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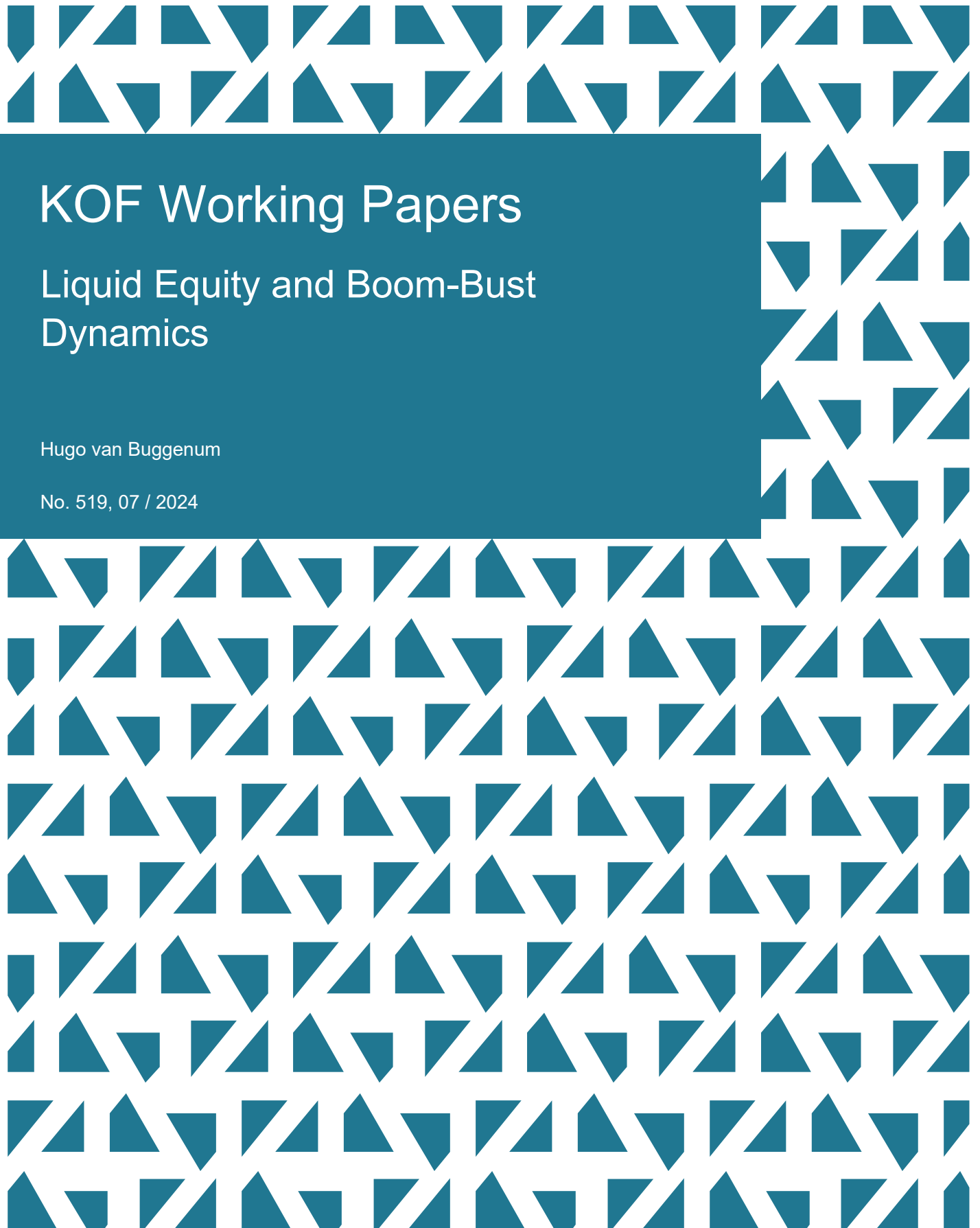
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Liquid Equity and Boom-Bust Dynamics

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Liquid Equity and Boom-Bust Dynamics*

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Abstract

I develop and analyze a monetary model with liquid equity. Equity is a claim on the profits of firms acting as sellers in the search-and-matching market. Buyers in that market devote search to obtain matches with firms, and use the equity to relax a liquidity constraint. The dual nature of equity in the search-and-matching market entails a strategic complementarity in search operating through buyers' liquidity constraint, and it gives rise to endogenous booms and busts. The economy is stable in an inflation-targeting regime if combined with asset purchases, meaning the government effectively puts a floor below the value of equity.

Keywords: coordination; equity; liquidity; money-search; sunspots.

JEL Classification: E32, E40, E44, E52, G10

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1 Introduction

The rapid advance of exchange-traded funds (ETFs) is making it easier to trade listed firms' equity and debt swiftly and cheaply (Lettau and Madhavan, 2018). This trend allows claims on firms' cash flows to become more liquid. The use of privately created assets as means of liquid wealth is however perceived to facilitate financial panics and amplify real recessions.¹ A sudden reduction in the market value of liquid wealth, for instance, severely impedes economic activity if credit is imperfect, and it could be self-fulfilling if such wealth comprises assets with payoffs that ultimately depend on real economy activity. ETFs in particular could open the door to this adverse feedback loop exactly because they are liquid and, ultimately, also a claim on the cash flow of all listed firms in the economy so that their underlying payoffs co-move with economic activity.²

Interestingly, central banks have unorthodoxly bought commercial-bond and equity ETFs to stabilize markets and, among other reasons, to break the adverse feedback loop. While central banks are normally reluctant to buy anything but high-grade government debt, the U.S. Federal Reserve bought about USD 8 billion of commercial bonds amid the 2020 crash. The Bank of Japan started purchasing domestic stocks in 2010 and held about USD 366 billion of them mid 2023, amounting to 6% of the Japanese stock market. Such purchases are not innocuous though; one can imagine that if the purchased assets loose value, which is not unlikely given their riskiness, the central bank may become insolvent. This may pose a threat to the credibility of the central bank and price stability (Reis, 2015), potentially forcing fiscal authorities to step in and recapitalize the central bank.³

The aim of this paper is twofold. First, to gain a better theoretic understanding of how liquid equity, or more generally liquid claims on firms' cash flows, can be a source of financial and macroeconomic instability. Second, what policy can do in response, particularly by buying assets to stabilize markets as observed in reality, and what the fiscal implications thereof might be. I develop for this purpose a money-search model à la Lagos and Wright (2005), modified to include endogenous search and liquid equity. The novelty lies in the modeling of equity as a liquid claim on firms' operating profits.⁴

¹This idea goes back to Fisher (1936) and other proponents of 100% fractional reserve banking.

²ETFs have been posed as a threat to financial and macroeconomic stability due to ETFs' perceived liquidity. See, for instance, Pagano *et al.* (2019).

³In case of the ECB, Council Regulation (EC) No. 1009/2000 sets out explicitly that the governing council of the ECB can force member states to recapitalize their national central banks.

⁴Because Modigliani and Miller's (1958) theorem applies in the model, equity is to be interpreted

It generates, together with endogenous search, a strategic complementarity producing endogenous cycles. The complementarity is reminiscent of that in Diamond (1982) but operates through liquid wealth rather than increasing returns in matching. Specifically, if agents search intensely, firms obtain more matches and earn higher profits, and so the value of their equity increases. This drives down the liquidity premium—the return forgone by holding liquid rather than illiquid assets—due to a greater supply of liquid wealth, in turn making liquid wealth cheaper to hold, entailing greater benefits of intense search through a looser liquidity constraint.

The strategic complementarity turns out to be a novel source of self-fulfilling dynamics in money-search models. Specifically, self-fulfilling cyclical dynamics now survive when ex-ante demand for liquid wealth is decreasing in the liquidity premium; an assumption in line with many money-search models calibrated to match monetary data, but that simultaneously rules out endogenous cycles in most of these existent frameworks. Adding currency to the picture, the results also indicate inflation targeting entails different outcomes than money-growth targeting due to liquid equity. With a currency-supply target for instance, endogenous cycles can exist and exhibit boom-bust dynamics with time-varying inflation, while if an inflation target is implemented and inflation is perfectly stabilized, deterministic cycles vanish but stochastic ones remain to exist. Further, while in an inflation-targeting regime the economy is perfectly stabilized at the Friedman rule—a slight deflation eliminating the opportunity cost of holding currency—, implementing the Friedman rule with currency-supply targeting leaves room for self-fulfilling transitional dynamics due to equity’s endogenous value. This suggests not only that targeting narrow-money growth may be undesirable, but also that broader monetary targets can be unreliable in times of financial innovation which would lead to unpredictable changes in the economic significance of monetary aggregates.⁵

Away from the Friedman rule, the economy is stabilized by combining inflation targeting with equity purchases. It does, however, require what Hall and Reis (2015) call fiscal support; potential losses from asset purchases by the central bank must be passed on to the fiscal authority, i.e., to the taxpayer. Analyzing asset purchases is particularly relevant since major central banks have used them in response to the global financial

broadly—it comprises the firm’s entire capital structure, i.e., not only shares of stock, but also bonds.

⁵McCallum (1985) mentions this as one of the criticisms against the U.S. Federal Reserve’s money-stock targets strategy, being used from 1979 to 1982.

crisis and the 2020 COVID-19 crash as mentioned before, thereby buying risky claims on firms' cash flows in the form of either commercial-bond ETFs, as the U.S. Federal Reserve did, or equity ETFs, as the Bank of Japan did.

Model and results. I analyze the joint role of endogenous search and liquid equity in a Lagos and Wright (2005) framework unifying: bilateral matching in a market where buyers devote endogenous search effort and matches materialize according to a constant-returns matching function as in Pissarides (1984); a transactions-based demand for assets originating from a liquidity constraint imposed on the buyers within bilateral matches; an asset resembling the equity of firms which act as sellers in the search-and-matching market; and intrinsically-worthless fiat currency. One interpretation of the framework runs as follows. Buyers can be thought of as small, private firms having imperfect access to credit markets. They seek to purchase tailor-made goods, e.g., machines, requiring them to devote time (search) to find other firms with the required expertise, and to subsequently work out blueprints, *etcetera*, for the tailor-made goods. The more time devoted to this process, the more likely the goods can be obtained. Imperfect credit means the private firms need liquid wealth to pay for the goods. The firms producing tailor-made goods can be thought of as big, public firms with tradable equity. This tradable equity is, in turn, used as liquid wealth by the small, private firms.

I consider as a benchmark a version of the model in which liquid wealth comprises only currency, supplied at a constant growth rate as commonly assumed in the literature. I verify that if ex-ante demand for liquid wealth is decreasing in the liquidity premium, there are no endogenous cycles; the monetary equilibrium is generically unique. Adding an asset paying an exogenous dividend as in Lucas (1978) does not change this result.

I then let liquid wealth comprise only equity, paying an inherently endogenous dividend. If search is exogenous, only a wealth channel is operative; a higher equity value relaxes buyers' liquidity constraint so firms earn greater profits, feeding back into a higher value for equity. This channel is too weak to generate equilibrium multiplicity, although it can amplify real shocks as in Guerrieri and Lorenzoni (2009). A search channel arises with endogenous search: if the value of equity increases, buyers are more likely to increase their search because they face a looser liquidity constraint, so firms are matched more frequently, leading to an increase in the value of equity. This channel is strong enough

to generate multiplicity of monetary equilibria and endogenous cycles, even if ex-ante demand for liquid wealth is decreasing in the liquidity premium. In that sense, the model isolates a novel joint role of search and liquid equity for endogenous dynamics.

If liquid wealth comprises both intrinsically-worthless currency—with supply growing at a constant rate—and equity, endogenous cycles survive and they exhibit boom-bust dynamics with time-varying inflation. If currency supply adjusts endogenously to stabilize inflation, stochastic cycles remain to exist, but only when policy is away from the Friedman rule. One way to then stabilize the economy is by means of equity purchases undertaken by the government. These purchases are effective but require commitment to pass potential losses from them on to taxpayers. Losses do not materialize if the price at which equity is bought is sufficiently high, since the mere fiscal commitment then suffices to stabilize the economy. The economy cannot be stabilized though if the purchase price is set too low, meaning there are contingencies in which purchases materialize and losses are indeed passed on to taxpayers. In that sense, setting the purchase price slightly too low due to worries about potential losses may backfire; too coordinate on a good equilibrium, markets must be convinced that the government will defend a sufficiently high floor below the value of equity. Otherwise, they may remain to coordinate on a bad equilibrium, triggering losses for the government which are much larger than those motivating only a slightly lower purchase price.

Related literature. Papers with a role for liquid assets other than fiat currency are abundant in the money-search literature (see Lagos *et al.*, 2017, for a review). Some, following Lucas (1978), treat dividends paid by such assets as exogenous (e.g., Geromichalos *et al.*, 2007; Lagos, 2010; Rocheteau and Wright, 2013; Geromichalos and Herrenbrueck, 2016, 2017). Others let dividends be determined in frictionless markets (e.g., Lagos and Rocheteau, 2008; Andolfatto *et al.*, 2016; Altermatt, 2022). Altermatt *et al.* (2023) analyze a rich model to study endogenous cycles if both fiat currency and exogenous-dividend assets comprise liquid wealth. Endogenous cycles can arise in the aforementioned papers, but only if assets are infinitely lived since the cycles rely on an infinite chain of asset-price expectations. Further, cycles are ruled out when ex-ante liquid-wealth demand is decreasing in liquidity premia; a property satisfied by most empirically-calibrated money-search models (e.g., Craig and Rocheteau, 2008; Lagos and Wright, 2005; Berentsen *et al.*, 2011;

Ait Lahcen *et al.*, 2022).

Rocheteau and Wright (2013) briefly analyze, in an extension, a money-search setup in which the fundamental value of assets is determined in markets in which these assets are used in payment. Their analysis lacks endogenous search though and focuses on firm entry instead, known to generate equilibrium multiplicity regardless of the nature of liquid assets (see, e.g., Rocheteau and Wright, 2005; Berentsen *et al.*, 2011; Nosal and Rocheteau, 2011). I instead uncover a strategic complementarity in search arising only if liquid wealth comprises equity. The complementarity is strong enough to entail endogenous cycles, even if a higher liquidity premium negatively affects ex-ante liquid-wealth demand since the mechanism does not rely on an infinite chain of expectations; the result is derived deliberately for one-period lived equity to elucidate its novelty.

The framework also relates to Guerrieri and Lorenzoni (2009), who study a model in which producers' earning prospects matter for consumers' spending, producing a feedback effect amplifying shocks. Angeletos and La'O (2013) show how limited communication can produce rational heterogeneous beliefs and endogenous booms and busts in a similar setup. My contribution is to show how a strategic complementarity in search can produce endogenous booms and busts in an environment with homogeneous rational beliefs.

A strand of the labor-search literature studies self-fulfilling prophecies regarding unemployment. Howitt and McAfee (1987) show that if the labor-market matching technology has increasing returns, there are multiple equilibria. Howitt and McAfee (1992) and Kaplan and Menzio (2016) consider constant returns in matching; they instead incorporate a positive demand effect of low unemployment to produce multiplicity. Branch and Silva (2022) study an economy à la Mortensen and Pissarides (1994) with households using government bonds and the equity of firms as liquid wealth. Their model features a demand channel working through firm entry as in Berentsen *et al.* (2011). My focus is on a setup with endogenous search and constant returns in matching, showing that multiplicity can arise if liquid wealth comprises firms' equity.

My analysis of a stable inflation regime contributes to the question whether a central bank should pay attention to financial developments over and above the extent to which these affect inflation. Some argue in favor (e.g., Smets, 1997; White and Borio, 2004; Bordo and Jeanne, 2002; Roubini, 2006), while others argue against (e.g., Bernanke and Gertler, 2001; Greenspan, 2007; Schwartz, 2003; Woodford, 2012). I show inflation sta-

bility is insufficient for financial stability; asset purchases are also necessary. The analysis of asset purchases contributes to the literature spurred by Sargent and Wallace (1981), studying the interaction between fiscal and monetary policy. It received renewed attention due to unconventional monetary policies, as losses from them may be inflationary, calling for bailout of the central bank (Reis, 2015; Tanaka, 2021). I contribute by showing that equity purchases require fiscal support, and that such support can occur on the equilibrium path if the price at which equity is bought is set too conservatively.

Finally, my work fits a theoretic literature on how various aspects of financial intermediation, e.g., the provision of liquidity insurance (Peck and Shell, 2003), market making (Rubinstein and Wolinsky, 1987), the role of intermediaries' reputation (Gu *et al.*, 2013), and the creation of information-insensitive liabilities (Gorton and Ordoñez, 2014), generate instability. Gu *et al.* (2020) review many of these aspects analytically. My contribution is to focus on the creation of liquid claims on firms' equity in a framework unifying liquidity constraints and search.

Outline. Section 2 lays out the model and Section 3 revisits the scope for endogenous dynamics if liquid wealth comprises only currency. Section 4 uncovers endogenous dynamics when liquid wealth comprises only equity and Section 5 adds currency. Section 6 studies stabilization policies and Section 7 concludes. Proofs are in Appendix E.

2 Model

Time $t \in \mathbb{N}_0$ is discrete and continues forever. The time subscript is suppressed; subscripts -1 and $+1$ are used to denote previous- and next-period variables. Two markets convene sequentially in every period: a decentralized market (DM) and a centralized market (CM). The DM is frictional with a key role for liquid wealth and search. The CM is frictionless and allows agents to re-balance their asset positions. There are two fully perishable and perfectly divisible goods, DM goods and CM goods, traded in the DM and the CM, respectively. CM goods are used as the numeraire throughout.

There is a unit mass of infinitely-lived buyers, overlapping generations of finitely-lived firms, and a government. Buyers' preferences are described by the flow-utility function $\mathcal{U}(q, x, \sigma) = u(q) + x - s(\sigma)$, where: $q \in \mathbb{R}_+$ is consumption of DM goods; $x \in \mathbb{R}$ is net consumption of CM goods; and $\sigma \in \Sigma \subseteq [0, 1]$ is search effort. Function s is increasing,

continuous, and convex; and u is twice continuously differentiable and satisfies $u(0) = 0$, $u' > 0$, $u'' < 0$, $\lim_{q \rightarrow 0} u'(q) = \infty$, and $\lim_{q \rightarrow \infty} u'(q) = 0$. For the search levels, let $\Sigma = \{l, h\}$, with $0 < l < h \leq 1$ and $s(h) - s(l) = k$. This makes the mechanism more transparent and is not critical—Appendix C shows existence of s so that $\Sigma = \{l, h\}$ produces the same results as $\Sigma = [0, 1]$. Utility between periods is discounted at $\beta \in (0, 1)$.

A unit mass of firms is born in the CM, which are owned by the buyers and live until CM_{+1} . They have an endowment of y CM_{+1} goods in DM_{+1} from which they can produce q DM_{+1} goods by using $c(q)$ CM_{+1} goods as an input, where $c(0) = 0$, $c' > 0$, and $c'' \geq 0$. CM_{+1} goods unused in DM_{+1} are stored until CM_{+1} . Firms do not devote search in the DM in contrast to buyers, keeping things simple without loss—Appendix D lays out a DM with two-sided search for which the main results derived below hold true.

There are two perfectly divisible assets. First, ownership shares of the firms, which are bundled into an ETF-like asset. The amount of shares issued by each firm is normalized to one, and the ETF shares are simply referred to as *equity*. The second asset is intrinsically-worthless *currency*, which is issued by the government. The finitely-lived nature of firms implies equity is one-period lived. This is arguably unrealistic, but ensures dynamics cannot emerge from an infinite chain of self-fulfilling asset-price expectations, which is normally key in money-search models. In that sense, the feature provides a clean laboratory to isolate the joint role of liquid equity and search.

Aggregate uncertainty comes from a sunspot generating a realization $s \in \mathcal{S}$ in every period before markets convene. Because prices, quantities, and values are indexed with t rather than the sunspot's history $H_t \equiv \{s_0, s_1, \dots, s_t\}$, variables and functions are (potentially) stochastic objects. To analyze the role of search and liquidity in isolation from inabilities to contract on aggregate states, Arrow securities allow the buyers to choose the amount of currency and equity carried into the DM contingent on s . Note s is the current realization of the sunspot, s_{+1} the realization one period ahead (a random variable), and \mathbb{P} the probability law for s_{+1} . As implied by the omitted index t , \mathbb{P} is conditional on H_t .

Markets. In the CM the incumbent firms pay dividends and subsequently die, shares in the new firms are issued and then traded, and buyers adjust their asset positions by producing or consuming CM goods. The CM prices of currency and newly issued equity are Φ and Ψ . A portfolio of Arrow securities for currency delivered in DM_{+1} with nominal

payoff schedule $m_{+1} \in \mathbb{R}^{\mathcal{S}}$ is priced at $\int_{\mathcal{S}} \phi_{+1}(s') m_{+1}(s') \mathbb{P}(ds')$, and likewise, a portfolio $e_{+1} \in \mathbb{R}^{\mathcal{S}}$ for equity is priced at $\int_{\mathcal{S}} \psi_{+1}(s') e_{+1}(s') \mathbb{P}(ds')$. Here, $\phi_{+1}, \psi_{+1} \in \mathbb{R}_+^{\mathcal{S}}$ are pricing kernels (see Hansen and Renault, 2010) and $\mathbb{R}^{\mathcal{S}}$ denotes the space of real-valued functions with domain \mathcal{S} . Treated as functions, $\phi_{+1}, \psi_{+1}, m_{+1}$, and e_{+1} are deterministic objects in the CM. When no confusion arises, let $\phi_{+1} \in \mathbb{R}$ be $\phi_{+1}(s_{+1})$, i.e. the value of ϕ_{+1} under the realization s_{+1} of the sunspot in the next time period, and do the same for ψ_{+1}, m_{+1} , and e_{+1} . There should be no arbitrage, so:⁶

$$\Phi = \int_{\mathcal{S}} \phi_{+1}(s') \mathbb{P}(ds') \equiv \mathbb{E}\{\phi_{+1}\} \quad \text{and} \quad \Psi = \int_{\mathcal{S}} \psi_{+1}(s') \mathbb{P}(ds') \equiv \mathbb{E}\{\psi_{+1}\}, \quad (1)$$

where expectation \mathbb{E} is taken w.r.t. probability law \mathbb{P} . No arbitrage thus means the price of currency (equity) equals the price of a portfolio of Arrow securities delivering exactly one unit of currency (resp. equity) in DM_{+1} regardless of the realization of s_{+1} .

The newborn firm issues a unit mass of shares, yielding Ψ CM goods paid to the buyers—the initial owners of the firm. The idiosyncratic risk faced by the firms in DM_{+1} is diversified away through bundling their shares into the ETF-like asset.

An incumbent firm—born in CM_{-1} —holding assets worth p CM goods and an inventory o of CM goods pays a dividend of $\delta = p + o$ CM goods and then dies. The incumbent equity pays a dividend of Δ CM goods, where Δ is the aggregated dividend of the underlying incumbent firms and also the cum-dividend value of the equity. Equity matures after this dividend payment takes place; the ex-dividend value is zero.

The government is only active in the CM. Its supply of currency at the end of a period is M and buyers are paid lump-sum transfer $\tau = \Phi(M - M_{-1})$ to close the budget.

Buyers are randomly and bilaterally matched to the firms in the DM and negotiate the terms of trade (q, p) , with q the amount of DM goods received by the buyer and p the value of the assets received by the firm. Utility surplus for the buyer from this trade is $u(q) - p$ and for the firm it is $p - c(q)$ (Appendix B provides details). The underlying negotiation process between the buyer and the firm determining (p, q) is, following Gu and Wright (2016), summarized by an exogenous payment protocol v , mapping $q \mapsto p$. The buyer's surplus is then $L(q) = u(q) - v(q)$ and that of the firm $\Pi(q) = v(q) - c(q)$.

Assumption 1. v is twice continuously differentiable and such that: (i) $v(0) = 0$, $v' > 0$;

⁶See Footnote 10 for how this no-arbitrage condition can be rationalized.

(ii) $L(q)$ attains a unique global maximum at $\hat{q} \in (0, q^*]$, where q^* solves $u'(q) = c'(q)$ and $c(\hat{q}) < y$, and is strictly increasing in q for $q \in (0, \hat{q})$; (iii) $L'(q)/v'(q)$ is strictly decreasing in q for $q \in (0, \hat{q})$, (iv) $\Pi(q) > 0$ for $q \in (0, \hat{q}]$; and (v) $\Pi'(q) > 0$ for $q \in (0, \hat{q})$.

Under Assumption 1: L and Π are increasing; Π will be strictly positive in equilibrium; buyers' asset demand will be well-behaved; and buyers will never demand more DM goods than the firm can produce from its endowment y .⁷

Buyer's maximization. Appendix B derives the buyer's Bellman equation:

$$V(m, e) = \max_{\sigma \in \{l, h\}} \left\{ \sigma \max_{q \geq 0} \{L(q) \mid \text{s.t. } v(q) \leq z(m, e)\} - s(\sigma) \right\} + \Phi m + \Delta e + \tau + \Psi \\ + \max_{(m_{+1}, e_{+1}) \in (\mathbb{R}_+^S)^2} \{ \mathbb{E} \{ -(\phi_{+1} m_{+1} + \psi_{+1} e_{+1}) + \beta V_{+1}(m_{+1}, e_{+1}) \} \}, \quad (2)$$

where: $z(m, e) = \Phi m + \chi \Delta e$ is liquid wealth; $(m, e) \in \mathbb{R}_+^2$ is currency and equity carried into the DM;⁸ $(m_{+1}, e_{+1}) \in (\mathbb{R}_+^S)^2$ is the portfolio of Arrow securities carried out of the CM; and $\chi \in \{0, 1\}$ indicates whether equity is liquid.

The Bellman equation comprises the following. In the DM, the buyer first determines search effort σ , which equals the probability of being matched to a firm.⁹ If matched, it chooses q to maximize $L(q) = u(q) - v(q)$ subject to a liquidity constraint, and the resulting terms of trade are

$$(q, p) = \begin{cases} (v^{-1} \circ z(m, e), z(m, e)) & \text{if } z(m, e) < v(\hat{q}), \\ (\hat{q}, v(\hat{q})) & \text{if } z(m, e) \geq v(\hat{q}). \end{cases} \quad (3)$$

The buyer thus ideally consumes \hat{q} , but needs liquid wealth $z \geq v(\hat{q})$ for that. If it does not command over $v(\hat{q})$, it spends all liquid wealth on DM consumption. To characterize

⁷Assumption 1 is satisfied for a broad set of bargaining protocols, including Nash (1950) bargaining, proportional bargaining à la Kalai (1977), and gradual bargaining as in Rocheteau *et al.* (2021), as well as a payment protocol representing constant-markup pricing.

⁸Following the convention on notation introduced earlier, $(m, e) \in \mathbb{R}_+^2$ is simply $(m(s), e(s))$ where s is the realization of the sunspot for the current period.

