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## An exponential continuous time GARCH process

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## AN EXPONENTIAL CONTINUOUS TIME GARCH PROCESS

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### Abstract

In this paper we introduce an exponential continuous time GARCH( $p, q$ ) process. It is defined in such a way that it is a continuous time extension of the discrete time EGARCH( $p, q$ ) process. We investigate stationarity and moment properties of the new model. An instantaneous leverage effect can be shown for the exponential continuous time GARCH( $p, p$ ) model.

*Keywords:* exponential continuous time GARCH process; EGARCH, Lévy process; stationarity; stochastic volatility

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Secondary 91B84

### 1. Introduction

GARCH type processes have become very popular in financial econometrics to model returns of stocks, exchange rates and other series observed at equidistant time points. They have been designed (see Engle [8] and Bollerslev [3]) to capture so-called *stylized facts* of such data, which are e.g. volatility clustering, dependence without correlation and tail heaviness. Another characteristic is that stock returns seem to be negatively correlated with changes in the volatility, i.e. that volatility tends to increase after negative shocks and to fall after positive ones. This effect is called *leverage effect* and can not be modeled by a GARCH type process without further extensions. This finding led Nelson [15] to introduce the exponential GARCH process, which is able to model this asymmetry in stock returns. The log-volatility of the EGARCH( $p, q$ ) process was modeled as a self-exciting ARMA( $q, p - 1$ ) process.

The availability of high frequency data, which increased enormously in the last years, is one reason to consider continuous time models with similar behaviour as discrete time GARCH models. The reason for this is ofcourse that at the highest available frequency the observations of the price process occur at irregularly spaced time points and therefore it is kind of natural to assume an underlying continuous time model. Different approaches have been taken to set up a continuous time model, which has the same features as discrete time GARCH processes. Recently Klüppelberg et al.[11] developed a continuous time GARCH(1,1) model, shortly called COGARCH(1,1).

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Their approach differs fundamentally from previous attempts, which could be summarized as diffusion approximations (see e.g. Nelson [14]), by the fact that their model is driven by only one source of randomness (like discrete time GARCH) instead of two (like in the diffusion approximations). They replaced the noise process of discrete time GARCH by the jumps of a Lévy process. The COGARCH(1,1) was then extended by Brockwell et al.[5] to a continuous time GARCH( $p, q$ ) process for general orders  $p, q \in \mathbb{N}$ ,  $q \geq p$ , henceforth called COGARCH( $p, q$ ).

In this paper a continuous time analogue of the EGARCH( $p, q$ ) model is introduced. The noise processes will also be modeled by the increments of a Lévy process. As in the discrete time case we describe the log-volatility process as a linear process, more precisely a continuous time ARMA( $q, p - 1$ ) process.

The paper is now organized as follows. In section 2 we review the definition of the discrete time EGARCH process. After a short review of elementary properties of Lévy processes we define the *exponential continuous time* GARCH( $p, q$ ) process at the beginning of section 3. In addition we state stationarity conditions for the log-volatility and volatility process of our model and give a strong mixing condition for the return process. In section 4 we investigate the second order properties of the volatility process. Section 5 is devoted to the analysis of the second order behaviour of the return process. We derive expressions for the first and second moment of the return process. The stylized fact of zero correlation in the return process but correlation of the squared returns is also shown. In the second part of section 5 we characterize the leverage effect in our model.

## 2. The discrete time EGARCH process

Motivated by empirical evidence that stock returns are negatively correlated with changes in returns volatility Nelson [15] defined the exponential GARCH process (EGARCH) to model this effect, which is called *leverage effect* (see also section 5.1).

*The process  $(X_n)_{n \in \mathbb{Z}}$  of the form  $X_n = \sigma_n \epsilon_n, n \in \mathbb{Z}$ , where  $(\epsilon_n)_{n \in \mathbb{Z}}$  is an i.i.d. sequence with  $\mathbb{E}(\epsilon_1) = 0$  and  $\text{Var}(\epsilon_1) = 1$ , is called an EGARCH process, if the volatility process  $(\sigma_n^2)_{n \in \mathbb{Z}}$  satisfies*

$$\log(\sigma_n^2) = \mu + \sum_{k=1}^{\infty} \beta_k f(\epsilon_{n-k}),$$

*where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is some measurable real valued deterministic function,  $\mu \in \mathbb{R}$  and  $(\beta_k)_{k \in \mathbb{N}}$  are real coefficients such that  $\mathbb{E}(|f(\epsilon_n)|) < \infty$ ,  $\text{Var}(f(\epsilon_n)) < \infty$  and  $\sum_{k=1}^{\infty} |\beta_k| < \infty$ .*

To achieve the asymmetric relation between the stock returns and the volatility,  $f(\epsilon_n)$  must be a function of the magnitude and the sign of  $\epsilon_n$  as noted by Nelson [15]. Therefore he proposed the following function:

$$f(\epsilon_n) := \theta \epsilon_n + \gamma [|\epsilon_n| - \mathbb{E}(|\epsilon_n|)], \quad (2.1)$$

with real coefficients  $\theta$  and  $\gamma$ . We see that  $f(\epsilon_n)$  is linear in  $\epsilon_n$  and has slope  $\theta + \gamma$  for positive shocks  $\epsilon_n$  and slope  $\theta - \gamma$  for negative ones. Therefore  $f(\epsilon_n)$  allows the

volatility process  $(\sigma_n^2)_{n \in \mathbb{Z}}$  to respond asymmetrically to positive and negative jumps in the stock price.

Nelson [15] also suggested a finite parameter model by modeling the log-volatility as an ARMA( $q, p-1$ ) process instead of an infinite moving average process. This leads to the EGARCH( $p, q$ ) model, which is defined in the following way.

Let  $p, q \in \mathbb{N}, \mu, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p \in \mathbb{R}$ , suppose  $\alpha_q \neq 0, \beta_p \neq 0$  and that the autoregressive polynomial  $\phi(z) := 1 - \alpha_1 z - \dots - \alpha_q z^q$  and the moving average polynomial  $\psi(z) := \beta_1 + \beta_2 z + \dots + \beta_p z^{p-1}$  have no common zeros and that  $\phi(z) \neq 0$  on  $\{z \in \mathbb{C} \mid |z| \leq 1\}$ . Let  $(\epsilon_n)_{n \in \mathbb{Z}}$  be an i.i.d. sequence with  $\mathbb{E}(\epsilon_1) = 0$  and  $\text{Var}(\epsilon_1) = 1$ , and let  $f(\cdot)$  be such that  $\mathbb{E}(|f(\epsilon_n)|) < \infty$  and  $\text{Var}(f(\epsilon_n)) < \infty$ . Then  $(X_n)_{n \in \mathbb{Z}}$ , where  $X_n = \sigma_n \epsilon_n$  and

$$\log(\sigma_n^2) = \mu + \sum_{k=1}^p \beta_k f(\epsilon_{n-k}) + \sum_{k=1}^q \alpha_k \log(\sigma_{n-k}^2)$$

is called an EGARCH( $p, q$ ) process.

### 3. Exponential COGARCH

The goal of this section is to construct a continuous time analogue of the discrete time EGARCH( $p, q$ ) process. Therefore we will use the idea of Klüppelberg et al. [11] to replace the noise variables  $\epsilon_n$  by the increments of a Lévy process  $L = (L_t)_{t \geq 0}$ . Any Lévy process  $L$  on  $\mathbb{R}$  has a characteristic function of the form  $\mathbb{E}(e^{iuL_t}) = \exp\{t\psi_L(u)\}$ ,  $t \geq 0$ , with

$$\psi_L(u) := i\gamma_L u - \frac{\tau_L^2}{2} u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\chi_{\{|x| \leq 1\}}) \nu_L(dx), \quad u \in \mathbb{R},$$

where  $\tau_L^2 \geq 0$ ,  $\gamma_L \in \mathbb{R}$ , the measure  $\nu_L$  satisfies

$$\nu_L(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(x^2, 1) \nu_L(dx) < \infty$$

and  $\chi_A(\cdot)$  denotes the indicator function of the set  $A \subset \mathbb{R}$ . The measure  $\nu_L$  is called the *Lévy measure* of  $L$  and the triplet  $(\gamma_L, \tau_L^2, \nu_L)$  is called the *characteristic triplet* of  $L$ . The map  $\psi_L$  is called the *Lévy symbol*. For more details on Lévy processes we refer to Sato [17] or Applebaum [1].

