of a half-space of a linear subspace with dimension  $m_i - f_{j^*}^i \geq j^* \geq 2$  with the unit sphere, it is a path connected

set on  $\mathcal{S}^{n-1}$ , and the convexity of the identified set immediately follows.

In a similar way we can prove the result for condition (3) of the theorem. In this respect we can use Lemma A.2 in Giacomini and Kitagawa (2014), that, based on our definition of  $\mathcal{F}_j^{i\perp}(\phi)$ , allows to derive the set of potential  $q_{j^*}^i$  subject to condition (3), i.e.  $\mathcal{E}_{j^*}^i(\phi) = \mathcal{F}_{j^*}^{i\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:k}^i) \cap \mathcal{S}^{n-1}$ . According to this definition,  $\mathcal{E}_{j^*}^i(\phi)$  is the intersection of a half-space of a linear subspace of dimension  $n - (n - m_i) - f_{j^*}^i - k > j^* - k \geq 2$  and a unit sphere, and, thus, it is a path connected set on  $\mathcal{S}^{n-1}$ . The convexity of the identified set, hence, clearly holds.

In all cases, the convexity of the identified set depends on  $\phi \in \Phi$ , being the multiplicity of  $\lambda_i$  equal to  $m_i$  only  $\phi$ -a.s. Thus, convexity of the identified set holds  $\phi$ -a.s.  $\square$

*Proof of Theorem 3.* The proof builds on Lemma A.2 in Giacomini and Kitagawa (2014). Let first  $j^* = 1$  and  $f_1^i < m_i - 1$ . According to the notation introduced in Eq. (36), the set of admissible  $q_1^i$  becomes  $V_1^i(\phi) \cap \{x \in \mathbb{R}^n : S_1^i(\phi) x \geq 0\}$ . Moreover, let  $\tilde{q}_1^i 1$  be another arbitrary unit length vector satisfying the zero, sign normalization and sign restrictions. Clearly, according to the sign restrictions, it must hold that  $q_1^i \neq \tilde{q}_1^i 1$ . The intuition for proving the result consists in observing that any weighted average of the two admissible vectors, with positive weights summing to one, continues to belong to the set. Then, if we define

$$q_1^i(\delta) = \frac{\delta q_1^i + (1 - \delta)\tilde{q}_1^i 1}{\|\delta q_1^i + (1 - \delta)\tilde{q}_1^i 1\|}, \quad \delta \in [0, 1] \quad (37)$$

it represents a connected path in  $V_1^i(\phi) \cap \{x \in \mathbb{R}^n : S_1^i(\phi) x \geq 0\}$ , as the denominator is always different than zero, given that  $q_1^i \neq \tilde{q}_1^i 1$ . Any generic couple of admissible vectors, thus, can be connected by a connected path. The convexity of the impulse response, thus, immediately follows. We now assume that condition (2) in Theorem 2 holds. Now, let  $\mathcal{E}_{j^*}^i(\phi)$  be the set of admissible  $q_{j^*}^i$  satisfying zero, sign normalization and sign restrictions. Let  $q_{j^*}^i$  and  $\tilde{q}_{j^*}^i$  be two arbitrary vectors belonging to  $\mathcal{E}_{j^*}^i(\phi)$ . Clearly, due to the sign restrictions,  $q_{j^*}^i \neq \tilde{q}_{j^*}^i$ . As before, we consider a path between these two vectors as follows

$$q_{j^*}^i(\delta) = \frac{\delta q_{j^*}^i + (1 - \delta)\tilde{q}_{j^*}^i}{\|\delta q_{j^*}^i + (1 - \delta)\tilde{q}_{j^*}^i\|}, \quad \delta \in [0, 1] \quad (38)$$

which is a continuous path on the unit sphere as the denominator is always different than zero, being  $q_{j^*}^i \neq \tilde{q}_{j^*}^i$ . Now, the path connectedness of  $\mathcal{E}_{j^*}^i(\phi)$  depends on whether it is possible to obtain an admissible set of vectors  $Q^i(\delta) = (q_1^i(\delta), \dots, q_{m_i}^i(\delta))$  whose  $j^*$ -th element is represented by the  $q_{j^*}^i(\delta)$  vector. Conditional on a basis  $(v_1^i, \dots, v_{(n-m_i)}^i)$  for the space  $Q^\perp(\lambda_i)$ , the first  $k$  vectors in  $Q^i(\delta)$ ,  $k = 1, \dots, j^* - 1$  can be obtained through the solutions of the recursive system of equations

$$\begin{pmatrix} F_s^i(\phi) \\ v_1^i \\ \vdots \\ v_{n-m_i} \\ q_1^i(\delta) \\ \vdots \\ q_{k-1}^i(\delta) \\ q_{j^*}^i(\delta) \end{pmatrix} q_k^i(\delta) = 0, \quad \delta \in [0, 1] \quad (39)$$

satisfying the further sign normalization restriction. As the rank of the matrix in the system is at most  $n - m_i + k + f_k^i$ , that is always less than  $n$  because  $f_k^i < m_i - k$ , a solution always exists. The remaining vectors for  $j^* + 1, \dots, m_i$  can be obtained recursively by extending the system in Eq. (39). The set  $\mathcal{E}_{j^*}^i(\phi)$ , thus, is non empty and path connected. The convexity of the impulse response identified set immediately follows.

Concerning point (2) of the theorem, let the zero restriction satisfy condition (3) of Theorem 2, and let  $(q_1^i, \dots, q_k^i)$  be the exactly identified vectors, common to all admissible  $Q(\lambda)$  matrices. As before, we chose two arbitrary vectors  $q_{j^*}^i$  and  $\tilde{q}_{j^*}^i$ , both satisfying the zero, sign normalization and sign restrictions, and obtain the further vector  $\tilde{q}_{j^*}^i(\delta)$  as in Eq. (38). We can thus construct the set  $Q^i(\delta)$ , whose first  $k$  columns are given by  $(q_1^i, \dots, q_k^i)$ . Conditional

on the choice of  $\delta$ , for  $s = k + 1, \dots, j^* - 1$ , we can recursively derive  $q_s^i(\delta)$  by solving

$$\begin{pmatrix} F_s^i(\phi) \\ v_1^i \\ \vdots \\ v_{n-m_i} \\ q_1^i \\ \vdots \\ q_k^i \\ q_{k+1}^i(\delta) \\ \vdots \\ q_{s-1}^i(\delta) \\ q_{j^*}^i(\delta) \end{pmatrix} q_s^i(\delta) = 0, \quad \delta \in [0, 1] \quad (40)$$

where  $q_s^i(\delta)$  satisfies the sign normalization restriction, and where  $(v_1^i, \dots, v_{(n-m_i)}^i)$  is a basis for the space  $Q^\perp(\lambda_i)$ . The system always admits a solution, being the rank of the matrix less than  $n$  by the assumption on the number of zero restrictions on  $q_{k+1}, \dots, q_{j^*-1}^i$ , being  $f_s^i < m_i - s$ , for  $s = 1, \dots, j^* - 1$ . The remaining  $q_{j^*+1}^i(\delta), \dots, q_{m_i}^i(\delta)$  vectors can be recursively derived by extending the system in Eq. (40). Once proved on how to derive  $Q^i(\delta)$  as a function of  $\delta \in [0, 1]$ , the set  $\mathcal{E}_{j^*}^i(\phi)$  is path connected, and the associated impulse response identified set is convex for every variable at any horizon.  $\square$

*Proof of Lemma 1.* The (squared of the) Frobenius norm states that

$$\left\| \Omega - \tilde{\Omega} \right\|_F^2 = \sum_{i,j}^n \left( \Omega_{ij} - \tilde{\Omega}_{ij} \right)^2 = \text{tr} \left[ (\Omega - \tilde{\Omega})' (\Omega - \tilde{\Omega}) \right].$$

However, given the definition of  $\Omega$  and  $\tilde{\Omega}$

$$\Omega = QAQ' \quad \text{and} \quad \tilde{\Omega} = Q\tilde{A}Q',$$

we have that

$$\begin{aligned}
\|\Omega - \tilde{\Omega}\|_F^2 &= \text{tr} \left[ (Q\Lambda Q' - Q\tilde{\Lambda}Q')'(Q\Lambda Q' - Q\tilde{\Lambda}Q') \right] \\
&= \text{tr} \left[ (Q(\Lambda - \tilde{\Lambda})Q')'(Q(\Lambda - \tilde{\Lambda})Q') \right] \\
&= \text{tr} \left[ Q(\Lambda - \tilde{\Lambda})(\Lambda - \tilde{\Lambda})Q' \right] \\
&= \text{tr} \left[ QQ'(\Lambda - \tilde{\Lambda})^2 \right] \\
&= \text{tr} \left[ (\Lambda - \tilde{\Lambda})^2 \right] \\
&= \sum_{h=1}^m (\lambda_h - \tilde{\lambda})^2.
\end{aligned}$$

Clearly, this is minimized when  $\tilde{\lambda} = \frac{1}{m} \sum_{h=1}^m \lambda_h$ , i.e. the mean of the eigenvalues corresponding to those restricted to be equal.  $\square$