⁹The setup can be microfounded with a constant-returns-to-scale matching function $\min\{b, f\}$, where f is the mass of firms (equal to one) and b is the effective mass of buyers—the mass of buyers multiplied by their average search $\tilde{\sigma}$. The mass of realized matches is then $\min\{\tilde{\sigma}, 1\}$, the probability a buyer finds a match is $\sigma \min\{\tilde{\sigma}, 1\}/\tilde{\sigma} = \sigma$, and the probability a firm finds a match is $\min\{\tilde{\sigma}, 1\} = \tilde{\sigma}$.

the tightness of the liquidity constraint, write the Lagrange multiplier as

$$\lambda(z) = (L'/v') \circ \min\{v^{-1}(z), \hat{q}\}. \quad (4)$$

Assumption 1 implies $\lambda(z)$ is continuous in z , equal to zero for $z \geq v(\hat{q})$, and strictly decreasing on the domain $(0, v(\hat{q}))$, i.e., the marginal value of liquidity is decreasing.

In the CM, the Arrow securities allow the buyer to choose next-period asset holdings $(m_{+1}(s'), e_{+1}(s'))$ for every potential realization $s' \in \mathcal{S}$ of s_{+1} .¹⁰ The CM cost of acquiring $(m_{+1}, e_{+1}) \in (\mathbb{R}_+^{\mathcal{S}})^2$ is $\int_{\mathcal{S}} [\phi_{+1}(s')m_{+1}(s') + \psi_{+1}(s')e_{+1}(s')] \mathbb{P}(ds')$ $\equiv \mathbb{E}\{\phi_{+1}m_{+1} + \psi_{+1}e_{+1}\}$ and the CM value of initial portfolio $(m, e) \in \mathbb{R}_+^2$ is $\Phi m + \Delta e$. The buyer chooses (m_{+1}, e_{+1}) independently of its trading history due to quasi-linear utility, and it also receives $\tau + \Psi$ CM goods in a lump-sum way from the government and the newborn firms (initially owned by buyers).

To close, although the buyer chooses its asset portfolio $(m_{+1}, e_{+1}) \in (\mathbb{R}_+^{\mathcal{S}})^2$ to be carried into DM_{+1} already in the preceding CM, i.e., in the form of two real-valued functions, quasi-linear utility and the Arrow securities imply the portfolio problem has an equivalent yet simpler representation with $(m_{+1}, e_{+1}) \in \mathbb{R}_+^2$, i.e., two real-valued scalars, being chosen upon entering DM_{+1} once the realized value of s_{+1} is known:

$$\begin{aligned} & \max_{(m_{+1}, e_{+1}) \in (\mathbb{R}_+^{\mathcal{S}})^2} \left\{ \mathbb{E} \left\{ -(\phi_{+1}m_{+1} + \psi_{+1}e_{+1}) + \beta V_{+1}(m_{+1}, e_{+1}) \right\} \right\} \\ & = \mathbb{E} \left\{ \max_{(m_{+1}, e_{+1}) \in \mathbb{R}_+^2} \left\{ -(\phi_{+1}m_{+1} + \psi_{+1}e_{+1}) + \beta V_{+1}(m_{+1}, e_{+1}) \right\} \right\}. \quad (5) \end{aligned}$$

Iterating back one period on (5) and using the recursive nature of the Bellman Equation (2) implies all the buyer's decisions that are relevant to what happens in the DM, can be expressed as resulting from a single maximization problem. In particular $(\sigma, m, e) \in$

¹⁰For every realization $s' \in \mathcal{S}$ for s_{+1} , the non-negativity constraint imposes positive currency and equity holdings. This does not rule out short selling *per se*, which is relevant for rationalizing no-arbitrage Condition (1). A buyer can for instance short one unit of equity and buy a portfolio of Arrow securities for equity delivering one unit regardless of s_{+1} , as the net effect of this trading strategy on DM_{+1} equity holdings is zero for every $s' \in \mathcal{S}$. The payoff in the CM is $\Psi - \int_{\mathcal{S}} \psi_{+1}(s') \mathbb{P}(ds')$. This strategy (and its converse) should not be profitable, which is exactly what the no-arbitrage condition imposes. An analogous argument applies to currency.

$\{l, h\} \times \mathbb{R}_+^2$ —search effort and the asset portfolio carried into the DM—results from¹¹

$$\max_{(\sigma, m, e) \in \{l, h\} \times \mathbb{R}_+^2} \left\{ \sigma L \left(\min\{v^{-1} \circ z(m, e), \hat{q}\} \right) - s(\sigma) + (\Phi - \phi/\beta)m + (\Delta - \psi/\beta)e \right\}. \quad (6)$$

Firm dividends. Conditionally expected dividend $\mathbb{E}_{-1}\{\delta|s\}$ paid by an incumbent firm equals the aggregate dividend payment Δ of equity. If $G(\sigma, m, e)$ is the mass of buyers devoting search $\sigma' \leq \sigma$, and holding currency $m' \leq m$ and equity $e' \leq e$, then

$$\Delta = \iiint \sigma \Pi \left(\min\{v^{-1} \circ z(m, e), \hat{q}\} \right) G(d\sigma, dm, de) + y. \quad (7)$$

I.e., firms receive an endowment of y CM goods upon entering the DM. Each firm then draws a buyer from G . If that buyer devotes search σ and carries portfolio $(m, e) \in \mathbb{R}_+^2$, a match occurs with probability σ and yields surplus $\Pi \left(\min\{v^{-1} \circ z(m, e), \hat{q}\} \right)$.

Equilibrium characterization. Distribution G must be in line with the maximization problem in Equation (6) and transversality condition $\lim_{T \rightarrow \infty} \beta^T [\Phi_T m_T + \Delta_T e_T] = 0$ (see Rocheteau and Wright, 2013), and it must also entail market clearance:

$$\iiint m G(d\sigma, dm, de) = M_{-1} \quad \text{and} \quad \iiint e G(d\sigma, dm, de) = 1. \quad (8)$$

Definition 1. *Given a (stochastic) process $\{M_{t-1}\}_{t=0}^\infty$ for currency supply, an equilibrium is a (stochastic) process $\{G_t : \mathbb{R}^3 \rightarrow [0, 1], (\phi_t, \Phi_{t-1}, \psi_t, \Psi_{t-1}, \Delta_t) \in \mathbb{R}_+^5\}_{t=0}^\infty$ such that: (i) the no-arbitrage condition (1) holds; (ii) buyers maximize utility, i.e., any (σ, m, e) on the support of G must solve (6) and satisfy $\lim_{T \rightarrow \infty} \beta^T [\Phi_T m_T + \Delta_T e_T] = 0$; (iii) the aggregate dividend payment Δ satisfies (7); and (iv) markets clear, i.e., (8) holds.*

Equilibrium asset prices. In equilibrium $\beta\Phi \leq \phi$ and $\beta\Delta \leq \psi$ must hold to induce boundedness of the buyer's portfolio (m, e) , as follows from (6). The optimality of (m, e) implies Euler equations

$$\phi = \beta [1 + \sigma \lambda (\Phi m + \chi \Delta e)] \Phi \quad \text{and} \quad \psi = \beta [1 + \chi \sigma \lambda (\Phi m + \chi \Delta e)] \Delta, \quad (9)$$

¹¹Following the convention on notation introduced earlier, $(\phi, \psi) \in \mathbb{R}_+^2$ is simply $(\phi(s), \psi(s))$ where s is the realization of the sunspot for the current period.

i.e., the time-discounted benefits of the marginal asset equal the acquisition cost. The benefits comprise: a savings component, being the price Φ for currency and the dividend Δ for equity; and a liquidity component, being $\Phi\sigma\lambda(\Phi m + \chi\Delta e)$ for currency and $\Delta\chi\sigma\lambda(\Phi m + \chi\Delta e)$ for equity. From (9) it is now useful to let

$$\iota \equiv \phi/\beta\Phi - 1, \quad (10)$$

be the *liquidity premium*. It is non-negative in equilibrium to induce bounded asset portfolios and entails

$$\Phi = \mathbb{E}\{\beta(1 + \iota_{+1})\Phi_{+1}\} \quad \text{and} \quad \Psi = \mathbb{E}\{\beta(1 + \chi\iota_{+1})\Delta_{+1}\}. \quad (11)$$

Currency is thus priced using (stochastic) discount factor $\beta(1 + \iota_{+1})$, where only the CM_{+1} price matters since currency pays zero dividend. Equity is priced using $\beta(1 + \chi\iota_{+1})$, where only the CM_{+1} dividend matters since the CM_{+1} ex-dividend price is zero.

Equilibrium search and liquid wealth holdings. Liquid wealth z_σ held by a buyer searching at σ relates to ι through λ , as follows from Equations (9) and (10):

$$\lambda(z_\sigma) = \iota/\sigma. \quad (12)$$

Due to the properties of λ , z_σ is pinned down as a continuous and strictly decreasing function of ι/σ for $\iota/\sigma \in (0, \bar{\lambda})$, where $\bar{\lambda} \equiv \lim_{q \rightarrow 0} L'(q)/v'(q)$.¹² If $\iota/\sigma = 0$, z_σ is only pinned down up to a lower bound $v(\hat{q})$, but there is no discontinuity in that $\lim_{z \nearrow v(\hat{q})} \lambda(z) = 0$; and if $\iota/\sigma \geq \bar{\lambda}$, $z_\sigma = 0$, again without discontinuity.

Equation (6) and $k = s(h) - s(l)$ imply buyers are willing to search at $\sigma = h$ ($\sigma = l$) if and only if

$$\max_{z_h \geq 0} \{hL(\min\{v^{-1}(z_h), \hat{q}\}) - \iota z_h\} - \max_{z_l \geq 0} \{lL(\min\{v^{-1}(z_l), \hat{q}\}) - \iota z_l\} \geq (\leq) k. \quad (13)$$

Here, the cost of carrying liquid wealth z is ι . Buyers intensify search when ι is low since the LHS is decreasing in ι , as search is more attractive when DM match surplus is large, which requires the buyer to command of much liquid wealth.

¹² $\lim_{q \rightarrow 0} L'(q)/v'(q)$ is either infinity or bounded depending on the negotiation generating v .

Assumption 2. $\max_{z \geq 0} \{hL \circ v^{-1}(z) - l\bar{\lambda}z\} < k < (h - l)L(\hat{q})$.

Assumption 2 generates meaningful variation in σ in that buyers choose $\sigma = h$ if $\iota = 0$ and switch to $\sigma = l$ —while still demanding a strictly positive amount of liquid wealth—if ι crosses a threshold $\tilde{\iota} \in (0, l\bar{\lambda})$, which is determined by k . The fraction of intensely searching buyers is thus

$$\eta \in \begin{cases} \{1\} & \text{if } \iota < \tilde{\iota}, \\ [0, 1] & \text{if } \iota = \tilde{\iota}, \\ \{0\} & \text{if } \iota > \tilde{\iota}. \end{cases} \quad (14)$$

Liquid wealth in equilibrium. Buyers' aggregate ex-post demand for liquid wealth—liquid wealth held in the DM—is

$$Z_d = \eta z_h + (1 - \eta)z_l. \quad (15)$$

Ex-post demand is decreasing in ι and indeterminate but subject to the lower bound $v(\hat{q})$ when $\iota = 0$. Further, it jumps down when ι crosses the threshold $\tilde{\iota}$, as $\iota \leq \tilde{\iota}$ implies $z_h \geq \tilde{z}_h \equiv \lambda^{-1}(\tilde{\iota}/h)$ and $\iota \geq \tilde{\iota}$ implies $z_l \leq \tilde{z}_l \equiv \lambda^{-1}(\tilde{\iota}/l)$. Assumptions 1 and 2 jointly imply $0 < \tilde{z}_l < \tilde{z}_h < v(\hat{q})$, so if a buyer reduces search from h to l , it will also demand strictly less liquid wealth. The liquidity premium relates to aggregate ex-post demand through the inverse demand function:

$$\iota = \Lambda(Z_d) \equiv \begin{cases} l\lambda(Z_d) & \text{if } Z_d < \tilde{z}_l, \\ \tilde{\iota} & \text{if } \tilde{z}_l \leq Z_d \leq \tilde{z}_h, \\ h\lambda(Z_d) & \text{if } Z_d > \tilde{z}_h. \end{cases} \quad (16)$$

Ex-ante demand—the cost of acquiring Z_d in CM_{-1} —is $W_{d,-1} = \mathbb{E}_{-1}\{\beta(1 + \iota)Z_d\}$ and it can be increasing or decreasing in ι . A higher ι reduces ex-ante demand through a substitution effect—ex-post demand Z_d decreases in ι —but it increases ex-ante demand through an income effect—the cost of acquiring Z_d in CM_{-1} is increasing in ι . Which effect dominates affects existence of endogenous dynamics absent of liquid equity. Ex-post demand is however key to most of the analysis, and is simply referred to as *demand*.

Ex-post liquid-wealth supply Z_s , simply referred to as *supply*, consists of currency and

equity (if liquid):

$$Z_s = \Phi M_{-1} + \chi [\eta h \Pi(\min\{v^{-1}(z_h), \hat{q}\}) + (1 - \eta) l \Pi(\min\{v^{-1}(z_l), \hat{q}\}) + y]. \quad (17)$$

Equations (12), (14), (15), (16) and (17) transpire a key point—demand and supply are interwoven if equity is liquid. Specifically, a higher supply reduces the liquidity premium through the inverse demand function (16) since demand must equal supply. The lower liquidity premium, first, leads to higher search-contingent demands z_h and z_l through the search-contingent demand function (12). This increases supply further through firms' dividends (17). Second, there is a search boost through (14) if the liquidity premium drops below \tilde{v} , and this increases supply further because (i) a greater mass of firms is then matched and (ii) matches are more profitable if buyers search intensely since they then hold more liquid wealth, i.e., $\tilde{z}_h > \tilde{z}_l$.

3 Liquidity with only currency

Self-fulfilling dynamics can arise if liquid wealth comprises intrinsically-worthless currency, as is well-known in the literature. The scope for such dynamics remains largely unaffected with endogenous search if equity is illiquid, as established in this section.

Let currency supply M develop as $M = \mu M_{-1}$, with $\mu > \beta$, which is a common assumption. Together $\chi = 0 \Rightarrow Z_d = Z_s = \Phi M_{-1}$, (1), (10), and (16), imply $\phi = \beta [1 + \Lambda(\Phi M_{-1})] \mathbb{E}\{\phi_{+1}\}$. Defining $x \equiv \phi M_{-1}/\mu$ and using $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$ then yields

$$x = f_m(\mathbb{E}\{x_{+1}\}) \equiv \beta [1 + \Lambda(\mathbb{E}\{x_{+1}\})] \mathbb{E}\{x_{+1}\}/\mu; \quad (18)$$

a difference equation describing the dynamic equilibrium, where subscript m elucidates liquid wealth comprises only currency. The focus is on bounded monetary equilibria, meaning there exist $\underline{N}, \bar{N} \in \mathbb{R}_{++}$ such that $\Phi M_{-1} (= \mathbb{E}\{x_{+1}\}) \in [\underline{N}, \bar{N}] \forall t$.¹³

One bounded monetary equilibrium is the monetary steady state. It features $x = x_{+1} = x_{ss} = (\Phi M_{-1})_{ss} > 0$, entailing $\Lambda(x_{ss}) = \iota_{ss} = \mu/\beta - 1$ and $\Phi_{+1} = \Phi/\mu$; inflation

¹³Equilibria in which ΦM_{-1} goes to zero entail a hyperinflation. Existence of such equilibria is already well-known in the literature since currency is intrinsically worthless. Also, it can be shown the transversality condition is violated if ΦM_{-1} would not be bounded from above. Hence the focus on monetary equilibria with $\Phi M_{-1} \in [\underline{N}, \bar{N}] \forall t$ for $\underline{N}, \bar{N} \in \mathbb{R}_{++}$.

equals the money growth rate and ι_{ss} is positive, which is why $\mu > \beta$ is commonly assumed.¹⁴ Figure 1 depicts various parameterized examples of f_m . Monetary steady-states are intersection of f_m with the 45-degree line, and f_m must always intersect from above. The monetary steady state is unique, unless $\mu = \beta(1 + \tilde{l})$; for that knife edge case, all $x_{ss} \in [\tilde{z}_l, \tilde{z}_h]$ are steady states as illustrated in Figure 1e. Buyers are then indifferent between high and low search, so any $\eta \in [0, 1]$ can be part of a steady state.

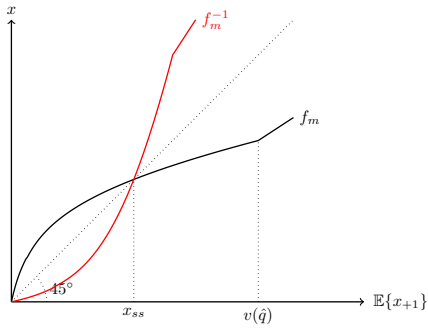
Equation (18) highlights that not much changes compared to a plain-vanilla model with exogenous search. The only substantial difference is that $f_m(x)$ is no longer differentiable at $x = \tilde{z}_l$ and $x = \tilde{z}_h$, which causes the continuum of steady states for $\mu = \beta(1 + \tilde{l})$. A sufficient condition to have self-fulfilling bounded dynamics, stochastic or deterministic, is $f'_m(x_{ss}) < -1$. This follows from the method of flip-bifurcations—mirroring f_m in the 45-degree line to obtain f_m^{-1} (see Azariadis, 1993). Intersections between f_m and f_m^{-1} that do not lie on the 45-degree line then constitute a two cycle. Figures 1b, 1f, and 1g illustrate two cycles. Even three different two cycles can exist here, as illustrated in Figure 1g, since f_m is not differentiable at $x = \tilde{z}_l$ and $x = \tilde{z}_h$.

Bounded monetary equilibria other than steady states do not exist if f is monotone increasing. The intuition is depicted in Figure 1h. Particularly, then $x < x_{ss} \Rightarrow \mathbb{E}\{x_{+1}\} < x$, so it must be that there is an equilibrium realization for x_{+1} such that $x_{+1} < x < x_{ss}$. Forward iterating the argument, x goes to zero with positive probability, so also ΦM_{-1} goes to zero with positive probability. Likewise, $x > x_{ss} \Rightarrow \mathbb{E}\{x_{+1}\} > x$, so ΦM_{-1} will go to infinity with some probability.

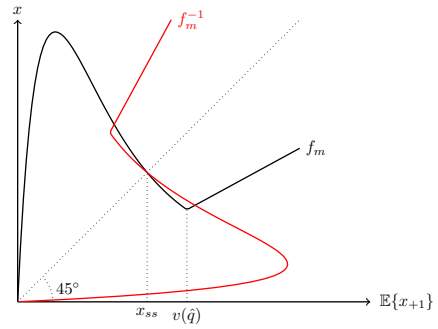
A similar argument applies when $\mu = \beta$, a policy know as the *Friedman rule*. Then, $f_m(x) \geq x$ on the domain \mathbb{R}_{++} , with equality if and only if $x \geq v(\hat{q})$. It follows $x < v(\hat{q})$ cannot be a bounded monetary equilibrium, as x would go to zero with positive probability. All $x \geq v(\hat{q})$ are part of an equilibrium, and induce identical real allocations since they all imply $\iota = 0$, entailing a slack liquidity constraint and intense search.

To close, here, bounded monetary equilibria other than steady states rest on a chain of expectations rationalized by the infinitely-lived nature of currency. This result carries over to infinitely-lived assets paying exogenous dividends as in Lucas (1978), as such assets approach currency if the dividend is infinitesimal (see Altermatt *et al.*, 2023). To

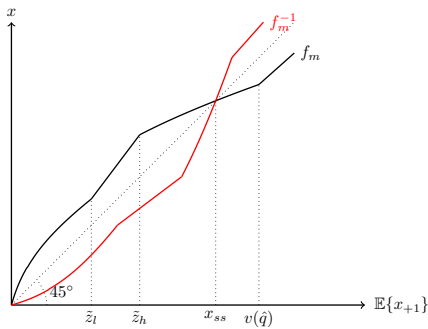
¹⁴Existence of the monetary steady state requires $\mu < \beta(1 + \bar{l})$; otherwise, $(\Phi M_{-1})_{ss} = 0$.



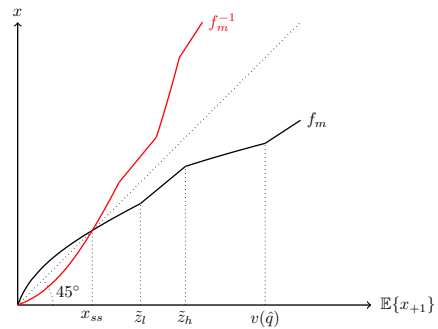
(a) Fixed search.



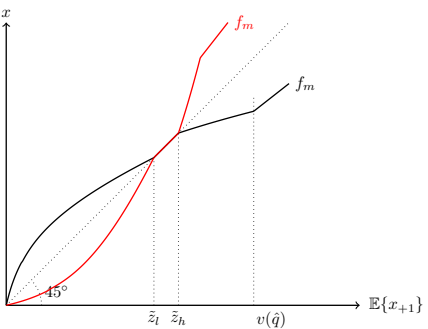
(b) Fixed search with two cycle.



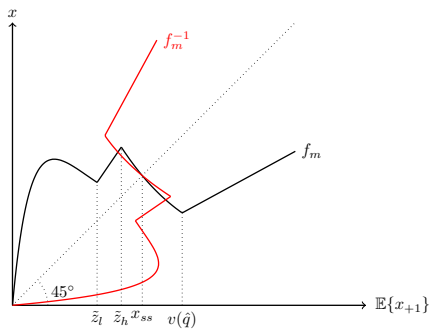
(c) Endogenous search; high search ss.



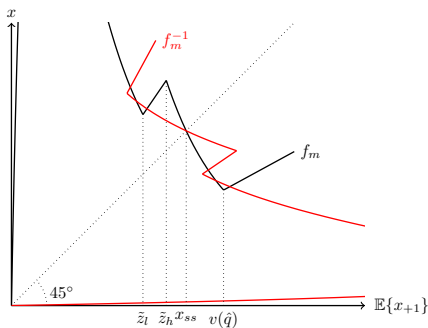
(d) Endogenous search; low search ss.



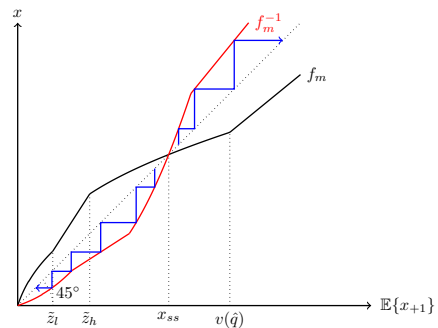
(e) Endogenous search; continuum of ss.



(f) Endogenous search; one two-cycle.



(g) Endogenous search; three two-cycles.



(h) Dynamics with monotone increasing f_m .

Figure 1: Depiction of f_m and f_m^{-1} . Parametrizations are in Section A.

contrast such self-fulfilling equilibria with those arising with liquid equity, introduce:

Assumption 3. $1 + \Lambda(z) + \Lambda'(z)z \geq 0 \forall z$.

Assumption 3 rules out bounded monetary equilibria other than steady states if liquid wealth comprises only currency— f_m is monotone increasing—and it relates directly to how ex-ante liquid-wealth demand and the liquidity premium move together; it holds true if and only if ex-ante liquid-wealth demand is monotonically decreasing in the premium since $d[\beta(1 + \iota)z]/d\iota = \beta(1 + \Lambda(z) + \Lambda'(z)z)/\Lambda'(z)$. The substitution effect in ex-ante demand thus dominates under Assumption 3 since $\Lambda'(z) < 0$.

4 Liquidity with only equity

Before analyzing liquid wealth comprising both currency and equity, consider liquid wealth comprising only equity. The liquidity premium ι is then the key equilibrium object since it determines z_h , z_l , and η (see Equations (12) and (14)). Demand and supply of liquid wealth (subscript e refers to the current environment) can thus be written as

$$Z_d(\iota) = \eta(\iota)z_h(\iota) + (1 - \eta(\iota))z_l(\iota), \quad (19)$$

$$Z_{s,e}(\iota) = h\eta(\iota)\Pi(\min\{z_h(\iota), \hat{q}\}) + l(1 - \eta(\iota))\Pi(\min\{z_l(\iota), \hat{q}\}) + y, \quad (20)$$

where z_h and z_l are uniquely pinned down by (ι, η) unless $\iota = 0$ —they are then indeterminate up to the lower bound $v(\hat{q})$ —and η is uniquely determined by ι unless $\iota = \tilde{\iota}$ —it can then take any value in $[0, 1]$. An equilibrium occurs when ι entails zero excess demand $R(\iota) \equiv Z_d(\iota) - Z_{s,e}(\iota)$. There is no need to consider inter-temporal conditions here due to the combination of quasi-linear preferences and one-period lived equity; the economy resets itself every t . An equilibrium exists since: $R(\iota)$ is continuous in ι ; excess demand can take any value $R \geq R(0) \equiv v(\hat{q}) - h\Pi(\hat{q}) - y$ if $\iota = 0$; and $\lim_{\iota \nearrow \bar{\iota}} R(\iota) = -y$.

The more relevant question though is whether multiple ι can clear the liquid-wealth market. Both demand and supply of liquid wealth are after all decreasing in ι through two channels: (i) a *wealth channel* operating through reduced demand when ι increases, rendering matches less profitable for firms; and (ii) a *search channel* operating through a reduction in search when ι increases beyond $\tilde{\iota}$, entailing fewer matched firms.

The wealth channel is evaluated by:

$$R'(\iota)\Big|_{\iota \neq \tilde{\iota}} = \left(1 - \frac{\sigma \Pi'(\min\{v^{-1} \circ z_\sigma(\iota), \hat{q}\})}{v'(\min\{v^{-1} \circ z_\sigma(\iota), \hat{q}\})}\right) z'_\sigma(\iota)\Big|_{\iota \neq \tilde{\iota}}, \quad \text{with } \sigma = \begin{cases} h & \text{if } \iota < \tilde{\iota}, \\ l & \text{if } \iota > \tilde{\iota}. \end{cases} \quad (21)$$

The term in brackets is positive because $\Pi'(q) < v'(q)$ on the domain $(0, \hat{q}]$; if buyers increase their liquid wealth by a dollar, the firms' profits cannot increase by more than a dollar. The overall effect is therefore negative since z_σ depends negatively on ι ; a higher liquidity premium reduces search-contingent demand for liquid wealth. There can thus be only one market-clearing ι if σ is treated as exogenous—the wealth channel cannot generate equilibrium multiplicity. It can however amplify shocks as in Guerrieri and Lorenzoni (2009). For instance, fixing σ , an increase in the firms' endowment y leads to a more than one-to-one increase in value of equity Δ if the liquidity constraint binds. A higher endowment namely directly leads to a higher equity value, in turn loosening the buyers' liquidity constraint, which then increases the value of equity further.