We consider zero mean Lévy processes  $L$  defined on a probability space  $(\Omega, \mathcal{F}, P)$  with jumps  $\Delta L_t := L_t - L_{t-}$ . Since  $\mathbb{E}(L_t) = t(\gamma_L + \int_{|x| > 1} x \nu_L(dx))$ , a zero mean implies that  $\gamma_L = - \int_{|x| > 1} x \nu_L(dx)$  and hence the corresponding Lévy symbol is of the form

$$\psi_L(u) = -\tau_L^2 \frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) \nu_L(dx),$$

and the Lévy-Itô decomposition (see e.g. Theorem 2.4.16 of Applebaum [1]) of  $L$  is

$$L_t = B_t + \int_{\mathbb{R} - \{0\}} x \tilde{N}_L(t, dx), \quad t \geq 0,$$

where  $B$  is a Brownian motion with variance  $\tau_L^2$  and  $\tilde{N}_L(t, dx) = N_L(t, dx) - t\nu_L(dx)$ ,  $t \geq 0$ , is the compensated random measure associated to the Poisson random measure

$$N_L(t, A) = \#\{0 \leq s < t; \Delta L_s \in A\} = \sum_{0 < s \leq t} \chi_A(\Delta L_s), \quad A \in \mathcal{B}(\mathbb{R} - \{0\}),$$

on  $\mathbb{R}_+ \times \mathbb{R} - \{0\}$ , which is independent of  $B$ .

Now we define the *exponential continuous time GARCH(p, q) process* by specifying the log-volatility process as a continuous time ARMA( $q, p - 1$ ) process, henceforth called CARMA( $q, p - 1$ ) process (see e.g. Brockwell and Marquardt [6] for details on CARMA processes), which is the continuous time analogue of an ARMA( $q, p - 1$ ) process. The driving noise process of the CARMA( $q, p - 1$ ) process will be defined similarly to (2.1).

**Definition 3.1.** Let  $L = (L_t)_{t \geq 0}$  be a zero mean Lévy process with Lévy measure  $\nu_L$  such that  $\int_{|x| \geq 1} x^2 \nu_L(dx) < \infty$ . Then we define the exponential COGARCH( $p, q$ ) process  $G$ , shortly ECOGARCH( $p, q$ ), as the stochastic process satisfying,

$$dG_t := \sigma_t dL_t, \quad t > 0, \quad G_0 = 0,$$

where the log-volatility process  $\log(\sigma^2) = (\log(\sigma_t^2))_{t \geq 0}$  is a CARMA( $q, p - 1$ ) process,  $1 \leq p \leq q$ , with mean  $\mu \in \mathbb{R}$  and state space representation

$$\log(\sigma_t^2) := \mu + \mathbf{b}^T \mathbf{X}_{t-}, \quad t > 0, \quad \log(\sigma_0^2) = \mu + \mathbf{b}^T \mathbf{X}_0 \quad (3.1)$$

$$d\mathbf{X}_t = A\mathbf{X}_t + \mathbf{1}_q dM_t, \quad t > 0 \quad (3.2)$$

where  $\mathbf{X}_0 \in \mathbb{R}^q$  is independent of the driving Lévy process  $L$  and

$$M_t := \int_{\mathbb{R} - \{0\}} h(x) \tilde{N}_L(t, dx), \quad t > 0, \quad (3.3)$$

is a zero mean Lévy process (see Remark 3.2) with  $h(x) := \theta x + \gamma|x|$  and parameters  $\theta, \gamma \in \mathbb{R}$ . The  $q \times q$  matrix  $A$  and the vectors  $\mathbf{b} \in \mathbb{R}^q$  and  $\mathbf{1}_q \in \mathbb{R}^q$  are defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_q & -a_{q-1} & -a_{q-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{q-1} \\ b_q \end{bmatrix}, \quad \mathbf{1}_q = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with coefficients  $a_1, \dots, a_q, b_1, \dots, b_p \in \mathbb{R}$ , where  $a_q \neq 0, b_p \neq 0$ , and  $b_{p+1} = \dots = b_q = 0$ .

One has to observe that (3.1) implies independence between  $\sigma_t^2$  and the jump  $\Delta L_t$  at time  $t$ , i.e.  $\sigma^2$  is left continuous. Returns over a time interval of length  $r > 0$  are described by the increments of  $G$

$$G_t^{(r)} := G_t - G_{t-r} = \int_{(t-r, t]} \sigma_s dL_s, \quad t \geq r > 0. \quad (3.4)$$

On the other hand an equidistant sequence of such non-overlapping returns of length  $r$  is given by  $(G_{nr}^{(r)})_{n \in \mathbb{N}}$ . Thus this gives us the possibility to model ultra high frequency data, which consists of returns over varying time intervals.

In the rest of the paper the following terminology will be used:

$G$	(log-) price process
$G^{(r)}$	(log-) return process
$\sigma^2$	volatility process
$\log(\sigma^2)$	log-volatility process.

**Remark 3.2.** (i) The process  $M$  defined by (3.3) is by construction a process with independent and stationary increments and by Theorem 4.3.4 in Applebaum [1] well defined if

$$\int_{\mathbb{E}} |h(x)|^2 \nu_L(dx) < \infty. \quad (3.5)$$

Condition (3.5) is satisfied since  $\nu_L$  is a Lévy measure and  $L$  has finite variance. By equation (2.9) of Applebaum [1] the characteristic function of  $M$  at time  $t \geq 0$  is given by

$$\begin{aligned} \mathbb{E}(e^{iuM_t}) &= \exp\left(t \int_{\mathbb{R}} [e^{iux} - 1 - iux] \nu_M(dx)\right) \\ &= \exp\left(t \left\{ iu\gamma_M + \int_{\mathbb{R}} [e^{iux} - 1 - iux\chi_{|x| \leq 1}] \nu_M(dx) \right\}\right) \\ &=: \exp(t\psi_M(u)), \end{aligned}$$

where  $\nu_M := \nu_L \circ h^{-1}$  is the Lévy measure of  $M$  and  $\gamma_M := -\int_{|x| > 1} x \nu_M(dx)$ . The precise form of  $\nu_M$  depends on the sign and size of  $\theta$  and  $\gamma$  and is given in the following:

$$\nu_M((-\infty, -x]) = \begin{cases} \nu_L([\frac{-x}{\theta+\gamma}, \infty)) + \nu_L((-\infty, -\frac{x}{\theta-\gamma}]), & \theta + \gamma < 0 \text{ and } \theta - \gamma > 0 \\ \nu_L((-\infty, -\frac{x}{\theta-\gamma}]), & \theta - \gamma > 0 \text{ and } \theta + \gamma > 0 \\ \nu_L([\frac{-x}{\theta+\gamma}, \infty)), & \theta + \gamma < 0 \text{ and } \theta - \gamma < 0 \\ 0 & \theta + \gamma > 0 \text{ and } \theta - \gamma < 0 \end{cases}$$

and

$$\nu_M([x, \infty)) = \begin{cases} \nu_L([\frac{x}{\theta+\gamma}, \infty)) + \nu_L((-\infty, \frac{x}{\theta-\gamma}]), & \theta + \gamma > 0 \text{ and } \theta - \gamma < 0 \\ \nu_L((-\infty, \frac{x}{\theta-\gamma}]), & \theta - \gamma < 0 \text{ and } \theta + \gamma < 0 \\ \nu_L([\frac{x}{\theta+\gamma}, \infty)), & \theta + \gamma > 0 \text{ and } \theta - \gamma > 0 \\ 0 & \theta + \gamma < 0 \text{ and } \theta - \gamma > 0 \end{cases}$$

for  $x > 0$ . One recognises that for  $\theta + \gamma < 0 \vee \theta - \gamma > 0$   $M$  is a spectrally negative Lévy process, i.e.  $M$  has only negative jumps, and for  $\theta + \gamma > 0 \vee \theta - \gamma < 0$   $M$  is a spectrally positive Lévy process. Therefore  $M$  has the characteristic triplet  $(\gamma_M, 0, \nu_M)$ .

(ii) The model can of course also be defined for a different choice of  $h$ , as long as condition (3.5) is satisfied.

(iii) In case the jump part of  $L$  is of finite variation  $M$  is a Lévy process of finite variation with Lévy-Itô decomposition

$$M_t := \sum_{0 < s \leq t} [\theta \Delta L_s + \gamma |\Delta L_s|] - Ct, \quad t > 0,$$

where  $C := \gamma \int_{\mathbb{R}} |x| \nu_L(dx)$ .