## Appendix D Proofs of Theorems on bivariate SVARs and HSVARs

*Proof of Theorem 8.* Given the decomposition of  $A_0$  as in Eq. (26), then

$$\beta = -[A_0]_{(1,2)} / [A_0]_{(1,1)} = -(q'_1 \omega_2) / (q'_1 \omega_1) \quad (41)$$

$$\alpha = -[A_0]_{(2,1)} / [A_0]_{(2,2)} = -(q'_2 \omega_1) / (q'_2 \omega_2) \quad (42)$$

where  $\omega_1$  and  $\omega_2$  are defined as in Eq. (27) while  $q_1$  and  $q_2$  are the two columns of the orthogonal matrix  $Q$ . The proof of the theorem is extremely intuitive when observing the two graphs in Figure B.3 and Figure B.4. Consider the situation of  $\omega_{pq} \geq 0$  (Case I, Figure B.3), first. In both panels we report the observable  $\omega_1$  and  $\omega_2$  vectors, compatible with  $\omega_{pq} \geq 0$ . In the left panel we focus on all the admissible  $\beta$ . First of all, if we look at the definition of  $\beta$  in Eq. (41), it is very simple to obtain the two values of  $q_1$  featuring the extreme values of  $\beta = -\infty$  and  $\beta = \infty$ . In both cases,  $q_1$  must be orthogonal to  $\omega_1$ ; however, in one case the numerator of  $\beta$  is negative (solid line), while in the other the numerator is positive (dotted line). The vector  $q_1$  featuring  $\beta = 0$  has to be orthogonal to  $\omega_2$ . This generates two potential  $q_1$  vectors, i.e.  $q_1 = (0, 1)$  and  $q_1 = (0, -1)$ . The latter, however, has to be discarded given

Assumption 1. It can be deduced, thus, that the admissible  $\beta$  are those generated by the  $q_1$  vectors lying on the right half of the unit circle (light red area) and, without any restriction,  $\beta \in (-\infty, \infty)$ . The arc in red, instead, highlight the feasible  $q_1$  vectors consistent with the sign restriction  $\beta \leq 0$ , as reported in Assumption 2. In the right panel, instead, we focus on the coefficient  $\alpha$ . Even in this case we report the two observable vectors  $\omega_1$  and  $\omega_2$ . Following the same strategy as before, we determine all the admissible vectors  $q_2$  compatible with the  $\alpha$  coefficient. However, keeping in mind that  $q_1$  and  $q_2$  must be orthogonal by construction, it can be remarked that, when  $q_1$  reaches the two extreme values  $\beta = -\infty$  and  $\beta = \infty$ , it can no longer rotate counterclockwise, as it is at odds with Assumption 1. This implies that  $q_2$  can not reach the limit case of  $\alpha = \infty$ . Without any further restriction,  $\alpha \in (-\infty, \omega_q^2/\omega_{pq})$ , where this upper bound of the interval is obtained by substituting in the definition of  $\alpha$  in Eq. (42) the value of  $q_2$  that is orthogonal to  $q_1$  featuring  $\beta = -\infty$  (or, equivalently, the one featuring  $\beta = \infty$ ), i.e.  $q_2 = -\frac{\omega_1}{\|\omega_1\|}$ , being  $\|\omega_1\|$  the Euclidean norm of  $\omega_1$ . Thus, with simple algebra, we obtain

$$\alpha = -(q_2'\omega_1)/(q_2'\omega_2) = -\left(\frac{1}{\|\omega_1\|}(\omega_1'\omega_1)\right) \Big/ \left(\frac{1}{\|\omega_1\|}(\omega_1'\omega_2)\right) = \frac{\omega_q^2}{\omega_{pq}}. \quad (43)$$

If, instead, we consider the sign restrictions in Assumption 2, then the admissible vectors for  $q_1$  and  $q_2$  are indicated in red and green, respectively. Specifically, being  $q_2$  orthogonal to  $q_1$ , it must be in between  $\omega_2$  and  $-\omega_1$  (green arc), providing thus a ‘natural’ restriction on the set of possible  $\alpha$ ’s consistent with the two sign restrictions. In particular, the lower bound of the identified set can be obtained through the definition of  $\alpha$  in Eq. (42) when  $q_2$  is the unitary vector parallel to  $\omega_2$ , i.e.  $q_2 = (0 \ 1)'$ . This leads to

$$\alpha = -(q_2'\omega_1)/(q_2'\omega_2) = -\left((0 \ 1)'\omega_1\right) \Big/ \left((0 \ 1)'\omega_2\right) = \frac{\omega_{pq}}{\omega_p^2}. \quad (44)$$

The second case, when  $\omega_{pq} < 0$ , can be addressed in the same way, but now the two sign restrictions in Assumption 2 induce an identified set for  $\beta$ . In particular, the upper bound of the identified set for  $\beta$  can be obtained by using the definition in Eq. (41), when  $q_1$  is a unitary vector parallel to  $\omega_1$ , while the lower bound can be obtained when  $q_1$  is a unitary vector parallel to  $\omega_2$ . Simple algebra provides the result for Case II in the theorem.  $\square$

*Proof of Corollary 1.* We know that when  $\omega_{pq} \geq 0$  and  $\beta = 0$ , from Eq. (41),  $q'_1 \omega_2 = 0$ . As a consequence, from Figure B.3 (left panel), this implies  $q_2 = (0 \ 1)'$ . Thus, substituting for  $q_2$  in Eq. (eq:alpha) and using the definition of  $\omega_1$  and  $\omega_2$  in Eq. (27) leads to the same result as in Eq. (44), proving thus the first part of the corollary.

When  $\omega_{pq} < 0$  and  $\alpha = 0$ , then  $q'_2 \omega_1 = 0$ . This implies that  $q_1$  will be the unit vector parallel to  $\omega_1$ , i.e.  $q_1 = \frac{\omega_1}{\|\omega_1\|}$ . Substituting in the definition of  $\beta$  in Eq. (41), leads to

$$\beta = - (q'_1 \omega_2) / (q'_1 \omega_1) = - \left( \frac{1}{\|\omega_1\|} (\omega'_1 \omega_2) \right) / \left( \frac{1}{\|\omega_1\|} (\omega'_1 \omega_1) \right) = \frac{\omega_{pq}}{\omega_q^2}. \quad (45)$$

which proves the second part of the corollary.  $\square$

*Proof of Theorem 10.* Based on the definition of  $\Omega_1$  and  $\Omega_2$ , the first step is to calculate the analytical expression for  $\Omega_{i,tr}^{-1} \Omega_2 \Omega_{i,tr}^{-1'}$ , i.e.

$$\begin{aligned} \Omega_{i,tr}^{-1} \Omega_2 \Omega_{i,tr}^{-1'} &= \begin{pmatrix} \frac{1}{\omega_{p,1}} & 0 \\ -\frac{\omega_{pq,1}}{\omega_{p,1}^2} \gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \omega_{p,2}^2 & \omega_{pq,2} \\ \omega_{pq,2} & \omega_{p,2}^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\omega_{p,1}} & -\frac{\omega_{pq,1}}{\omega_{p,1}^2} \gamma_1 \\ 0 & \gamma_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\omega_{p,2}^2}{\omega_{p,1}} & \frac{-\omega_{p,2}^2 \omega_{pq,1} + \omega_{pq,2} \omega_{p,1}^2}{\omega_{p,1}^3} \gamma_1 \\ \frac{-\omega_{p,2}^2 \omega_{pq,1} + \omega_{pq,2} \omega_{p,1}^2}{\omega_{p,1}^3} \gamma_1 & \frac{\omega_{pq,1}^2 \omega_{p,2}^2 - 2\omega_{pq,1} \omega_{pq,2} \omega_{p,1}^2 + \omega_{p,1}^4 \omega_{p,2}^2}{\omega_{p,1}^4} \gamma_1^2 \end{pmatrix} \end{aligned}$$

with  $\gamma_1$  defined as in Eq. (31). The following step is to calculate the eigenvalues of the previous matrix, that, after some algebra, corresponds to find the solutions of the following quadratic equation of the standard form  $a\lambda^2 + b\lambda + c = 0$ :

$$\lambda^2 + \left( \frac{-\omega_{p,1}^2 \omega_{q,2}^2 - \omega_{p,2}^2 \omega_{q,1}^2 + 2\omega_{pq,1} \omega_{pq,2}}{\omega_{p,1}^2 \omega_{q,1}^2 - \omega_{pq,1}^2} \right) \lambda + \left( \frac{\omega_{p,2}^2 \omega_{q,2}^2 - \omega_{pq,2}^2}{\omega_{p,1}^2 \omega_{q,1}^2 - \omega_{pq,1}^2} \right) = 0. \quad (46)$$

In solving the quadratic equation it is crucial to focus on the discriminant  $\Delta = b^2 - 4ac$  of the equation

$$\Delta = \frac{(\omega_{p,1}^2 \omega_{q,2}^2 - \omega_{p,2}^2 \omega_{q,1}^2)^2 + 4(\omega_{p,1}^2 \omega_{pq,2} - \omega_{p,2}^2 \omega_{pq,1})(\omega_{q,1}^2 \omega_{pq,2} - \omega_{q,2}^2 \omega_{pq,1})}{(\omega_{p,1}^2 \omega_{q,1}^2 - \omega_{pq,1}^2)^2}. \quad (47)$$

Given that the original matrix  $\Omega_{i,tr}^{-1} \Omega_2 \Omega_{i,tr}^{-1'}$  is symmetric, then the two eigenvalues are clearly real, and this implies that the discriminant will be not negative. However, if we want the



solutions to be distinct (distinct eigenvalues), then we need to find the conditions for  $\Delta$  to be strictly positive. Firstly, the denominator in Eq. (47) is clearly a real positive number being the square of the determinant of  $\Omega_1$ , that is clearly different from zero. The condition of distinct eigenvalues, thus, has to be found on the positiveness of the numerator of Eq. (47). The first term of the sum is clearly non-negative, and, if we show that the second one cannot be negative, too, then the eigenvalues will be clearly distinct as  $\Delta > 0$ . The non-negativeness of the second term of Eq. (47), with simple algebra, can be seen as:

$$(\omega_{p,1}^2 \omega_{pq,2} - \omega_{p,2}^2 \omega_{pq,1}) (\omega_{q,1}^2 \omega_{pq,2} - \omega_{q,2}^2 \omega_{pq,1}) \geq 0. \quad (48)$$