Inverse excess liquid-wealth demand can be pinned down contingent on search level σ because of the weak wealth effect, which turns out useful later on. If $\iota = \Gamma_\sigma(R)$ is the ι entailing excess demand R given search level σ , then $\Gamma_\sigma(R)$ uniquely solves

$$R = \lambda^{-1}(\Gamma_\sigma/\sigma) - \sigma \Pi(\min\{v^{-1} \circ \lambda^{-1}(\Gamma_\sigma/\sigma), \hat{q}\}) - y, \quad \sigma \in \{l, h\}. \quad (22)$$

The search channel can be evaluated by comparing

$$\lim_{\iota \searrow \tilde{\iota}} R(\iota) = \tilde{z}_l - l \Pi \circ v^{-1}(\tilde{z}_l) - y \equiv \tilde{R}_l \quad \text{vs.} \quad \lim_{\iota \nearrow \tilde{\iota}} R(\iota) = \tilde{z}_h - h \Pi \circ v^{-1}(\tilde{z}_h) - y \equiv \tilde{R}_h; \quad (23)$$

the right- and the left-hand limit of $R(\iota)$ at $\tilde{\iota}$, where two opposing forces are at play. First, $\tilde{z}_l < \tilde{z}_h$ drives a negative wedge between \tilde{R}_l and \tilde{R}_h ; liquid wealth jumps down if ι increases beyond $\tilde{\iota}$, generating fall in both firm profit and demand for liquid wealth, and the latter effect dominates since $\Pi'(q) < v'(q)$, so that excess demand drops. Second, $l < h$ drives a positive wedge between \tilde{R}_l and \tilde{R}_h ; firms find fewer matches because search drops if ι moves above $\tilde{\iota}$, reducing the supply of liquid wealth and increasing excess demand. The latter effect dominates for sure if $\tilde{\iota} \rightarrow 0$ since \tilde{z}_l and \tilde{z}_h are then almost the same. This allows for multiplicity when $\tilde{R}_h < 0 < \tilde{R}_l$, illustrated in Figure 2.

Lemma 1. $\tilde{R}_h < 0 < \tilde{R}_l \Leftrightarrow (k, y) \in \mathcal{S}_e$, where \mathcal{S}_e has positive mass.

The proof of Lemma 1 characterizes the set \mathcal{S}_e explicitly. Figure 2 illustrates market clearance if $\tilde{R}_h < 0 < \tilde{R}_l$: there is an $\iota = \Gamma_h(0) < \tilde{\iota}$ inducing high search—a *boom*; an $\iota = \Gamma_l(0) > \tilde{\iota}$ inducing low search—a *bust*; and $\iota = \tilde{\iota}$ for which some buyers devote high and others devote low search—a *mix* with $\eta = \frac{\tilde{R}_l}{\tilde{R}_l - \tilde{R}_h}$. The liquidity premium can freely fluctuate over time between these levels, entailing endogenous dynamics.

Lemma 1 holds true under Assumption 3, pointing towards a different type of equilibrium multiplicity and self-fulfilling dynamics than in Section 3. It also contrasts the common perception that with a finitely-lived asset, there cannot be self-fulfilling dynamics. This perception is based on models in which, following Lucas (1978), an asset earns an exogenous dividend. Since a finitely-lived asset is priced fundamentally when it matures, through backwards induction, a chain of self-fulfilling expectations is ruled out. In particular, in their model of a monetary economy, Altermatt *et al.* (2023) show there cannot be cycles if the only liquid asset is finitely lived.

Equity in the setup above is the sole means of liquidity, finitely lived, and priced fundamentally when traded in the DM—its value equals the firms’ aggregate dividend (see Equation (17)). Yet, the dividend depends on DM trade, and DM trade depends on the dividend through the buyers’ liquidity constraint. This intricate relationship entails a strong strategic complementarity in search which is reminiscent of that in Diamond (1982) but operates through liquid wealth rather than increasing returns in the matching technology. Specifically, if other buyers search intensely, liquid-wealth supply increases, driving down the liquidity premium in order to clear the market for liquid wealth. This makes carrying liquid wealth cheaper, in turn making it more attractive for the individual buyer to relax its liquidity constraint, entailing higher match surplus and thus greater benefits of search.

The mechanism above is also different from the entry mechanism found in money-search models with endogenous seller participation (see, e.g., Rocheteau and Wright, 2005; Berentsen *et al.*, 2011; Nosal and Rocheteau, 2011). Fixing the liquidity premium in such models, buyers increase their demand for liquid wealth amid a greater mass of active sellers. This is because the probability of meeting a seller, and thus being able to use costly liquid wealth to relax a liquidity constraint, is then larger. Although sellers’

profitability is usually decreasing in the mass of active sellers due to market congestion, for some parametrizations this effect can be reversed exactly because matches become more profitable if buyers hold more liquid wealth, giving rise to a strategic complementarity in entry.

The entry complementarity is independent of the assets comprising liquid wealth because it works for a fixed liquidity premium. For instance, Nosal and Rocheteau (2011, Chapter 9.2) prove for a model with endogenous seller entry the existence of multiple steady states if liquid wealth comprises only currency, which, as in Section 3, implies the liquidity premium is pinned down by the currency-supply growth rate. In contrast, the strategic complementarity in buyers' search at the core of the current paper depends critically on the type of assets comprising liquid wealth. Particularly, it relies on the dual nature of equity in the DM. The value of equity is on the one hand determined by firms' profitability in the DM, while on the other hand firms' profitability depends on the value of equity through buyers' liquid wealth, but only if liquid wealth comprises equity. This is further elucidated by evaluating Lemma 1 in light of Section 3. Assumption 3 implies a generically unique bounded monetary equilibrium if liquid wealth comprises only fiat currency, while the search channel, under the same assumption, allows for multiple bounded monetary equilibria for a set of parameters with positive mass if liquid wealth comprises only equity.

5 Liquidity with currency and equity

Reconsider the scope for self-fulfilling bounded monetary equilibria as in Section 3, but with liquid wealth comprising *both* currency and equity (subscript *me*). Supply of currency again develops as $M = \mu M_{-1}$, and the first-order difference equation for x is

$$x \in f_{me}(\mathbb{E}\{x_{+1}\}) \equiv \beta [1 + \Lambda \circ (1 + \Delta) \circ \mathbb{E}\{x_{+1}\}] \mathbb{E}\{x_{+1}\} / \mu, \quad (24)$$

where the equity value $\Delta(\mathbb{E}\{x_{+1}\})$ depends endogenously on $\mathbb{E}\{x_{+1}\}$ through:

$$\Delta(\mathbb{E}\{x_{+1}\}) = h\eta(\iota)\Pi(\min\{z_h(\iota), \hat{q}\}) + l(1 - \eta(\iota))\Pi(\min\{z_l(\iota), \hat{q}\}) + y, \quad (25)$$

where $\iota = \Lambda \circ (1 + \Delta) \circ \mathbb{E}\{x_{+1}\}$.

Equation (24) differs from (18) as inverse demand Λ now depends also on the value of equity, which, in turn, is a function of $\iota = \Lambda$. Equation (25) captures this intricacy and implies f_{me} can be a correspondence rather than a function.

Exogenous dividend. Consider first equity with an exogenous dividend $\bar{\Delta}$:

$$x = \overline{f_{me}}(\mathbb{E}\{x_{+1}\}) \equiv \beta[1 + \Lambda(\mathbb{E}\{x_{+1}\} + \bar{\Delta})]\mathbb{E}\{x_{+1}\}/\mu. \quad (26)$$

Proposition 1. $\overline{f'_{me}} > 0$; *bounded monetary equilibria must be steady states, which are generically unique if $\mu > \beta$; and if $\mu = \beta$, there is a unique real allocation entailing a slack liquidity constraint and intense search.*

The implications of Assumption 3 for the existence of multiple bounded monetary equilibria thus remain unchanged when adding a one-period, fixed-dividend asset. In fact, the proof of Proposition 1 demonstrates the assumption can be slightly relaxed once such an asset is added.

Endogenous dividend and exogenous search. Consider next endogenous-dividend equity, but with exogenous search level σ . The search-contingent value of equity is

$$\Delta_{\sigma}(\mathbb{E}\{x_{+1}\}) = \sigma\Pi(\min\{v^{-1} \circ (1 + \Delta_{\sigma}) \circ \mathbb{E}\{x_{+1}\}, \hat{q}\}) + y. \quad (27)$$

This equation pins down $\Delta_{\sigma}(\mathbb{E}\{x_{+1}\})$ uniquely since $\Pi'(q) < v'(q)$, i.e., because of the weak wealth effect. With the search level fixed at σ , the difference equation in x is

$$x = f_{\sigma,me}(\mathbb{E}\{x_{+1}\}) \equiv \beta[1 + \sigma\lambda \circ (1 + \Delta_{\sigma}) \circ \mathbb{E}\{x_{+1}\}]\mathbb{E}\{x_{+1}\}/\mu. \quad (28)$$

Proposition 2. $f'_{\sigma,me} > 0$; *with fixed search, bounded monetary equilibria must be steady states, which are unique if $\mu > \beta$; and if $\mu = \beta$, a unique real allocation prevails entailing a slack liquidity constraint.*

Although equity now earns an endogenous dividend which depends on itself through the liquidity constraint, this entails only a wealth channel if search is fixed. As in Section 4, the wealth effect is too weak to generate bounded dynamics under Assumption 3.

Endogenous dividend and endogenous search. The endogenous search level depends on ι , and ι relates to $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$ through clearance of the liquid-wealth market:

$$\begin{aligned} & \eta(\iota)z_h(\iota) + (1 - \eta(\iota))z_l(\iota) \\ & \leq \mathbb{E}\{x_{+1}\} + h\eta(\iota)\Pi(\min\{z_h(\iota), \hat{q}\}) + l(1 - \eta(\iota))\Pi(\min\{z_l(\iota), \hat{q}\}) + y, \end{aligned} \quad (29)$$

with $=$ if $\iota > 0$. Equation (29) holds if and only if $\mathbb{E}\{x_{+1}\} = R(\iota)$, with $R(\iota)$ the excess liquid-wealth demand if liquid wealth comprises only equity (see Section 4). Given ι , currency balances $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$ thus absorb this excess demand.

To relate to search, recall a boom ($\eta = 1$) requires $\iota \leq \tilde{\iota}$ and thus entails $R(\iota) \geq \tilde{R}_h$ since $R(\iota)$ is locally decreasing in ι on the domain $(0, \tilde{\iota})$ (see Equation (21)). Given $\mathbb{E}\{x_{+1}\}$, a boom can thus occur if $\mathbb{E}\{x_{+1}\} \geq \tilde{R}_h$ since $R(\iota) = \mathbb{E}\{x_{+1}\}$ in equilibrium. Analogously, a bust ($\eta = 0$) requires $\iota \geq \tilde{\iota}$ and entails $R(\iota) \leq \tilde{R}_l$. Given, $\mathbb{E}\{x_{+1}\}$, a bust can thus occur if $\mathbb{E}\{x_{+1}\} \leq \tilde{R}_l$. Finally, a mix ($\eta \in (0, 1)$) requires $\iota = \tilde{\iota}$ and through Equation (29) it implies η must solve $\mathbb{E}\{x_{+1}\} = \eta\tilde{R}_h + (1 - \eta)\tilde{R}_l$. Given $\mathbb{E}\{x_{+1}\}$, a mix can thus occur if $0 < \frac{\mathbb{E}\{x_{+1}\} - \tilde{R}_l}{\tilde{R}_h - \tilde{R}_l} < 1$. Whether η is pinned down by $\mathbb{E}\{x_{+1}\} > 0$, where non-negativity is required in bounded monetary equilibrium, thus depends on whether $\tilde{R}_l \leq \max\{0, \tilde{R}_h\}$.

Lemma 2. $\max\{0, \tilde{R}_h\} < \tilde{R}_l \Leftrightarrow (k, y) \in \mathcal{S}_{me}$, with $\mathcal{S}_e \subseteq \mathcal{S}_{me}$.

Lemma 2 (the proof characterizes \mathcal{S}_{me}) generalizes Lemma 1 and implies search is not pinned down by currency balances $\Phi M_{-1} = \mathbb{E}\{x_{+1}\} > 0$ if there is a strong search channel. Phrased in spirit of Section 4, given currency balances, the market for liquid wealth clears for a unique ι if $\max\{0, \tilde{R}_h\} \geq \tilde{R}_l$, while it can clear for multiple ι if $\max\{0, \tilde{R}_h\} < \tilde{R}_l$. This affects the dynamic development of currency balances, as analyzed below.

Currency balances pin down search. This case has $\max\{0, \tilde{R}_h\} > \tilde{R}_l$, entailing

$$\eta = \begin{cases} 0 & \text{if } 0 < \mathbb{E}\{x_{+1}\} \leq \tilde{R}_l, \\ \frac{\mathbb{E}\{x_{+1}\} - \tilde{R}_l}{\tilde{R}_h - \tilde{R}_l} & \text{if } \max\{0, \tilde{R}_l\} < \mathbb{E}\{x_{+1}\} \leq \tilde{R}_h, \\ 1 & \text{if } \mathbb{E}\{x_{+1}\} > \max\{0, \tilde{R}_h\}; \end{cases} \quad (30)$$

i.e., currency balances pin down search (see Figure 3a). Building on insights from the case with exogenous search, particularly Equation (28), the difference equation in x becomes

$$x = f_{me}(\mathbb{E}\{x_{+1}\}) \equiv \begin{cases} f_{l,me}(\mathbb{E}\{x_{+1}\}) & \text{if } 0 < \mathbb{E}\{x_{+1}\} \leq \tilde{R}_l, \\ \beta(1 + \tilde{\iota})\mathbb{E}\{x_{+1}\}/\mu & \text{if } \max\{0, \tilde{R}_l\} < \mathbb{E}\{x_{+1}\} \leq \tilde{R}_h, \\ f_{h,me}(\mathbb{E}\{x_{+1}\}) & \text{if } \mathbb{E}\{x_{+1}\} > \max\{0, \tilde{R}_h\}. \end{cases} \quad (31)$$

Proposition 3. $\max\{0, \tilde{R}_h\} > \tilde{R}_l$ implies $f'_{me} > 0$; bounded monetary equilibria must be steady states, which are generically unique if $\mu > \beta$; and if $\mu = \beta$, a unique real allocation prevails entailing a slack liquidity constraint and intense search.

Thus, given currency balances, η and ι are pinned down and in combination with Assumption 3, there is a strong substitution effect in liquid-wealth demand. So, despite the presence of fiat currency and liquid equity, there are no self-fulfilling bounded dynamics.

Currency balances do not pin down search. This case has $\max\{0, \tilde{R}_h\} < \tilde{R}_l$, entailing¹⁵

$$\eta \in \begin{cases} 0 & \text{if } 0 < \mathbb{E}\{x_{+1}\} < \max\{0, \tilde{R}_h\}, \\ \left\{0, \frac{\tilde{R}_l - \mathbb{E}\{x_{+1}\}}{\tilde{R}_l - \tilde{R}_h}, 1\right\} & \text{if } \max\{0, \tilde{R}_h\} \leq \mathbb{E}\{x_{+1}\} \leq \tilde{R}_l, \\ 1 & \text{if } \mathbb{E}\{x_{+1}\} > \tilde{R}_l. \end{cases} \quad (32)$$

Thus, for currency balances $\Phi M_{-1} = \mathbb{E}\{x_{+1}\} \in [\tilde{R}_h, \tilde{R}_l] \cap \mathbb{R}_{++}$, there can be a boom, bust, or mix because of a strong strategic complementary in search—see Figures 3b–3c—, entailing the market for liquid wealth clears for three different ι , particularly $\iota = \Gamma_h(\mathbb{E}\{x_{+1}\}) < \tilde{\iota}$, $\iota = \tilde{\iota}$, and $\iota = \Gamma_l(\mathbb{E}\{x_{+1}\}) > \tilde{\iota}$. Hence, f_{me} reads as

$$x \in f_{me}(\mathbb{E}\{x_{+1}\}) \equiv \begin{cases} f_{l,me}(\mathbb{E}\{x_{+1}\}) & \text{if } 0 < \mathbb{E}\{x_{+1}\} < \max\{0, \tilde{R}_h\}, \\ \left\{ \begin{array}{l} f_{l,me}(\mathbb{E}\{x_{+1}\}), \\ \beta(1 + \tilde{\iota})\mathbb{E}\{x_{+1}\}/\mu, \\ f_{h,me}(\mathbb{E}\{x_{+1}\}) \end{array} \right\} & \text{if } \max\{0, \tilde{R}_h\} \leq \mathbb{E}\{x_{+1}\} \leq \tilde{R}_l, \\ f_{h,me}(\mathbb{E}\{x_{+1}\}) & \text{if } \mathbb{E}\{x_{+1}\} > \tilde{R}_l. \end{cases} \quad (33)$$

¹⁵The knife-edge case $\tilde{R}_l = \tilde{R}_h$ is ignored. For that case, η is pinned down unless $\mathbb{E}\{x_{+1}\} = \tilde{R}_l = \tilde{R}_h$, in which case any value for $\eta \in [0, 1]$ goes. However, f_{me} remains a monotonically increasing function, implying bounded monetary equilibria other than steady states do not exist.

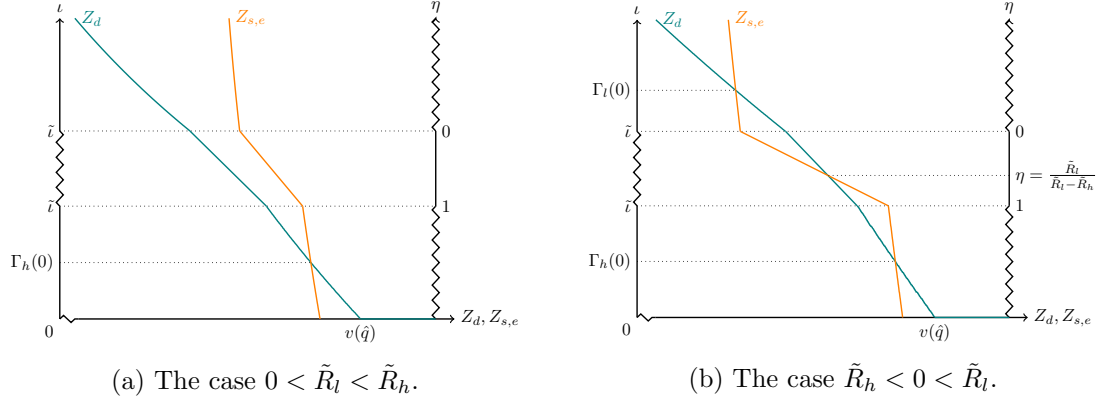


Figure 2: Depiction of liquid-wealth demand Z_d and supply $Z_{s,e}$ if liquid wealth comprises only equity. Parametrizations are in Section A.

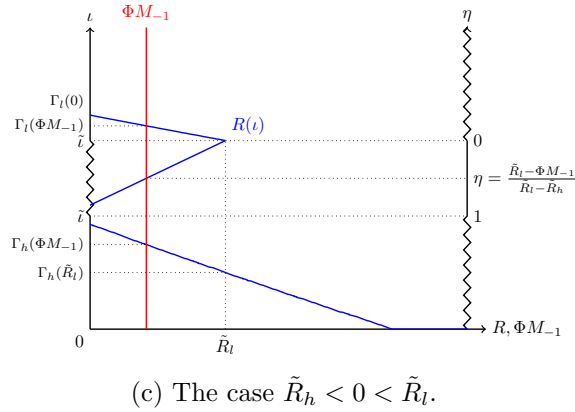
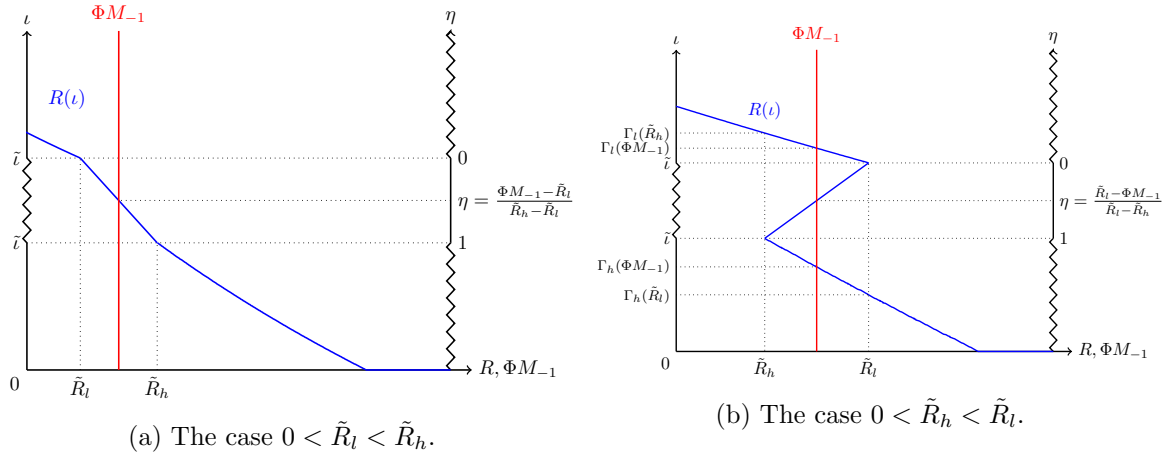


Figure 3: Sketched excess liquid-wealth demand and currency supply.

Multiple bounded monetary equilibria, particularly cyclical ones, now exist for some growth rates of currency supply. To see which, recall $\Gamma_\sigma(R)$ is the search-contingent inverse excess-demand function. Since $R = \mathbb{E}\{x_{+1}\}$, the lowest and highest value for ι which are consistent with market clearance given $\mathbb{E}\{x_{+1}\} \in [\max\{\varepsilon, \tilde{R}_h\}, \tilde{R}_l]$, where $\varepsilon > 0$ but infinitesimal, are $\Gamma_h(\tilde{R}_l)$ and, respectively, $\Gamma_l(\max\{\varepsilon, \tilde{R}_h\})$, as illustrated by Figures 3b and 3c. Let $\mathcal{I} \equiv \{\iota \geq 0 : \exists \varepsilon > 0 \text{ s.t. } \Gamma_h(\tilde{R}_l) \leq \iota \leq \Gamma_l(\max\{\varepsilon, \tilde{R}_h\})\}$ contain all values in between the extrema, with \mathcal{I} having positive mass if $\max\{0, \tilde{R}_h\} < \tilde{R}_l$ and containing $\tilde{\iota}$ in its interior (see the proof of Proposition 4).

Proposition 4. *If $\max\{0, \tilde{R}_h\} < \tilde{R}_l$, then there exist a two cycle if $\mu/\beta - 1 \in \text{int}(\mathcal{I})$. The cycle represents boom-bust dynamics with counter-cyclical inflation.*

The proof of Proposition 4 is illustrated by Figures 4, 5, and 6, sketching hypothetical f_{me} in the $(\mathbb{E}\{x_{+1}\}, x)$ -space, where $f_{l,me}$ and $f_{h,me}$ are monotonically increasing due to Assumption 3. If $\mu/\beta - 1 \in \text{int}(\mathcal{I})$, the strategic complementarity in search implies there is some level of currency balances $\Phi M_{-1} = \mathbb{E}\{x_{+1}\} \in (\max\{0, \tilde{R}_h\}, \tilde{R}_l)$ for which: both $\eta = 0$ and $\eta = 1$ are feasible; if $\eta = 0$ then $f_{me}(\mathbb{E}\{x_{+1}\}) = f_{l,me}(\mathbb{E}\{x_{+1}\}) > \mathbb{E}\{x_{+1}\}$; and if $\eta = 1$ then $f_{me}(\mathbb{E}\{x_{+1}\}) = f_{h,me}(\mathbb{E}\{x_{+1}\}) < \mathbb{E}\{x_{+1}\}$. These properties are illustrated in Figures 3b and 3c, and they imply there are levels for currency balances for which f_{me} can be both above and below the 45-degree line. The former entails a bust ($\eta = 0$) with a high level for ι , so that $x > \mathbb{E}\{x_{+1}\}$; and the latter entails a boom ($\eta = 1$) with a low level for ι , so that $x < \mathbb{E}\{x_{+1}\}$. Calling such a level for currency balances \hat{x} , consider the modified difference equation

$$x \in g(\mathbb{E}\{x_{+1}\}) \equiv \begin{cases} \{f_{l,me}(\mathbb{E}\{x_{+1}\})\} & \text{if } \mathbb{E}\{x_{+1}\} < \hat{x}, \\ [f_{h,me}(\mathbb{E}\{x_{+1}\}), f_{l,me}(\mathbb{E}\{x_{+1}\})] & \text{if } \mathbb{E}\{x_{+1}\} = \hat{x}, \\ \{f_{h,me}(\mathbb{E}\{x_{+1}\})\} & \text{if } \mathbb{E}\{x_{+1}\} > \hat{x}. \end{cases} \quad (34)$$

Here, g satisfies $g(0) = 0$, $g(x) > x \forall x \in (0, \hat{x})$, and $g(x) < x \forall x > \hat{x}$, as illustrated in Figures 4b, 5b, and 6b. The key point is that the graph of g is a vertical line at \hat{x} and intersects 45-degree line from above in that point. It follows from the method of flip bifurcations that there exist two points $(\mathbb{E}\{x_{+1}\}, x) = (x', x'')$ and $(\mathbb{E}\{x_{+1}\}, x) = (x'', x')$ where the graphs of g and g^{-1} intersect offside the 45-degree line, meaning $x' \neq x''$.

Now, if g were a correct difference equation in that it guarantees $x \in g(\mathbb{E}\{x_{+1}\}) \Rightarrow$

$x \in f_{me}(\mathbb{E}\{x_{+1}\})$, then $\{x_t\}_{t=0}^\infty = \{x', x'', x', x'', \dots\}$ would satisfy all equilibrium properties. Obviously, g does not exhibit that property, but if points (x', x'') and (x'', x') do not lie on the vertical or horizontal part of g and g^{-1} , see Figure 4b, then $x' < \hat{x} < x''$ and it can be proven $\{x_t\}_{t=0}^\infty = \{x', x'', x', x'', \dots\}$ is still an equilibrium. Particularly, it is a deterministic two-cycle in which $(\Phi M_{-1}, \eta)$ alternates between $(x', 0)$ and $(x'', 1)$, i.e., a boom-bust cycle with counter-cyclical inflation as in Figure 4c.