**Proposition 3.3.** *Let  $\sigma^2$  and  $G$  be as in Definition 3.1, with  $\theta$  and  $\gamma$  not both equal to zero. If the eigenvalues of  $A$  all have negative real parts and  $\mathbf{X}_0$  has the same distribution as  $\int_0^\infty e^{Au} \mathbf{1}_q dM_u$ , then  $\log(\sigma^2)$  and  $\sigma^2$  are strictly stationary.*

**Proof:** The strict stationarity of  $\log(\sigma^2)$  follows from Proposition 2 in Brockwell and Marquardt [6], since it is a CARMA( $q, p-1$ ) process. Since strict stationarity is invariant under continuous transformations,  $\sigma^2$  also has this property.  $\square$

**Remark 3.4.** *The solution of the continuous time state space model (3.1) and (3.2) has the representation*

$$\log(\sigma_{t+}^2) = \mu + \mathbf{b}^T e^{At} \mathbf{X}_0 + \int_0^t \mathbf{b}^T e^{A(t-u)} \mathbf{1}_q dM_u, \quad t > 0.$$

If we choose a second Lévy process  $(\tilde{L}_t)_{t \geq 0}$  independent of  $L$  and with the same distribution as  $L$ , then we can define an extension  $(L_t^*)_{t \in \mathbb{R}}$  of  $L$  to the real line by:

$$L_t^* := L_t \chi_{(0, \infty)}(t) - \tilde{L}_{-t} \chi_{(-\infty, 0)}(t), \quad t \in \mathbb{R},$$

where  $\chi_A(\cdot)$  denotes the indicator function of the set  $A$ . Using  $L^*$  instead of  $L$  in (3.3) we get an extension  $M^*$  of  $M$ . In the following we will write for simplicity  $L$  and  $M$  instead of  $L^*$  and  $M^*$ . In the strictly stationary case the log-volatility process can be defined on the whole real line

$$\log(\sigma_{t+}^2) = \mu + \int_{-\infty}^t g(t-u) dM_u, \quad t \in \mathbb{R}, \quad (3.6)$$

with kernel function

$$g(t) = \mathbf{b}^T e^{At} \mathbf{1}_q \chi_{(0, \infty)}(t) \quad (3.7)$$

(see section 2 of Brockwell and Marquardt [6] for more details).

From (3.4) it follows directly that the increments  $G^{(r)} = \int_{(-r, \cdot]} \sigma_s dL_s$  of  $G$  are stationary if the volatility  $\sigma^2$  is stationary, since the increments of  $L$  are stationary and independent by definition.

**Corollary 3.5.** *If  $\sigma^2$  is strictly stationary, then  $G$  has strictly stationary increments.*

**Remark 3.6.** (i) *If  $q \geq p+1$  the log-volatility process is  $(q-p-1)$  times differentiable, which follows from the state space representation of  $\log(\sigma^2)$ , and hence the volatility process has continuous sample path. In particular the volatility will only contain jumps for  $p=q$ .*

(ii) *The volatility of the ECOGARCH( $p, q$ ) process is positive by definition. Therefore*

the parameters do not need to satisfy any constraints to assure positivity of the volatility. This is not the case for the COGARCH( $p, q$ ) model. For higher order COGARCH( $p, q$ ) processes these condition become quite difficult to check (see Theorem 5.1 in Brockwell et al. [5]).

**Example 3.7.** As a first illustrative example we consider an ECOGARCH(1, 1) process driven by a Lévy process  $L$  with Lévy symbol

$$\psi_L(u) = -\frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1) \lambda \Phi_{0,1/\lambda}(dx),$$

where  $\Phi_{0,1/\lambda}(\cdot)$  is the distribution function of a normal distribution with mean 0 and variance  $1/\lambda$ . This means that  $L$  is the sum of a standard Brownian motion  $W$  and the compound Poisson process  $J_t = \sum_{k=1}^{N_t} Z_k$ ,  $J_{-t} = \sum_{k=1}^{-N_{-t}} Z_{-k}$ ,  $t \geq 0$ , where  $(N_t)_{t \in \mathbb{R}}$  is an independent Poisson process with intensity  $\lambda > 0$  and jump times  $(T_k)_{k \in \mathbb{Z}}$ . The Poisson process  $N$  is also independent from the i.i.d. sequence of jump sizes  $(Z_k)_{k \in \mathbb{Z}}$ , with  $Z_1 \sim N(0, 1/\lambda)$ . The Lévy process  $M$  is in this case given by the following expression

$$M_t = \sum_{k=1}^{N_t} [\theta Z_k + \gamma |Z_k|] - Ct, \quad t > 0,$$

with  $C = \gamma \int_{\mathbb{R}} |x| \lambda \Phi_{0,1/\lambda}(dx) = \sqrt{\frac{2\lambda}{\pi}} \gamma$ .  $M_{-t}$ ,  $t \geq 0$  is defined analogously. If we just consider the case that  $\theta < -\gamma < 0$  then the Lévy measure  $\nu_M$  of  $M$  is defined by

$$\nu_M((-\infty, -x]) = \lambda \Phi_{0,1/\lambda}([\frac{-x}{\theta + \gamma}, \infty)), \quad x > 0,$$

on the negative half real line and by

$$\nu_M([x, \infty)) = \lambda \Phi_{0,1/\lambda}((-\infty, \frac{x}{\theta - \gamma}]), \quad x > 0,$$

on the positive half real line. In the top row of Figure 1 a simulated sample path of the compound Poisson process  $J$ , with  $N(0, 1/2)$  distributed jumps, can be seen over three time scales. The corresponding Lévy process  $M$ , with parameters  $\theta = -0.3$  and  $\gamma = 0.1$ , can be seen in the bottom row. Over all three time intervals one can recognise the desired asymmetry for this set of parameters. If  $J$  jumps up, then  $M$  jumps down and vice versa. If  $J$  does not move, then one observes the downwards drift of  $M$ , which can be seen on the right hand side of Figure 1.

The log-volatility process is then of the form

$$\begin{aligned} \log(\sigma_{t+}^2) &= \mu + \int_{-\infty}^t b_1 e^{-a_1(t-s)} dM_s \\ &= \mu + \sum_{\substack{k=-\infty \\ k \neq 0}}^{N_t} b_1 e^{-a_1(t-T_k)} [\theta Z_k + \gamma |Z_k|] - C \frac{b_1}{a_1}, \quad t > 0 \end{aligned}$$

and the log-price process is given by

$$G_t = \int_0^t \sigma_s dW_s + \sum_{k=1}^{N_t} \sigma_{T_k} Z_k, \quad t > 0, \quad G_0 = 0.$$



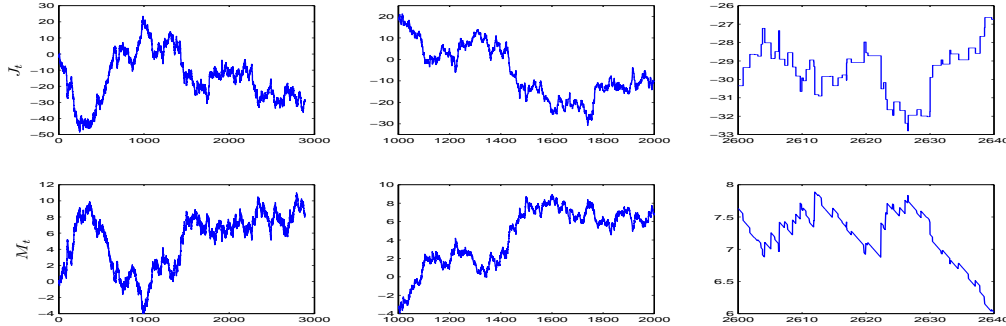


FIGURE 1: Simulated sample paths of  $J$  (top row) and  $M$  (bottom row), with parameters  $\theta = -0.3$  and  $\gamma = 0.1$ , over three different time scales.

with jump times  $T_k$ ,  $k \in \mathbb{N}$ .

The simulation of a sample path of the log-price process  $G$  and the log-volatility process  $\log(\sigma^2)$  over a time interval  $[0, T]$  is done in the following steps.

1. Choose observation times  $0 = t_0 < t_1 < \dots < t_n \leq T$ , possibly random.
2. Simulate the jump times  $(T_k)$ ,  $k = 1, \dots, n_T$ , with  $n_T := \max\{k \in \mathbb{N} : T_k \leq T\}$ , of the compound Poisson process  $J$ .
3. Approximate the state process (3.2) of the log-volatility by a stochastic Euler scheme.
4. Compute an approximation  $\hat{G}$  via the recursion

$$\hat{G}_{t_i} = \hat{G}_{t_{i-1}} + \sigma_{t_{i-1}} \tilde{W}_i + \sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} \sqrt{\exp\{\mu + \mathbf{b}^T \hat{\mathbf{X}}_{T_k-}\}} Z_k,$$

where  $\tilde{W}_i \sim N(0, t_i - t_{i-1})$  and  $\hat{\mathbf{X}}_{T_k-}$  is the Euler approximation without the jump  $\Delta M_{T_k}$ .

In Figure 2 the results of the above simulation procedure are shown. For exponentially distributed interarrival times  $\Delta t_i := t_i - t_{i-1} \sim \text{expo}(1)$  the sample path of the log-price  $G$ , the return process  $G^{(\Delta t)}$  and the volatility process  $\sigma^2$  are displayed in the first three rows of Figure 2. The sample path of the driving Lévy process  $L$  is shown in the last row. The jumps of the compound Poisson process  $J$  are  $N(0, 1/2)$  distributed. From the plots of the return and volatility process we see the negative correlation between the two processes. We recognise increases in the volatility after large negative returns.