It immediately emerges that if  $\omega_{pq,1}$  and  $\omega_{pq,2}$  are of different sign, the previous quantity becomes negative. If, instead, they maintain the same sign, we need to consider the two terms separately

$$\omega_{p,1}^2 \omega_{pq,2} - \omega_{p,2}^2 \omega_{pq,1} \geq 0 \iff \frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{p,2}^2}{\omega_{p,1}^2} \quad (49)$$

$$\omega_{q,1}^2 \omega_{pq,2} - \omega_{q,2}^2 \omega_{pq,1} \geq 0 \iff \frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{q,2}^2}{\omega_{q,1}^2}. \quad (50)$$

In order to prove when these quantities are positive, it can be useful to consider the definition of  $\omega_1$  and  $\omega_2$  as a function of the structural parameters contained in  $C = A_0^{-1}$ , i.e.

$$\Omega_1 = CC' = \begin{pmatrix} c_{11}^2 + c_{12}^2 & c_{11}c_{21} + c_{12}c_{22} \\ c_{11}c_{21} + c_{12}c_{22} & c_{21}^2 + c_{22}^2 \end{pmatrix} \quad (51)$$

$$\Omega_2 = CAC' = \begin{pmatrix} c_{11}^2 \lambda_{11} + c_{12}^2 \lambda_{22} & c_{11}c_{21} \lambda_{11} + c_{12}c_{22} \lambda_{22} \\ c_{11}c_{21} \lambda_{11} + c_{12}c_{22} \lambda_{22} & c_{21}^2 \lambda_{11} + c_{22}^2 \lambda_{22} \end{pmatrix} \quad (52)$$

where

$$A = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{\varepsilon_2}^2 / \sigma_{\varepsilon_1}^2 & 0 \\ 0 & \sigma_{\eta_2}^2 / \sigma_{\eta_1}^2 \end{pmatrix}$$

and collects the relative shifts in the variances of the structural shocks across the two regimes.

From these relationships we obtain

$$\begin{aligned}\frac{\omega_{pq,2}}{\omega_{pq,1}} &= \frac{c_{11}c_{21}A_{11} + c_{12}c_{22}A_{22}}{c_{11}c_{21} + c_{12}c_{22}} \\ \frac{\omega_{p,2}^2}{\omega_{p,1}^2} &= \frac{c_{11}^2A_{11} + c_{12}^2\lambda_{22}}{c_{11}^2 + c_{12}^2} \\ \frac{\omega_{q,2}^2}{\omega_{q,1}^2} &= \frac{c_{21}^2A_{11} + c_{22}^2A_{22}}{c_{21}^2 + c_{22}^2}\end{aligned}$$

that allow to investigate the previous inequalities in Eq.s (49)-(50) as follows

$$\begin{aligned}\frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{p,2}^2}{\omega_{p,1}^2} &\iff \frac{c_{11}c_{21}A_{11} + c_{12}c_{22}A_{22}}{c_{11}c_{21} + c_{12}c_{22}} \geq \frac{c_{11}^2A_{11} + c_{12}^2\lambda_{22}}{c_{11}^2 + c_{12}^2} \\ &\iff c_{12}c_{11}(c_{12}c_{21} - c_{11}c_{22})(\lambda_{11} - \lambda_{22}) \geq 0\end{aligned}\quad (53)$$

$$\begin{aligned}\frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{q,2}^2}{\omega_{q,1}^2} &\iff \frac{c_{11}c_{21}A_{11} + c_{12}c_{22}A_{22}}{c_{11}c_{21} + c_{12}c_{22}} \geq \frac{c_{21}^2A_{11} + c_{22}^2A_{22}}{c_{21}^2 + c_{22}^2} \\ &\iff c_{21}c_{22}(c_{11}c_{22} - c_{12}c_{21})(\lambda_{11} - \lambda_{22}) \geq 0.\end{aligned}\quad (54)$$

Given that  $c_{11}$  and  $c_{22}$  are positive by construction, the system of inequalities becomes

$$c_{12}(c_{12}c_{21} - c_{11}c_{22})(\lambda_{11} - \lambda_{22}) \geq 0 \quad (55)$$

$$c_{21}(c_{11}c_{22} - c_{12}c_{21})(\lambda_{11} - \lambda_{22}) \geq 0. \quad (56)$$

At this point it is important to remember that, from the definition of  $C = A_0^{-1}$ ,  $c_{12} \geq 0$  and  $c_{21} \leq 0$ , due to the sign restrictions on  $\alpha$  and  $\beta$ . The previous inequalities, thus, are either always jointly satisfied or jointly never, depending on the sign of  $(\lambda_{11} - \lambda_{22})$ . This result shows that the inequality in Eq. (48) is always satisfied and thus, being the two addends of the discriminant in Eq. (47) both non-negative, the only possibility we have to exclude for having distinct eigenvalues is when both of them are null, i.e. we have the following system of equations:

$$\omega_{p,1}^2\omega_{q,2}^2 - \omega_{p,2}^2\omega_{q,1}^2 = 0 \quad (57)$$

$$(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1})(\omega_{q,1}^2\omega_{pq,2} - \omega_{q,2}^2\omega_{pq,1}) = 0. \quad (58)$$

for which the solutions are:

$$\frac{\omega_{p,1}^2}{\omega_{p,2}^2} = \frac{\omega_{q,1}^2}{\omega_{q,2}^2}, \quad \frac{\omega_{p,1}^2}{\omega_{p,2}^2} = \frac{\omega_{pq,1}}{\omega_{pq,2}}, \quad \frac{\omega_{pq,1}}{\omega_{pq,2}} = \frac{\omega_{q,1}^2}{\omega_{q,2}^2} \quad (59)$$

that corresponds to the case  $\Omega_1 = a\Omega_2$ , that has been excluded in the theorem.

The eigenvectors, instead, can be calculated from the two systems

$$(\Omega_{i,tr}^{-1}\Omega_2\Omega_{i,tr}^{-1'} - I_2\lambda_i) q_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = \{1, 2\} \quad (60)$$

where  $\lambda_i$  is the  $i$ -th eigenvalue and  $q_i$  is the  $i$ -th eigenvector. Tedious algebra, not reported here to save space, but available from the authors upon request, proves the following result:

$$q_1 = \begin{pmatrix} \frac{\Delta\omega_{p,1}^2 + D_1}{2D_2[(\Delta\omega_{p,1}^2 + D_1)^2 / (2D_3^2\Delta^2) + 1]^{1/2} D_3} \\ \frac{1}{[(\Delta\omega_{p,1}^2 + D_1)^2 / (2D_3^2\Delta^2) + 1]^{1/2}} \end{pmatrix}, \quad q_2 = \begin{pmatrix} \frac{-\Delta\omega_{p,1}^2 + D_1}{2D_2[(\Delta\omega_{p,1}^2 - D_1)^2 / (2D_3^2\Delta^2) + 1]^{1/2} D_3} \\ \frac{1}{[(\Delta\omega_{p,1}^2 - D_1)^2 / (2D_3^2\Delta^2) + 1]^{1/2}} \end{pmatrix}. \quad (61)$$

where

$$\begin{aligned} \Delta &= \left[ (\omega_{p,1}^2\omega_{q,2}^2 - \omega_{p,2}^2\omega_{q,1}^2)^2 + 4(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1})(\omega_{q,1}^2\omega_{pq,2} - \omega_{q,2}^2\omega_{pq,1}) \right]^{1/2} \\ D_1 &= -2\omega_{p,2}^2\omega_{pq,1} - \omega_{p,1}^4\omega_{q,2}^2 + 2\omega_{p,1}^2\omega_{pq,1}\omega_{pq,2} + \omega_{p,1}^2\omega_{p,2}^2\omega_{q,1}^2 \\ D_2 &= (\omega_{p,1}^2\omega_{q,1}^2 - \omega_{pq,1}^2)^{1/2} \\ D_3 &= (\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1}). \end{aligned}$$

The  $q_1$  and  $q_2$  unit vectors form the columns of the orthogonal matrix  $Q = (q_1, q_2)$  such that  $C = \Omega_{1,tr}Q$ .

If all the  $\lambda$ s are equal, the relation  $Q'\Omega_{1tr}^{-1}\omega_{2tr}\Omega'_{2tr}\Omega_{1tr}^{-1'}Q = \Lambda$  will become

$$\begin{aligned} \Omega_{1tr}^{-1}\Omega_{2tr}\Omega'_{2tr}\Omega_{1tr}^{-1'} &= \lambda QQ' \\ \Rightarrow \Omega_{1tr}^{-1}\Omega_{2tr}\Omega'_{2tr}\Omega_{1tr}^{-1'} &= \lambda I_n \\ \Rightarrow \Omega_{2tr}\Omega'_{2tr} &= \lambda\Omega_{1tr}\Omega'_{1tr} \\ \Rightarrow \Omega_2 &= \lambda\Omega_1, \end{aligned}$$

which implies that the condition for identification fails. In other words, the two covariance matrices, once rescaled for the factor  $\lambda$ , contain the same amount of information. While  $\lambda$  can be used for estimating the variance of the structural shocks, the remaining part of information will be used for estimating the parameters of the conditional expected value of the structural form, i.e.  $A_0$ . However, this amount of information is the same as in standard bivariate SVARs, indicating that the results of Theorem 8 can be applied. This completes the proof.  $\square$

## Appendix E Frequentist and Bayesian estimators for reduced-form HVAR

We present three estimators for the parameters of the HVAR: two frequentist ones and a Bayesian one. This last is at the heart of our procedure for making inference in the case of set identification.