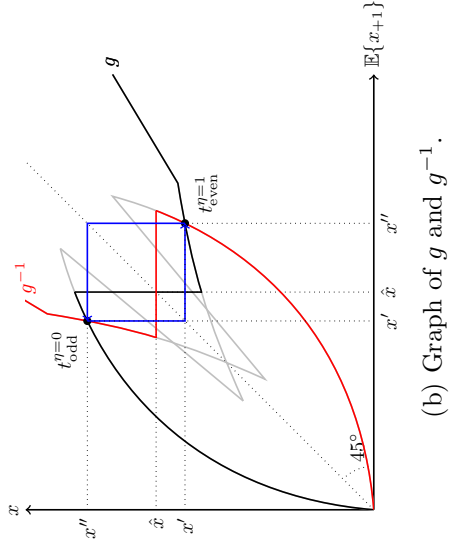
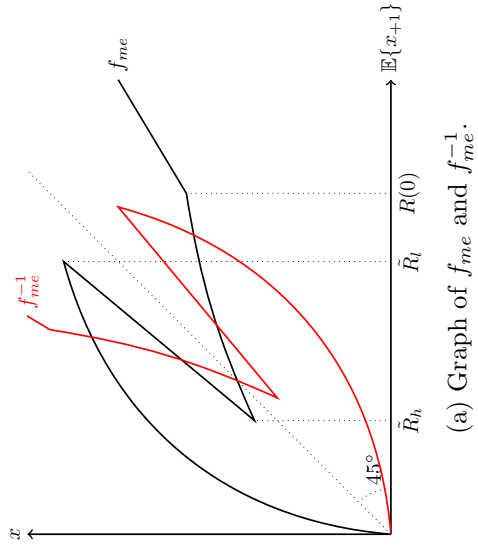
If the intersections between g and g^{-1} lie on the vertical part of g or the horizontal part of g^{-1} , then $\{x_t\}_{t=0}^\infty = \{x', x'', x', x'', \dots\}$ is no longer an equilibrium, but it can be used to construct a stochastic two cycle with boom-bust dynamics. This is because if $\mathbb{E}\{x_{+1}\} = \hat{x}$, both $\eta = 0$ and $\eta = 1$ are consistent with equilibrium since $\hat{x} \in (\max\{0, \tilde{R}_h\}, \tilde{R}_l)$. The resulting dynamics again feature counter-cyclical inflation (see Figures 5c and 6c).

Proposition 5. *If $\max\{0, \tilde{R}_h\} < \tilde{R}_l$ and $\mu/\beta - 1 \in \mathcal{I}/\text{int}(\mathcal{I})$, there exist bounded monetary equilibria that converge to the monetary steady state with a boom-bust cycle.*

Proposition 5 applies to the knife-edge cases: $\mu = \beta(1 + \Gamma_h(\tilde{R}_l))$; and $\mu = \beta(1 + \Gamma_l(\tilde{R}_h))$ if $\tilde{R}_h > 0$. They are characterized by a unique monetary steady state involving a boom respectively bust. However, the steady state currency balances are also consistent with having a bust, respectively, boom. Figure 7 and 8 illustrates how this allows for transition to the steady state in a boom-bust-boom respectively bust-boom-bust fashion. Interestingly, Proposition 5 can apply as well at the Friedman rule (see Figure 9), i.e. $\mu = \beta$. Particularly, this happens when $\tilde{R}_l \geq R(0)$, i.e., if a level for currency balances rendering the liquidity constraint slack in a boom also allows for a bust.

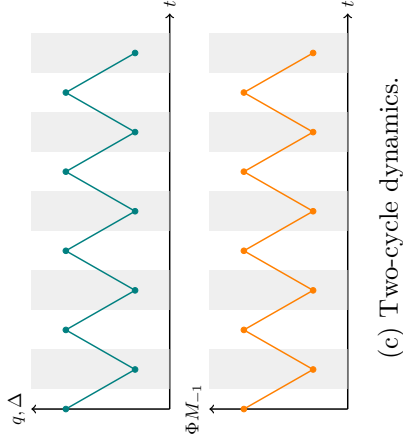
Proposition 6. *If $\max\{0, \tilde{R}_h\} < \tilde{R}_l$ and $\mu/\beta - 1 \notin \mathcal{I}$, then bounded monetary equilibria must be steady states, which are generically unique if $\mu > \beta$; and if $\mu = \beta$, a unique real allocation prevails entailing intense search and a slack liquidity constraint.*

Despite $\max\{0, \tilde{R}_h\} < \tilde{R}_l$ implying there are values for currency balances inducing both a boom, bust, and mix, these values are too far away from the steady state if $\mu/\beta - 1 \notin \mathcal{I}$. More technically, for some x there are still multiple $\mathbb{E}\{x_{+1}\}$ for which $x \in f_{me}(\mathbb{E}\{x_{+1}\})$ since f_{me} is a correspondence, but these $\mathbb{E}\{x_{+1}\}$ are, because of Assumption 3, either: too far below the monetary steady state if μ is high, so x would go to zero anyhow; or too far above the monetary steady state if μ is low, so x would go to infinity anyhow.

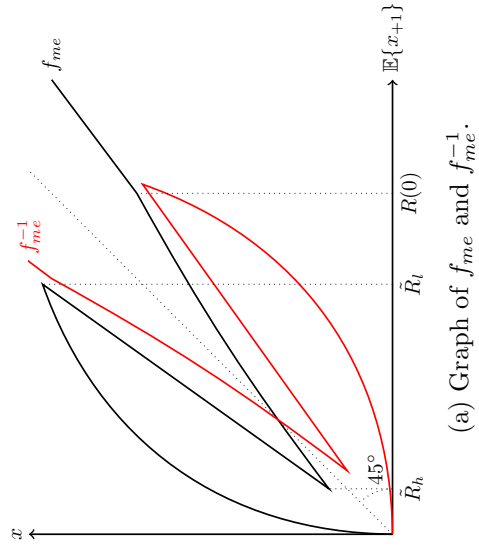


(a) Graph of f_{me} and f_{me}^{-1} .

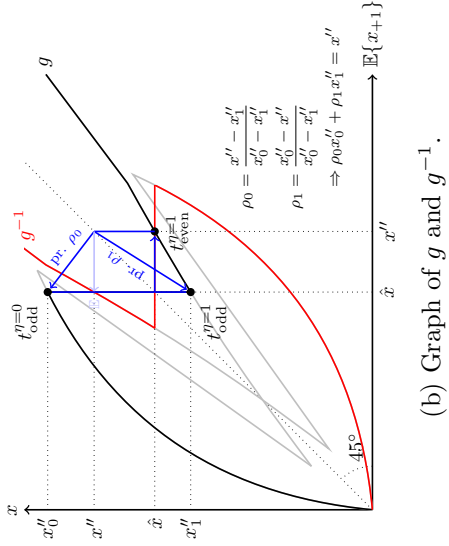
Figure 4: The sketched case for a deterministic two cycle. Gray shaded areas in panel 4c are busts (i.e., $\eta = 0$).



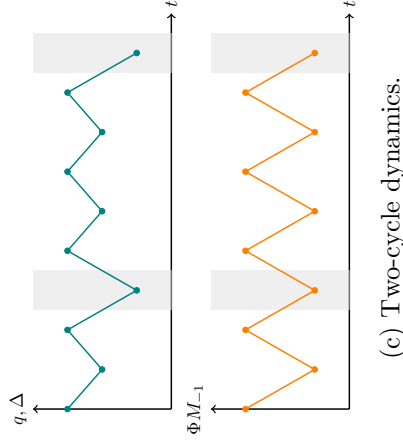
(c) Two-cycle dynamics.



(a) Graph of f_{me} and f_{me}^{-1} .



(b) Graph of g and g^{-1} .



(c) Two-cycle dynamics.

Figure 5: The sketched case for a stochastic two cycle with occasional busts. Gray shaded areas in panel 5c are busts.

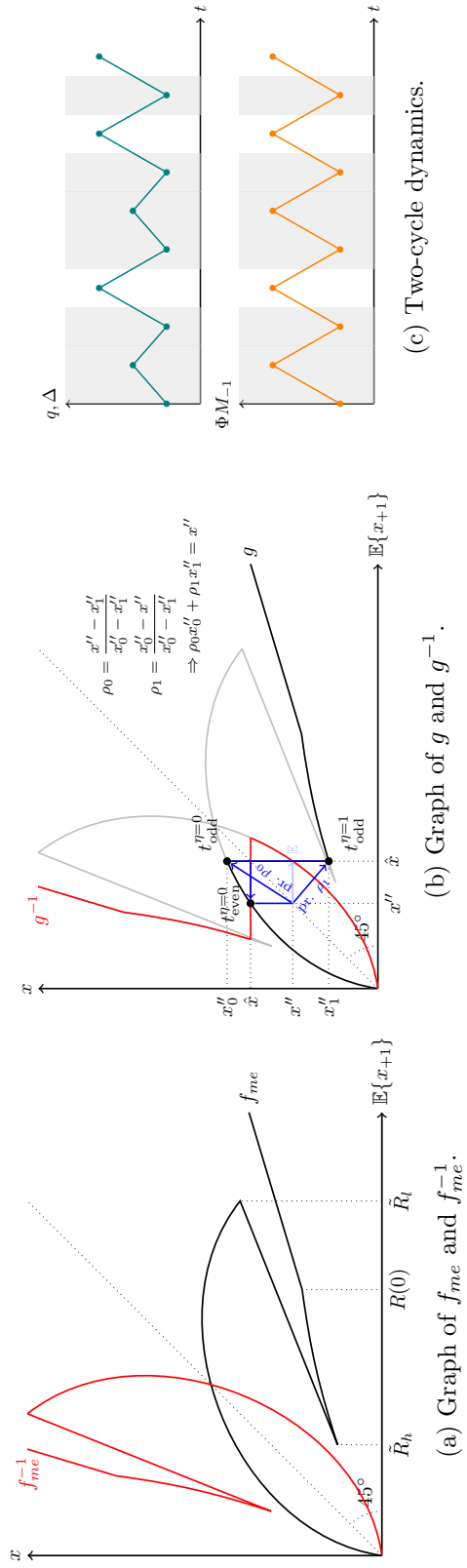
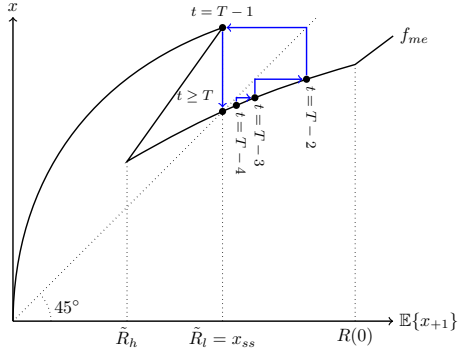
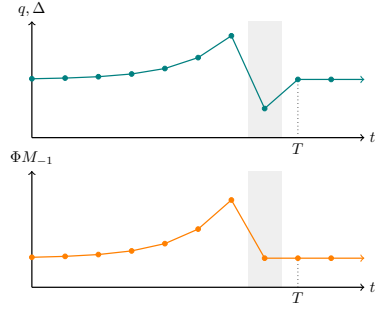


Figure 6: The sketched case for a stochastic two cycle with occasional booms. Gray shaded areas in panel 5c are busts.

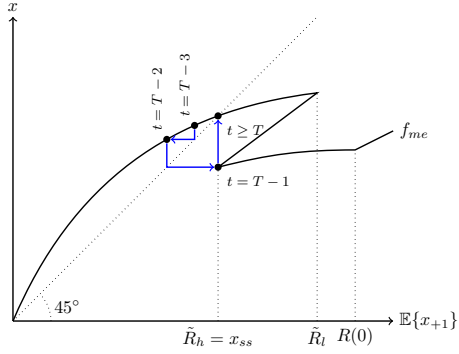


(a) Graph of f_{me} .

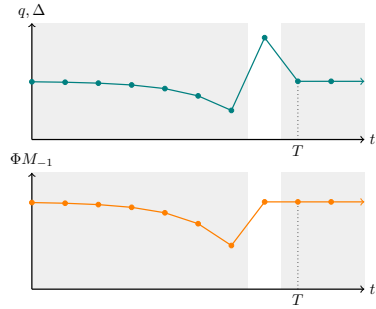


(b) Transition dynamics.

Figure 7: The sketched case for transition dynamics to the steady state when $\mu = \beta(1 + \Gamma_h(\tilde{R}_l))$ and $\tilde{R}_h < \tilde{R}_l$. Gray shaded areas in Panel 7b are busts.

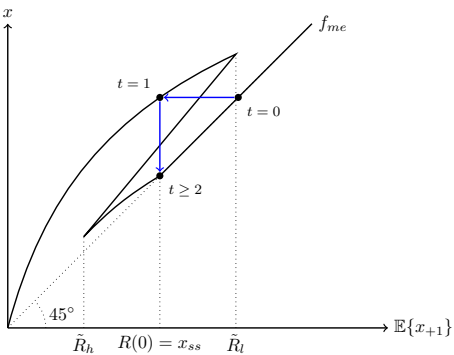


(a) Graph of f_{me} .

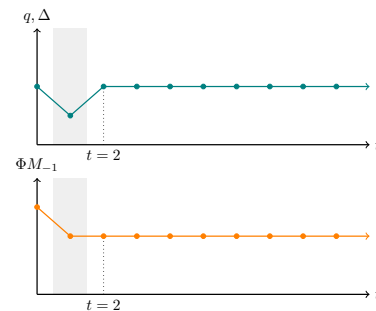


(b) Transition dynamics.

Figure 8: The sketched case for transition dynamics to the steady state when $\mu = \beta(1 + \Gamma_l(\tilde{R}_h))$ and $0 < \tilde{R}_h < \tilde{R}_l$. Gray shaded areas in Panel 8b are busts.



(a) Graph of f_{me} .



(b) Transition dynamics.

Figure 9: The sketched case for transition dynamics to the steady state at the Friedman rule if $\tilde{R}_l > R(0)$. Gray shaded areas in Panel 9b are busts.

Taking stock, under Assumption 3 only steady states can be bounded monetary equilibria if liquid wealth comprises only currency, or currency and an exogenous-dividend asset; while bounded equilibria with self-fulfilling cycles can exist under Assumption 3 if liquid wealth comprises only equity, or both equity and currency. Proposition 2 elucidates endogenous search is critical to this novel insight; if search were exogenous, Assumption 3 would still rule out self-fulfilling cycles despite the endogeneity of firms' dividend. In that sense, the results point to the joint role of search and liquid equity as a novel source of self-fulfilling dynamics in money-search models.

Propositions 4 and 5 indicate such dynamics arise under, loosely speaking, two conditions. First, the search channel should be strong, meaning the market for liquid-wealth can clear for multiple search levels. Second, the money-growth rate must entail steady state currency balances for which that is indeed the case. Proposition 5 furthermore indicates a subtlety when the Friedman rule is implemented with a currency-growth target. Under such a target, the Friedman rule implies a steady state with intense search and a slack liquidity constraint, leading to maximized economic activity as is well-known in the literature. However, the steady state may not be reached immediately; there could be a boom-bust-boom like transition path to it due to endogenous liquid-wealth supply.

6 Inflation targeting and stabilization policies

Central banks nowadays target inflation instead of currency-supply growth. At first sight this may help eliminate the equilibria with endogenous dynamics identified earlier since these feature fluctuating real currency balances and thus fluctuating inflation. To investigate whether this is true, suppose the government implements gross inflation target π , with currency supply adjusting endogenously. The optimal price index is the nominal price of CM goods due to quasi-linear preferences, so $1/\Phi$ should grow at a gross rate π ; $\Phi_{+1} = \Phi/\pi$. From Equation (11) this implies

$$\mathbb{E}_{-1} \{\iota\} = (\pi - \beta)/\beta \equiv i. \quad (35)$$

Here, i is the *Fisher rate*: the nominal interest rate compensating exactly for inflation and time discounting. In equilibrium, i pins down only the expected value for ι . This precludes inflation targeting alone may not suffice to stabilize the economy.

Deterministic environment. If all uncertainty about time t , particularly the realization of s , is already revealed in the previous period, then $\iota = \mathbb{E}_{-1}\{\iota\}$ and the inflation target pins down ι through the Fisher rate. The real allocations are then pinned down uniquely, except for the knife-edge case $i = \tilde{i}$, as is clear from Equations (12) and (14).

Stochastic environment. If the outcomes at time t are not perfectly predictable one period in advance, ι can fluctuate in that $\iota \neq \mathbb{E}_{-1}\{\iota\}$. Yet, currency balances ΦM_{-1} act as an exogenous variable in the DM since $\Phi = \Phi_{-1}/\pi$ if the inflation target is implemented. Thus, $\Phi M_{-1} = \mathbb{E}_{-1}\{\Phi M_{-1}\}$; currency balances are perfectly predictable one period in advance, so write $\mathbb{M}_{-1} = \mathbb{E}_{-1}\{\Phi M_{-1}\}$ to capture this. Demand and supply of liquid wealth are functions of ι , where supply now includes \mathbb{M}_{-1} :

$$Z_{s,\pi e}(\iota) = h\eta(\iota)\Pi(\min\{z_h(\iota), \hat{q}\}) + l(1 - \eta(\iota))\Pi(\min\{z_l(\iota), \hat{q}\}) + y + \mathbb{M}_{-1}. \quad (36)$$

Not much changes compared to the analysis in Section 4 (cf. Equation (20)); given some $\mathbb{M}_{-1} > 0$, market clearance now occurs if $R(\iota) = \mathbb{M}_{-1}$. Earlier results directly imply:

Corollary 1. *If $(k, y) \in \mathcal{S}_{me}$ and $\mathbb{M}_{-1} \in [\max\{0, \tilde{R}_h\}, \tilde{R}_l]$, then $R(\iota) = \mathbb{M}_{-1}$ for $\iota = \Gamma_h(\mathbb{M}_{-1}) < \tilde{i}$, $\iota = \tilde{i}$, and $\iota = \Gamma_l(\mathbb{M}_{-1}) > \tilde{i}$. Otherwise, ι is pinned down by \mathbb{M}_{-1} .*

With multiple market clearing ι , the corresponding Fisher rate is

$$i = \mathbb{E}_{-1}\{\iota\} = \mathbb{P}_{-1}\{\iota = \Gamma_h(\mathbb{M}_{-1})\}\Gamma_h(\mathbb{M}_{-1}) + \mathbb{P}_{-1}\{\iota = \tilde{i}\}\tilde{i} + \mathbb{P}_{-1}\{\iota = \Gamma_l(\mathbb{M}_{-1})\}\Gamma_l(\mathbb{M}_{-1}). \quad (37)$$

If there is multiplicity, an equilibrium can thus be thought of as currency balances $\mathbb{M}_{-1} \in [\max\{0, \tilde{R}_h\}, \tilde{R}_l]$ and probabilities $(\mathbb{P}_{-1}\{\iota = \Gamma_h(\mathbb{M}_{-1})\}, \mathbb{P}_{-1}\{\iota = \tilde{i}\}, \mathbb{P}_{-1}\{\iota = \Gamma_l(\mathbb{M}_{-1})\})$ satisfying Equation (37) given $i = \pi/\beta - 1$. It has been established in Section 5 that the lowest and highest value for ι which can be observed for $\mathbb{M}_{-1} \in [\max\{0, \tilde{R}_h\}, \tilde{R}_l]$ are $\Gamma_h(\tilde{R}_l)$ and $\Gamma_l(\max\{0, \tilde{R}_h\})$. It follows rather directly that if and only if i is strictly in between these extrema, there exists an $\mathbb{M}_{-1} \in [\max\{0, \tilde{R}_h\}, \tilde{R}_l]$ and probabilities $(\mathbb{P}_{-1}\{\iota = \Gamma_h(\mathbb{M}_{-1})\}, \mathbb{P}_{-1}\{\iota = \tilde{i}\}, \mathbb{P}_{-1}\{\iota = \Gamma_l(\mathbb{M}_{-1})\})$ for which Equation (37) holds and ι is non-degenerate, i.e., $\mathbb{P}_{-1}\{\iota = \mathbb{E}_{-1}\{\iota\}\} \neq 1$.

Proposition 7. *If $\max\{0, \tilde{R}_h\} < \tilde{R}_l$, there is equilibrium multiplicity if $i \in \text{int}(\mathcal{I})$, i.e., the probability distribution for ι can either be degenerate at i or non-degenerate, mean-*

ing ι fluctuates stochastically. If either $\max\{0, \tilde{R}_h\} \geq \tilde{R}_i$ or $i \notin \text{int}(\mathcal{I})$, the probability distribution for ι is degenerate at i .

Proposition 7 implies the following in comparison to Proposition 4. If two cycles exist for a currency-growth regime entailing steady-state inflation $\pi = \mu$, then if the same inflation rate is implemented successfully in an inflation-targeting regime, there is still scope for stochastic dynamics. Only the transition dynamics for $i \in \mathcal{I}/\text{int}(\mathcal{I})$ identified in a currency-growth regime vanish under inflation targeting. In that sense, there is an almost one-to-one overlap between the set of parameters for which there is generic multiplicity of bounded monetary equilibria under currency-growth targeting and inflation targeting. The implications of the search complementarity thus do not hinge too much on the conduct of monetary policy, although there is a notable exception at the Friedman rule as discussed below.

With a successful inflation target, currency acts as a risk-free liquid asset since its real return is perfectly predictable. Buyers can thus use it as a substitute for risky equity, and if they would only rely on currency as a form of liquid wealth, equilibrium multiplicity would vanish. A positive Fisher rate however implies currency is costly to hold, so buyers also use equity as liquid wealth. The intricate relationship between liquid-wealth demand and supply uncovered in Section 4 thus remains present. If the Fisher rate approaches zero, the scope for equilibrium multiplicity disappears as buyers then have access to a risk-free and costless form of liquid wealth:

Corollary 2. $\lim_{i \rightarrow 0} \mathbb{P}_{-1}\{\iota = 0\} = 1$.

Compared to Proposition 5, Corollary 2 points to the desirability of running the Friedman rule in an inflation-targeting environment, i.e., setting $\pi = \beta$ to eliminate the opportunity cost of holding currency. Particularly, over and above the fact that the Friedman rule is consistent with maximizing economic activity, it also fosters financial and macroeconomic stability more in an inflation-targeting regime than in a currency-growth regime. Implementing the Friedman rule with currency-supply growth namely leaves room for self-fulfilling transitional dynamics involving a boom-bust-boom pattern, while this is not the case if it is implemented with an inflation target. This indicates narrow-money growth targets are unreliable when liquid wealth comprises also assets whose value is closely tied

to macroeconomic activity. It is more effective to target the Fisher rate directly by accordingly adjusting currency supply in line with demand, as currency demand endogenously adjusts for the liquid wealth provided by other assets.

A variety of reasons may motivate why it is optimal to deviate from the Friedman rule, e.g.: to alleviate congestion effects in other decentralized markets (Shi, 1997; Nosal, 2011); to reduce real borrowing rates (through inflation) for borrowing-constrained entrepreneurs (Altermatt, 2022); or to improve the distribution of wealth across agents with heterogeneous preferences (Andolfatto, 2011; Kocherlakota, 2003; Uras and van Buggenum, 2022; van Buggenum, 2023). To close the analysis, consider how the economy can be stabilized away from the Friedman rule, without explicitly motivating why the Friedman rule is not being implemented, as this can be done by building on the aforementioned papers.

Stabilization under a successful inflation target implies the government must intervene in the DM, as otherwise currency balances ΦM_{-1} act as a predetermined variable, leading to exactly the same findings as before. Focus on inflation targets $\pi < \beta(1 + \tilde{t})$, i.e., the deterministic equilibrium features intensive search. Stabilization then requires preventing busts, which can be done with an asset purchasing program to back equity. Consider the government stands ready to purchase equity at real price $\underline{\Delta}$ during the DM. Although currency is nominal, guaranteeing a real price is feasible but has fiscal implications as detailed later. The fraction ω of equity sold to the government in the DM is

$$\omega \begin{cases} = 0 & \text{if } \Delta > \underline{\Delta}, \\ \in [0, 1] & \text{if } \Delta = \underline{\Delta}, \\ = 1 & \text{if } \Delta < \underline{\Delta}; \end{cases} \quad (38)$$

where $\Delta = h\eta(\iota)\Pi(\min\{z_h(\iota), \hat{q}\}) + l(1 - \eta(\iota))\Pi(\min\{z_l(\iota), \hat{q}\}) + y$ is the actual equity value. Buyers thus sell equity if the value falls short of $\underline{\Delta}$. The resulting supply of liquid wealth is $Z_s = \max\{\Delta, \underline{\Delta}\} + \mathbb{M}_{-1}$, where $\mathbb{M}_{-1} = \Phi M_{-1}$ are currency balances brought into the DM measured before equity has been sold to the government. The government thus effectively puts a floor below the value of equity.

Proposition 8. *The lower bound on $\underline{\Delta}$ to rule out stochastic equilibrium multiplicity is*

$$\Delta' \equiv \eta' h \Pi \circ v^{-1}(\tilde{z}_h) + (1 - \eta') l \Pi \circ v^{-1}(\tilde{z}_l) + y, \quad \text{where} \quad \eta' \equiv \frac{\tilde{R}_l - R(i)}{\tilde{R}_l - \tilde{R}_h}. \quad (39)$$

Note the purchase price $\underline{\Delta}$ can be set below the deterministic-equilibrium value of equity because of the weak wealth effect.¹⁶

Equity purchases affect the lump-sum transfer off the equilibrium path, indicating such a program requires fiscal support.¹⁷ With inflation targeting and equity purchases, the transfer is $\tau = \Phi(M - M_{purch,-1})$, where $M_{purch,-1} \equiv M_{-1} + \omega(\underline{\Delta} - \Delta)/\Phi$ is nominal currency brought into the CM net of the nominal value of equity bought by the government. With inflation targeting, in the CM the government passively supplies real currency balances $\Phi M = \pi \Phi_{+1} M = \pi \mathbb{M}$ that buyers carry out of the CM given the Fisher rate, so that $\tau = \pi \mathbb{M} - \mathbb{M}_{-1} - \max\{\underline{\Delta} - \Delta, 0\}$, where \mathbb{M} is determined by buyers' next-period demand for currency balances.

The expression for τ elucidates purchases materialize only if Δ drops below $\underline{\Delta}$; purchases entail a loss for the government since equity is bought above fundamental value. This loss has to be passed on to the taxpayer if the inflation target is to be achieved in all contingencies. There would be excess currency supply otherwise since $M_{purch,-1} > M_{-1}$, causing inflationary pressure that would also nullify the effect of equity purchases. To see this, let $D \equiv \Delta/\Phi$ denote the nominal value of equity and \underline{D} a nominal purchase price, which can be guaranteed for sure. Suppose now there is no fiscal support for asset purchases in that $\tau = (\pi - 1)R(\pi/\beta - 1)$, i.e., the transfer equals the level observed in the deterministic equilibrium without asset purchases, where $\iota = \pi/\beta - 1$ and $\mathbb{M} = \mathbb{M}_{-1} = R(\pi/\beta - 1)$, with π the inflation target.¹⁸ If $\iota_{+1} = \pi/\beta - 1$ with probability one and there are no asset purchases in the next period, i.e., it is business-as-usual in the next period, then $\mathbb{M} = R(\pi/\beta - 1)$, so the lack of fiscal support implies

¹⁶It suffices to use as purchasing price the value that would prevail if $\iota = \tilde{\iota}$ and the real currency supply is at the deterministic-equilibrium level $R(i)$.

¹⁷Fiscal support differs from fiscal backing as Del Negro and Sims (2015) point out. The latter requires that the fiscal authority adjusts its primary surpluses to ensure it can repay its debt in a way consistent with the central bank's inflation target. The former means both profits and losses of the central bank are ultimately passed on to the fiscal authority. In the context of the model, τ can be thought of as a transfer of the central bank to the fiscal authority. The fiscal authority takes τ as given and immediately passes it on to the taxpayer. Fiscal support in this context entails the fiscal authority allows τ to take any positive or negative value.