Mixing properties (see Doukhan [7] for a comprehensive treatment of mixing properties) are useful for a number of applications. In particular for asymptotic statistics,

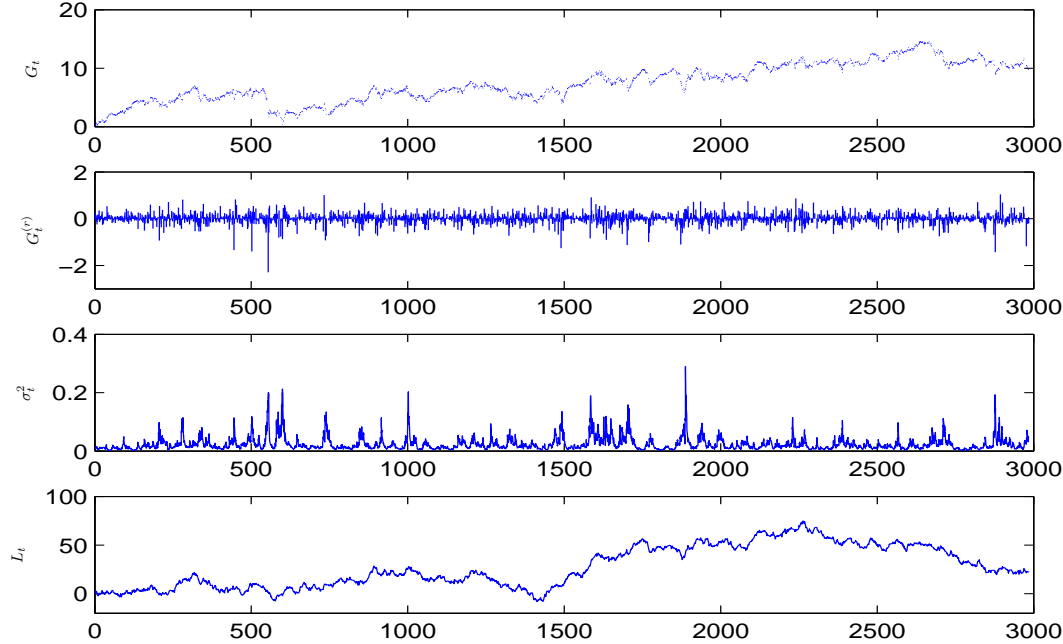


FIGURE 2: 3 000 observations with exponentially distributed interarrival times of the log-price process  $G_t$  (top row), the return process  $G_t^{(r)}$  (second row), the volatility process  $\sigma_t^2$  (third row), with parameters  $b_1 = 1, a_1 = 0.1, \mu = -4, \theta = -0.3$  and  $\gamma = 0.1$  and the driving Lévy process  $L_t$  (last row).

since central limit theorems exist for mixing processes. Thus we will derive mixing properties of the strictly stationary volatility process and the return process over equidistant time intervals. First we recall from Masuda [13] the definition of two mixing concepts, strong mixing, which is also called  $\alpha$ -mixing and  $\tilde{\alpha}$ -mixing for a stationary process with continuous time parameter.

**Definition 3.8.** For a stationary process  $Y = (Y_s)_{s \geq 0}$  define the  $\sigma$ -algebras  $\mathcal{F}_{[0,u]}^Y := \sigma((Y_s)_{s \in [0,u]})$  and  $\mathcal{F}_{[u+t,\infty)}^Y := \sigma((Y_s)_{s \geq u+t})$  for all  $u \geq 0$ . Then  $f \in b\mathcal{F}_{[u,\infty)}^Y$  means that  $f$  is a bounded  $\mathcal{F}_{[u,\infty)}^Y$ -measurable random variable. Let  $\|\cdot\|_{L^1(P)}$  and  $\|\cdot\|_\infty$  be the  $L^1$ -norm under  $P$  and the supnorm, respectively. Then  $Y$  is called

(i) strongly or  $\alpha$ -mixing, if

$$\begin{aligned} \alpha(t) &= \alpha(\mathcal{F}_{[0,u]}^Y, \mathcal{F}_{[u+t,\infty)}^Y) \\ &:= \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_{[0,u]}^Y, B \in \mathcal{F}_{[u+t,\infty)}^Y\} \rightarrow 0, \end{aligned}$$

as  $t \rightarrow \infty$ , for all  $u \geq 0$ .

(ii)  $\tilde{\alpha}$ -mixing, if

$$\begin{aligned}\tilde{\alpha}(t) &= \tilde{\alpha}(\mathcal{F}_{[0,u]}^Y, \mathcal{F}_{[u+t,\infty)}^Y) \\ &:= \sup\{\|\mathbb{E}(f|\mathcal{F}_{[0,u]}^Y) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}_{[u+t,\infty)}^Y, \|f\|_\infty \leq 1\} \rightarrow 0,\end{aligned}$$

as  $t \rightarrow \infty$ , for all  $u \geq 0$ .

Throughout the paper all  $\sigma$ -algebras are assumed to include all  $P$ -null sets. From equation (2) in Masuda [13] we know that for all  $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) \leq \tilde{\alpha}(\mathcal{F}_1, \mathcal{F}_2) \leq 6\alpha(\mathcal{F}_1, \mathcal{F}_2). \quad (3.8)$$

Therefore the  $\alpha$ -mixing property is equivalent to the  $\tilde{\alpha}$ -mixing property. In Theorem 3.10 we will use this fact to show that  $(G_{nr}^{(r)})_{n \in \mathbb{N}}$  is  $\alpha$ -mixing.

The strong mixing property with exponential rate of the log-volatility and volatility process is subject of the next proposition. Thereby strongly mixing with exponential rate (exponentially  $\alpha$ -mixing) means that  $\alpha(t)$  decays to zero exponentially fast for  $t \rightarrow \infty$ .

**Proposition 3.9.** *Let  $\log(\sigma^2)$  be defined by (3.1) and (3.2) with  $\theta$  and  $\gamma$  not both equal to zero. Assume that  $\mathbb{E}(L_1^2) < \infty$ , the eigenvalues of  $A$  all have negative real parts and  $\mathbf{X}_0$  has the same distribution as  $\int_0^\infty e^{Au} \mathbf{1}_q dM_u$ , hence  $\log(\sigma^2)$  and  $\sigma^2$  are strictly stationary. Then there exist constants  $K > 0$  and  $a > 0$  such that*

$$\alpha_{\log(\sigma^2)}(t) \leq K \cdot e^{-at} \quad \text{and} \quad \alpha_{\sigma^2}(t) \leq K \cdot e^{-at}, \quad \text{as } t \rightarrow \infty, \quad (3.9)$$

where  $\alpha_{\log(\sigma^2)}(t)$  and  $\alpha_{\sigma^2}(t)$  are the  $\alpha$ -mixing coefficients of the log-volatility and volatility process, respectively.

**Proof:** The log-volatility process is a CARMA( $q, p-1$ ) process, which is equal to the first component of the  $q$ -dimensional OU process  $\mathbf{V} := (V^1, \dots, V^q)^T \in \mathbb{R}^q$  (see e.g. section 4 of Brockwell [4]) where for fixed  $t$

$$\mathbf{V}_t = e^{BAB^{-1}(t-s)} \mathbf{V}_s + \int_s^t e^{A(t-u)} B \mathbf{1}_q dM_u \quad a.s., \quad (3.10)$$

with

$$B = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_q \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since  $L$ , hence  $M$ , has finite second moment  $\mathbf{V}$  also has finite second moment. Therefore the condition (4.5) in Masuda [12] is satisfied. By Theorem 4.3 in Masuda [12]  $\mathbf{V}$  is then exponentially  $\alpha$ -mixing. Since every component of a multidimensional strongly mixing process is strongly mixing, the log-volatility process is also exponentially  $\alpha$ -mixing. The property of  $\alpha$ -mixing is invariant under continuous transformations, which implies that  $\sigma^2$  also has this property.  $\square$

**Theorem 3.10.** Assume that  $L$  is a Lévy process with finite variance. Let the volatility process  $\sigma^2$  be strictly stationary and strongly mixing under  $P$ . Then the discrete time process  $(G_{nr}^{(r)})_{n \in \mathbb{N}}$ ,

$$G_{nr}^{(r)} := G_{nr} - G_{(n-1)r} = \int_{((n-1)r, nr]} \sigma_s dL_s, n \in \mathbb{N},$$

is strongly mixing with geometric rate and since strict stationarity of  $\sigma^2$  implies strict stationarity of  $(G_{nr}^{(r)})_{n \in \mathbb{N}}$  also ergodic.