### E.1 GLS and ML estimators

Let the  $nm \times 1$  vector of parameters  $\phi_B = \text{vec}(B)$  and the  $n \times T$  matrices  $Y = [y_1, y_2 \dots, y_T]$  containing the data, and  $U = [u_1, u_1 \dots, u_t]$  containing the error terms. We can define  $y = \text{vec}(Y)$  and  $u = \text{vec}(U)$ . Now, the presence of volatility clusters allows to write

$$V(U) = \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix} \quad (62)$$

where  $T_1 = T_B$  and  $T_2 = T - T_B$ . Given the initial observations  $y_{-l+1}, \dots, y_0$ , the  $m \times T$  matrix  $X = [x_1, \dots, x_t \dots, x_T]$ , with  $x_t = (1, y'_{t-1}, \dots, y'_{t-l})'$ .

Given these definitions, the reduced-form HVAR in Eq. (3) can be written as

$$y = (X' \otimes I_n)\phi_B + u \quad \text{or} \quad Y = BX + U. \quad (63)$$

These compact notations, as well as a suitable partitioning of  $y$  and  $X$  as follows

$$y = \begin{pmatrix} y_1 \\ nT_1 \times 1 \\ y_2 \\ nT_2 \times 1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_1 & X_2 \\ m \times T_1 & m \times T_2 \end{pmatrix} \quad (64)$$

allow to define a feasible generalized least squares (GLS) estimator. In particular, using the well known formula for the GLS estimator

$$\hat{\phi}_{B,GLS} = \left( (X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} (X' \otimes I_n) \right)^{-1} (X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y. \quad (65)$$

and according to the partitioning of  $y$  and  $X$  as given in Eq. (64), the formula for the GLS estimator becomes:

$$\hat{\phi}_{B,GLS} = \left[ (X_1 X_1' \otimes \hat{\Omega}_1^{-1}) + (X_2 X_2' \otimes \hat{\Omega}_2^{-1}) \right]^{-1} \left[ (X_1 \otimes \hat{\Omega}_1^{-1}) y_1 + (X_2 \otimes \hat{\Omega}_2^{-1}) y_2 \right] \quad (66)$$

where  $\hat{\Omega}_i$ ,  $i = \{1, 2\}$ , is the covariance matrix of the residuals when Eq. (63) is estimated with equation-wise ordinary least squares in a first step.

Moreover, apart from a constant term, and conditional on the initial observations  $y_{-l+1}, \dots, y_0$ , the reduced-form Gaussian likelihood function can be written as

$$L(Y|\phi_B, \Omega_1, \Omega_2) \propto |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} \exp \left\{ -\frac{1}{2} [y - (X' \otimes I_n)\phi_B]' \dots \right. \\ \left. \dots \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} [y - (X' \otimes I_n)\phi_B] \right\} \quad (67)$$

If the data generating process is Gaussian, maximizing  $L(Y|\phi_B, \Omega_1, \Omega_2)$  with respect to the parameters  $(\phi_B, \Omega_1, \Omega_2)$  gives the maximum likelihood (ML) estimators. Instead, if departures from gaussianity do arise, the resulting estimators are quasi-ML estimators.

The Gaussian likelihood function reported in Eq. (67) can be transformed in a more

convenient way to derive the posterior distributions discussed in the next section. The term in the exponent can be written as

$$y' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y - 2\phi'_B(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y + \phi'_B(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} (X' \otimes I_n)\phi_B.$$

Using the mentioned partitioning of the  $X$  and  $y$  allows to write the addends as follows

$$\begin{aligned} y' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y &= y_1(I_{T_1} \otimes \Omega_1)^{-1}y_1 + y_2(I_{T_2} \otimes \Omega_2)^{-1}y_2 \\ \phi'_B(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y &= \phi'_B(X'_1 \otimes \Omega_1^{-1})'y_1 + \phi'_B(X'_2 \otimes \Omega_2^{-1})'y_2 \\ \phi'_B(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} (X' \otimes I_n)\phi_B &= \phi'_B(X_1X'_1 \otimes \Omega_1^{-1})\phi_B + \phi'_B(X_2X'_2 \otimes \Omega_2^{-1})\phi_B. \end{aligned}$$

Thus, an alternative expression for the likelihood function becomes

$$\begin{aligned} L(Y|\phi_B, \Omega_1, \Omega_2) &\propto |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} \exp \left\{ -\frac{1}{2} [y'_1(I_{T_1} \otimes \Omega_1)^{-1}y_1 + y'_2(I_{T_2} \otimes \Omega_2)^{-1}y_2 + \right. \\ &\quad \left. -2\phi'_B(X'_1 \otimes \Omega_1^{-1})'y_1 - 2\phi'_B(X'_2 \otimes \Omega_2^{-1})'y_2 + \right. \\ &\quad \left. + \phi'_B(X_1X'_1 \otimes \Omega_1^{-1})\phi_B + \phi'_B(X_2X'_2 \otimes \Omega_2^{-1})\phi_B] \right\} \end{aligned} \quad (68)$$

## E.2 Bayesian estimators

Combining the likelihood function in Eq. (68) with the following Normal and inverse Wishart priors for  $\phi_B$ ,  $\Omega_1$  and  $\Omega_2$

$$\begin{aligned} \phi_B &\sim \mathcal{N}(\mu_\phi, V_\phi) \\ \Omega_1 &\sim i\mathcal{W}(S_1, d_1) \\ \Omega_2 &\sim i\mathcal{W}(S_2, d_2) \end{aligned}$$

allows to obtain the following posterior distributions for  $\phi_B$ ,  $\Omega_1$  and  $\Omega_2$

$$\begin{aligned}
P(\phi_B, \Omega_1, \Omega_2 | Y) &\propto |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} |\Omega_1|^{-\frac{d_1+n+1}{2}} |\Omega_2|^{-\frac{d_2+n+1}{2}} \\
&\exp \left\{ -\frac{1}{2} [y - (X' \otimes I_n) \phi_B]' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} [y - (X' \otimes I_n) \phi_B] \right\} \\
&\exp \left\{ -\frac{1}{2} \text{tr} [\Omega_1^{-1} S_1] \right\} \exp \left\{ -\frac{1}{2} \text{tr} [\Omega_2^{-1} S_2] \right\}. \tag{69}
\end{aligned}$$

This joint distribution is not of a known form and drawing directly from it is very hard. However, given the conditional distributions for  $\phi_B$  given  $\Omega_1$  and  $\Omega_2$ , and those of  $\Omega_1$  and  $\Omega_2$  given  $\phi_B$ , derived in the following subsections, we can explore the posterior joint distribution by using a Gibbs sampler.

### E.2.1 Case I) Inference on $\phi_B$ with $\Omega_1$ and $\Omega_2$ known

If  $\Omega_1$  and  $\Omega_2$  are known parameters, the kernel of the likelihood that is relevant for  $\phi_B$  is

$$\begin{aligned}
L(Y | \phi_B) &\propto \exp \left\{ -\frac{1}{2} \left[ -2\phi_B'(X_1' \otimes \Omega_1^{-1})' y_1 - 2\phi_B'(X_2' \otimes \Omega_2^{-1})' y_2 + \right. \right. \\
&\quad \left. \left. + \phi_B'(X_1 X_1' \otimes \Omega_1^{-1}) \phi_B + \phi_B'(X_2 X_2' \otimes \Omega_2^{-1}) \phi_B \right] \right\}.
\end{aligned}$$

As a prior distribution for  $\phi_B$  we can use

$$\phi_B \sim \mathcal{N}(\mu_\phi, V_\phi),$$

where  $P(\phi_B) \propto \exp \left\{ -\frac{1}{2} [(\phi_B - \mu_\phi)' V_\phi^{-1} (\phi_B - \mu_\phi)] \right\}$ , with the argument of the exponential function that can be written as

$$[(\phi_B - \mu_\phi)' V_\phi^{-1} (\phi_B - \mu_\phi)] = \phi_B' V_\phi^{-1} \phi_B - 2\phi_B' V_\phi^{-1} \mu_\phi + \mu_\phi' V_\phi^{-1} \mu_\phi,$$

where the last addend of the sum is not informative about  $\phi_B$ .