¹⁸Assuming $\tau \geq (\pi - 1)R(\pi/\beta - 1)$ yields similar results, what matters is that the transfer cannot drop below the value in deterministic equilibrium.

$\Phi [M_{-1} + \omega(\underline{D} - D)] = \Phi M_{purch,-1} = R(\pi/\beta - 1)$. Thus, if the real value $\Phi D = \Delta$ of purchased equity goes down, keeping everything else constant, the real value Φ of currency must go down as well, i.e., losses on asset purchases must be inflated away.

By construction, real liquidity supply in the CM is $Z_s = \Phi [M_{-1} + \omega \underline{D} + (1 - \omega)D] = \Phi M_{purch,-1} + \Delta$. If fiscal support is lacking, it thus holds $Z_s = R(\pi/\beta - 1) + \Delta$. This transpires that the government now cannot prevent a self-fulfilling drop in liquid wealth; Corollary 1 implies there are multiple market clearing ι if $\pi/\beta - 1 \in \text{int}(\mathcal{I})$. The bust equilibrium thus cannot be prevented. This is because the government can purchase equity at a high nominal price, but because fiscal support is lacking, it cannot guarantee a real price. In particular, despite the government putting a floor below the nominal value of equity, the real value of equity may still drop since this would trigger inflation which renders the nominal floor ineffective.

Returning to the case with fiscal support, the government's commitment to pass on losses to the taxpayer ensures currency injected into the economy with equity purchases has real value. This commitment is strong enough to stabilize the economy if $\underline{\Delta} \geq \Delta'$. Purchases then never materialize on the equilibrium path; there is no reason for the value of equity to drop below $\underline{\Delta}$.

Proposition 9. *If $\underline{\Delta} \in (\Delta'', \Delta')$, where Δ'' solves $\Delta'' = l\Pi \circ v^{-1}(R(i) + \Delta'') + y$, then purchases can materialize with positive probability since they fail to stabilize the economy. If $\underline{\Delta} \leq \Delta''$, then purchases never materialize and the economy is not stabilized either.*

Using equity purchases too conservatively can however fiscally backfire, as Proposition 9 indicates. If the purchase price is set slightly below the threshold $\underline{\Delta}$, i.e., because the policymaker is worried about potential losses, then purchases fail to stabilize the economy, which allows the value of equity to drop strictly below $\underline{\Delta}$ because of self-fulfilling beliefs. If that happens, buyers sell their equity to the government, which then indeed runs a loss since it buys equity at a price above the correct value. Counter intuitive at first sight, the loss can be avoided by setting the purchase price slightly higher rather than lower, so as to unwind the self-fulfilling beliefs rationalizing the drop in equity value. If, on the other hand, the purchase price is set very low, the premise of equity purchases still fails to stabilize the economy but, exactly because the price is set very low, the value of equity cannot drop below $\underline{\Delta}$. The economy can thus experience a bust, but the government never actually buys equity so it never experiences a loss either.

7 Conclusion

This paper introduces liquid equity in a money-search model. Equity is a claim on the profits of firms that sell goods in the search-and-matching market, and simultaneously, equity is used in payment by the buyers in the search-and-matching market. This interwovenness entails a strong strategic complementarity in search, entailing self-fulfilling bounded dynamics. The joint role of liquid equity and search is elucidated by assuming ex-ante liquid wealth-demand is decreasing in the liquidity premium. While this rules out self-fulfilling bounded dynamics in plain-vanilla models, such dynamics are preserved with liquid equity and endogenous search. The economy is stable at the Friedman rule in an inflation-targeting regime, or, if away from the Friedman rule, if inflation targeting is combined with equity purchases.

Directions for future research are at least twofold. First, the current setup views equity as a one-period lived asset, which is a useful assumption to derive transparent results but also arguably unrealistic. Relaxing it by modeling equity as a long-lived asset is a useful extension to bring the model to the data. Second, it can be useful to distinguish between direct and indirect liquidity as in Geromichalos and Herrenbrueck (2016, 2017) and Geromichalos *et al.* (2021). The current model has directly-liquid equity; it can be used to purchase goods in the search-and-matching market. In reality, equity is rather indirectly liquid; it must first be sold for directly-liquid assets (currency, deposits, *etcetera*) in a financial market, after which these assets can be used for real transactions. In the current model these two steps can be thought of as occurring simultaneously; the financial market can be accessed when in a bilateral match. If the steps occur sequentially, indirectly-liquid assets typically inherit the properties of their liquid counterparts. It would be interesting to investigate if indirectly-liquid equity and search interact similarly as directly-liquid equity and search.

A Figure parametrizations

Parameters for Figure 1. Flow utility from DM consumption is

$$u(q) = \frac{(q+b)^{1-\rho} - b^{1-\rho}}{1-\rho}, \quad \rho > 0, \quad (\text{A.1})$$

where b is a positive constant ensuring $u(0) = 0$. The pricing protocol is simply $v(q) = q$, and the specification of $c(q)$ and y are irrelevant since equity is illiquid. Panels 1a and 1b have fixed search effort with parameter values specified in Table 1.

Fig	β	μ	σ	ρ	b
1a	0.98	1.5	0.8	0.8	0.50
1b	0.98	1.8	0.8	6.0	5.25

Table 1: Parameter values for Figures 1a and 1b.

Panels 1c-1h have endogenous search with parameter values specified in Table 2.

Fig	β	μ	h	l	\tilde{t}	ρ	b
1c	0.98	1.10	0.8	0.4	$4.0 \times (\mu/\beta - 1)$	0.80	0.50
1d	0.98	1.50	0.8	0.4	$0.5 \times (\mu/\beta - 1)$	0.80	0.50
1e	0.98	1.25	0.9	0.6	$1.0 \times (\mu/\beta - 1)$	0.80	0.50
1f	0.98	1.8	0.8	0.4	$2.0 \times (\mu/\beta - 1)$	6.00	5.25
1g	0.98	1.8	0.8	0.4	$2.0 \times (\mu/\beta - 1)$	7.25	5.25
1h	0.98	1.2	0.8	0.4	$4.0 \times (\mu/\beta - 1)$	0.80	0.50

Table 2: Parameter values for Figures 1c-1h. The value for k can be backed out by using \tilde{t} in Equation (E.4).

Parameters for Figure 2. $u(q)$ is as in Equation (A.1), $c(q) = q$ and $v(q) = \sigma q$, with $\sigma > 1$; the pricing protocol v represents constant markup pricing. The parameter values are in Table 3 and β is irrelevant due to the static nature of the equilibrium if only equity is liquid. The illiquidity of currency entails the irrelevance of μ .

Fig	σ	y	h	l	\tilde{t}	ρ	b
2a	1.3	0.5407	0.8	0.4	0.06	0.50	0
2b	1.3	0.5654	0.8	0.4	0.02	0.50	0

Table 3: Parameter values for Figure 2. The value for k can be backed out by using \tilde{t} in Equation (E.4), and it holds true that $y > c(\hat{q})$ for both parametrizations.

B Value functions and bargaining

This appendix details the derivation of the buyers' Bellman Equation (2), the firms' dividend (7), and the surplus of bilateral matches.

Centralized market. An incumbent firm, born in CM_{-1} , pays dividend and subsequently dies. An incumbent holding an asset portfolio worth p CM goods as well as an inventory o of CM goods will pay a dividend

$$\delta(p, o) = p + o. \quad (\text{B.1})$$

Let $(m_{+1}, e_{+1}) \in (\mathbb{R}_+^S)^2$ describe the portfolio of Arrow securities for currency and equity the buyer carries out of the CM. Let $V_{+1}(m_{+1}(s_{+1}), e_{+1}(s_{+1}))$ be the value of entering DM_{+1} given s_{+1} , to be characterized later. To simplify the notation, let $V_{+1}(m_{+1}, e_{+1}) = V_{+1}(m_{+1}(s_{+1}), e_{+1}(s_{+1}))$, which is in line with the notation used in Section 2. The utility value of entering the CM with currency and equity $(m, e) \in \mathbb{R}_+^2$ is

$$\begin{aligned} W(m, e) = & \max_{(x, (m_{+1}, e_{+1})) \in \mathbb{R} \times (\mathbb{R}_+^S)^2} \{x + \beta \mathbb{E}\{V_{+1}(m_{+1}, e_{+1})\}\} \\ \text{s.t. } & x + \int_S [\phi_{+1}(s')m_{+1}(s') + \psi_{+1}(s')e_{+1}(s')] \mathbb{P}(ds') \leq \Phi m + \Delta e + \tau + \Psi, \end{aligned} \quad (\text{B.2})$$

where τ is the government transfer, Ψ the lump-sum transfer arising from the issuance of new equity, and expectation \mathbb{E} is taken w.r.t. probability law \mathbb{P} . The budget constraint in (B.2) binds for the optimal choices and since utility is linear in x , we can write W as

$$\begin{aligned} W(m, e) = & \Phi m + \Delta e + \tau + \Psi \\ & + \max_{(m_{+1}, e_{+1}) \in (\mathbb{R}_+^S)^2} \left\{ - \int_S [\phi_{+1}(s')m_{+1}(s') + \psi_{+1}(s')e_{+1}(s')] \mathbb{P}(ds') \right. \\ & \left. + \beta \mathbb{E}\{V_{+1}(m_{+1}, e_{+1})\} \right\}. \end{aligned} \quad (\text{B.3})$$

With \mathbb{E} being the expectation operator w.r.t. probability law \mathbb{P} , we have

$$\int_S [\phi_{+1}(s')m_{+1}(s') + \psi_{+1}(s')e_{+1}(s')] \mathbb{P}(ds) \equiv \mathbb{E}\{\phi_{+1}m_{+1} + \psi_{+1}e_{+1}\} \quad (\text{B.4})$$

and thus (B.3) simplifies as

$$W(m, e) = \Phi m + \Delta e + \tau + \Psi + \max_{(m_{+1}, e_{+1}) \in (\mathbb{R}_+^S)^2} \{\mathbb{E}\{-(\phi_{+1}m_{+1} + \psi_{+1}e_{+1}) + \beta V_{+1}(m_{+1}, e_{+1})\}\}. \quad (\text{B.5})$$

Decentralized market. Buyers are randomly matched to the firms and the probability a buyer ends up in a match with a firm equals the buyer's search level σ . Communication within bilateral matches is limited due to spatial separation; the buyer-firm pair cannot observe what happens in other matches.

Bargaining. The buyer-firm pair negotiates terms of trade (q, p) , with q the DM goods received by the buyer and p the payment (in terms of CM goods) received by the firm. This payment must be made with liquid assets. The utility surplus for the buyer is $u(q) - p$, as follows from the linearity of W in Equation (B.5). The trade increases the firm's dividend in (B.1) by $p - c(q)$ since the firm uses $c(q)$ CM goods from its endowment y to produce q DM goods in exchange for liquid wealth worth p CM goods.

Firms are interested in maximizing the utility of their shareholders. The firm and the buyer disregard the effects of changes in the firm's dividend on other matches due to limited communication. Changes in the dividend of the firm also leave the buyer's (with which the firm negotiates) wealth unaffected because there is a continuum of firms and matching is random. The dividend change from the transaction thus directly represents the shareholders' utility gain since it is expressed in CM goods.

The total surplus from negotiated terms of trade (q, p) is $u(q) - c(q)$. With payment protocol v , mapping $q \mapsto p$, the buyer's surplus is $L(q) = u(q) - v(q)$ and the firm's surplus is $\Pi(q) = v(q) - c(q)$. A buyer chooses q to maximize $L(q)$ subject to $v(q) \leq z(m, e) \equiv \Phi m + \chi \Delta e$ and $c(q) \leq y$, where $(m, e) \in \mathbb{R}_+^2$ are its currency and equity holdings carried into the DM. It follows that the negotiated terms of trade are given by Equation (3) if the constraint $c(q) \leq y$ is slack. Since $q \leq \hat{q}$ and Assumption 1 implies $c(\hat{q}) \leq y$, the slackness of the capacity constraint is indeed guaranteed.

Value functions and dividends. Expected dividend an incumbent firm will pay in the CM, contingent on the aggregate uncertainty being resolved, i.e., $\mathbb{E}_{-1}\{\delta(p, o)|s\}$, equal the dividend payment Δ of equity by the law of large numbers. If a firm is matched to a

buyer with currency and equity holdings $(m, e) \in \mathbb{R}_+^2$, its CM dividend payment will be

$$\delta = \Pi \left(\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\} \right) + y; \quad (\text{B.6})$$

its endowment of y CM goods plus the match surplus, where $q = \min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\}$ as implied by Equation (3). Accounting for the distribution G of search and asset holdings across buyers, the firm's expected dividend payment $\Delta = \mathbb{E}_{-1}\{\delta(p, o)|s\}$ upon entering the DM is then given by Equation (7).

If a buyer holds assets $(m, e) \in \mathbb{R}_+^2$, its value when matched to a firm is

$$L \left(\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\} \right) + \Phi m + \Delta e + W(0, 0), \quad (\text{B.7})$$

as follows from the linearity of (B.5) and the specification of q in Equation (3). The buyer chooses search σ optimally and since σ equals the probability of being matched, the value of entering the DM with assets $(m, e) \in \mathbb{R}_+^2$ is

$$V(m, e) = \max_{\sigma \in \{l, h\}} \left\{ \sigma L \left(\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\} \right) - s(\sigma) \right\} + \Phi m + \Delta e + W(0, 0). \quad (\text{B.8})$$

Bellman equation. Using (B.5) to substitute out the term $W(0, 0)$ in Equation (B.8) gives a recursive expression for $V(m, e)$:

$$\begin{aligned} V(m, e) = & \max_{\sigma \in \{l, h\}} \left\{ \sigma L \left(\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\} \right) - s(\sigma) \right\} \\ & + \Phi m + \Delta e + \tau + \Psi \\ & + \max_{(m_{+1}, e_{+1}) \in (\mathbb{R}_+^S)^2} \left\{ \mathbb{E} \left\{ -(\phi_{+1} m_{+1} + \psi_{+1} e_{+1}) + \beta V_{+1}(m_{+1}, e_{+1}) \right\} \right\}, \end{aligned} \quad (\text{B.9})$$

Since $q = \min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\}$ solves $\max_{q \geq 0} L(q)$ subject to $v(q) \leq z(m, e) \equiv \Phi m + \chi \Delta e$, we have

$$L(\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\}) = \max_{q \geq 0} \{L(q) \mid \text{s.t. } v(q_t) \leq z(m, e)\}. \quad (\text{B.10})$$

Using (B.10) in (B.9) gives the Bellman Equation (2).

C Convex set for search effort

By choosing the right search-cost function, the results from the model can be generalized to a setup in which the set of feasible search-effort levels is $\Sigma = [0, 1]$.

The objective is to construct a search-cost function s which entails buyers are willing to restrict attention only to search levels $l, h \in (0, 1]$, with $l < h$, in equilibrium. To achieve this, let s be

$$s(\sigma) = \begin{cases} 0 & \text{if } \sigma \leq l, \\ \frac{k(\sigma-l)}{h-l} & \text{if } \sigma \in (l, h], \\ \frac{\tilde{k}(\sigma-h)}{h-l} + k & \text{if } \sigma \geq h; \end{cases} \quad \text{where } \tilde{k} > k. \quad (\text{C.1})$$

The specification ensures s is increasing, continuous, and convex. Further, it features $s(h) - s(l) = k$. Equation (B.8) implies a buyer will choose σ to solve

$$\max_{\sigma \in \Sigma} \{\sigma L - s(\sigma)\}, \quad (\text{C.2})$$

where the buyer's asset portfolio (m, e) entails $L = L(\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\})$. Under Specification (C.1), the solution satisfies

$$\sigma \in \begin{cases} [0, l] & \text{if } (h-l)L = 0, \\ \{l\} & \text{if } 0 < (h-l)L < k, \\ [l, h] & \text{if } k = (h-l)L < \tilde{k}, \\ \{h\} & \text{if } k < (h-l)L < \tilde{k}, \\ [h, 1] & \text{if } \tilde{k} = (h-l)L, \\ \{1\} & \text{if } \tilde{k} < (h-l)L. \end{cases} \quad (\text{C.3})$$

By construction, $L \leq L(\hat{q})$. Imposing $\tilde{k} > L(\hat{q})$ thus implies

$$\max_{\sigma \in \Sigma} \{\sigma L - s(\sigma)\} = \max_{\sigma \in \{l, h\}} \{\sigma L - s(\sigma)\}. \quad (\text{C.4})$$

In other words, buyers lose nothing if they restrict attention only to $\sigma \in \{l, h\}$.¹⁹ It

¹⁹In fact, combining with the optimal portfolio problem, it is easy to show that if $\Sigma = [0, 1]$, optimal

then follows directly that as long as $L(\hat{q}) < \tilde{k}$, all results from the model with $\Sigma = \{l, h\}$ hold true in a model with $\Sigma = [0, 1]$ and s given by Equation (C.1).

D Two-sided search

The results from the model can be generalized to a setup with two-sided search in the DM. To do so, introduce a unit mass of identical, infinitely-lived *workers* who value the net consumption $x_w \in \mathbb{R}$ of CM goods and who devote search $\sigma_w \in \Sigma \subseteq [0, 1]$ on behalf of the firms. A worker's flow utility is $\mathcal{U}(\sigma_w, x_w) = -s(\sigma_w) + x_w$ and the time-discount rate is β . The CM is as in the baseline model and workers have no reason to hold assets since they do not consume DM goods.

Workers and firms form worker-firm pairs in the DM which disband after the DM has convened. Every worker is matched to a firm and *vice versa*. The workers devote search σ_w on behalf of the worker-firm pair. The mass of matches between buyers and workers in the DM is given by a constant-returns-to-scale matching function $\mathcal{N}(\tilde{\sigma}_b, \tilde{\sigma}_w)$, where $\tilde{\sigma}_b$ and $\tilde{\sigma}_w$ is average search across the buyers respectively the workers.

A buyer devoting search σ_b in the DM finds a match with a worker with probability $\sigma_b \mathcal{N}(1, 1/\kappa)$, where $\kappa = \tilde{\sigma}_b/\tilde{\sigma}_w$ is *market tightness*. A worker devoting search σ_w likewise finds a match with a buyer with probability $\sigma_w \mathcal{N}(\kappa, 1)$. Once matched with a buyer, the worker can connect the buyer to the firm.

Assumption D.1. *Search devoted by the worker is private information and the firm cannot incentive the worker to search. Moreover, the worker's decision to connect the buyer to the firm cannot be contracted ex ante.*

Assumption D.1 implies firms negotiate with workers after the matching of buyers to workers has taken place. A worker matched to a buyer negotiates a real payment w from the firm in return for connecting the buyer with the firm. The buyer's liquid wealth is observable to both the worker and the firm during the negotiation process. The firm can settle the payment w instantaneously with ownership shares in its profits, and assume w follows from a protocol $\omega : \Pi \rightarrow w$, mapping the firm's surplus Π from being connected

asset portfolios are never such that $L = (h - l)k$.

with the buyer into w . Hence, σ_w follows from

$$\max_{\sigma_w \in \Sigma} \left\{ \frac{\sigma_w \mathcal{N}(\kappa, 1)}{\tilde{\sigma}_b} \iiint \sigma [\omega \circ \Pi (\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\})] G(d\sigma, dm, de) - s(\sigma_w) \right\}, \quad (\text{D.1})$$

and σ_b follows from

$$\max_{\sigma_b \in \Sigma} \left\{ \sigma_b \mathcal{N}(1, 1/\kappa) L (\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\}) - s(\sigma_b) \right\}, \quad (\text{D.2})$$

with (m, e) the buyer's asset holdings.

The dividend paid by equity becomes

$$\Delta = \mathcal{N}(1, 1/\kappa) \iiint \sigma [(1 - \omega) \circ \Pi (\min\{v^{-1}(\Phi m + \chi \Delta e), \hat{q}\})] G(d\sigma, dm, de) + y. \quad (\text{D.3})$$

Assumption D.2. *Buyers and workers obtain the same share $\theta < 1/2$ of total match surplus $u(q) - c(q)$. That means, $v(q) = (1 - \theta)u(q) + \theta c(q)$ and $\omega \circ \Pi(q) = \theta[u(q) - c(q)]$.*

Given Assumption D.2, $\kappa = 1$ in a symmetric equilibrium, i.e., when $\sigma_b = \tilde{\sigma}_b$ for all buyers and $\sigma_w = \tilde{\sigma}_w$ for all workers. To see this, note all buyers then carry liquid wealth worth Z_d and workers anticipate $q = \min\{v^{-1}(Z_d), \hat{q}\}$ and therefore choose σ_w to maximize $\sigma_w \mathcal{N}(\kappa, 1) \theta [u(q) - c(q)] - s(\sigma_w)$. Buyers choose σ_b to maximize $\sigma_b \mathcal{N}(1, 1/\kappa) \theta [u(q) - c(q)] - s(\sigma_b)$. This results in a unique σ_b and σ_w , except for knife-edge cases. If $\kappa = 1$, then $\sigma_b = \sigma_w$ and this rationalizes $\kappa = 1$ as an equilibrium outcome. When $\kappa > 1$, it is required that $\sigma_b > \sigma_w$. But high κ is especially beneficial for the workers—they get matched to a buyer with a high probability so the search incentives imply $\sigma_b < \sigma_w$ instead. Likewise, when $\kappa < 1$, it is required that $\sigma_b < \sigma_w$ but a low κ is especially beneficial for the buyers. The search incentives would then imply $\sigma_b > \sigma_w$ instead.

Taking stock, in symmetric equilibria a buyer is matched to a worker with probability $\sigma_b \mathcal{N}(1, 1)$. Imposing the normalization $\mathcal{N}(1, 1) = 1$ then results in the same Bellman equation for buyers as in the baseline model. The only difference arises when calculating the value of equity since firms now earn lower profits due to the payment w to workers, but this does not affect the main properties of the baseline model. Further, the main results about equilibrium multiplicity and endogenous dynamics do not rely on equilibria

in which buyers use mixed strategies for their search. These results therefore hold true under the setup with two-sided search laid out above.

E Proofs

Proof of Lemma 1. I first characterize the set \mathcal{S}_e and prove $\tilde{R}_h < 0 < \tilde{R}_l \Leftrightarrow (k, y) \in \mathcal{S}_e$, after which I prove $(k, y) \in \mathcal{S}_e$ implies Assumption 2. Finally I show \mathcal{S}_e has positive mass under both the parameter restriction $c(\hat{q}) \leq y$ (see Assumption 1) and Assumption 3. Throughout, I restrict attention to $(k, y) \in \mathbb{R}_+^2$.

Define $q_\sigma(\iota) : \iota = \sigma L'(q_\sigma)/v'(q_\sigma)$. Also define $q_\sigma(y) \leq \hat{q}$ as the unique solution of $\sigma\Pi(q_\sigma) + y \geq v(q_\sigma)$ with $=$ if $q_\sigma < \hat{q}$. Note: $q_\sigma(\iota)$ is strictly decreasing in ι ; and $q_\sigma(y)$ is strictly increasing in y if $y < v(\hat{q}) - \sigma\Pi(\hat{q})$ and constant in y for $y \geq v(\hat{q}) - \sigma\Pi(\hat{q})$. Further, $q_\sigma(y) \geq 0$ with $=$ if and only if $y = 0$. By the definition of \tilde{R}_l and \tilde{R}_h :

$$\tilde{R}_l = (v - l\Pi) \circ q_l(\tilde{\iota}) - y \quad \text{and} \quad \tilde{R}_h = (v - h\Pi) \circ q_h(\tilde{\iota}) - y. \quad (\text{E.1})$$

The properties of $q_\sigma(\iota)$ and $q_\sigma(y)$ then directly imply

$$\tilde{R}_l > 0 \Leftrightarrow \tilde{\iota} < \frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)} \quad \text{and} \quad \tilde{R}_h < 0 \Leftrightarrow \tilde{\iota} > \frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)}. \quad (\text{E.2})$$

Hence, $\tilde{R}_h < 0 < \tilde{R}_l \Leftrightarrow \frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)} < \tilde{\iota} < \frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)}$. Define the set

$$\mathcal{Y} \equiv \left\{ y : \frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)} < \frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)} \right\}. \quad (\text{E.3})$$

Recall $\bar{\lambda} \equiv \lim_{q \rightarrow 0} [L'(q)/v'(q)]$ so that $0 \notin \mathcal{Y}$ since $h > l$. Thus $\mathcal{Y} \subseteq \mathbb{R}_{++}$.

Parameter k can be backed out from $\tilde{\iota}$ by using that (13) holds with equality at $\tilde{\iota}$:

$$k = \kappa(\tilde{\iota}) \equiv \max_{q_h \geq 0} \{hL(q_h) - \tilde{\iota}v(q_h)\} - \max_{q_l \geq 0} \{lL(q_l) - \tilde{\iota}v(q_l)\}, \quad (\text{E.4})$$

where I use $\max_{q_\sigma \geq 0} \{\sigma L(q_\sigma) - \iota v(q_\sigma)\} = \max_{z_\sigma \geq 0} \{\sigma L(\min\{v^{-1}(z_\sigma), \hat{q}\}) - \iota z_\sigma\}$. It follows that: $\kappa(\tilde{\iota})$ is a strictly decreasing function of $\tilde{\iota}$ on the domain $(0, h\bar{\lambda})$; and $\kappa(\tilde{\iota}) \geq 0$ with $>$ if and only if $\tilde{\iota} < h\bar{\lambda}$. Further note that: $\kappa(\tilde{\iota}) > \max_{z_h} \{hL \circ v^{-1}(z_h) - l\bar{\lambda}z_h\}$ if and only if $\tilde{\iota} < l\bar{\lambda}$; and $\kappa(\tilde{\iota}) < (h - l)L(\hat{q})$ if and only if $\tilde{\iota} > 0$.