**Proof:** Consider again the  $q$ -dimensional OU process  $\mathbf{V}$  defined in the proof of Proposition 3.9. Equation (3.10) shows that  $\mathbf{V}$  is a Markov process. The process  $\mathbf{V}$  is also strictly stationary and  $\tilde{\alpha}$ -mixing, since  $\alpha$ -mixing and  $\tilde{\alpha}$ -mixing are equivalent. The same is true for the  $q$ -dimensional process  $\boldsymbol{\sigma}^2 := (\exp(V^1), \dots, \exp(V^q))$ , since both mixing properties are invariant under continuous transformations. Now define the discrete time  $q$ -dimensional process

$$\mathbf{G}_{nr}^{(r)} = \int_{((n-1)r, nr]} \sqrt{\boldsymbol{\sigma}_s^2} dL_s, \quad n \in \mathbb{N}, \quad (3.11)$$

which is also strictly stationary under  $P$ . Here we should mention that the integration of vectors in (3.11) is understood componentwise.

Define the  $\sigma$ -algebra  $\mathcal{F}_I^{dL} := \sigma(L_t - L_s : s, t \in I)$  for  $I \subset \mathbb{R}$ . Further we denote the  $\sigma$ -algebra generated by the volatility process and the increments of  $L$  over the interval  $[0, t]$  by  $\mathcal{F}_{[0, t]}^{\boldsymbol{\sigma}^2, dL} := \sigma((\boldsymbol{\sigma}_s^2)_{s \in [0, t]} \vee \mathcal{F}_{[0, t]}^{dL})$ .

From  $\mathcal{F}_{\{1, 2, \dots, l\}}^{\mathbf{G}^{(r)}} \subset \mathcal{F}_{[0, lr]}^{\boldsymbol{\sigma}^2, dL}$  and  $\mathcal{F}_{\{k+l, k+l+1, \dots\}}^{\mathbf{G}^{(r)}} \subset \mathcal{F}_{[(k+l-1)r, \infty]}^{\boldsymbol{\sigma}^2, dL}$  we get that

$$\begin{aligned} \tilde{\alpha}_{\mathbf{G}^{(r)}}(k) &= \sup \left\{ \|\mathbb{E}(f | \mathcal{F}_{\{1, 2, \dots, l\}}^{\mathbf{G}^{(r)}}) - \mathbb{E}(f)\|_{L^1(P)} : \right. \\ &\quad \left. f \in b\mathcal{F}_{\{k+l, k+l+1, \dots\}}^{\mathbf{G}^{(r)}}, l \in \mathbb{N}, \|f\|_\infty \leq 1 \right\} \\ &\leq \sup \left\{ \|\mathbb{E}(f | \mathcal{F}_{[0, lr]}^{\boldsymbol{\sigma}^2, dL}) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}_{[(k+l-1)r, \infty]}^{\boldsymbol{\sigma}^2, dL}, l \in \mathbb{N}, \|f\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \|\mathbb{E}(f | \mathcal{F}_{[0, lr]}^{\boldsymbol{\sigma}^2, dL}) - \mathbb{E}(f)\|_{L^1(P)} : \right. \\ &\quad \left. f \in b\mathcal{F}_{(k+l-1)r}^{\boldsymbol{\sigma}^2} \vee b\mathcal{F}_{[(k+l-1)r, \infty]}^{dL}, l \in \mathbb{N}, \|f\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \|\mathbb{E}(f | \mathcal{F}_{[0, lr]}^{\boldsymbol{\sigma}^2}) - \mathbb{E}(f)\|_{L^1(P)} : \right. \\ &\quad \left. f \in b\mathcal{F}_{(k+l-1)r}^{\boldsymbol{\sigma}^2} \vee b\mathcal{F}_{[(k+l-1)r, \infty]}^{dL}, l \in \mathbb{N}, \|f\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \|\mathbb{E}(f | \mathcal{F}_{[0, lr]}^{\boldsymbol{\sigma}^2}) - \mathbb{E}(f)\|_{L^1(P)} : \right. \\ &\quad \left. f \in b\mathcal{F}_{(k+l-1)r}^{\boldsymbol{\sigma}^2} \vee b\mathcal{F}_{[(k+l-1)r, \infty]}^{dM}, l \in \mathbb{N}, \|f\|_\infty \leq 1 \right\} \\ &= \tilde{\alpha}_{\boldsymbol{\sigma}^2}((k-1)r) \end{aligned}$$

for any  $k \in \mathbb{N}$ . The second equality is due to the fact that the driving process of  $\boldsymbol{\sigma}^2$  is defined through the jumps of  $L$ .  $\mathcal{F}_{[0, lr]}^{dL}$  and  $b\mathcal{F}_{(k+l-1)r}^{\boldsymbol{\sigma}^2} \vee b\mathcal{F}_{[(k+l-1)r, \infty]}^{dL}$  are conditionally independent given  $\mathcal{F}_{[0, lr]}^{\boldsymbol{\sigma}^2}$ , which is due to the Markov property of  $\boldsymbol{\sigma}^2$  and

the independence between  $\mathcal{F}_{[0,lr]}^{dL}$  and  $b\mathcal{F}_{[(k+l-1)r,\infty)}^{dL}$ . This shows the third equality. The fourth one follows from the independence of  $b\mathcal{F}_{[(k+l-1)r,\infty)}^{dL}$  and  $b\mathcal{F}_{[(k+l-1)r,\infty)}^{dM}$  from  $\mathcal{F}_{[0,lr]}^{\sigma^2}$  and again the Markov property of  $\sigma^2$ , whereas the last one is due to the fact that  $M$  is the driving process of  $\sigma^2$ .

Therefore  $(\mathbf{G}_{nr}^{(r)})_{n \in \mathbb{N}}$  is  $\tilde{\alpha}$ -mixing and by (3.8) also strongly mixing. The strict stationarity of  $(\mathbf{G}_{nr}^{(r)})_{n \in \mathbb{N}}$  implies that it is also ergodic. Since both properties have to hold componentwise, we have shown that  $(G_{nr}^{(r)})_{n \in \mathbb{N}}$  as the first component of  $(\mathbf{G}_{nr}^{(r)})_{n \in \mathbb{N}}$  is strongly mixing and ergodic. From (3.8) and (3.9) we further get that

$$\alpha_{G^{(r)}}(k) \leq 6\alpha_{\sigma^2}((k-1)r) \leq 6Ke^{-a(k-1)r},$$

as  $k \rightarrow \infty$ . Hence  $(G_{nr}^{(r)})_{n \in \mathbb{N}}$  is strongly mixing with exponential rate and ergodicity follows from the strict stationarity.  $\square$

#### 4. Second order properties of the volatility process

In this section we derive moments and the autocovariance function of the volatility process  $\sigma^2$ . Since it is a non-linear transformation of a CARMA( $q, p-1$ ) process, we will first recall the moment structure and conditions for weak stationarity of a CARMA( $q, p-1$ ) process.

**Proposition 4.1.** *If  $\mathbf{X}_0$  has the same mean vector and covariance matrix as  $\int_0^\infty e^{Au} \mathbf{1}_q dM_u$ , then  $\log(\sigma^2)$  is weakly stationary. In the weakly stationary case the mean and autocovariance function of  $\log(\sigma^2)$  are given by*

$$\mathbb{E}(\log(\sigma_t^2)) = \mu \quad \text{and} \quad \text{Cov}(\log(\sigma_{t+h}^2), \log(\sigma_t^2)) = \mathbb{E}(M_1^2) \mathbf{b}^T e^{Ah} \Sigma \mathbf{b}, \quad h, t \geq 0, \quad (4.1)$$

where  $\Sigma := \int_0^\infty e^{As} \mathbf{1}_q \mathbf{1}_q^T e^{A^T s} ds$ .

**Proof:** The condition for weak stationarity is given in Proposition 1 in Brockwell and Marquardt [6]. The moment expressions follow from Remark 4 in Brockwell and Marquardt [6] and the fact that  $\int_{\mathbb{R}} g(u-h)g(u)du = \mathbf{b}^T e^{Ah} \Sigma \mathbf{b}$ , with  $g$  defined in (3.7).  $\square$

The moments of the strictly stationary volatility process are exponential moments of the limit distribution of the log-volatility process. Therefore we characterise the limit distribution of the log-volatility process in the following Proposition.