The posterior distribution, thus, can be written as

$$P(\phi_B|Y) \propto \exp \left\{ -\frac{1}{2} \left[ -2\phi'_B(X'_1 \otimes \Omega_1^{-1})'y_1 - 2\phi'_B(X'_2 \otimes \Omega_2^{-1})'y_2 + \right. \right. \\ \left. \left. + \phi'_B(X_1X'_1 \otimes \Omega_1^{-1})\phi_B + \phi'_B(X_2X'_2 \otimes \Omega_2^{-1})\phi_B + \right. \right. \\ \left. \left. + \phi'_B V_\phi^{-1} \phi_B - 2\phi'_B V_\phi^{-1} \mu_\phi \right] \right\}.$$

However, it is possible to show that:

$$\begin{aligned} \phi'_B(X_1X'_1 \otimes \Omega_1^{-1})\phi_B + \phi'_B(X_2X'_2 \otimes \Omega_2^{-1})\phi_B + \phi'_B V_\phi^{-1} \phi_B &= \phi'_B \underbrace{\left[ (X_1X'_1 \otimes \Omega_1^{-1}) + (X_2X'_2 \otimes \Omega_2^{-1}) + V_\phi^{-1} \right]}_{V_\phi^{*-1}} \phi_B \\ &= \phi'_B V_\phi^{*-1} \phi_B. \end{aligned}$$

Moreover:

$$\begin{aligned} &-2\phi'_B(X'_1 \otimes \Omega_1^{-1})'y_1 - 2\phi'_B(X'_2 \otimes \Omega_2^{-1})'y_2 - 2\phi'_B V_\phi^{-1} \mu_\phi \\ &= -2\phi'_B \left[ (X'_1 \otimes \Omega_1^{-1})'y_1 + (X'_2 \otimes \Omega_2^{-1})'y_2 + V_\phi^{-1} \mu_\phi \right] \\ &= -2\phi'_B V_\phi^{*-1} \underbrace{V_\phi^* \left[ (X'_1 \otimes \Omega_1^{-1})'y_1 + (X'_2 \otimes \Omega_2^{-1})'y_2 + V_\phi^{-1} \mu_\phi \right]}_{\mu_\phi^*} \\ &= -2\phi'_B V_\phi^{*-1} \underbrace{\left[ (X_1X'_1 \otimes \Omega_1^{-1}) + (X_2X'_2 \otimes \Omega_2^{-1}) + V_\phi^{-1} \right]^{-1} \left[ (X'_1 \otimes \Omega_1^{-1})'y_1 + (X'_2 \otimes \Omega_2^{-1})'y_2 + V_\phi^{-1} \mu_\phi \right]}_{\mu_\phi^*} \\ &= -2\phi'_B V_\phi^{*-1} \mu_\phi^*. \end{aligned}$$

If we add the term  $\mu_\phi^{*'} V_\phi^{*-1} \mu_\phi^*$ , that is however not informative for the parameter  $\phi$ , the posterior is proportional to a Normal distribution

$$\phi_B|Y \sim \mathcal{N}(\mu_\phi^*, V_\phi^*)$$

where

$$\begin{aligned} V_\phi^* &= \left[ (X_1X'_1 \otimes \Omega_1^{-1}) + (X_2X'_2 \otimes \Omega_2^{-1}) + V_\phi^{-1} \right]^{-1} \\ \mu_\phi^* &= V_\phi^* (V_\phi^{-1} \mu_\phi + (X'_1 \otimes \Omega_1^{-1})'y_1 + (X'_2 \otimes \Omega_2^{-1})'y_2). \end{aligned}$$

### E.2.2 Case II) Inference on $\Omega_1$ and $\Omega_2$ with $\phi_B$ known

In the literature it is quite common to use inverse Wishart priors for covariance matrices. In our case we follow this approach and proceed in the same way for each of the two covariance



matrices  $\Omega_1$  and  $\Omega_2$ :

$$\begin{aligned}\Omega_1 &\sim i\mathcal{W}(S_1, d_1) \\ \Omega_2 &\sim i\mathcal{W}(S_2, d_2)\end{aligned}\tag{70}$$

with  $P(\Omega_i) \propto |\Omega_i|^{-\frac{d_i+n+1}{2}} \exp\left\{-\frac{1}{2}\text{tr}[\Omega_i^{-1}S_i]\right\}$ , for  $i = 1, 2$ , and where  $E(\Omega_i) = \frac{S_i}{d_i-n-1}$ .

The first step consists in re-writing the likelihood function in a more convenient way. If the underlying model is written as

$$y = (X' \otimes I_n)\phi_B + u$$

the likelihood function is as in Eq. (67) before. The exponent can be re-written as

$$\begin{aligned}&[y_1(I_{T_1} \otimes \Omega_1^{-1})y_1 + \phi'_B(X_1X_1') \otimes \Omega_1^{-1})\phi_B - y_1(X_1' \otimes \Omega_1^{-1})\phi_B - \phi'_B(X_1 \otimes \Omega_1^{-1})y_1] \\ &[y_2(I_{T_2} \otimes \Omega_2^{-1})y_2 + \phi'_B(X_2X_2') \otimes \Omega_2^{-1})\phi_B - y_2(X_2' \otimes \Omega_2^{-1})\phi_B - \phi'_B(X_2 \otimes \Omega_2^{-1})y_2].\end{aligned}\tag{71}$$

Using simple properties of the trace and vec operators, the first part under brackets can be written as

$$\begin{aligned}y_1(I_{T_1} \otimes \Omega_1^{-1})y_1 &= \text{tr}\{Y_1'\Omega_1^{-1}Y_1\} = \text{tr}\{\Omega_1^{-1}Y_1Y_1'\} \\ \phi'_B(X_1X_1' \otimes \Omega_1^{-1})\phi_B &= \text{tr}\{B'\Omega_1^{-1}BX_1X_1'\} = \text{tr}\{\Omega_1^{-1}BX_1X_1'B'\} \\ y_1(X_1' \otimes \Omega_1^{-1})\phi_B &= \text{tr}\{Y_1'\Omega_1^{-1}BX_1\} = \text{tr}\{\Omega_1^{-1}BX_1Y_1'\} \\ \phi'_B(X_1 \otimes \Omega_1^{-1})y_1 &= \text{tr}\{B'\Omega_1^{-1}Y_1X_1'\} = \text{tr}\{\Omega_1^{-1}Y_1X_1'B\}\end{aligned}$$

where we have used the decomposition

$$Y_{n \times T} = \begin{bmatrix} Y_1 & Y_2 \\ n \times T_1 & n \times T_2 \end{bmatrix}$$

and the fact that  $\phi_B = \text{vec}(B)$ . Obviously, these transformations can be replicated for the second part under the brackets of the exponent of the likelihood function. Definitely, the two exponents in Eq. (71) become

$$\begin{aligned}&\text{tr}[\Omega_1^{-1}(Y_1Y_1' + BX_1X_1'B' - BX_1Y_1' - Y_1X_1'B)] + \text{tr}[\Omega_2^{-1}(Y_2Y_2' + BX_2X_2'B' - BX_2Y_2' - Y_2X_2'B)] \\ &= \text{tr}[\Omega_1^{-1}(Y_1 - BX_1)(Y_1 - BX_1)'] + \text{tr}[\Omega_2^{-1}(Y_2 - BX_2)(Y_2 - BX_2)'].\end{aligned}$$

Now, if we combine the likelihood function with the two priors in Eq. (70) it becomes rather simple to derive the posterior distributions for the two variables  $\Omega_1$  and  $\Omega_2$ . Overall, the joint posterior distribution for  $\Omega_1$  and  $\Omega_2$  can be written as

$$\begin{aligned}
P(\Omega_1, \Omega_2|Y) &\propto P(\Omega_1) P(\Omega_2) P(Y|\Omega_1, \Omega_2) \\
&= |\Omega_1|^{-\frac{d_1+n+1}{2}} |\Omega_2|^{-\frac{d_2+n+1}{2}} |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} \\
&\quad \exp\left\{-\frac{1}{2}\text{tr}(\Omega_1^{-1}S_1)\right\} \exp\left\{-\frac{1}{2}\text{tr}(\Omega_2^{-1}S_2)\right\} \\
&\quad \exp\left\{-\frac{1}{2}\text{tr}[\Omega_1^{-1}(Y_1 - BX_1)(Y_1 - BX_1)']\right\} \\
&\quad \exp\left\{-\frac{1}{2}\text{tr}[\Omega_2^{-1}(Y_2 - BX_2)(Y_2 - BX_2)']\right\}. \tag{72}
\end{aligned}$$

However, focusing on the posterior distribution of each of the two covariance matrices, we obtain that

$$\begin{aligned}
P(\Omega_1|Y) &\propto |\Omega_1|^{-\frac{T_1+d_1+n+1}{2}} \exp\left\{-\frac{1}{2}\text{tr}[\Omega_1^{-1}(S_1 + (Y_1 - BX_1)(Y_1 - BX_1)')]\right\} \\
P(\Omega_2|Y) &\propto |\Omega_2|^{-\frac{T_2+d_2+n+1}{2}} \exp\left\{-\frac{1}{2}\text{tr}[\Omega_2^{-1}(S_2 + (Y_2 - BX_2)(Y_2 - BX_2)')]\right\},
\end{aligned}$$

or, more compactly

$$\begin{aligned}
\Omega_1 &\sim i\mathcal{W}(S_1^*, d_1^*) \\
\Omega_2 &\sim i\mathcal{W}(S_2^*, d_2^*),
\end{aligned}$$

where

$$\begin{aligned}
S_1^* &= S_1 + (Y_1 - BX_1)(Y_1 - BX_1)' = S_1 + \hat{\Omega}_{1,OLS}(T_1 - nm) \\
S_2^* &= S_2 + (Y_2 - BX_2)(Y_2 - BX_2)' = S_2 + \hat{\Omega}_{2,OLS}(T_2 - nm) \\
d_1^* &= T_1 + d_1 \\
d_2^* &= T_2 + d_2
\end{aligned}$$

and where

$$\hat{\Omega}_{i,OLS} = (Y_i - BX_i)(Y_i - BX_i)' / (T_i - nm), \quad \text{with } i = \{1, 2\}. \tag{73}$$

# Appendix F The test for identification via heteroskedasticity of Lütkepohl et al. (2020)

Lütkepohl et al. (2020) develop their test for identification via heteroskedasticity under the assumption that reduced-form error terms  $u_t$  have an elliptically symmetric distribution with density  $(\sqrt{\det \Omega_m})^{-1} g(u_t' \Omega_m^{-1} u_t)$  where  $\Omega_m$  is the covariance matrix in regime  $m = 1, 2$ ,  $g(\cdot)$  is positive function such that the density integrates to one and the fourth moments of the distribution exist. A characteristic of elliptical distributions is that to impose the same kurtosis parameter for all the  $n$  elements of  $u_t$ . Formally, if we denote with  $\omega_{im}^2$  the  $i$ -th diagonal element of  $\Omega_m$ , the kurtosis parameter  $\kappa_m = [E(u_{it}^4)/3\omega_{im}^4] - 1$  is the same for all  $i = 1, \dots, n$  but can be different for different volatility regimes,  $m$ .