From the above it follows that for $\iota', \iota'' \in (0, h\bar{\lambda})$, it holds that $\kappa(\iota') < (>) \kappa(\iota'') \Leftrightarrow \iota' > (<) \iota''$. Therefore define the set

$$\mathcal{K}(y) \equiv \left(\kappa \left(\frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)} \right), \kappa \left(\frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)} \right) \right). \quad (\text{E.5})$$

The set $\mathcal{K}(y)$ has positive mass if and only if $y \in \mathcal{Y}$. Also, since $k = \kappa(\tilde{\iota})$, we have $\frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)} < \tilde{\iota} < \frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)}$ if and only if $k \in \mathcal{K}(y)$. This proves that

$$\tilde{R}_h < 0 < \tilde{R}_l \Leftrightarrow (k, y) \in \mathcal{S}_e \equiv \{(k, y) : y \in \mathcal{Y} \text{ and } k \in \mathcal{K}(y)\}. \quad (\text{E.6})$$

I next show $(k, y) \in \mathcal{S}_e$ implies Assumption 2 is satisfied. Since $\mathcal{Y} \subseteq \mathbb{R}_{++}$, we have $\frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)} < l\bar{\lambda} \forall y \in \mathcal{Y}$, where I use that $q_\sigma(y)$ is strictly increasing in y on the domain $(0, v(\hat{q}) - \sigma\Pi(\hat{q}))$ and satisfies $q_\sigma(0) = 0$. Because $(k, y) \in \mathcal{S}_e \Rightarrow \kappa \left(\frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)} \right) < k$, we have $\frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)} < l\bar{\lambda} \Rightarrow \kappa(l\bar{\lambda}) < \kappa \left(\frac{l\Pi' \circ q_l(y)}{v' \circ q_l(y)} \right) < k$, where I use that for $\iota', \iota'' \in (0, h\bar{\lambda})$ we have $\kappa(\iota') < (>) \kappa(\iota'') \Leftrightarrow \iota' > (<) \iota''$. Since $\kappa(l\bar{\lambda}) = \max_{z_h \geq 0} \{hL \circ v^{-1}(z_h) - l\bar{\lambda}z_h\}$, we thus have $(k, y) \in \mathcal{S}_e \Rightarrow \max_{z_h \geq 0} \{hL \circ v^{-1}(z_h) - l\bar{\lambda}z_h\} < k$. Further, we have that $\frac{\sigma\Pi' \circ q_\sigma(y)}{v' \circ q_\sigma(y)} \geq 0$, with $>$ if and only if $y < v(\hat{q}) - \sigma\Pi(\hat{q})$, so we also have $(k, y) \in \mathcal{S}_e \Rightarrow \frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)} \geq 0$. Also, $(k, y) \in \mathcal{S}_e \Rightarrow k < \kappa \left(\frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)} \right)$. Thus, $k < \kappa \left(\frac{h\Pi' \circ q_h(y)}{v' \circ q_h(y)} \right) \leq \kappa(0)$, where I use that $\kappa(\iota)$ is weakly increasing in ι . Since $\kappa(0) = (h-l)L(\hat{q})$, it follows $(k, y) \in \mathcal{S}_e \Rightarrow k < (h-l)L(\hat{q})$. This proves that $(k, y) \in \mathcal{S}_e \Rightarrow \max_{z_h \geq 0} \{hL \circ v^{-1}(z_h) - l\bar{\lambda}z\} < k < (h-l)L(\hat{q})$, i.e., Assumption 2 is satisfied if $(k, y) \in \mathcal{S}_e$.

It remains to show \mathcal{S}_e has positive mass under both the parameter restriction $c(\hat{q}) \leq y$ and Assumption 3. For this, it suffices to show the set $\mathcal{Y}' \equiv \mathcal{Y} \cap [c(\hat{q}), \infty)$ has positive mass under Assumption 3.

Define $\mathcal{Y}'' \equiv (v(\hat{q}) - h\Pi(\hat{q}), v(\hat{q}) - l\Pi(\hat{q}))$. From the definition of $q_\sigma(y)$, it follows directly that $y \in \mathcal{Y}'' \Rightarrow q_l(y) < q_h(y) = \hat{q}$. Since $L'(q)/v'(q) = 0$ for $q = \hat{q}$ and $L'(q)/v'(q) > 0$ for $q < \hat{q}$, it follows directly that $\mathcal{Y}'' \subseteq \mathcal{Y}$. Moreover, $y \in \mathcal{Y}'' \Rightarrow y > v(\hat{q}) - h\Pi(\hat{q})$, and in turn $v(\hat{q}) - h\Pi(\hat{q}) = v(\hat{q}) - h[v(\hat{q}) - c(\hat{q})] = (1-h)v(\hat{q}) + hc(\hat{q}) \geq c(\hat{q})$, where the first equality uses $\Pi(q) = v(q) - c(q)$ and the inequality follows because $\Pi(q) = v(q) - c(q) > 0$ since $q \in (0, \hat{q}]$. It follows $\mathcal{Y}'' \subseteq [c(\hat{q}), \infty)$ and combining with the previous result, we have $\mathcal{Y}'' \subseteq \mathcal{Y}'$.

The set \mathcal{Y}'' has positive mass since $h > l$ and $\Pi(q) > 0$ on the relevant domain

$(0, \hat{q}]$. This result holds true under Assumption 3; the result only requires that $\lambda(z) \equiv (L'/v') \circ \min\{v^{-1}(z), \hat{q}\} \geq 0$ (with equality if and only if $z \geq v(\hat{q})$), which does not rule out Assumption 3. Concluding, \mathcal{Y}' has positive mass under Assumption 3 since $\mathcal{Y}'' \subseteq \mathcal{Y}'$ and \mathcal{Y}'' has positive mass under Assumption 3. Q.E.D.

Proof of Proposition 1. We have that

$$\mu \overline{f'_{me}}(\mathbb{E}\{x_{+1}\})/\beta = 1 + \Lambda(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) + \Lambda'(\mathbb{E}\{x_{+1}\} + \overline{\Delta})\mathbb{E}\{x_{+1}\}. \quad (\text{E.7})$$

If $\mathbb{E}\{x_{+1}\} + \overline{\Delta} \geq v(\hat{q})$, then buyers' liquidity constraint is slack so $\Lambda(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) = \Lambda'(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) = 0$. It follows $\overline{f'_{me}} = \beta/\mu > 0$.

If $\mathbb{E}\{x_{+1}\} + \overline{\Delta} < v(\hat{q})$, then buyers' liquidity constraint binds so $\Lambda(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) > 0$ and $\Lambda'(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) < 0$. Therefore,

$$\begin{aligned} \mu \overline{f'_{me}}(\mathbb{E}\{x_{+1}\})/\beta &= 1 + \Lambda(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) + \Lambda'(\mathbb{E}\{x_{+1}\} + \overline{\Delta})\mathbb{E}\{x_{+1}\} \\ &> 1 + \Lambda(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) + \Lambda'(\mathbb{E}\{x_{+1}\} + \overline{\Delta})(\mathbb{E}\{x_{+1}\} + \overline{\Delta}) \geq 0, \end{aligned} \quad (\text{E.8})$$

where the last step uses Assumption 3. Again, we find $\overline{f'_{me}} > 0$.

Summarizing, the difference equation is monotonically increasing so bounded monetary equilibria must be steady states. The steady state features $\iota_{ss} = \mu/\beta - 1$ and entails generically unique real allocations; if $\mu > \beta$, search is given by Equation (14) and currency balances are determined by $\mu/\beta - 1 = \Lambda(\Phi M_{-1} + \overline{\Delta})$; and if $\mu = \beta$, all buyers devote intense search and the slack liquidity constraint entails unique real allocations, although currency balances ΦM_{-1} are only determined up to a lower bound $\max\{0, v(\hat{q}) - \overline{\Delta}\}$. If $\overline{\Delta} < v(\hat{q})$, non-negativity of currency balances implies existence of the monetary steady state if and only if $\beta \leq \mu < \beta[1 + \Lambda(\overline{\Delta})]$. If $\overline{\Delta} \geq v(\hat{q})$, monetary steady states exist if and only if $\mu = \beta$. Q.E.D.

Proof of Proposition 2. In a model with exogenous search at level σ , the inverse demand function for liquid wealth is $\Lambda(z) = \sigma\lambda(z)$. Hence, in this context, Assumption 3 entails $1 + \sigma\lambda(z) + \sigma\lambda'(z)z \geq 0 \forall z$.

With an endogenous-dividend asset, the difference equation $f_{\sigma,me}$ implies

$$\begin{aligned} \mu f'_{\sigma,me}(\mathbb{E}\{x_{+1}\})/\beta &= 1 + \sigma\lambda \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} \\ &\quad + [\sigma\lambda' \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\}] [1 + \Delta'_\sigma(\mathbb{E}\{x_{+1}\})] \mathbb{E}\{x_{+1}\}. \end{aligned} \quad (\text{E.9})$$

This expression can be rewritten as

$$\begin{aligned} \mu f'_{\sigma,me}(\mathbb{E}\{x_{+1}\})/\beta &= 1 + \sigma\lambda \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} \\ &\quad + [\sigma\lambda' \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\}] [(1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\}] \\ &\quad + o(\mathbb{E}\{x_{+1}\}), \end{aligned} \quad (\text{E.10})$$

where

$$o(\mathbb{E}\{x_{+1}\}) \equiv - [\sigma\lambda' \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\}] [\Delta_\sigma(\mathbb{E}\{x_{+1}\}) - \Delta'_\sigma(\mathbb{E}\{x_{+1}\})\mathbb{E}\{x_{+1}\}]. \quad (\text{E.11})$$

In case $(1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} \geq v(\hat{q})$, buyers' liquidity constraint is slack so $\lambda \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} = \lambda' \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} = 0$ and $\Delta'_\sigma(\mathbb{E}\{x_{+1}\}) = 0$. Hence, $o(\mathbb{E}\{x_{+1}\}) = 0$ and $f'_{\sigma,me} = \beta/\mu > 0$.

If $(1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} < v(\hat{q})$, buyers' liquidity constraint binds and $\lambda' \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} < 0$ and

$$\begin{aligned} \Delta_\sigma(\mathbb{E}\{x_{+1}\}) - \Delta'_\sigma(\mathbb{E}\{x_{+1}\})\mathbb{E}\{x_{+1}\} &= \sigma\Pi(q) + y - \frac{\sigma\Pi'(q)}{v'(q) - \sigma\Pi'(q)} (v(q) - \sigma\Pi(q) - y) \\ &= \frac{(\sigma\Pi(q) + y)v'(q) - \sigma\Pi'(q)v(q)}{v'(q) - \sigma\Pi'(q)} \\ &= \frac{(\sigma\Pi(q) + y)v'(q) - \sigma[v'(q) - c'(q)][\Pi(q) + c(q)]}{v'(q) - \sigma\Pi'(q)} \quad (\text{E.12}) \\ &= \frac{v'(q)[y - \sigma c(q)] + \sigma c'(q)[\Pi(q) + c(q)]}{v'(q) - \sigma\Pi'(q)} \\ &\geq \frac{\sigma[\Pi(q) + c(q)]c'(q)}{v'(q) - \sigma\Pi'(q)}, \end{aligned}$$

where: the first equality exploits that given a binding liquidity constraint,

$$\Delta_\sigma(\mathbb{E}\{x_{+1}\}) = \sigma\Pi(q) + y, \quad \Delta'_\sigma(\mathbb{E}\{x_{+1}\}) = \frac{\sigma\Pi'(q)}{v'(q) - c'(q)}, \quad \mathbb{E}\{x_{+1}\} = v(q) - \sigma\Pi(q) - y, \quad (\text{E.13})$$

(this follows from Equation (27)); the third equality uses that $v(q) = \Pi(q) + c(q)$; and the last inequality uses that $y \geq \sigma c(q)$, being implied by $\sigma \leq 1$ and the firms' slack capacity constraint. Since $c'(q) > 0$ and $\Pi'(q) < v'(q)$, it follows $o(\mathbb{E}\{x_{+1}\}) > 0$. At the same time, Assumption 3 implies

$$1 + \sigma\lambda \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\} + [\sigma\lambda' \circ (1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\}] [(1 + \Delta_\sigma) \circ \mathbb{E}\{x_{+1}\}] \geq 0, \quad (\text{E.14})$$

so we again have $f'_{\sigma,me} > 0$.

Summarizing, $f'_{\sigma,me} > 0$. Therefore $f_{\sigma,me}$ is a monotonically increasing function and only steady states can be bounded monetary equilibria. The steady state features $\iota_{ss} = \mu/\beta - 1$ and entails a unique real allocation; if $\mu > \beta$, currency balances are given by $\mu/\beta - 1 = \sigma\lambda \circ (1 + \Delta_\sigma) \circ (\Phi M_{-1})$; and if $\mu = \beta$, the slack liquidity constraint entails unique real allocations, although currency balances ΦM_{-1} are only determined up to a lower bound $\max\{0, v(\hat{q}) - \sigma\Pi(\hat{q}) - y\}$. If $\sigma\Pi(\hat{q}) + y < v(\hat{q})$, non-negativity of currency balances implies existence of the monetary steady state if and only if $\beta \leq \mu < \beta[1 + \sigma\lambda \circ \Delta_\sigma(0)]$. If $\sigma\Pi(\hat{q}) + y \geq v(\hat{q})$, monetary steady states exist if and only if $\mu = \beta$. Q.E.D.

Proof of Lemma 2. First note $\max\{0, \tilde{R}_h\} < \tilde{R}_l \Leftrightarrow \exists n > 0$ s.t. $\tilde{R}_h - n < 0 < \tilde{R}_l - n$. From the proof of Lemma 1 it is immediate that $\tilde{R}_h - n < 0 < \tilde{R}_l - n \Leftrightarrow (k, y + n) \in \mathcal{S}_e$. Hence

$$\max\{0, \tilde{R}_h\} < \tilde{R}_l \Leftrightarrow (k, y) \in \mathcal{S}_{me} \equiv \{(k, y) : \exists n > 0 \text{ s.t. } (k, y + n) \in \mathcal{S}_e\}. \quad (\text{E.15})$$

Because \mathcal{S}_e is an open set, it follows $\mathcal{S}_e \subseteq \mathcal{S}_{me}$. Therefore \mathcal{S}_{me} has positive mass under both the parameter restriction $c(\hat{q}) \leq y$ and Assumption 3 since \mathcal{S}_e exhibits this property as well. Also, $(k, y) \in \mathcal{S}_e$ implies Assumption 2 is satisfied. We have $(k, y) \in \mathcal{S}_{me} \Rightarrow \exists n > 0$ such that $(k, y + n) \in \mathcal{S}_e$. For that n , it must hold $y + n > 0$ and $k \in \mathcal{K}(y + n)$, as otherwise $(k, y + n) \notin \mathcal{S}_e$. It then follows directly from the proof of Lemma 1 that indeed Assumption 2 is satisfied. Q.E.D.

Proof of Proposition 3. Since $\tilde{R}_l < \max\{0, \tilde{R}_h\}$, it holds f_{me} is a function. From Equation (31) it follows that for $\mathbb{E}\{x_{+1}\} \in (\tilde{R}_l, \tilde{R}_h)$, f_{me} is strictly increasing; while for other $\mathbb{E}\{x_{+1}\}$,

$$\begin{aligned} \mu f'_{me}(\mathbb{E}\{x_{+1}\})/\beta &= 1 + \Lambda \circ (1 + \Delta) \circ \mathbb{E}\{x_{+1}\} \\ &+ [\Lambda' \circ (1 + \Delta) \circ \mathbb{E}\{x_{+1}\}][1 + \Delta'(\mathbb{E}\{x_{+1}\})]\mathbb{E}\{x_{+1}\}. \end{aligned} \quad (\text{E.16})$$

In spirit of the proof of Proposition 2, this expression can be rewritten as

$$\begin{aligned} \mu f'_{me}(\mathbb{E}\{x_{+1}\})/\beta &= 1 + \Lambda \circ (1 + \Delta) \circ \mathbb{E}\{x_{+1}\} \\ &+ [\Lambda' \circ (1 + \Delta) \circ \mathbb{E}\{x_{+1}\}][(1 + \Delta) \circ \mathbb{E}\{x_{+1}\}] + o(\mathbb{E}\{x_{+1}\}), \end{aligned} \quad (\text{E.17})$$

where

$$o(\mathbb{E}\{x_{+1}\}) \equiv -[\Lambda' \circ (1 + \Delta) \circ \mathbb{E}\{x_{+1}\}][\Delta(\mathbb{E}\{x_{+1}\}) - \Delta'(\mathbb{E}\{x_{+1}\})\mathbb{E}\{x_{+1}\}]. \quad (\text{E.18})$$

Because search is either $\sigma = l$ or $\sigma = h$, depending on whether $\mathbb{E}\{x_{+1}\} < \tilde{R}_l$ or $\mathbb{E}\{x_{+1}\} > \tilde{R}_h$, it follows directly from the proof of Proposition 2 that $f'_{me} > 0$.

Summarizing, $f'_{me} > 0$. Therefore f_{me} is a monotonically increasing function and only steady states can be bounded monetary equilibria. The steady state features $\iota = \mu/\beta - 1$ and entails generically unique real allocations; if $\mu > \beta$, search is determined by Equation (14) and currency balances are given by $\mu/\beta - 1 = \Lambda \circ (1 + \Delta) \circ (\Phi M_{-1})$; and if $\mu = \beta$, the slack liquidity constraint entails unique real allocations, although currency balances ΦM_{-1} are only determined up to a lower bound $\max\{0, v(\hat{q}) - h\Pi(\hat{q}) - y\}$. If $h\Pi(\hat{q}) + y \geq v(\hat{q})$, the monetary steady state exists if and only if $\mu = \beta$. In the other case, if $\tilde{R}_l \geq 0$, monetary steady states exist if and only if $\beta \leq \mu < \beta[1 + l\lambda \circ \Delta_l(0)]$. If $\tilde{R}_l < 0$, monetary steady states exist if and only if $\beta \leq \mu < \beta[1 + \min\{h\lambda \circ \Delta_h(0), \tilde{\iota}\}]$. Q.E.D.

Proof of Propositions 4, 5, and 6. Since $\max\{0, \tilde{R}_h\} < \tilde{R}_l$, it holds f_{me} is a correspondence on the domain $[\max\{0, \tilde{R}_h\}, \tilde{R}_l]$, which has positive mass if $(k, y) \in \mathcal{S}_{me}$. The steady state features $\iota = \mu/\beta - 1$ and entails generically unique real allocations; if $\mu > \beta$, search is determined by Equation (14) and currency balances are given by

$\mu/\beta - 1 = \Lambda \circ (1 + \Delta) \circ (\Phi M_{-1})$; and if $\mu = \beta$, the slack liquidity constraint entails unique real allocations, although currency balances ΦM_{-1} are only determined up to a lower bound $\max\{0, v(\hat{q}) - h\Pi(\hat{q}) - y\}$. If $h\Pi(\hat{q}) + y \geq v(\hat{q})$, the monetary steady state exists if and only if $\mu = \beta$. In the other case, if $\tilde{R}_h \geq 0$, monetary steady states exist if and only if $\beta \leq \mu < \beta[1 + l\lambda \circ \Delta_l(0)]$. If $\tilde{R}_h < 0$, monetary steady states exist if and only if $\mu \in [\beta, h\lambda \circ \Delta_h(0)] \cup [\beta(1 + \tilde{\iota}), h\lambda \circ \Delta_l(0)]$.

To characterize bounded monetary equilibria other than steady states, define $i \equiv \mu/\beta - 1$. To elucidate how search behaves in equilibrium, note a dynamic equilibrium is characterized by a process $\{(x_t, \eta_t)\}_{t=0}^\infty$ satisfying

$$x = \begin{cases} f_{l,me}(\mathbb{E}\{x_{+1}\}) & \text{if } \eta = 0, \\ \frac{1+\tilde{\iota}}{1+i}\mathbb{E}\{x_{+1}\} & \text{if } \eta \in (0, 1), \\ f_{h,me}(\mathbb{E}\{x_{+1}\}) & \text{if } \eta = 1; \end{cases} \quad \eta \in \begin{cases} \{0\} & \text{if } \mathbb{E}\{x_{+1}\} < \tilde{R}_h, \\ \left\{0, \frac{\tilde{R}_l - \mathbb{E}\{x_{+1}\}}{\tilde{R}_l - \tilde{R}_h}, 1\right\} & \text{if } \tilde{R}_h \leq \mathbb{E}\{x_{+1}\} \leq \tilde{R}_l, \\ \{1\} & \text{if } \mathbb{E}\{x_{+1}\} > \tilde{R}_l. \end{cases} \quad (\text{E.19})$$

This follows directly from Equations (32) and (33). As established in the proof of Proposition 3, $f_{l,me}$ and $f_{h,me}$ are monotonically increasing functions given Assumption 3, which I use throughout the proof.

Next, note $\Gamma_\sigma(R)$, as defined in Equation (22), is: continuous; strictly decreasing in R on the domain $(-y, \hat{R}_\sigma)$, where $\hat{R}_\sigma \equiv v(\hat{q}) - \sigma\Pi(\hat{q}) - y$; and satisfies $\Gamma_\sigma(R) > (=) 0 \Leftrightarrow R < (\geq) \hat{R}_\sigma$ and $\Gamma_\sigma(R) = \sigma\bar{\lambda} \Leftrightarrow R = -y$. Further, $\Gamma_h(R) \leq (<) \tilde{\iota} \Leftrightarrow R \geq (>) \tilde{R}_h$ and $\tilde{\iota} \leq (<) \Gamma_l(R) \Leftrightarrow R \leq (<) \tilde{R}_l$ since $\Gamma_h(\tilde{R}_h) = \Gamma_l(\tilde{R}_l) = \tilde{\iota}$ and $\tilde{\iota} \in (0, \bar{\lambda})$.

Then, note the set $\mathcal{I} \equiv \{\iota \geq 0 : \exists \varepsilon > 0 \text{ s.t. } \Gamma_h(\tilde{R}_l) \leq \iota \leq \Gamma_l(\max\{\varepsilon, \tilde{R}_h\})\}$ has positive mass. This is the case since $(k, y) \in \mathcal{S}_{me}$ implies $\max\{0, \tilde{R}_h\} < \tilde{R}_l$. Particularly, note: $\Gamma_l(\cdot)$ and $\Gamma_h(\cdot)$ are strictly decreasing on the domain $(-y, \hat{R}_l)$ and $(-y, \hat{R}_h)$, respectively; and $\tilde{R}_l < \hat{R}_l$ and $\tilde{R}_h < \hat{R}_h$ (since $\tilde{\iota} > 0$). These two properties imply $\Gamma_h(\tilde{R}_l) < \Gamma_h(\tilde{R}_h) = \tilde{\iota} = \Gamma_l(\tilde{R}_l) < \Gamma_l(\max\{0, \tilde{R}_h\})$. Note $\text{int}(\mathcal{I}) = (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\}))$ then also has positive mass and contains $\tilde{\iota}$.

It is now useful to study separately the cases: (a) $0 = i \leq \Gamma_h(\tilde{R}_l)$; (b) $0 < i \leq \Gamma_h(\tilde{R}_l)$; (c) $i \in (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\}))$; and (d) $i \geq \Gamma_l(\max\{0, \tilde{R}_h\})$.

Case a: $0 = i \leq \Gamma_h(\tilde{R}_l)$. We have on the relevant domain \mathbb{R}_{++} that $f_{h,me}(x) > x \forall x < \tilde{R}_h$ and $f_{h,me}(x) = x \forall x \geq \tilde{R}_h$. Likewise, $f_{l,me}(x) > x \forall x < \hat{R}_l$ and $f_{l,me}(x) = x \forall x \geq \hat{R}_l$.

Hence, since $\hat{R}_h < \hat{R}_l$, we have $\mathbb{E}\{x_{+1}\} < \hat{R}_h \Rightarrow \mathbb{E}\{x_{+1}\} < x$ by Equation (E.19).

If $\Gamma_h(\tilde{R}_l) > 0$, we must also have $\eta = 1$ if $\mathbb{E}\{x_{+1}\} \geq \hat{R}_h$ since $\Gamma_h(\tilde{R}_l) > 0 \Rightarrow \tilde{R}_l < \hat{R}_h$. Thus $\mathbb{E}\{x_{+1}\} \geq \hat{R}_h \Rightarrow x = \mathbb{E}\{x_{+1}\} \geq \hat{R}_h$. Hence, $x < \hat{R}_h \Rightarrow \mathbb{E}\{x_{+1}\} < \hat{R}_h (< \hat{R}_l) \Rightarrow \mathbb{E}\{x_{+1}\} < x$. Thus, we cannot have $x < \hat{R}_h$ in a bounded monetary equilibrium since it would imply x would go to zero with positive probability, which directly implies ΦM_{-1} goes to zero with positive probability since $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$, and in turn $\mathbb{E}\{x_{+1}\} < x$ if $x < \hat{R}_h$. If $x \geq \hat{R}_h$, we can have $\eta < 1$ only if $\mathbb{E}\{x_{+1}\} < \hat{R}_h$ since $\tilde{R}_l < \hat{R}_h$, so $x \geq \hat{R}_h$ and $\eta < 1$ would likewise imply x would go to zero with positive probability, which applies to ΦM_{-1} as well for the same reason as above. Thus, in a bounded monetary equilibrium we must have $x \geq \hat{R}_h$ and $\eta = 1$. From Equation (E.19) it follows $\mathbb{E}\{x_{+1}\} = x$; in a bounded equilibrium we may have x developing stochastically over time but $\eta = 1 \forall t$ and $x \geq \hat{R}_h \forall t$, which implies the real allocation is pinned down. I.e., all buyers search intensely ($\sigma = h$) and they consume \hat{q} DM goods if matched to a firm since the liquidity constraint is slack.

If $\Gamma_h(\tilde{R}_l) = 0$, illustrated in Figure 9, we have $\tilde{R}_l \geq \hat{R}_h$. Consider the sequence

$$\{(x_t, \eta_t)\}_{t=0}^{\infty} = \{(f_{l,me}(\hat{R}_h), 1), (f_{l,me}(\hat{R}_h), 0), (\hat{R}_h, 1), (\hat{R}_h, 1), \dots\}. \quad (\text{E.20})$$

Clearly, $\{(x_t, \eta_t)\}_{t=2}^{\infty}$ satisfies Equation (E.19) since it is a steady state. Further, we have $\eta_1 = 0$ is feasible since $x_2 = \hat{R}_h \leq \tilde{R}_l$. From Equation (E.19) this then indeed implies $x_1 = f_{l,me}(\hat{R}_h)$; $\{(x_t, \eta_t)\}_{t=1}^{\infty}$ satisfies Equation (E.19) as well. Then, note $x_1 = f_{l,me}(\hat{R}_h) > \hat{R}_h > \tilde{R}_h$ because $\tilde{R}_h < \hat{R}_h < \hat{R}_l$. Therefore we can have $\eta_0 = 1$, which through Equation (E.19) then implies $x_0 = f_{h,me} \circ f_{l,me}(\hat{R}_h) = f_{l,me}(\hat{R}_h)$, were the last equality uses $f_{h,me}(x) = x \forall x \geq \hat{R}_h$; the proposed sequence in Equation (E.20) is indeed an equilibrium. Note the equilibrium features a one-time boom-bust cycle; the economy starts in a boom, then experiences a bust, and subsequently remains in a boom forever.

Case b: $0 < i \leq \Gamma_h(\tilde{R}_l)$. It follows $i < \tilde{i}$ because $\tilde{i} \in \text{int}(\mathcal{I}) \equiv (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\}))$. We have a unique monetary steady state at $x_{ss} \equiv R(i) \in (\max\{0, \tilde{R}_h\}, \hat{R}_h)$, entailing high search. Moreover, $x_{ss} \geq \tilde{R}_l$, with $=$ if and only if $i = \Gamma_h(\tilde{R}_l)$ since $\Gamma_h(\cdot)$ is strictly decreasing on $(-y, \hat{R}_h)$ and $0 < \Gamma_h(x_{ss}) = i \leq \Gamma_h(\tilde{R}_l)$. The unique monetary steady state implies $f_{h,me}(x) > (<)x \Leftrightarrow x < (>)x_{ss}$ on the relevant domain \mathbb{R}_{++} , as well as $f_{l,me}(x) > x$ on the relevant domain $(0, \tilde{R}_l]$.