**Proposition 4.2.** *Let  $(\gamma_M, 0, \nu_M)$  be the characteristic triplet of the Lévy process  $M$ , where  $M$  is defined in (3.3), and  $P_t(x, B) = P(\log(\sigma_t^2) \in B | \log(\sigma_0^2) = x)$ ,  $x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R})$ . If the eigenvalues of  $A$  only have negative real parts, then there exists a limit distribution  $F$  such that*

$$P_t(x, B) \rightarrow F(B), \quad (4.2)$$

as  $t \rightarrow \infty$ . The distribution  $F$  is infinitely divisible with characteristic triplet  $(\gamma_\infty, 0, \nu_\infty)$ , where

$$\begin{aligned} \gamma_\infty &= \int_0^\infty g(s) \gamma_M ds + \int_0^\infty \int_{\mathbb{R}} g(s)x [\chi_{\{|g(s)x| \leq 1\}} - \chi_{\{|x| \leq 1\}}] \nu_M(dx) ds \\ \nu_\infty(B) &= \int_0^\infty \int_{\mathbb{R}} \chi_B(g(s)x) \nu_M(dx) ds, \quad B \in \mathcal{B}(\mathbb{R}), \end{aligned}$$

with  $g(s) = \mathbf{b}^T e^{As} \mathbf{1}_q \chi_{(0, \infty)}(s)$ . Let the limit of the log-volatility process be denoted by  $\log(\sigma_\infty^2)$ . The characteristic function of  $\log(\sigma_\infty^2)$  can then be written as

$$\begin{aligned} \mathbb{E}(e^{iu \log(\sigma_\infty^2)}) &= \exp \left\{ iu\gamma_\infty + \int_{\mathbb{R}} (e^{iux} - 1 - iux\chi_{|x| \leq 1}) \nu_\infty(dx) \right\} \\ &= \exp \left\{ \int_0^\infty \psi_M(g(s)u) ds \right\} = \exp \{ \psi_\infty(u) \}. \end{aligned}$$

**Proof:** Since  $M$  has finite variance, this implies  $\int_{|x| > 1} \log(|x|) \nu_M(dx) < \infty$ . The existence of a limit distribution  $F$  and the infinitely divisibility of  $F$  now follows from Theorem 4.1 in Sato and Yamazato [18]. The second equality in the representation of the characteristic function follows from the fact that  $\int_{\mathbb{R}} H(y) \nu_\infty(dy) = \int_0^\infty \int_{\mathbb{R}} H(g(s)x) \nu_M(dx) ds$  for all  $h$  integrable with respect to  $\nu_\infty$  (see e.g. Theorem 41 in Rocha-Arteaga and Sato [16]).  $\square$

With the limit distribution  $F$  at hand, we can now apply Theorem 25.17 of Sato [17] to calculate the exponential moments of  $\log(\sigma_\infty^2)$ , i.e. the moments of  $\sigma_\infty^2$ , in the next Proposition.

**Proposition 4.3.** *Let the distribution function of  $\log(\sigma_\infty^2)$  be  $F$  with characteristic triplet  $(\gamma_\infty, 0, \nu_\infty)$ . Then the  $k$ -th moment of  $\sigma_\infty^2$  is finite, if*

$$k \in K_\infty = \left\{ s \in \mathbb{R} : \int_{|x| > 1} e^{sx} \nu_\infty(dx) < \infty \right\}.$$

In this case

$$\Psi_\infty(k) = \psi_\infty(-ik) = \int_0^\infty \Psi_M(g(s)k) ds, \quad (4.3)$$

with  $\Psi_M(u) := \psi_M(-iu)$ ,  $u \in \mathbb{R}$ , is well defined and

$$\mathbb{E}(\sigma_\infty^{2k}) = e^{\mu k} e^{\Psi_\infty(k)}. \quad (4.4)$$

**Proof:** Since  $F$  is infinitely divisible we can apply Theorem 25.17 of Sato [17]. This theorem gives an expression for the exponential moment of a Lévy process. It is straightforward to apply this result to the case that we just have an infinitely divisible distribution. Therefore the  $k$ -th exponential moment of  $\log(\sigma_\infty^2)$  is given by

$$\mathbb{E}(\exp(\log(\sigma_\infty^2))^k) = e^{\mu k} e^{\Psi_\infty(k)},$$

with  $\Psi_\infty(k) = \gamma_\infty k + \int_{\mathbb{R}} (e^{kx} - 1 - kx\chi_{|x| \leq 1}) \nu_\infty(dx)$ . Since  $\log(\sigma_\infty^2)$  has the representation  $\log(\sigma_\infty^2) = \int_0^\infty g(s) dM_s$  (compare to (3.6)), with  $g(s) = \mathbf{b}^T e^{As} \mathbf{1}_q \chi_{(0, \infty)}(s)$  and  $\mathbb{E}(M_1) = 0$ , we get  $\Psi_\infty(k) = \int_0^\infty \Psi_M(g(s)k) ds$ .  $\square$

**Remark 4.4.** *By (4.2)  $F$  is also the distribution of the strictly stationary solution of (3.1) and (3.2), hence*

$$\mathbb{E}(\sigma_t^{2k}) = e^{\mu k} e^{\Psi_\infty(k)}, \quad t \geq 0, \quad (4.5)$$

for all  $k \in K_\infty$ .

**Proposition 4.5.** *Let  $\log(\sigma_t^2)$  be the strictly stationary solution of (3.1) and (3.2). Assume that  $\mathbb{E}(\sigma_t^4) < \infty$  for all  $t \geq 0$ . Let  $\Psi_\infty^h(1)$  and  $\Psi^h(1)$  be defined by (4.3) with kernel function  $g$  replaced by  $g_\infty^h(s) = \mathbf{b}^T(I_q + e^{Ah})e^{As}\mathbf{1}_q$  and  $g^h(s) = \mathbf{b}^T e^{As}\mathbf{1}_q \chi_{(0,h)}(s)$ , respectively. Then the autocovariance function of  $\sigma^2$  is given by the following expression*

$$\text{Cov}(\sigma_{t+h}^2, \sigma_t^2) = e^{2\mu}(e^{\Psi_\infty^h(1)}e^{\Psi^h(1)} - e^{2\Psi_\infty(1)}), \quad h > 0, t \geq 0. \quad (4.6)$$

**Proof:** Let  $\mu = 0$  and  $(\mathcal{F}_t^{\sigma^2})_{t \geq 0}$  be the filtration generated by  $(\sigma_t^2)_{t \geq 0}$ , then  $\mathbb{E}(\sigma_{t+h}^2 | \mathcal{F}_t^{\sigma^2}) = \exp \left\{ \int_{-\infty}^t \mathbf{b}^T e^{Ah} e^{A(t-s)} \mathbf{1}_q dM_s \right\} \mathbb{E} \left( \exp \left\{ \int_t^{t+h} g(t+h-s) dM_s \right\} \middle| \mathcal{F}_t^{\sigma^2} \right)$ . Therefore we get

$$\begin{aligned} \mathbb{E}(\sigma_{t+h}^2 \sigma_t^2) &= \mathbb{E}(\mathbb{E}(\sigma_{t+h}^2 \sigma_t^2 | \mathcal{F}_t^{\sigma^2})) = \mathbb{E}(\sigma_t^2 \mathbb{E}(\sigma_{t+h}^2 | \mathcal{F}_t^{\sigma^2})) \\ &= \mathbb{E} \left( \sigma_t^2 \exp \left\{ \int_{-\infty}^t \mathbf{b}^T e^{Ah} e^{A(t-s)} \mathbf{1}_q dM_s \right\} \mathbb{E} \left( \exp \left\{ \int_t^{t+h} g(t+h-s) dM_s \right\} \middle| \mathcal{F}_t^{\sigma^2} \right) \right) \\ &= \mathbb{E} \left( \exp \left\{ \int_{-\infty}^t \mathbf{b}^T (I_q + e^{Ah}) e^{A(t-s)} \mathbf{1}_q dM_s \right\} \right) \mathbb{E} \left( \exp \left\{ \int_0^h g(s) dM_s \right\} \right) \\ &= \mathbb{E} \left( \exp \left\{ \int_0^\infty \mathbf{b}^T (I_q + e^{Ah}) e^{As} \mathbf{1}_q dM_s \right\} \right) \mathbb{E} \left( \exp \left\{ \int_0^\infty \mathbf{b}^T e^{As} \mathbf{1}_q \chi_{(0,h)}(s) dM_s \right\} \right) \\ &= e^{\Psi_\infty^h(1)} e^{\Psi^h(1)}, \end{aligned}$$

where the last equality follows from (4.4). This together with (4.5) yields (4.6). The case  $\mu \neq 0$  results in multiplying the expressions by  $e^{2\mu}$ .  $\square$

**Remark 4.6.**

*In Proposition 3.9 we have seen that the volatility process is strongly mixing with exponential rate. A consequence of this property (see e.g. section 1.2.2 in Doukhan [7]) is that*

$$|\text{Cov}(\sigma_{t+h}^2, \sigma_t^2)| \leq K \cdot e^{-ah}, \quad \forall h > 0.$$

*In particular this means that the autocovariance function of the volatility process will decay to zero at an exponential rate.*

## 5. Second order properties of the return process

In this section we derive the moment structure of the *return process*

$$G_t^{(r)} := G_t - G_{t-r} = \int_{(t-r, t]} \sigma_s dL_s, \quad t \geq r > 0.$$

We will only consider the case of a strictly stationary volatility process. In the second part of the section we discuss the *leverage effect* for our model.