To implement the test of Lütkepohl et al. (2020), estimates of the kurtosis parameters are obtained as follows:

$$\hat{\kappa}_m = \frac{1}{3n} \sum_{i=1}^n \frac{z_i^m}{w_i^m} - 1, \quad m = 1, 2$$

with

$$z_i^m = \frac{\sum_{t \in T_m} (\hat{u}_{it} - \bar{u}_i^m)^4 - 6\hat{\omega}_i^4}{T_m - 4} \quad \text{and} \quad w_i^m = \frac{T_m}{T_m - 1} \left( \hat{\omega}_i^4 - \frac{z_i^m}{T_m} \right) \quad m = 1, 2$$

where  $\bar{u}_i = T_m^{-1} \sum_{t \in T_m} \hat{u}_{it}$  is the sample average of reduced-form residuals,  $\hat{u}_{it}^m$ , for the  $m$ -th regime,  $T_1 = 1, \dots, T_B$  and  $T_2 = T_B + 1, \dots, T$ .

Denoting the estimated eigenvalues – ordered from largest to smallest – as  $\hat{\lambda}_i$  for  $i = 1, \dots, n$  we write the test statistic as:

$$\begin{aligned} H_r(\hat{\kappa}_1, \hat{\kappa}_2) &= -c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2 Tr \log \left( \frac{\prod_{k=s+1}^{s+r} \hat{\lambda}_k^{1/r}}{\frac{1}{r} \sum_{k=s+1}^{s+r} \hat{\lambda}_k} \right) \\ &= -c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2 \left[ T \sum_{k=s+1}^{s+r} \log \hat{\lambda}_k - Tr \log \left( \frac{1}{r} \sum_{k=s+1}^{s+r} \hat{\lambda}_k \right) \right] \end{aligned}$$

with  $s = 0, \dots, n - 1$  and  $r = 2, \dots, n - s$ . Note that the first line of the equation highlights that the statistic is based on the ratio of the geometric mean to the arithmetic mean of the estimators of the eigenvalues assumed to be identical under the null. The term  $c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2$  is

defined as follows:

$$c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2 = \left( \frac{1 + \hat{\kappa}_1}{\tau} + \frac{1 + \hat{\kappa}_2}{1 - \tau} \right)^{-1} \quad \text{with } \tau \equiv T_B/T$$

Note that the fraction  $\tau$  is assumed to be known and fixed. The test statistic converges in distribution to a  $\chi^2((r+2)(r-1)/2)$  and involves the following pair of hypotheses:

$$H_0 : \lambda_{s+1} = \lambda_{s+2} = \dots = \lambda_{s+r} \text{ against } H_1 : \neg H_0$$

where “ $\neg$ ” denotes negation.

Let us consider the case of testing identification via heteroskedasticity with  $n = 3$  variables. Then we rely on  $H_3(\hat{\kappa}_1, \hat{\kappa}_2)$  with a  $\chi^2(5)$  distribution to test:  $H_0 : \lambda_1 = \lambda_2 = \lambda_3$ . If the null is rejected we test  $H_0 : \lambda_1 = \lambda_2$  and  $H_0 : \lambda_2 = \lambda_3$  using  $H_2(\hat{\kappa}_1, \hat{\kappa}_2)$  with a  $\chi^2(2)$  distribution. If also these hypotheses are rejected, the SVAR model is fully identified via heteroskedasticity.

## Appendix G Empirical application – Further details and results

### G.1 Data

The data entering the VAR model in Section V are the following:

- $\Delta prod_t$  is percent change in world crude oil production and is defined as  $100 \times \ln(prod_t/prod_{t-1})$ . World oil production,  $prod_t$ , is sourced from the Monthly Energy Review maintained by the U.S. Energy Information Administration.
- The index of real economic activity,  $rea_t$ , is based on dry cargo ocean shipping rates and is available on the website of Lutz Kilian. It is used to proxy monthly changes in the world demand for industrial commodities, including crude oil.
- The real price of crude oil,  $rpo_t$ , is the refiner’s acquisition cost of imported crude oil and it is available from the U.S. EIA. Deflation is carried out using the CPI for All Urban Consumers, as reported by the Bureau of Labor Statistics.

The time series included in the VAR and the reduced-form residuals of the VAR(6) are shown in Figures G.6 and G.7 that also displays a vertical bar in correspondence of the break date, October 1987.

Figure G.6: Data used in the SVAR model for the global market of crude oil (January 1973-December 2007)

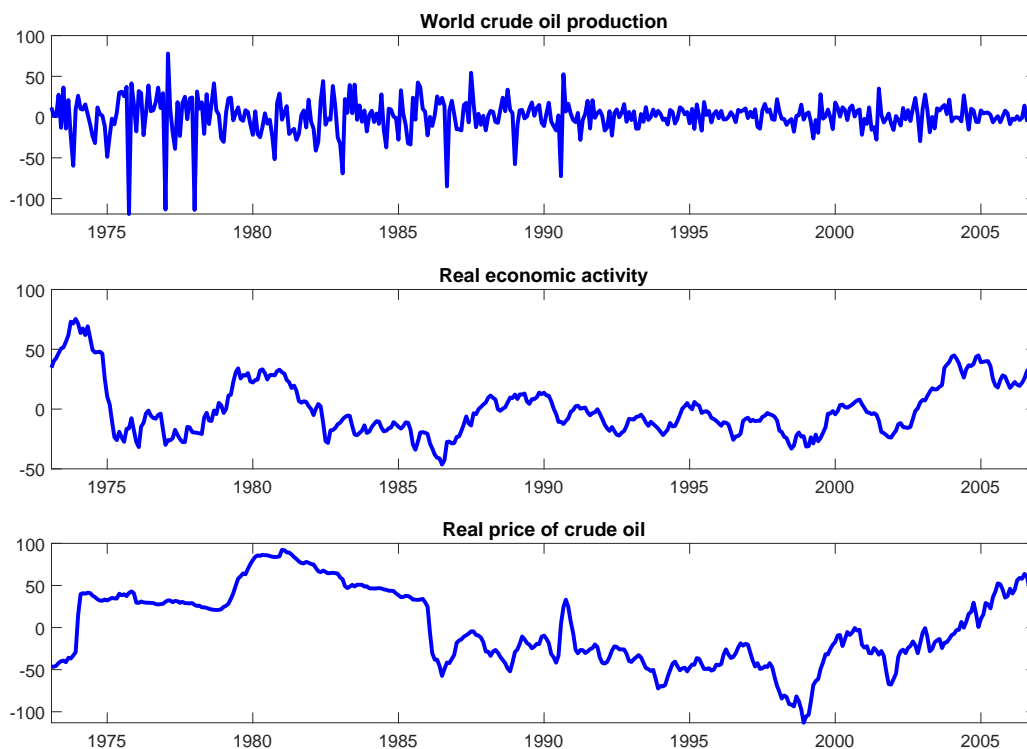


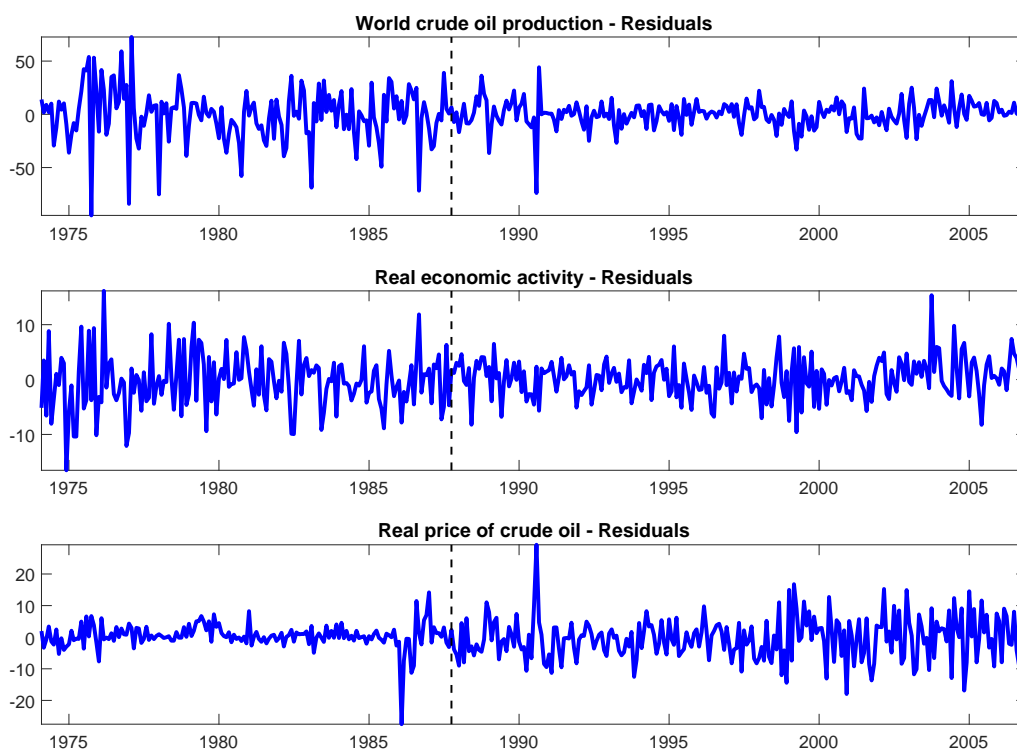
Table 2(a) shows the estimated eigenvalues and their standard errors. Recall that the variances of structural shocks are normalized to unity before the break and hence estimates in Table 2(a) represent the change in variances from the first to the second volatility regime. We see that the volatility of the structural shock associated with the first eigenvalue is larger after the break, while the remaining structural shocks have relative variances lower than unity in the second regime.