Equation (E.19) implies $\eta = 1 \forall \mathbb{E}\{x_{+1}\} > x_{ss}$ since $x_{ss} \geq \tilde{R}_l$. This property implies $x_{ss} < x < \mathbb{E}\{x_{+1}\} \forall \mathbb{E}\{x_{+1}\} > x_{ss}$ due to monotonicity of $f_{h,me}$. At the same time, since $f_{l,me}(x) > x \forall x \in (0, \tilde{R}_l]$ and $f_{h,me}(x) > x \forall x \in (0, x_{ss})$, we have $x < x_{ss} \Rightarrow \mathbb{E}\{x_{+1}\} < x_{ss} \Rightarrow \mathbb{E}\{x_{+1}\} < x$. Thus, it must be that x goes to zero with positive probability if $x < x_{ss}$, which likewise implies that ΦM_{-1} goes to zero with positive probability since $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$ and $x < x_{ss}$ implies $\mathbb{E}\{x_{+1}\} < x$. We must thus have $x \geq x_{ss}$ in a bounded monetary equilibrium.

On the other hand, if $x > x_{ss}$, then if $\eta = 1$ (feasible since $x_{ss} > \tilde{R}_h$) we have for sure $\mathbb{E}\{x_{+1}\} > x$ by the monotonicity of $f_{h,me}$. Since $x_{ss} \geq \tilde{R}_l$, other $\mathbb{E}\{x_{+1}\}$ that satisfy Equation (E.19) for $x > x_{ss}$ must induce $\eta < 1$ and thus $\mathbb{E}\{x_{+1}\} \leq \tilde{R}_l$, which in turn satisfies $\tilde{R}_l \leq x_{ss}$. If $\tilde{R}_l < x_{ss}$ it therefore follows directly that $x > x_{ss}$ implies x grows either unbounded or goes to zero with positive probability. If it goes to zero with positive probability, we know from the reasoning above that ΦM_{-1} goes to zero with positive probability as well. If x grows unbounded with positive probability, then at some time t we have $x_t > \max\{f_{l,me}(\tilde{R}_l), x_{ss}\}$, so that we must have $\eta_t = 1$ and $\mathbb{E}_t\{x_{t+1}\} > x_t > \max\{f_{l,me}(\tilde{R}_l), x_{ss}\}$, which thus implies that ΦM_{-1} grows unbounded with positive probability as well. We must thus have $x \leq x_{ss}$ in a bounded monetary equilibrium if $\tilde{R}_l < x_{ss}$. For the knife edge case $\tilde{R}_l = x_{ss}$, we have $\mathbb{E}\{x_{+1}\} = x_{ss}$ only for $(x, \eta) = (x_{ss}, 1)$ and $(x, \eta) = (f_{l,me}(\tilde{R}_l), 0)$.

Taking stock, if $\tilde{R}_l < x_{ss}$, we must have $(x, \eta) = (x_{ss}, 1) \forall t$ in a bounded monetary equilibrium. For the special case $\tilde{R}_l = x_{ss}$, illustrated in Figure 7, we can also have a deterministic sequence

$$\{(x_t, \eta_t)\}_{t=0}^{\infty} = \left\{ (f_{h,me}^{T-1}(x_{T-1}), 1), (f_{h,me}^{T-2}(x_{T-1}), 1), \dots, (f_{h,me}(x_{T-1}), 1), (x_{T-1}, 0), (x_{ss}, 1), (x_{ss}, 1), \dots \right\}, \quad (\text{E.21})$$

where $x_{T-1} = f_{l,me}(x_{ss})$ and $T \in \mathbb{N}$. The sequence $\{(x_t, \eta_t)\}_{t=T}^{\infty}$ satisfies Equation (E.19) since it is the monetary steady state. Further, $x_T = x_{ss}$ implies we can have $\eta_{T-1} = 0$ since $\mathbb{E}_{T-1}\{x_T\} = x_{ss} = \tilde{R}_l$. In turn, to satisfy Equation (E.19), this requires $x_{T-1} = f_{l,me}(x_{ss})$; the sequence $\{(x_t, \eta_t)\}_{t=T-1}^{\infty}$ also satisfies Equation (E.19). Then note $f_{l,me}(x_{ss}) > x_{ss}$. In turn, this implies we can have $\eta_{T-2} = 1$ since $\mathbb{E}_{T-2}\{x_{T-1}\} = f_{l,me}(x_{ss}) > x_{ss} > \max\{0, \tilde{R}_h\}$. To satisfy Equation (E.19), this requires $x_{T-2} = f_{h,me}(x_{T-1})$; the sequence $\{(x_t, \eta_t)\}_{t=T-2}^{\infty}$ also satisfies Equation (E.19). Since $x_{T-1} > x_{ss} \Rightarrow x_{ss} < x_{T-2} < x_{T-1}$, as

established before, we can have $\eta_{T-3} = 1$ as well. We can then backward iterate further to conclude the entire sequence $\{(x_t, \eta_t)\}_{t=0}^\infty$ characterized in Equation (E.21) satisfies Equation (E.19) $\forall T \in \mathbb{N}$.

Case c: $i \in (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\}))$. The set

$$\mathcal{X} = \left\{ x \in (\max\{0, \tilde{R}_h\}, \tilde{R}_l) : f_{h,me}(x) < x < f_{l,me}(x) \right\} \quad (\text{E.22})$$

is non-empty. To see this, note

$$f_{\sigma,me}(x) = \frac{1 + \Gamma_\sigma(x)}{1 + i} x, \quad \sigma \in \{l, h\}. \quad (\text{E.23})$$

For some arbitrary $x \in (\max\{0, \tilde{R}_h\}, \tilde{R}_l)$ we therefore have $f_{h,me}(x) < x < f_{l,me}(x)$ if and only if $i \in (\Gamma_h(x), \Gamma_l(x))$, where it has to be noted: $x < \tilde{R}_l \Rightarrow \Gamma_l(x) > \tilde{i}$; and $x > \max\{0, \tilde{R}_h\} \Rightarrow \Gamma_h(x) < \tilde{i}$. The set $(\Gamma_h(x), \Gamma_l(x))$ therefore has positive mass for all $x \in (\max\{0, \tilde{R}_h\}, \tilde{R}_l)$. It follows that for an arbitrary $i \in (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\}))$, there exists an $x \in (\max\{0, \tilde{R}_h\}, \tilde{R}_l)$ such that $f_{h,me}(x) < x < f_{l,me}(x)$ since $\Gamma_l(\cdot)$ and $\Gamma_h(\cdot)$ are continuous and decreasing. Thus, \mathcal{X} has positive mass and is, in fact, a convex set.

Pick an arbitrary $\hat{x} \in \mathcal{X}$. Suppose first $f_{l,me} \circ f_{h,me}(\hat{x}) < \hat{x}$ (this case is illustrated in Figure 6). It follows $\exists x' \in (f_{h,me}(\hat{x}), \hat{x})$ such that $f_{l,me}(x') = x'$ by the intermediate value theorem since $f_{l,me}(\hat{x}) > \hat{x}$. Consider therefore the following dynamics process for (x, η) :

$$(x, \eta) = \begin{cases} \begin{cases} (f_{l,me}(\hat{x}), 0) & \text{with prob. } \rho, \\ (f_{h,me}(\hat{x}), 1) & \text{with prob. } 1 - \rho, \end{cases} & \text{if } t \text{ odd} \\ (\hat{x}, 0) & \text{if } t \text{ even;} \end{cases} \quad \rho \equiv \frac{x' - f_{h,me}(\hat{x})}{f_{l,me}(\hat{x}) - f_{h,me}(\hat{x})}. \quad (\text{E.24})$$

Note $\rho \in (0, 1)$ since $f_{h,me}(\hat{x}) < x' < \hat{x} < f_{l,me}(\hat{x})$. Given process (E.24), Equation (E.19) is satisfied for odd t by construction since we then have $\mathbb{E}\{x_{t+1}\} = \hat{x} \in \mathcal{X} \subseteq (\max\{0, \tilde{R}_h\}, \tilde{R}_l)$, thus allowing for both $(x, \eta) = (f_{l,me}(\hat{x}), 0)$ and $(x, \eta) = (f_{h,me}(\hat{x}), 1)$. Further, for even t , we have $\mathbb{E}\{x_{t+1}\} = x'$, as follows from the definition of ρ . Equation (E.19) is then satisfied for even t as well since $x' < \hat{x} < \tilde{R}_l$, thus allowing for $(x, \eta) = (f_{l,me}(\mathbb{E}\{x_{t+1}\}), 0) = (f_{l,me}(x'), 0) = (\hat{x}, 0)$. We have thus found a stochastic two cycle with $\Phi M_{-1} = \mathbb{E}\{x_{t+1}\} = x' < \hat{x}$ and $\eta = 0$ for even t ; and $\Phi M_{-1} = \{x_{t+1}\} = \hat{x}$ with $\eta = 0$

with prob. ρ and $\eta = 1$ with prob. $1 - \rho$ for odd t . Currency balances are thus pro cyclical and inflation is counter cyclical.

Suppose next $f_{l,me} \circ f_{h,me}(\hat{x}) \geq \hat{x}$ and $f_{h,me}^{-1}(\hat{x}) \geq f_{l,me}(\hat{x})$ (this case is illustrated in Figure 4). It follows that $\exists x' \in [f_{h,me}(\hat{x}), \hat{x}]$ such that $f_{h,me}^{-1}(x') = f_{l,me}(x')$ by the intermediate value theorem since $f_{l,me} \circ f_{h,me}(\hat{x}) \geq \hat{x} \Rightarrow f_{l,me} \circ f_{h,me}(\hat{x}) \geq f_{h,me}^{-1} \circ f_{h,me}(\hat{x})$. By construction, $x' = f_{h,me} \circ f_{l,me}(x')$, so consider the process

$$(x, \eta) = \begin{cases} (x', 1) & \text{if } t \text{ odd,} \\ (x'', 0) & \text{if } t \text{ even;} \end{cases} \quad \text{where } x'' \equiv f_{l,me}(x'). \quad (\text{E.25})$$

For even t , we have $\mathbb{E}\{x_{+1}\} = x'$. It follows $\eta = 0$ for even t is in line with (E.19) because $x' \leq \hat{x} < \tilde{R}_l$. Given $\eta = 0$ and $\mathbb{E}\{x_{+1}\} = x'$ for even t , it follows (E.19) implies $x = x''$ for even t since $x'' \equiv f_{l,me}(x')$. For odd t , we have $\mathbb{E}\{x_{+1}\} = x''$. It follows $\eta = 1$ for odd t is in line with (E.19) because $x'' \equiv f_{l,me}(x') \geq f_{l,me} \circ f_{h,me}(\hat{x}) \geq \hat{x} > \max\{0, \tilde{R}_h\}$, where the first inequality follows from the fact that $f_{l,me}$ is monotone increasing and $x' \geq f_{h,me}(\hat{x})$; the second is satisfied by supposition; and the third follows from $\hat{x} \in \mathcal{X} \subseteq (\max\{0, \tilde{R}_h\}, \tilde{R}_l)$. Given $\eta = 1$ and $\mathbb{E}\{x_{+1}\} = x''$ for odd t , it follows (E.19) implies $x = f_{h,me}(x'') = f_{h,me} \circ f_{l,me}(x') = x'$ for odd t . We have thus found a deterministic two cycle with $\Phi M_{-1} = \mathbb{E}\{x_{+1}\} = x'$ and $\eta = 0$ for even t ; and $\Phi M_{-1} = \mathbb{E}\{x_{+1}\} = x''$ and $\eta = 1$ for odd t . Further, $x' < x''$; if $x' = x''$ we must have $x' = x'' = \hat{x}$ since the previous steps implied $x' \leq \hat{x} \leq x''$, but $x' = \hat{x}$ implies $x'' = f_{l,me}(\hat{x}) > \hat{x}$ since $\hat{x} \in \mathcal{X}$. Currency balances are thus pro cyclical and inflation is counter cyclical.

Suppose finally $f_{l,me} \circ f_{h,me}(\hat{x}) \geq \hat{x}$ and $f_{h,me}^{-1}(\hat{x}) < f_{l,me}(\hat{x})$ (this case is illustrated in Figure 5). It follows $f_{h,me}(\hat{x}) < \hat{x} < f_{h,me}^{-1}(\hat{x}) < f_{l,me}(\hat{x})$ since $\hat{x} \in \mathcal{X} \Rightarrow f_{h,me}(\hat{x}) < \hat{x} < f_{l,me}(\hat{x}) \Rightarrow f_{l,me}^{-1}(\hat{x}) < \hat{x} < f_{h,me}^{-1}(\hat{x})$. Consider therefore the following process for (x, η) :

$$(x, \eta) = \begin{cases} \begin{cases} (f_{l,me}(\hat{x}), 0) & \text{with prob. } \rho, \\ (f_{h,me}(\hat{x}), 1) & \text{with prob. } 1 - \rho, \end{cases} & \text{if } t \text{ odd} \\ (\hat{x}, 1) & \text{if } t \text{ even;} \end{cases} \quad \rho \equiv \frac{f_{h,me}^{-1}(\hat{x}) - f_{h,me}(\hat{x})}{f_{l,me}(\hat{x}) - f_{h,me}(\hat{x})}. \quad (\text{E.26})$$

Note $\rho \in (0, 1)$ since $f_{h,me}(\hat{x}) < \hat{x} < f_{h,me}^{-1}(\hat{x}) < f_{l,me}(\hat{x})$. Given process (E.26), Equation (E.19) is satisfied for odd t by construction since we then have $\mathbb{E}\{x_{+1}\} = \hat{x} \in \mathcal{X} \subseteq$

$(\max\{0, \tilde{R}_h\}, \tilde{R}_l)$, thus allowing for both $(x, \eta) = (f_{l,me}(\hat{x}), 0)$ and $(x, \eta) = (f_{h,me}(\hat{x}), 1)$. Further, for even t , we have $\mathbb{E}\{x_{+1}\} = f_{h,me}^{-1}(\hat{x})$, as follows from the definition of ρ . Equation (E.19) is then satisfied for even t as well since $f_{h,me}^{-1}(\hat{x}) > \hat{x} > \max\{0, \tilde{R}_h\}$, thus allowing for $(x, \eta) = (f_{h,me}(\mathbb{E}\{x_{+1}\}), 1) = (f_{h,me} \circ f_{h,me}^{-1}(\hat{x}), 1) = (\hat{x}, 1)$. We have thus found a stochastic two cycle with $\Phi M_{-1} = \mathbb{E}\{x_{+1}\} = f_{h,me}^{-1}(\hat{x}) > \hat{x}$ and $\eta = 1$ for even t ; and $\Phi M_{-1} = \mathbb{E}\{x_{+1}\} = \hat{x}$ with $\eta = 0$ with prob. ρ and $\eta = 1$ with prob. $1 - \rho$ for odd t . Currency balances are thus pro cyclical and inflation is counter cyclical.

Case d: $i \geq \Gamma_l(\max\{0, \tilde{R}_h\})$. We have $i > \tilde{i}$, as $\tilde{i} \in \text{int}(\mathcal{I}) \equiv (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\}))$. There exists no monetary steady state if either $\tilde{R}_h \leq 0$, or $\tilde{R}_h > 0$ but $i \geq \Gamma_l(0)$. Both cases feature $i \geq \Gamma_l(0)$, entailing $f_{l,me}(x) < x \forall x > 0$ and $f_{h,me}(x) < x \forall x \geq \max\{\varepsilon, \tilde{R}_h\}$, where $\varepsilon > 0$ but infinitesimal. The monotonicity of $f_{l,me}$ and $f_{h,me}$ then imply $x > 0 \Rightarrow x < \mathbb{E}\{x_{+1}\}$ to satisfy (E.19). This, in turn, means that x grows unbounded with positive probability if $x > 0$. Hence, also ΦM_{-1} grows unbounded with positive probability since $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$ and $\mathbb{E}\{x_{+1}\} > x$ for $x > 0$. In other words, there is no bounded monetary equilibrium.

If $\tilde{R}_h > 0$ and $i < \Gamma_l(0)$, we have a unique monetary steady state at $x_{ss} \equiv R(i) \in (0, \tilde{R}_l)$, entailing low search. The procedure now develops analogously to case *b*. We have $x_{ss} \leq \tilde{R}_h$, with $=$ if and only if $i = \Gamma_l(\tilde{R}_h)$ since $\Gamma_l(\cdot)$ is strictly decreasing on $(-y, \hat{R}_l)$ and $\Gamma_l(\tilde{R}_h) \leq i = \Gamma_l(x_{ss}) < \Gamma_l(0)$, where $\tilde{R}_h < \hat{R}_h < \hat{R}_l$. The unique monetary steady state at x_{ss} implies we have $f_{l,me}(x) > (<)x \Leftrightarrow x < (>)x_{ss}$ on the relevant domain \mathbb{R}_{++} , as well as $f_{h,me}(x) < x \forall x \geq \tilde{R}_h$.

Equation (E.19) implies $\eta = 0 \forall \mathbb{E}\{x_{+1}\} < x_{ss}$ since $x_{ss} \leq \tilde{R}_h$. This property implies $\mathbb{E}\{x_{+1}\} < x < x_{ss} \forall \mathbb{E}\{x_{+1}\} < x_{ss}$ due to monotonicity of $f_{l,me}$. At the same time, since $f_{h,me}(x) < x \forall x \geq \tilde{R}_h$ and $f_{l,me}(x) < x \forall x > x_{ss}$, we have $x > x_{ss} \Rightarrow \mathbb{E}\{x_{+1}\} > x_{ss}$ so $x > x_{ss} \Rightarrow \mathbb{E}\{x_{+1}\} > x > x_{ss}$. I.e., it must be that x grows unbounded with positive probability if $x > x_{ss}$, which means ΦM_{-1} grows unbounded with positive probability as well since $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$ and $\mathbb{E}\{x_{+1}\} > x$ for $x > x_{ss}$. We must thus have $x \leq x_{ss}$ in a bounded monetary equilibrium.

On the other hand, if $x < x_{ss}$, then if $\eta = 0$ (feasible since $x_{ss} < \tilde{R}_l$) we have for sure that $\mathbb{E}\{x_{+1}\} < x$ by the monotonicity of $f_{l,me}$. Since $x_{ss} \leq \tilde{R}_h$, other $\mathbb{E}\{x_{+1}\}$ that satisfy Equation (E.19) for $x < x_{ss}$ must induce $\eta > 0$ and thus $\mathbb{E}\{x_{+1}\} \geq \tilde{R}_h$, which in turn

satisfies $\tilde{R}_h \geq x_{ss}$. If $\tilde{R}_h > x_{ss}$ it therefore follows directly that $x < x_{ss}$ implies x grows either unbounded or goes to zero with positive probability. This carries over to ΦM_{-1} as well. We have shown this for x growing unbounded already. If x goes to zero with positive probability, then by the monotonicity of $f_{h,me}$, we must have $\eta = 0$ once $x < f_{h,me}(\tilde{R}_h)$. For that case, we already established that $\mathbb{E}\{x_{+1}\} < x$ if also $x < x_{ss}$. It then follows ΦM_{-1} goes to zero with positive probability as well since $\Phi M_{-1} = \mathbb{E}\{x_{+1}\}$. We must thus have $x \geq x_{ss}$ in a bounded monetary equilibrium if $\tilde{R}_h > x_{ss}$. For the knife edge case $\tilde{R}_h = x_{ss}$, we have that $\mathbb{E}\{x_{+1}\} = x_{ss}$ only for $(x, \eta) = (x_{ss}, 0)$ and $(x, \eta) = (f_{h,me}(\tilde{R}_h), 1)$.

Taking stock, if $\tilde{R}_h > x_{ss}$, we must have $(x, \eta) = (x_{ss}, 0) \forall t$ in a bounded monetary equilibrium. For the special case $\tilde{R}_h = x_{ss}$, illustrated in Figure 8, we can also have a deterministic sequence

$$\{(x_t, \eta_t)\}_{t=0}^{\infty} = \left\{ \begin{array}{l} (f_{l,me}^{T-1}(x_{T-1}), 0), (f_{l,me}^{T-2}(x_{T-1}), 0), \\ \dots, (f_{l,me}(x_{T-1}), 0), (x_{T-1}, 1), (x_{ss}, 0), (x_{ss}, 0), \dots \end{array} \right\}, \quad (\text{E.27})$$

where $x_{T-1} = f_{h,me}(x_{ss})$ and $T \in \mathbb{N}$. The sequence $\{(x_t, \eta_t)\}_{t=T}^{\infty}$ satisfies Equation (E.19) since it is the monetary steady state. Further, $x_T = x_{ss}$ implies we can have $\eta_{T-1} = 1$ since $\mathbb{E}_{T-1}\{x_T\} = x_{ss} = \tilde{R}_h$. In turn, to satisfy Equation (E.19), this requires $x_{T-1} = f_{h,me}(x_{ss})$; the sequence $\{(x_t, \eta_t)\}_{t=T-1}^{\infty}$ also satisfies Equation (E.19). Then note $f_{h,me}(x_{ss}) < x_{ss}$. In turn, this implies we can have $\eta_{T-2} = 0$ since $\mathbb{E}_{T-2}\{x_{T-1}\} = f_{h,me}(x_{ss}) < x_{ss} < \tilde{R}_l$. To satisfy Equation (E.19), this requires $x_{T-2} = f_{l,me}(x_{T-1})$; the sequence $\{(x_t, \eta_t)\}_{t=T-2}^{\infty}$ also satisfies Equation (E.19). Since $x_{T-1} < x_{ss} \Rightarrow x_{T-1} < x_{T-2} < x_{ss}$, as established before, we can have $\eta_{T-3} = 0$ as well. We can then backward iterate further to conclude that the entire sequence $\{(x_t, \eta_t)\}_{t=0}^{\infty}$ characterized in Equation (E.27) satisfies Equation (E.19) $\forall T \in \mathbb{N}$.

Combining insights from the cases *a-d*, it holds: (i) two cycles with boom-bust dynamics and counter-cyclical inflation exist if $i \in \text{int}(\mathcal{I})$, proving Proposition 4; (ii) equilibria that converge to the monetary steady state with a boom-bust cycle on the transition path exist if $i \in \mathcal{I} / \text{int}(\mathcal{I})$, proving Proposition 5; and (iii) bounded monetary equilibria other than steady states do not exist if $i \notin \mathcal{I}$, proving Proposition 6. Q.E.D.

Proof of Proposition 7. The focus is on monetary equilibria, so assume $\mathbb{M}_{-1} > 0$. Clearance of the market for liquid wealth implies only ι which solve $R(\iota) = \mathbb{M}_{-1}$ can occur on the equilibrium path due to the perfect predictability of \mathbb{M}_{-1} .

If $(k, y) \in \mathcal{S}_{me}$, it follows from Corollary 1 that there are three ι for which $R(\iota) = \mathbb{M}_{-1}$ if and only if $\mathbb{M}_{-1} \in [\max\{0, \tilde{R}_h\}, \tilde{R}_l]$, namely $\Gamma_l(\mathbb{M}_{-1})$, $\Gamma_h(\mathbb{M}_{-1})$, and $\tilde{\iota}$. Because $\Gamma_\sigma(\mathbb{M}_{-1})$ is decreasing in \mathbb{M}_{-1} and $\Gamma_h(\mathbb{M}_{-1}) \leq \tilde{\iota} \leq \Gamma_l(\mathbb{M}_{-1})$, it follows that for a given i , there exists an $\mathbb{M}_{-1} \in [\max\{0, \tilde{R}_h\}, \tilde{R}_l]$ and a probability vector $(\mathbb{P}_{-1}\{\iota = \Gamma_h(\mathbb{M}_{-1})\}, \mathbb{P}_{-1}\{\iota = \tilde{\iota}\}, \mathbb{P}_{-1}\{\iota = \Gamma_l(\mathbb{M}_{-1})\}) \in \Delta^2/\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ that jointly solve Equation (37) if and only if

$$i \in (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\})) \equiv \text{int}(\mathcal{I}). \quad (\text{E.28})$$

Thus, if $i \in \text{int}(\mathcal{I})$, there is equilibrium multiplicity; besides $\iota = i$, there is also a non-degenerate equilibrium distribution for ι . If $i \notin \text{int}(\mathcal{I})$, then it must be $\iota = i$, i.e., ι must be perfectly predictable, entailing generically unique equilibrium allocations.

If $(k, y) \notin \mathcal{S}_{me}$, then $\# \mathbb{M}_{-1} > 0$ s.t. $\tilde{R}_h < \mathbb{M}_{-1} < \tilde{R}_l$. It follows there is a unique ι that solves $R(\iota) = \mathbb{M}_{-1} \forall \mathbb{M}_{-1} > 0$, as implied by Corollary 1. Since \mathbb{M}_{-1} is perfectly predictable one period in advance, ι must be perfectly predictable on period in advance as well. Equation (35) therefore implies $\iota = i$, entailing a degenerate distribution for ι and generically unique equilibrium allocations. Q.E.D.

Proof of Proposition 8. First observe there is only a scope for equilibrium multiplicity if $(k, y) \in \mathcal{S}_{me}$ and $i \in \text{int}(\mathcal{I}) \equiv (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\}))$ (see Proposition 7). Further, the focus is on the case $i < \tilde{\iota}$, meaning the objective is to implement the boom equilibrium with probability one. For future purposes, it is useful to define

$$\tilde{\Delta}_h \equiv h\Pi \circ v^{-1}(\tilde{z}_h) + y \quad \text{and} \quad \tilde{\Delta}_l \equiv l\Pi \circ v^{-1}(\tilde{z}_l) + y. \quad (\text{E.29})$$

Also, let $\mathbb{M}_{det,-1}$ solve $i = \Gamma_h(\mathbb{M}_{det,-1})$, i.e., $\mathbb{M}_{det,-1}$ is the value of currency balances in deterministic equilibrium where ι is degenerate at i . Note $\mathbb{M}_{det,-1} > \tilde{R}_h$ because $i < \tilde{\iota}$. We also have $\mathbb{M}_{det,-1} < \tilde{R}_l$ because: $i \in \text{int}(\mathcal{I}) \Rightarrow \Gamma_h(\tilde{R}_l) < i; i = \Gamma_h(\mathbb{M}_{det,-1})$; and $\Gamma_h(\cdot)$ is decreasing. Thus, $\mathbb{M}_{det,-1} \in (\tilde{R}_h, \tilde{R}_l)$.