### 5.1. Moments and autocovariance function of the return process

**Proposition 5.1.** *Let  $L$  be a Lévy process with  $\mathbb{E}(L_1) = 0$  and  $\mathbb{E}(L_1^2) < \infty$ . Assume that the volatility process  $\sigma^2$  is strictly stationary with finite mean. Then  $\mathbb{E}(G_t^2) < \infty$*

for all  $t \geq 0$ , and for every  $t, h \geq r > 0$  it holds

$$\mathbb{E}G_t^{(r)} = 0 \quad (5.1)$$

$$\mathbb{E}(G_t^{(r)})^2 = e^{\mu + \Psi_\infty(1)r} \mathbb{E}(L_1^2) \quad (5.2)$$

$$\mathbb{Cov}(G_t^{(r)}, G_{t+h}^{(r)}) = 0. \quad (5.3)$$

If further  $\mathbb{E}(L_1^4) < \infty$  and the volatility process has finite second moment, then  $\mathbb{E}(G_t^4) < \infty$  for all  $t \geq 0$  and for every  $t, h \geq r > 0$  we have

$$\mathbb{Cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = \mathbb{E}(L_1^2) \int_h^{h+r} \mathbb{Cov}(G_r^2, \sigma_s^2) ds. \quad (5.4)$$

**Proof:** If  $L$  has no Brownian component the proof of (5.1) - (5.3) is analogously to the proof of Proposition 5.1 in Klüppelberg et al. [11] and can be extended in the same as in the proof of Proposition 2.1 in Haug et al. [10] in case  $L$  has a Brownian component. Since  $G$  is a square integrable martingale we get

$$\mathbb{E}((G_0^{(r)})^2 (G_h^{(r)})^2) = \mathbb{E}(G_r^2 (G_{h+r} - G_h)^2) = \mathbb{E}(G_r^2 (G_{h+r}^2 - G_h^2)).$$

Using this result,  $G_t^2 = 2 \int_0^t G_{s-} \sigma_s dL_s + \int_0^t \sigma_s^2 d[L, L]_s$ ,  $t \geq 0$ , and the compensation formula (see e.g. section 0.5 in Bertoin [2]) we get

$$\begin{aligned} \mathbb{E}((G_0^{(r)})^2 (G_h^{(r)})^2) &= \mathbb{E} \left( 2 \int_h^{h+r} G_r^2 G_{s-} \sigma_s dL_s + \int_h^{h+r} G_r^2 \sigma_s^2 d[L, L]_s \right) \\ &= \mathbb{E} \left( \int_h^{h+r} G_r^2 \sigma_s^2 d[L, L]_s \right) \\ &= \int_h^{h+r} \mathbb{E}(G_r^2 \sigma_s^2) \tau_L^2 ds + \int_h^{h+r} \mathbb{E}(G_r^2 \sigma_s^2) ds \int_{\mathbb{R}} x^2 \nu_L(dx) \\ &= \mathbb{E}(L_1^2) \int_h^{h+r} \mathbb{E}(G_r^2 \sigma_s^2) ds \end{aligned}$$

Hence the covariance is equal to

$$\begin{aligned} \mathbb{Cov}((G_0^{(r)})^2 (G_h^{(r)})^2) &= \mathbb{E}((G_0^{(r)})^2 (G_h^{(r)})^2) - (\mathbb{E}(G_t^{(r)})^2)^2 \\ &= \mathbb{E}(L_1^2) \int_h^{h+r} (\mathbb{Cov}(G_r^2, \sigma_s^2) + \mathbb{E}(G_r^2) \mathbb{E}(\sigma_s^2)) ds - (\mathbb{E}(G_t^{(r)})^2)^2 \\ &= \mathbb{E}(L_1^2) \int_h^{h+r} \mathbb{Cov}(G_r^2, \sigma_s^2) ds. \end{aligned}$$

The covariance is finite if  $\mathbb{E}(G_t^4) < \infty$ ,  $\forall t \geq 0$ , and this follows with  $\mathbb{E}(L_1^4) < \infty$  and  $2 \in K_\infty$  analogously as in Proposition 1.1 in Haug et al. [10].  $\square$

**Example 5.2.** Let us consider again Example 3.7. From 50 000 equidistant observations of the simulated log-price we computed the empirical autocorrelation function of the returns and squared returns. In Figure 3 the first 40 lags of both empirical autocorrelation functions are shown. One recognises the GARCH like behaviour of zero correlation of the returns and significant correlation of the squared returns.



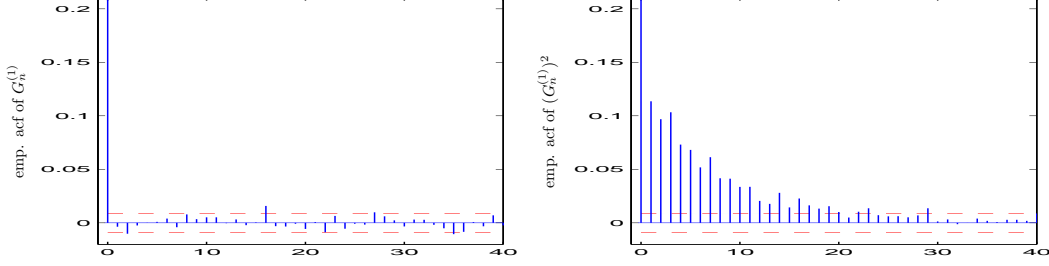


FIGURE 3: The first 40 lags of the empirical autocorrelation function of the return (*left*) and squared return (*right*) process.

In the next Proposition we want to consider the special case that  $p = q = 1$ . Under a further assumption on the mapping  $h$ , which has not to be the standard choice of Definition 3.1, and the Lévy measure  $\nu_L$  we can express the covariance of the squared returns in terms of the covariance of the *actual volatility*

$$\sigma^2(h) := \int_0^{h+r} \sigma_s^2 ds - \int_0^h \sigma_s^2 ds, \quad h \geq r.$$

**Proposition 5.3.** *Let  $p = q = 1$  and assume that the mapping  $h$  satisfies (3.5) and additionally  $\int_{\mathbb{R}} x \{\exp(b_1 e^{-a_1 t} h(x)) - 1\} \nu_L(dx) = 0$  for all  $t \geq 0$ . Then under the same conditions as in Proposition 5.1 we get*

$$\text{Cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = (\mathbb{E}(L_1^2))^2 \text{Cov}(\sigma^2(h), \sigma^2(0)).$$

**Proof:** From the proof of Proposition 5.1 we know that

$$\begin{aligned} \text{Cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) &= \mathbb{E}(L_1^2) \int_h^{h+r} \mathbb{E}(G_r^2 \sigma_s^2) ds - (r \mathbb{E}(L_1^2) \mathbb{E}(\sigma_1^2))^2 \\ &= \mathbb{E}(L_1^2) \mathbb{E} \left( \int_h^{h+r} \left\{ 2 \int_0^r G_u - \sigma_u dL_u + \int_0^r \sigma_u^2 d[L, L]_u \right\} \sigma^2 ds \right) - (r \mathbb{E}(L_1^2) \mathbb{E}(\sigma_1^2))^2. \end{aligned}$$

Hence the result follows if we can show that  $\mathbb{E}(\int_0^r G_u - \sigma_u \sigma_s^2 dL_u) = 0$ , for all  $s > r$ , since  $\mathbb{E}(\int_h^{h+r} \int_0^r \sigma_u^2 \sigma_s^2 d[L, L]_u ds) = (\mathbb{E}(L_1^2))^2 \mathbb{E}(\sigma^2(h) \sigma^2(0))$ . Define  $Y_t := \int_0^t G_u - \sigma_u dL_u$ ,  $t \geq 0$ , and  $\tilde{\sigma}_{s,t}^2 := \exp(b_1 e^{-a_1(s-t)} X_t)$  for all  $t \in [0, s]$ , where  $X$  is the state process (3.2). Then  $\mathbb{E}(Y_r \sigma_s^2) = \mathbb{E}(Y_r \tilde{\sigma}_{s,r+}^2) \mathbb{E}(\exp(\int_r^s b_1 e^{-a_1(s-u)} dM_u))$  and we have to show  $\mathbb{E}(Y_r \tilde{\sigma}_{s,r+}^2) = 0$ . An application of Itô's formula and substituting from (3.2) yields

$$\begin{aligned} \tilde{\sigma}_{s,r+}^2 &= \tilde{\sigma}_{s,0}^2 + \int_0^r \int_{\mathbb{R}-\{0\}} \tilde{\sigma}_{s,u}^2 b_1 e^{-a_1(s-u)} h(x) \tilde{N}_L(du, dx) \\ &\quad + \int_0^r \int_{\mathbb{R}-\{0\}} \tilde{\sigma}_{s,u}^2 \left\{ \exp(b_1 e^{-a_1(s-u)} h(x)) - 1 - b_1 e^{-a_1(s-u)} h(x) \right\} N_L(du, dx). \end{aligned}$$