Table 2(b) illustrates that the test for identification through heteroskedasticity of Lütkepohl et al. (2020) does not allow to reject the null hypothesis  $H_0 : \lambda_2 = \lambda_3$ . Eigenvalue multiplicity implies that standard identification through heteroskedasticity, presented in Theorem 5, fails.

Figure G.8 shows impulse responses<sup>18</sup> for the recursively identified model of Kilian (2009),  $\mathcal{M}_0$ , along with the highest posterior density (HPD) region with credibility 68%. An oil supply

<sup>18</sup>As for the estimation, we rely on a noninformative improper Jeffreys' prior that allows to draw reduced-form parameters from a normal-inverse-Wishart posterior.

Figure G.7: Reduced form residuals and break date



*Notes:* Reduced form residuals and time of the break (1st October 1987). Monthly data.

shock causes an immediate and long-lasting decline in global oil production, a decrease in real economic activity and a transitory increase in the real price of crude oil that peaks six months after the shock. Notice that the 68% HPD region of the price response does not include zero only for the first eight months. A shock boosting aggregate demand causes a small temporary increase in global oil production and large and persistent increase in the index of real economic activity and in the price of crude oil. For the latter two responses the 68% HPD region never contains zero. An unexpected rise in oil-specific demand generates a long-lasting increase in the real price of crude oil and a temporary jump in the index of real economic activity. Lastly, an oil market demand shock causes a small and only transitory positive effect on global oil production. Notice that in this case the 68% HPD always includes zero.

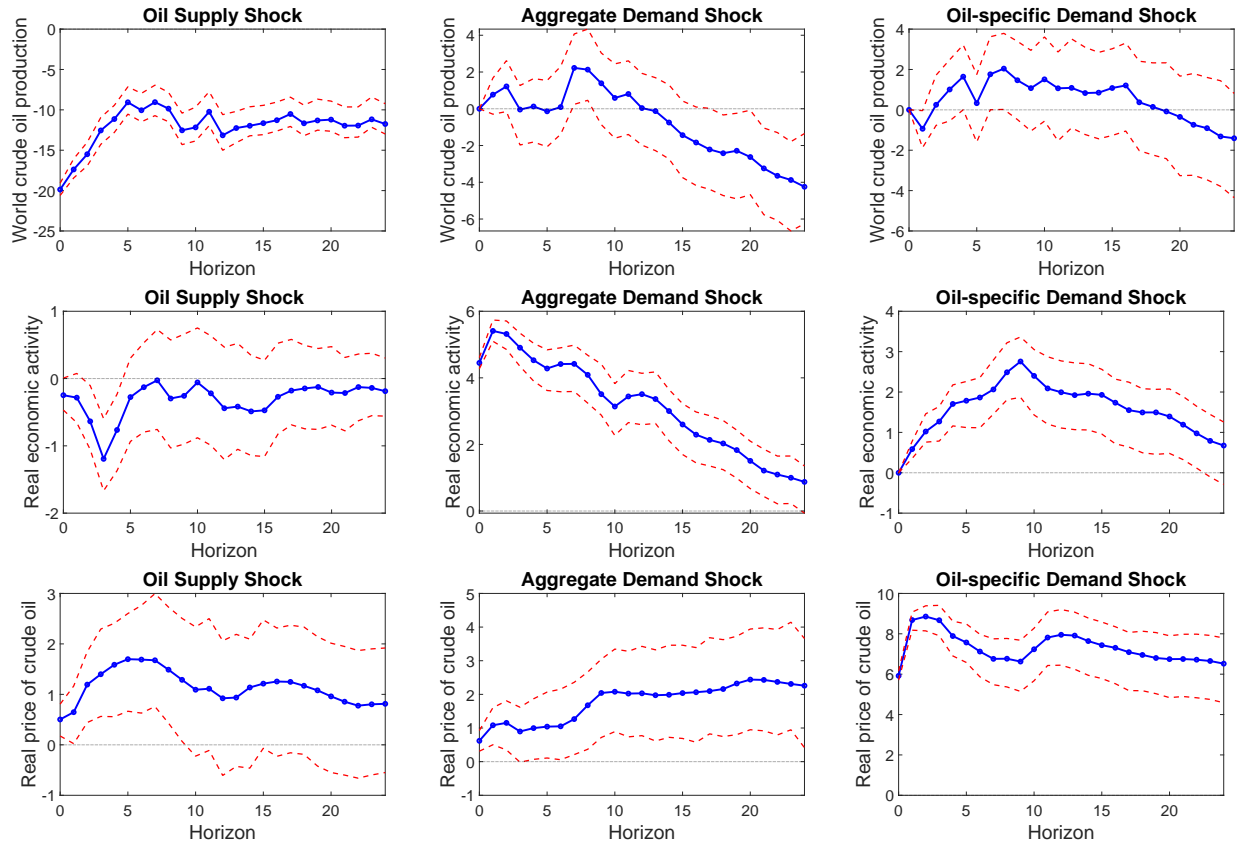
Model  $\mathcal{M}_1$  assumes that all eigenvalues are distinct and hence we estimate the reduced form of the model with the Gibbs sampler discussed in Appendix E. The last column of Figure G.9 shows the responses to the shock that is associated with the only distinct eigenvalue. Comparing the shape of these responses to the one in Figure G.8, we see that they are consistent with those following an oil-specific demand shock. Impulse responses in the second and

Table 2: Estimated eigenvalues and tests for identification through heteroskedasticity

Panel (a). Estimated eigenvalues			
$\hat{\lambda}_1$		3.712	(1.032)
$\hat{\lambda}_2$		0.341	(0.095)
$\hat{\lambda}_3$		0.159	(0.046)
Panel (b). Tests for identification through heteroskedasticity			
$H_0$	$H_r(\hat{\kappa}_1, \hat{\kappa}_2)$	Degrees of freedom ( $r$ )	$p$ -value
$\lambda_1 = \lambda_2 = \lambda_3$	79.166	5	0.0000
$\lambda_1 = \lambda_2$	35.569	2	0.0000
$\lambda_2 = \lambda_3$	4.2758	2	0.1179

Notes: Panel (a) shows the estimated eigenvalues,  $\hat{\lambda}_j$  for  $j = 1, 2, 3$  and their standard errors in brackets. Panel (b) shows the test for identification through heteroskedasticity of Lütkepohl, Meitz, Netšunajev, and Saikkonen (2021).  $H_r(\hat{\kappa}_1, \hat{\kappa}_2)$  is the test statistics with  $r - 1$  degrees of freedom, where  $\hat{\kappa}_m$  for  $m = 1, 2$  is an estimate of the kurtosis of reduced-form residuals in the  $m$ -th volatility regime. See Appendix F.

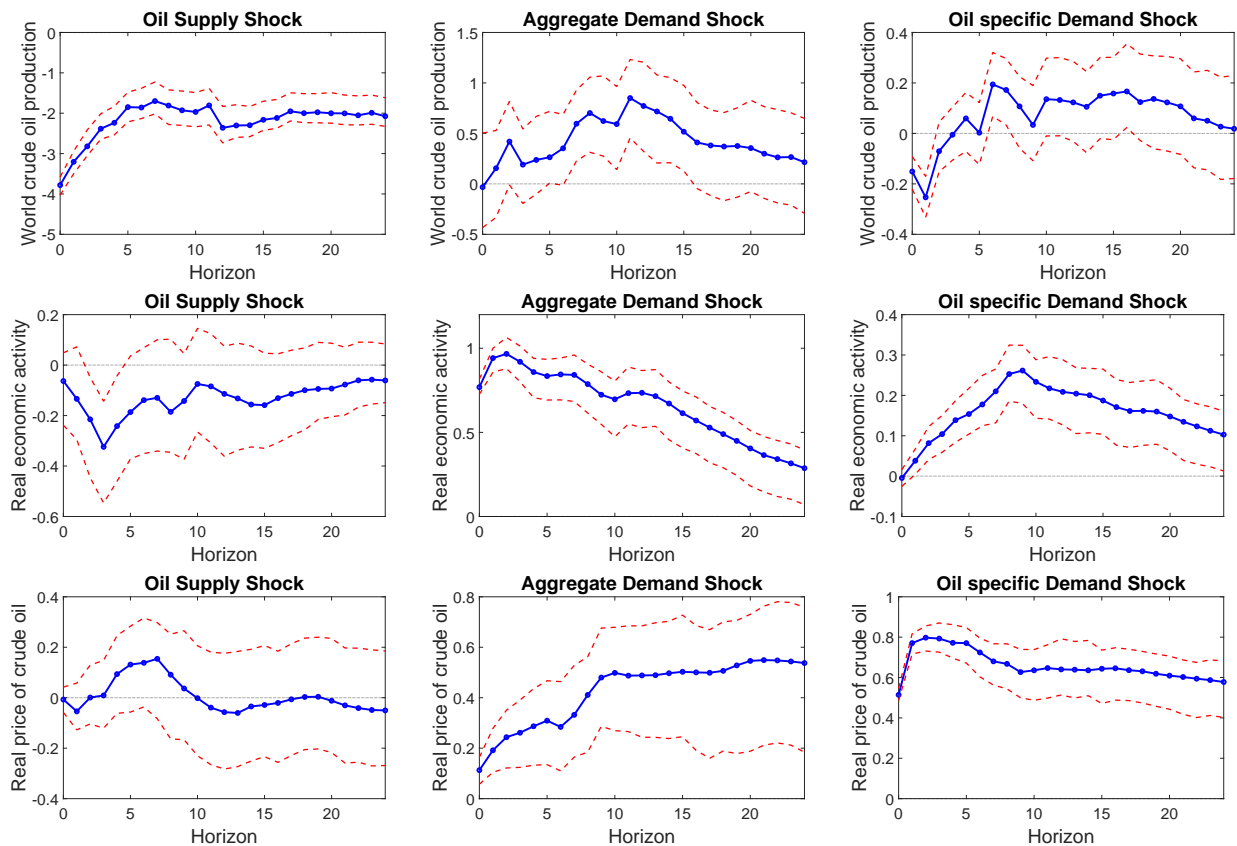
Figure G.8: Impulse response functions  $\mathcal{M}_0$