The first step is to prove $\mathbb{M}_{-1} > \mathbb{M}_{det,-1}$ if ι is non-degenerate and $\underline{\Delta} \leq \tilde{\Delta}_h$. Because Δ is decreasing in ι (see Equation (17)), it follows $Z_s(\iota < \tilde{\iota}) = \mathbb{M}_{-1} + \Delta$; equity purchases do not materialize if $\iota < \tilde{\iota}$. We therefore have $\iota = \Gamma_h(\mathbb{M}_{-1})$ if $\iota < \tilde{\iota}$. We also have $\mathbb{P}_{-1}\{\iota < i\} > 0$ if ι is non-degenerate since $i = \mathbb{E}_{-1}\{\iota\}$ by Equation (35). It follows that $\mathbb{M}_{-1} > \mathbb{M}_{det,-1}$ if ι is non-degenerate because for any $\iota < \tilde{\iota}$ we have $\iota = \Gamma_h(\mathbb{M}_{-1})$ and $\Gamma_h(\cdot)$ is decreasing.

The next step is to prove ι cannot be non-degenerate if

$$\underline{\Delta} = \Delta' \equiv \eta' h \Pi \circ v^{-1}(\tilde{z}_h) + (1 - \eta') l \Pi \circ v^{-1}(\tilde{z}_l) + y, \quad \text{where} \quad \eta' \equiv \frac{\tilde{R}_l - R(i)}{\tilde{R}_l - \tilde{R}_h}. \quad (\text{E.30})$$

Consider a proof by contradiction. First, note $\Delta' = \eta' \tilde{\Delta}_h + (1 - \eta') \tilde{\Delta}_l$. Second, note $\eta' \in (0, 1)$ because $\mathbb{M}_{det,-1} = R(i)$ and $\mathbb{M}_{det,-1} \in (\tilde{R}_h, \tilde{R}_l)$. Third, note $\tilde{\Delta}_l < \tilde{\Delta}_h$ since $l < h$ and $0 < \tilde{z}_l < \tilde{z}_h < v(\hat{q})$ (this is implied by Assumption 2). We thus have $\Delta' \in (\tilde{\Delta}_l, \tilde{\Delta}_h)$.

Because Δ is decreasing in ι , equity purchases do not materialize if $\iota < \tilde{\iota}$ since $\lim_{\iota \nearrow \tilde{\iota}} \Delta = \tilde{\Delta}_h$, while they are deployed in case $\iota > \tilde{\iota}$ since $\lim_{\iota \searrow \tilde{\iota}} \Delta = \tilde{\Delta}_l$. On the one hand we thus have $Z_s(\iota < \tilde{\iota}) = \mathbb{M}_{-1} + \Delta$, so that $\iota = \Gamma_h(\mathbb{M}_{-1})$ if $\iota < \tilde{\iota}$. On the other hand, $Z_s(\iota > \tilde{\iota}) = \mathbb{M}_{-1} + \Delta'$ and also, $Z_d(\iota > \tilde{\iota}) \leq \tilde{z}_l$ since Z_d is decreasing in ι and $\lim_{\iota \searrow \tilde{\iota}} Z_d(\iota) = \tilde{z}_l$. Since $\tilde{z}_l = \tilde{R}_l + \tilde{\Delta}_l$, it holds:

$$\begin{aligned} [Z_d(\iota) - Z_s(\iota)]_{\iota > \tilde{\iota}} &\leq \tilde{R}_l + \tilde{\Delta}_l - \Delta' - \mathbb{M}_{-1} \\ &< \tilde{R}_l + \tilde{\Delta}_l - \Delta' - \mathbb{M}_{det,-1} \\ &= \tilde{R}_l + \tilde{\Delta}_l - \eta' \tilde{\Delta}_h - (1 - \eta') \tilde{\Delta}_l - R(i) \\ &= (\tilde{R}_l - R(i)) \left(1 - \frac{\tilde{\Delta}_h - \tilde{\Delta}_l}{\tilde{R}_l - \tilde{R}_h} \right) \\ &= (\tilde{R}_l - R(i)) \frac{\tilde{z}_l - \tilde{z}_h}{\tilde{R}_l - \tilde{R}_h} \\ &< 0, \end{aligned} \quad (\text{E.31})$$

where the second-last line uses $\tilde{z}_l = \tilde{R}_l + \tilde{\Delta}_l$ and $\tilde{z}_h = \tilde{R}_h + \tilde{\Delta}_h$. The last line then follows from $R(i) = \mathbb{M}_{det,-1} \in (\tilde{R}_h, \tilde{R}_l)$ and $\tilde{z}_l < \tilde{z}_h$. Thus, for $\underline{\Delta} = \Delta'$, having $\iota > \tilde{\iota}$ is inconsistent with clearance of the market for liquid wealth, as this would require $R(\iota) \equiv Z_d(\iota) - Z_s(\iota) = 0$.

Likewise, $\iota = \tilde{\iota}$ can be proven to be inconsistent with market clearance by means of a contradiction. Thus, suppose $\iota = \tilde{\iota}$ and the market clears. Specifically, this means η solves

$$\begin{aligned}
0 &= \eta \tilde{z}_h + (1 - \eta) \tilde{z}_l - \max\{\Delta, \Delta'\} - \mathbb{M}_{-1} \\
&= \eta \tilde{z}_h + (1 - \eta) \tilde{z}_l - \max\{\eta \tilde{\Delta}_h + (1 - \eta) \tilde{\Delta}_l, \eta' \tilde{\Delta}_h + (1 - \eta') \tilde{\Delta}_l\} - \mathbb{M}_{-1} \\
&= \eta(\tilde{z}_h - \tilde{\Delta}_h) + (1 - \eta)(\tilde{z}_l - \tilde{\Delta}_l) - \max\{0, \eta' - \eta\}(\tilde{\Delta}_h - \tilde{\Delta}_l) - \mathbb{M}_{-1} \\
&= \eta \tilde{R}_h + (1 - \eta) \tilde{R}_l - \max\{0, \eta' - \eta\}(\tilde{\Delta}_h - \tilde{\Delta}_l) - \mathbb{M}_{-1}.
\end{aligned} \tag{E.32}$$

It follows that η satisfies

$$\begin{aligned}
\eta &= \frac{\tilde{R}_l - \mathbb{M}_{-1} - \max\{0, \eta' - \eta\}(\tilde{\Delta}_h - \tilde{\Delta}_l)}{\tilde{R}_l - \tilde{R}_h} \\
&< \frac{\tilde{R}_l - \mathbb{M}_{det, -1}}{\tilde{R}_l - \tilde{R}_h} \\
&= \frac{\tilde{R}_l - R(i)}{\tilde{R}_l - \tilde{R}_h} \equiv \eta'.
\end{aligned} \tag{E.33}$$

Equity purchases should thus materialize if the market clears at $\iota = \tilde{\iota}$ since $\eta < \eta' \Leftrightarrow \Delta < \Delta'$. It follows

$$\begin{aligned}
0 &= \eta \tilde{R}_h + (1 - \eta) \tilde{R}_l - (\eta' - \eta)(\tilde{\Delta}_h - \tilde{\Delta}_l) - \mathbb{M}_{-1} \\
&= \eta(\tilde{R}_h + \tilde{\Delta}_h) + (1 - \eta)(\tilde{R}_l + \tilde{\Delta}_l) - \frac{\tilde{R}_l - R(i)}{\tilde{R}_l - \tilde{R}_h}(\tilde{\Delta}_h - \tilde{\Delta}_l) - \mathbb{M}_{-1}.
\end{aligned} \tag{E.34}$$

Hence

$$\begin{aligned}
\eta &= \frac{\frac{\tilde{R}_l - R(i)}{\tilde{R}_l - \tilde{R}_h}(\tilde{\Delta}_h - \tilde{\Delta}_l) - (\tilde{R}_l - \mathbb{M}_{-1})}{\tilde{R}_h + \tilde{\Delta}_h - \tilde{R}_l - \tilde{\Delta}_l} \\
&> \frac{\frac{\tilde{R}_l - R(i)}{\tilde{R}_l - \tilde{R}_h}(\tilde{\Delta}_h - \tilde{\Delta}_l) - (\tilde{R}_l - \mathbb{M}_{det, -1})}{\tilde{R}_h + \tilde{\Delta}_h - \tilde{R}_l - \tilde{\Delta}_l} \\
&= \frac{\frac{\tilde{R}_l - R(i)}{\tilde{R}_l - \tilde{R}_h}(\tilde{\Delta}_h - \tilde{\Delta}_l) - (\tilde{R}_l - R(i))}{\tilde{R}_h + \tilde{\Delta}_h - \tilde{R}_l - \tilde{\Delta}_l} \\
&= \frac{\tilde{R}_l - R(i)}{\tilde{R}_h + \tilde{\Delta}_h - \tilde{R}_l - \tilde{\Delta}_l} \frac{\tilde{R}_h + \tilde{\Delta}_h - \tilde{R}_l - \tilde{\Delta}_l}{\tilde{R}_l - \tilde{R}_h} \\
&= \eta',
\end{aligned} \tag{E.35}$$

directly contradicting the earlier statement that $\eta < \eta'$.

Taking stock, if $\underline{\Delta} = \Delta'$ and ι is non-degenerate (so that $\mathbb{M}_{-1} > \mathbb{M}_{det, -1}$), then on

the one hand we can neither have $\iota > \tilde{\iota}$ nor $\iota = \tilde{\iota}$. On the other, we have $\iota = \Gamma_h(\mathbb{M}_{-1})$ if $\iota < \tilde{\iota}$. Hence, it must be that $\iota = \Gamma_h(\mathbb{M}_{-1})$ if $\underline{\Delta} = \Delta'$ and ι is non-degenerate, which contradicts that ι is non-degenerate since it is then perfectly predictable from \mathbb{M}_{-1} .

The last step is to prove ι can be non-degenerate if $\underline{\Delta} < \Delta'$. Consider

$$\iota = \begin{cases} \underline{\iota} & \text{with prob. } 1 - \rho \\ \tilde{\iota} & \text{with prob. } \rho; \end{cases} \quad \text{where } \rho \equiv \frac{\tilde{\iota} - i}{\tilde{\iota} - \underline{\iota}}, \quad (\text{E.36})$$

as an equilibrium distribution for ι if $\underline{\iota}$ satisfies

$$\underline{\iota} \in \mathcal{J} \equiv \left(\Gamma_h \left(\tilde{R}_l - \max \left\{ \frac{(\underline{\Delta} - \tilde{\Delta}_l)(\tilde{R}_l - \tilde{R}_h)}{\tilde{\Delta}_h - \tilde{\Delta}_l}, 0 \right\}, 0 \right), i \right). \quad (\text{E.37})$$

Note the set \mathcal{J} has positive mass since: $i \in \text{int}(\mathcal{I}) \equiv (\Gamma_h(\tilde{R}_l), \Gamma_l(\max\{0, \tilde{R}_h\})) \Rightarrow i > \Gamma_h(\tilde{R}_l)$ (we need $i \in \text{int}(\mathcal{I})$ to have a scope for equilibrium multiplicity in the first place); and

$$\underline{\Delta} < \Delta' \quad \Rightarrow \quad \tilde{R}_l - \max \left\{ \frac{(\underline{\Delta} - \tilde{\Delta}_l)(\tilde{R}_l - \tilde{R}_h)}{\tilde{\Delta}_h - \tilde{\Delta}_l}, 0 \right\} \in (R(i), \tilde{R}_l]. \quad (\text{E.38})$$

First, \mathbb{M}_{-1} solves $\Gamma_h(\mathbb{M}_{-1}) = \underline{\iota}$ since $\underline{\iota} < i < \tilde{\iota}$. I.e., with $\Gamma_h(\mathbb{M}_{-1}) = \underline{\iota}$ the market for liquid wealth clears for $\iota = \underline{\iota}$. With $\Gamma_h(\cdot)$ decreasing in \mathbb{M}_{-1} and $\underline{\iota} \in \mathcal{J}$ it follows

$$\mathbb{M}_{-1} \in \mathcal{R} \equiv \left(R(i), \tilde{R}_l - \max \left\{ \frac{(\underline{\Delta} - \tilde{\Delta}_l)(\tilde{R}_l - \tilde{R}_h)}{\tilde{\Delta}_h - \tilde{\Delta}_l}, 0 \right\} \right), \quad (\text{E.39})$$

which is a set with positive mass since $\underline{\Delta} < \Delta'$.

Second, with $\iota = \tilde{\iota}$, the market for liquid wealth clears if and only if there is an $\eta \in [0, 1]$ which solves

$$\eta \tilde{z}_h + (1 - \eta) \tilde{z}_l = \max\{\eta \tilde{\Delta}_h + (1 - \eta) \tilde{\Delta}_l, \underline{\Delta}\} + \mathbb{M}_{-1}. \quad (\text{E.40})$$

Suppose η is such that $\underline{\Delta} \leq \Delta' \equiv \eta \tilde{\Delta}_h + (1 - \eta) \tilde{\Delta}_l$. Then

$$\begin{aligned} \eta &= \frac{(\tilde{z}_l - \tilde{\Delta}_l) - \mathbb{M}_{-1}}{(\tilde{z}_l - \tilde{\Delta}_l) - (\tilde{z}_h - \tilde{\Delta}_h)}, \\ &= \frac{\tilde{R}_l - \mathbb{M}_{-1}}{\tilde{R}_l - \tilde{R}_h}. \end{aligned} \quad (\text{E.41})$$

and with $\mathbb{M}_{-1} \in \mathcal{R}$ it follows

$$\eta \in \mathcal{N} \equiv \left(\max \left\{ 0, \frac{\underline{\Delta} - \tilde{\Delta}_l}{\tilde{\Delta}_h - \tilde{\Delta}_l} \right\}, \eta' \right), \quad (\text{E.42})$$

which is a set with positive mass since

$$\underline{\Delta} < \Delta' \quad \Rightarrow \quad \max \left\{ 0, \frac{\underline{\Delta} - \tilde{\Delta}_l}{\tilde{\Delta}_h - \tilde{\Delta}_l} \right\} < \eta'. \quad (\text{E.43})$$

The last step is to verify $\eta \in \mathcal{N} \Rightarrow \Delta > \underline{\Delta}$. Here,

$$\Delta > \underline{\Delta} \quad \Leftrightarrow \quad \eta > \frac{\underline{\Delta} - \tilde{\Delta}_l}{\tilde{\Delta}_h - \tilde{\Delta}_l} \quad (\text{E.44})$$

if $\iota = \tilde{\iota}$. This is indeed satisfied since $\eta \in \mathcal{N} \Rightarrow \eta > \max \left\{ 0, \frac{\underline{\Delta} - \tilde{\Delta}_l}{\tilde{\Delta}_h - \tilde{\Delta}_l} \right\}$; there exists an $\eta \in [0, 1]$ that clears the market for liquid wealth if $\iota = \tilde{\iota}$, and it is such that $\Delta > \underline{\Delta}$.

Taking stock, the market for liquid wealth clears if $\iota = \tilde{\iota}$ and if $\iota = \underline{\iota}$, i.e., the ι on the support of the distribution in Equation (E.36) can both occur in equilibrium. From the definition of ρ it also follows $\mathbb{E}\{\iota\} = i$ given the probability distribution from Equation (E.36). I.e., there is an equilibrium with ι non-degenerate. Q.E.D.

Proof of Proposition 9. Focus on the relevant case in which there is indeed a stochastic equilibrium, i.e., a non-degenerate distribution for ι , and $i < \tilde{\iota}$, i.e., the deterministic equilibrium is a boom. From the proof of Proposition 8 it follows $\mathbb{M}_{det,-1} \in (\tilde{R}_h, \tilde{R}_l)$, where $\tilde{R}_h < \tilde{R}_l$, and $\mathbb{M}_{det,-1} = R(i)$.

First note Δ'' is determined uniquely and satisfies $\Delta'' < \tilde{\Delta}_l$, with $\tilde{\Delta}_l$ as defined in the proof of Proposition 8. Uniqueness follows from the fact that

$$\Delta - l\Pi \circ v^{-1}(R(i) + \Delta) - y \quad (\text{E.45})$$

is increasing in Δ since $l\Pi'(q) < v'(q)$. To prove $\Delta'' < \tilde{\Delta}_l$ it therefore suffices to show

$$0 < \tilde{\Delta}_l - l\Pi \circ v^{-1}(R(i) + \tilde{\Delta}_l) - y. \quad (\text{E.46})$$

Since $\tilde{\Delta}_l \equiv l\Pi \circ v^{-1}(\tilde{z}_l) + y$ and $\tilde{R}_l \equiv \tilde{z}_l - l\Pi \circ v^{-1}(\tilde{z}_l) - y$, it follows directly that

$$0 < \tilde{\Delta}_l - l\Pi \circ v^{-1}(R(i) + \tilde{\Delta}_l) - y \quad \Leftrightarrow \quad R(i) < \tilde{R}_l, \quad (\text{E.47})$$

where the latter is satisfied since $R(i) = \mathbb{M}_{det,-1} \in (\tilde{R}_h, \tilde{R}_l)$. It now follows from the proof of Proposition 8 that $\mathbb{M}_{-1} > \mathbb{M}_{det,-1}$, with \mathbb{M}_{-1} as defined before, since $\Delta'' < \tilde{\Delta}_l < \tilde{\Delta}_h$.

Next, consider the case $\underline{\Delta} \leq \Delta''$. It follows directly that equity purchases never materialize if $\iota \leq \tilde{l}$ since Δ is decreasing in ι and satisfies $\Delta \geq \tilde{\Delta}_l$ if $\iota \leq \tilde{l}$. It remains to consider $\iota > \tilde{l}$, for which it can be proven equity purchases do not materialize by means of a contradiction. I.e., suppose equity purchases do materialize, which, in turn, requires $\Delta \leq \underline{\Delta}$. With equity purchases, supply of liquid wealth equals $\mathbb{M}_{-1} + \underline{\Delta}$, entailing

$$\Delta = l\Pi \circ v^{-1}(\mathbb{M}_{-1} + \underline{\Delta}) + y. \quad (\text{E.48})$$

Hence

$$\begin{aligned} \Delta \leq \underline{\Delta} &\Leftrightarrow \underline{\Delta} \geq l\Pi \circ v^{-1}(\mathbb{M}_{-1} + \underline{\Delta}) + y \\ &> \Pi \circ v^{-1}(R(i) + \underline{\Delta}) + y \Leftrightarrow \underline{\Delta} > \Delta'', \end{aligned} \quad (\text{E.49})$$

where the last line uses $\mathbb{M}_{-1} > \mathbb{M}_{det,-1} = R(i)$ and that Equation E.45 is increasing in Δ . With $\underline{\Delta} \leq \Delta''$ it follows equity purchases never materialize in equilibrium, entailing the exact same result as in Proposition 7; i.e., the economy is not stabilized as a non-degenerate distribution for ι is feasible.

Then, consider $\underline{\Delta} \in (\Delta'', \tilde{\Delta}_l + \tilde{R}_l - R(i))$, for which it can be proven equity purchases can materialize with positive probability by supposing $\mathbb{M}_{-1} = \mathbb{M}_{det,-1} + \varepsilon$, where $\varepsilon > 0$ but infinitesimal. Consider a two-point distribution $\{\iota_h, \iota_l\}$ for ι , with $\iota_h < \tilde{l}$ and $\iota_l > \tilde{l}$.

For $\iota > \tilde{l}$, first show equity purchases materialize, for which it suffices to show $\Delta < \underline{\Delta}$. If purchases indeed materialize,

$$\Delta = l\Pi \circ v^{-1}(R(i) + \varepsilon + \underline{\Delta}) + y \quad (\text{E.50})$$

It follows $\Delta < \underline{\Delta}$ since $\underline{\Delta} > \Delta''$ and ε is infinitesimal; equity purchases indeed materialize if $\iota > \tilde{l}$. On the other hand, as follows from the proof of Proposition 8, purchases do not materialize if $\iota < \tilde{l}$ since $\underline{\Delta} \leq \tilde{\Delta}_l + \tilde{R}_l - R(i) < \Delta' < \tilde{\Delta}_h$, where $\tilde{\Delta}_l + \tilde{R}_l - R(i) < \Delta'$

follows from the definition of Δ' in Proposition 8.

Then show the market for liquid wealth indeed clears for some $\iota > \tilde{\iota}$, which, since equity purchases materialize in this case, requires existence of an $\iota > \tilde{\iota}$ solving

$$z_l(\iota) = R(i) + \varepsilon + \underline{\Delta}. \quad (\text{E.51})$$

Such an ι exists if $R(i) + \varepsilon + \underline{\Delta} < \tilde{R}_l + \tilde{\Delta}_l$ since $z_l(\iota) (= \lambda^{-1}(\iota/l))$ is decreasing in ι and $z_l(\tilde{\iota}) = \tilde{z}_l = \tilde{R}_l + \tilde{\Delta}_l$ by the definition of \tilde{R}_l and $\tilde{\Delta}_l$. In turn, $R(i) + \varepsilon + \underline{\Delta} < \tilde{R}_l + \tilde{\Delta}_l$ is satisfied because $\underline{\Delta} < \tilde{\Delta}_l + \tilde{R}_l - R(i)$ and ε is infinitesimal. Hence, set $\iota_l = \lambda^{-1} \left(\frac{R(i) + \varepsilon + \underline{\Delta}}{l} \right)$.

The market for liquid wealth also clears for $\iota_h = \Gamma_h(R(i) + \varepsilon) < \tilde{\iota}$. This follows directly from the fact that such an ι is decreasing ε and exist for sure if $\varepsilon = 0$; otherwise $i < \tilde{\iota}$ cannot hold. Particularly, we have $\iota_h = i - \delta$, where $\delta > 0$ but infinitesimal exactly because $\varepsilon > 0$ but infinitesimal and $i > 0$; otherwise we cannot have a non-degenerate distribution for ι in the first place.

It remains to construct a non-degenerate probability distribution over $\{\iota_h, \iota_l\}$ such that $i = \mathbb{E}_{-1}\{\iota\}$ (see Equation (35)) holds. This requires setting

$$\mathbb{P}_{-1}\{\iota = \iota_l\} = \frac{i - \iota_h}{\iota_l - \iota_h} \quad \text{and} \quad \mathbb{P}_{-1}\{\iota = \iota_h\} = 1 - \mathbb{P}_{-1}\{\iota = \iota_l\}. \quad (\text{E.52})$$

It follows $\mathbb{P}_{-1}\{\iota = \iota_l\} > 0$ but infinitesimal since $\frac{i - \iota_h}{\iota_l - \iota_h} = \frac{\delta}{\iota_l - i + \delta}$, where $\delta > 0$ but infinitesimal while $\iota_l - i > 0$ since $\iota_l > \tilde{\iota} > i$; we have that $\delta \rightarrow 0$ by letting $\varepsilon \rightarrow 0$, while $\iota_l - i$ remains fixed at some positive value. This proves existence of a stochastic equilibrium in which equity purchases materialize with positive probability, in which case $\Delta < \underline{\Delta}$, entailing losses for the taxpayer.

Finally, consider $\underline{\Delta} \in [\tilde{\Delta}_l + \tilde{R}_l - R(i), \Delta')$, for which it can be proven equity purchases can materialize with positive probability by supposing $\mathbb{M}_{-1} = \mathbb{M}_{det,-1} + \varepsilon$, where $\varepsilon > 0$ but infinitesimal. Now, consider two-point distribution $\{\iota_h, \tilde{\iota}\}$ for ι , with $\iota_h < \tilde{\iota}$.

For $\iota = \tilde{\iota}$, first prove equity purchases materialize, for which it suffices to show

$$\eta < \underline{\eta} \equiv \frac{\underline{\Delta} - \tilde{\Delta}_l}{\tilde{\Delta}_h - \tilde{\Delta}_l}. \quad (\text{E.53})$$

If equity purchases materialize,

$$0 = \eta \tilde{z}_h + (1 - \eta) \tilde{z}_l - R(i) - \varepsilon - \underline{\Delta} \Leftrightarrow \eta = \frac{\underline{\Delta} - \tilde{\Delta}_l - (\tilde{R}_l - R(i)) + \varepsilon}{\tilde{z}_h - \tilde{z}_l}. \quad (\text{E.54})$$

Using η' as defined in Proposition 8, it follows

$$\eta < \underline{\eta} \Leftrightarrow \underline{\eta} < \eta' - \frac{\varepsilon}{\tilde{R}_l - \tilde{R}_h}. \quad (\text{E.55})$$

Further, $\underline{\eta} < \eta' \Leftrightarrow \underline{\Delta} < \Delta'$ by the definition of Δ' in Proposition 8; it follows that indeed, $\eta < \underline{\eta}$ if ε is infinitesimal.

Next, for $\iota = \tilde{l}$, prove the market for liquid wealth clears. For this, it suffices to show η , as given by Equation (E.54), is in the interval $[0, 1]$. For $\eta \geq 0$ we need

$$0 \leq \underline{\Delta} - \tilde{\Delta}_l - (\tilde{R}_l - R(i)) + \varepsilon. \quad (\text{E.56})$$

Condition (E.56) is satisfied since $\varepsilon > 0$ and $\underline{\Delta} \geq \tilde{\Delta}_l + \tilde{R}_l - R(i)$ by assumption. On the other hand, note

$$\begin{aligned} \eta &= \frac{\underline{\Delta} - \tilde{\Delta}_l - (\tilde{R}_l - R(i)) + \varepsilon}{\tilde{z}_h - \tilde{z}_l} \\ &= \frac{\underline{\Delta} - \tilde{\Delta}_l - \eta'(\tilde{R}_l - \tilde{R}_h) + \varepsilon}{\tilde{z}_h - \tilde{z}_l} \\ &< \frac{\Delta' - \tilde{\Delta}_l - \eta'(\tilde{R}_l - \tilde{R}_h) + \varepsilon}{\tilde{z}_h - \tilde{z}_l} \\ &= \eta' + \frac{\varepsilon}{\tilde{z}_h - \tilde{z}_l}. \end{aligned} \quad (\text{E.57})$$

With ε infinitesimal and $\eta' \in (0, 1)$ since $R(i) \in (\tilde{R}_h, \tilde{R}_l)$, it follows $\eta < 1$.

The market for liquid wealth also clears for $\iota_h = \Gamma_h(R(i) + \varepsilon) (\equiv i - \delta)$ for the exact same reason as explained for the case $\underline{\Delta} \in (\Delta'', \tilde{\Delta}_l + \tilde{R}_l - R(i))$. A non-degenerate probability distribution over $\{\iota_h, \tilde{l}\}$ such that $i = \mathbb{E}_{-1}\{\iota\}$ (see Equation) is

$$\mathbb{P}_{-1}\{\iota = \tilde{l}\} = \frac{i - \iota_h}{\tilde{l} - \iota_h} \quad \text{and} \quad \mathbb{P}_{-1}\{\iota = \iota_h\} = 1 - \mathbb{P}_{-1}\{\iota = \tilde{l}\}. \quad (\text{E.58})$$

It follows $\mathbb{P}_{-1}\{\iota = \tilde{l}\} > 0$ but infinitesimal for the exact same reason as before, again proving existence of a stochastic equilibrium in which equity purchases materialize with positive probability, entailing losses for the taxpayer. Q.E.D.

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