Using this representation, integration by parts and taking expectation gives

$$\begin{aligned} \mathbb{E}(Y_r \tilde{\sigma}_{s,r+}^2) &= \\ & \int_0^r \int_{\mathbb{R}-\{0\}} \mathbb{E}(Y_u \tilde{\sigma}_{s,u}^2) \left\{ \exp(b_1 e^{-a_1(s-u)} h(x)) - 1 - b_1 e^{-a_1(s-u)} h(x) \right\} \nu_L(dx) du \\ & + \int_0^r \int_{\mathbb{R}-\{0\}} \mathbb{E}(\tilde{\sigma}_{s,u}^2 G_u - \sigma_u) x \left\{ \exp(b_1 e^{-a_1(s-u)} h(x)) - 1 \right\} \nu_L(dx) \\ & = \int_0^r \int_{\mathbb{R}-\{0\}} \mathbb{E}(Y_u \tilde{\sigma}_{s,u+}^2) \left\{ \exp(b_1 e^{-a_1(s-u)} h(x)) - 1 - b_1 e^{-a_1(s-u)} h(x) \right\} \nu_L(dx) du, \end{aligned}$$

where we used  $\int_{\mathbb{R}} x \left\{ \exp(b_1 e^{-a_1 t} h(x)) - 1 \right\} \nu_L(dx) = 0$  and  $Y_u \tilde{\sigma}_{s,u}^2 = Y_u \tilde{\sigma}_{s,u+}^2$  almost surely for fixed  $u$ . Solving this linear integral equation with initial condition  $\mathbb{E}(Y_0 \tilde{\sigma}_{s,0+}^2) = 0$ , it follows that  $\mathbb{E}(Y_r \tilde{\sigma}_{s,r+}^2) = 0$  for all  $r \geq 0$  is the only solution.  $\square$

**Remark 5.4.** In Theorem 3.10 we have seen that  $(G_{nr}^{(r)})_{n \in \mathbb{N}}$  is strongly mixing with exponential rate. A consequence of this property (see e.g. section 1.2.2 in Doukhan [7]) is that there exists a constant  $K_G > 0$  such that

$$|\text{Cov}((G_{(n+h)r}^{(r)})^2, (G_{nr}^{(r)})^2)| \leq K_G \cdot e^{-ah}, \quad \forall h > 0.$$

In particular this means that the autocovariance function of the squared returns will decay to zero at an exponential rate.

## 5.2. Leverage effect

In empirical return data researchers have found evidence (see e.g. section 1 in Nelson [15]) that current returns are negatively correlated with future volatility. This means that a negative shock increases the future volatility more than a positive one or increases it while a positive one even decreases the volatility. This phenomenon is called *leverage effect* in the literature.

If we take a look at the shocks of the state process  $\mathbf{X}$  in the ECOGARCH( $p, q$ ) model

$$\Delta M_t = \begin{cases} (\theta + \gamma) \Delta L_t, & \Delta L_t \geq 0 \\ (\theta - \gamma) \Delta L_t, & \Delta L_t < 0 \end{cases},$$

we see that:

a positive shock in the return data increases  $\mathbf{X}$

- (a) less than a negative one for  $-\gamma < \theta < 0$ ,
- (b) more than a negative one for  $0 < \theta < \gamma$ ,
- (c) while a negative one decreases it for  $\theta > |\gamma|$ ,

a positive shock in the return data decreases  $\mathbf{X}$

- (d) less than a negative one for  $0 < \theta < -\gamma$ ,
- (e) more than a negative one for  $\gamma < \theta < 0$ ,
- (f) while a negative one increases it for  $\theta < -|\gamma|$ .

The corresponding regions in a subset of the parameter space of  $\theta$  and  $\gamma$  can also be seen in Figure 4.

If we compare this to the COGARCH( $p, q$ ) process, we see that in the COGARCH model the innovations of the volatility process at time  $t$  are given by the squared innovations of the log-price process (see section 2 of Brockwell et al. [5]). Hence the volatility process of the COGARCH model reacts in the same way to positive and negative shocks. We will consider now an *instantaneous leverage effect*, which is defined as

$$\text{Cov}(\Delta G_t, \sigma_{t+}^2 \mid |\Delta L_t| > \epsilon)$$

being negative. Intuitively it is clear that this correlation can just be different from zero, if the sample

path of  $\sigma^2$  can have jumps. But from Remark 3.6 (i) we know that this is just the case for  $p = q$ . The reason is that for  $p < q$  the parameter  $b_q$  will be zero and therefore the jump  $\Delta L_t$  at time  $t$  just contributes to the  $(q - 1)$ th derivative of the state process  $\mathbf{X}$ , but is not taken into account for the log-volatility at that time point. Thus we will expect an instantaneous leverage effect only for the ECOGARCH( $p, p$ ) models. This will be shown in the next proposition, in particular we will show that the sign of the correlation is equal to the sign of  $\theta b_q$ . This result is similar to the discrete time case (see Proposition 2.9 in Surgailis and Viano [19]).

**Proposition 5.5.** *Let  $(G_t^{(r)})_{t \geq 0}$  be strictly stationary. Assume that the distribution of the jumps of  $L$  is symmetric, i.e. for all  $\epsilon > 0$ ,*

$$P(\Delta L_t \in dx \mid |\Delta L_t| > \epsilon) = P(\Delta L_t \in -dx \mid |\Delta L_t| > \epsilon), \quad t \geq 0.$$

*Conditionally on the event that  $|\Delta L_t| > \epsilon$ , the sign of  $\text{Cov}(\Delta G_t, \sigma_{t+}^2)$  is equal to the sign of  $\theta b_q$ .*

**Proof:** Since the distribution of the jumps of  $L$  is symmetric we get

$$\begin{aligned} \mathbb{E}(\Delta G_t \mid |\Delta L_t| > \epsilon) &= \mathbb{E}(\sigma_t) \mathbb{E}(\Delta L_t \mid |\Delta L_t| > \epsilon) \\ &= \mathbb{E}(\sigma_t) \left( \int_{x > \epsilon} x P(\Delta L_t \in dx \mid |\Delta L_t| > \epsilon) - \int_{x > \epsilon} x P(\Delta L_t \in dx \mid |\Delta L_t| > \epsilon) \right) \\ &= 0. \end{aligned}$$

This then implies

$$\begin{aligned} \text{Cov}(\Delta G_t, \sigma_{t+}^2 \mid |\Delta L_t| > \epsilon) &= \mathbb{E}(\Delta G_t \sigma_{t+}^2 \mid |\Delta L_t| > \epsilon) \\ &= \mathbb{E} \left( \Delta G_t \exp \left\{ \int_{(-\infty, t]} \mathbf{b}^T e^{A(t-s)} \mathbf{1}_q (\theta \Delta L_s + \gamma |\Delta L_s| - C ds) \right\} \mid |\Delta L_t| > \epsilon \right) \\ &= \mathbb{E} (\sigma_t^3 \Delta L_t \exp \{ \mathbf{b}^T \mathbf{1}_q (\theta \Delta L_t + \gamma |\Delta L_t|) \} \mid |\Delta L_t| > \epsilon) \end{aligned}$$

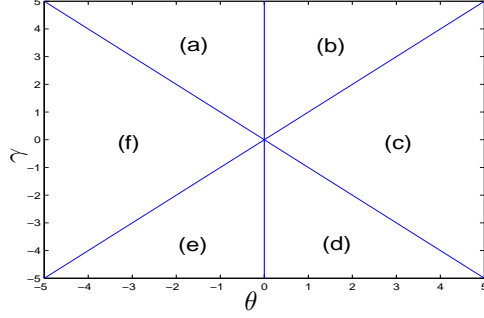


FIGURE 4: Subset of the parameter space of  $\theta$  and  $\gamma$ .

Since  $\Delta L_t$  is independent of  $\sigma_t^3$  we get

$$\begin{aligned} & \mathbb{Cov}(\Delta G_t, \sigma_{t+}^2 \mid |\Delta L_t| > \epsilon) \\ &= \mathbb{E}(\sigma_t^3) \mathbb{E}(\Delta L_t \exp\{b_q(\theta \Delta L_t + \gamma |\Delta L_t|)\} \mid |\Delta L_t| > \epsilon) \\ &= \mathbb{E}(\sigma_t^3) \int_{x>\epsilon} x \exp(b_q \gamma x) (\exp(\theta b_q x) - \exp(-\theta b_q x)) P(\Delta L_t \in dx \mid |\Delta L_t| > \epsilon). \end{aligned}$$

From  $\text{sgn}(\exp(\theta b_q x) - \exp(-\theta b_q x)) = \text{sgn}(\theta b_q)$  for all  $x > \epsilon$  the desired result follows.  $\square$

## 6. Conclusion

In this paper we have proposed a continuous time extension of the discrete time EGARCH( $p, q$ ) process. We gave conditions for stationarity of the volatility and return process and derived some moment expressions depending on the distribution of the driving Lévy process. Compared to the continuous time GARCH( $p, q$ ) process we do not need any parameter restriction to ensure positivity of the volatility. Further we demonstrated that our model is able to describe an instantaneous leverage effect. The model can also be extended to incorporate a long memory effect in the volatility, which was done by Haug and Czado [9] by characterising the log-volatility process as a fractionally integrated CARMA( $q, p - 1$ ) process.

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