Notes: the blue line with dots represents the posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Recursive identification imposing  $c_{12} = c_{13} = c_{23} = 0$ . The model is point-identified

third column of Figure G.9 are consistent with those induced by an aggregate demand shock and an oil supply shock respectively. Focusing on the response of the real price of crude oil to an oil supply shock, we see that the 68% HPD region always contains zero. Similarly, the response of real economic activity to an oil supply disruption is very modest. All in all, these results highlight how heteroskedasticity conveys information that is useful to the purpose of

Figure G.9: Impulse response functions  $\mathcal{M}_1$

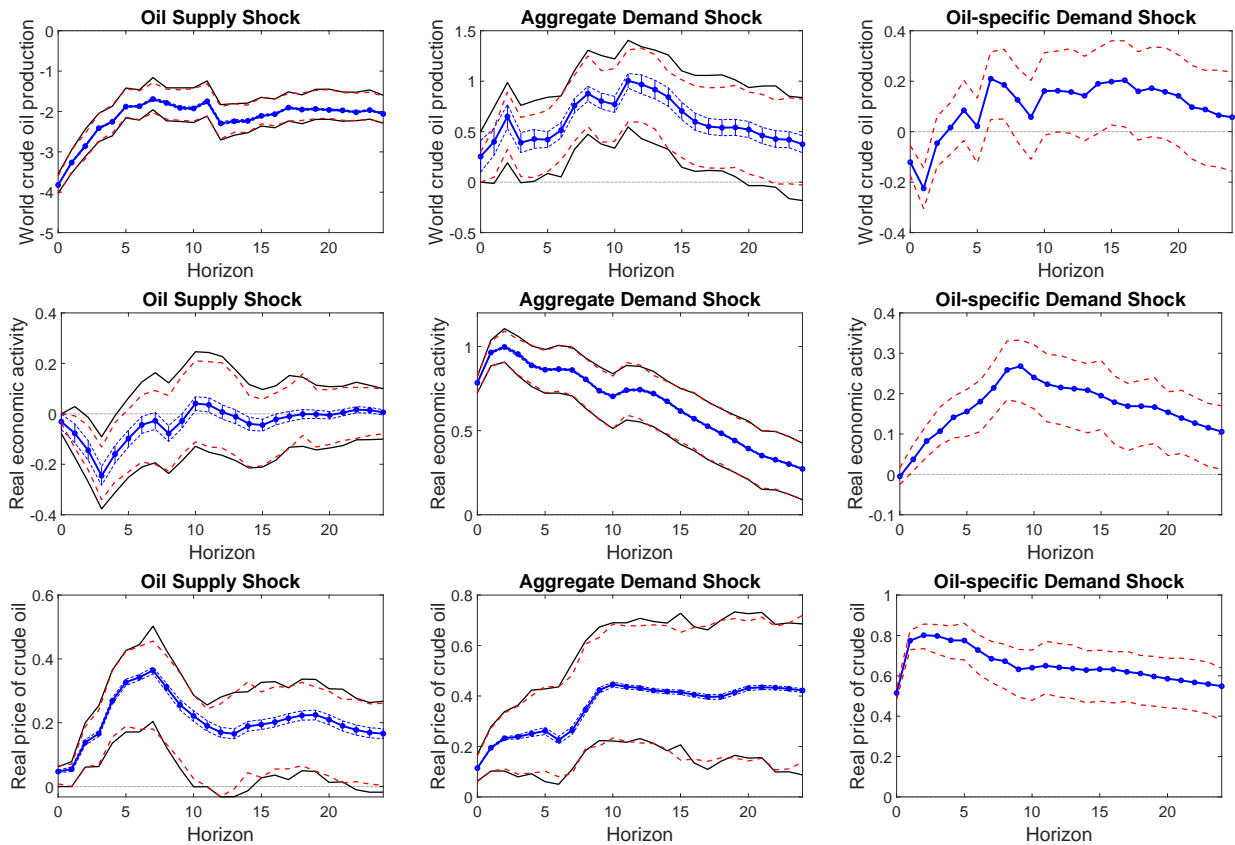


Notes: the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Identification is obtained via heteroskedasticity assuming distinct eigenvalues

identifying structural shocks.

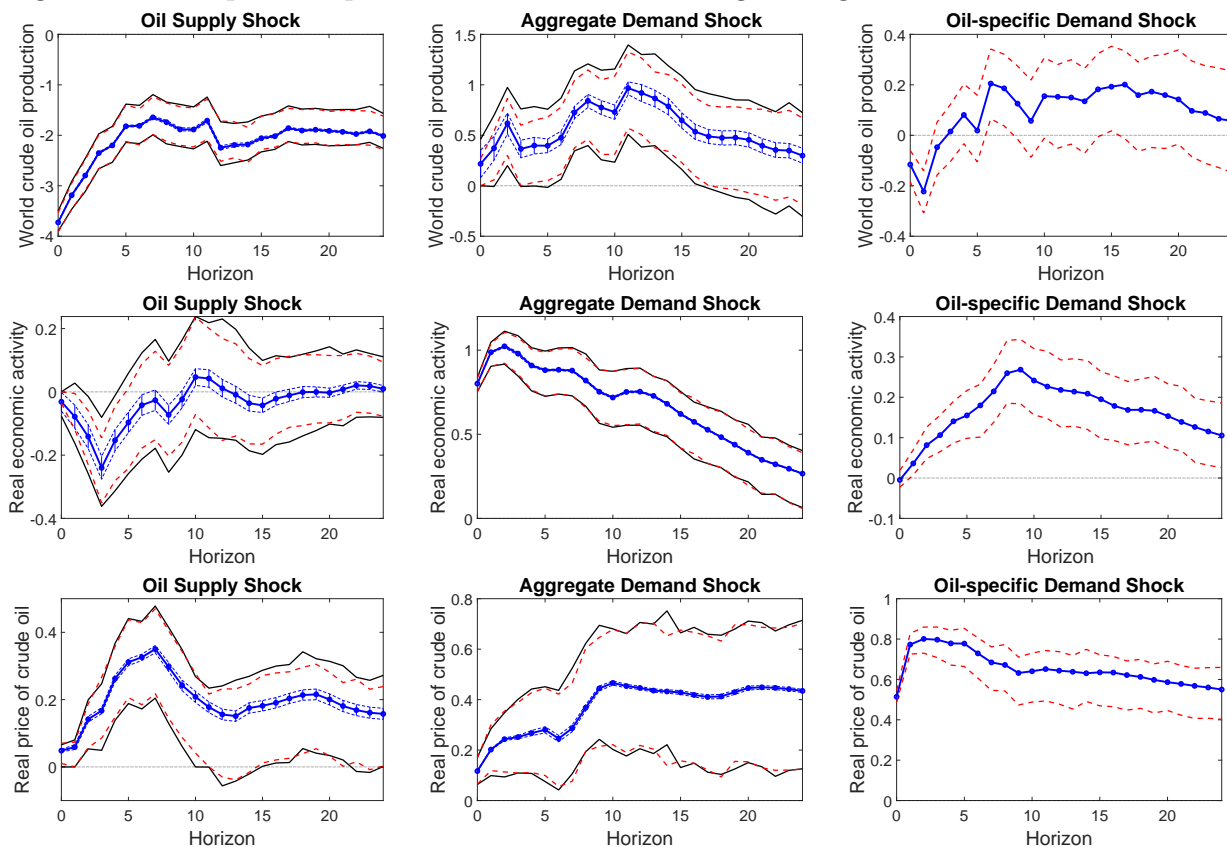


Figure G.10: Impulse response functions  $\mathcal{M}_2$  – Alternative implementation of Algorithm 1



*Notes:* the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Plots in first and second columns of the figure also report the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% (solid black curves). Identification via heteroskedasticity with multiple eigenvalues (i.e. only one shock is point identified), static and dynamic sign restrictions. We substitute Step 5 of Algorithm 1 with 10000 iterations of Step 4.1-Step 4.3. The interval  $[\ell(\phi_m), u(\phi_m)]$  is then approximated by the minimum and maximum values over such iterations.

Figure G.11: Impulse response functions  $\mathcal{M}_2$  - Testing the eigenvalues



*Notes:* the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Plots in first and second columns of the figure also report the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% (solid black curves). Identification via heteroskedasticity with multiple eigenvalues (i.e. only one shock is point identified), static and dynamic sign restrictions.

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