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Lower semicontinuity of monotone functionals in the mixed topology on C_b

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Abstract

The main result of this paper characterises the continuity from below of monotone functionals on the space C_b of bounded continuous functions on an arbitrary Polish space as lower semicontinuity in the mixed topology. In this particular situation, the mixed topology coincides with the Mackey topology for the dual pair (C_b, ca) , where ca denotes the space of all countably additive signed Borel measures of finite variation. Hence lower semicontinuity in the mixed topology is for convex monotone maps $C_b \rightarrow \mathbb{R}$ equivalent to a dual representation in terms of countably additive measures. Such representations are of fundamental importance in finance, e.g. in the context of risk measures and superhedging problems. Based on the main result, regularity properties of capacities and dual representations of Choquet integrals in terms of countably additive measures for 2-alternating capacities are studied. Moreover, a well-known characterisation of star-shaped risk measures on L^∞ is transferred to risk measures on C_b . In a second step, the paper provides a characterisation of equicontinuity in the mixed topology for families of convex monotone maps. As a consequence, for every convex monotone map on C_b taking values in a locally convex vector lattice, continuity in the mixed topology is equivalent to continuity on norm-bounded sets.

Keywords Risk measure · Monotone functional · Choquet integral · Continuity from below · Lower semicontinuity · Mixed topology · Mackey topology · Star-shaped

Mathematics Subject Classification 91G70 · 46A20 · 28A12

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1 Introduction

In this paper, we study continuity properties for monotone maps $C_b \rightarrow \mathbb{R}$, where $C_b = C_b(\Omega)$ denotes the space of all bounded continuous functions on a Polish

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space Ω with values in \mathbb{R} . Monotone functionals $C_b \rightarrow \mathbb{R}$ appear in many applications. Special instances of such maps, in the context of finance and actuarial science, are

- risk measures or nonlinear expectations; cf. Denis et al. [11], Föllmer and Schied [15, Chap. 4] and Peng [22, Chap. 1].
- superhedging functionals; cf. Cheridito et al. [7] and Cheridito et al. [8].
- robust expected utilities or loss functions; cf. Delbaen [9, 10].
- Choquet integrals, e.g. in the context of insurance premia; cf. Wang et al. [25].

In order to obtain dual representations of convex monotone functionals in terms of countably additive measures, additional continuity properties are usually required. The two most prominent continuity properties in this context are continuity from above and continuity from below; cf. Föllmer and Schied [15, Lemma 4.21 and Theorem 4.22]. For convex monotone functionals, continuity from below is usually a weaker requirement than continuity from above; see for instance Cheridito et al. [8].

In the field of mathematical finance, these continuity properties have been studied in many contexts in the past decades. For risk measures, continuity from above is (up to a different sign convention) related to the Lebesgue property, whereas continuity from below is closely tied to the Fatou property. The Lebesgue and Fatou properties refer to sequential continuity and lower semicontinuity, respectively, for uniformly bounded pointwise convergent sequences of measurable functions, and Fatou closedness is a fundamental ingredient in no-arbitrage theory; see e.g. Burzoni and Maggis [5] and Herdegen and Khan [18]. Fixing a reference measure and working on L^∞ , it is well known that continuity from below of convex risk measures is equivalent to a dual representation in terms of countably additive measures; cf. Föllmer and Schied [15, Theorem 4.33]. If the risk measure is also law-invariant, continuity from below is automatically satisfied if the underlying probability space is assumed to be atomless; cf. Jouini et al. [19].

On the other hand, monetary risk measures are closely linked to nonlinear expectations and the topic of model uncertainty in finance. In this context, risk measures which are not dominated by a single probability measure that deems events to be negligible or not play a crucial role. An example for such a risk measure is the G -expectation; cf. Denis et al. [11] and Peng [22, Chap. 2]. However, on the space B_b of bounded measurable functions without a reference probability, continuity from below alone is not sufficient to guarantee a dual representation of convex monotone functionals in terms of countably additive measures, despite the fact that it implies sequential lower semicontinuity of such functionals in the weak topology $\sigma(B_b, ca)$ of the dual pair (B_b, ca) . In Denk et al. [12, Example 3.6], an example is given for a coherent risk measure which is continuous from below on B_b , but does not have a single countably additive minorant. On the other hand, continuity from above of a risk measure on the space of bounded measurable functions already implies the existence of a dominating reference measure; see e.g. Denk et al. [12, Remark 3.3].

One way out of this dilemma is to restrict attention to continuous claims. For the space C_b of bounded continuous functions as an underlying function space, it is well known that continuity from above, for a convex monotone functional $C_b \rightarrow \mathbb{R}$, is sufficient but not necessary for a dual representation in terms of countably additive

measures; see e.g. Cheridito et al. [8]. However, the question whether such a representation is equivalent to the weaker notion of continuity from below on general Polish state spaces has remained unanswered for almost a decade, as discussed in the introduction of Delbaen [10]. In a series of papers, this question has been answered positively by Delbaen [9, 10], and as a consequence, convex monotone functionals on C_b which are continuous from below are lower semicontinuous in the Mackey topology $\mu(C_b, ca)$ of the dual pair (C_b, ca) , where ca denotes the space of countably additive signed Borel measures with finite variation. From a mathematical perspective, this is a remarkable result, since the Mackey topology $\mu(C_b, ca)$ is not metrisable and continuity from below is a requirement for sequences, so that nonmetrisability poses a major problem. Therefore in [9, 10], another path is chosen and the proofs there rely on compactification methods; more precisely, they use the fact that every Polish space can be embedded as a G_δ into a compact metric space.

Theorem 2.2 in the present paper generalises the main result of Delbaen [9] by showing that for *any* monotone functional $C_b \rightarrow \mathbb{R}$, continuity from below is equivalent to lower semicontinuity in the mixed topology. The latter is a classical general concept in analysis, cf. Wiweger [27], and coincides with the Mackey topology $\mu(C_b, ca)$ in this particular setting. Moreover, Theorem 2.2 shows that for monotone functionals $C_b \rightarrow \mathbb{R}$, upper or lower semicontinuity in the mixed topology are equivalent to sequential upper or lower semicontinuity in the mixed topology, respectively, despite the fact that the mixed topology is not metrisable. Since the Mackey topology $\mu(C_b, ca)$ is the finest topology leading to the dual space ca of countably additive signed Borel measures with finite variation, it is a natural choice for duality theory on C_b . In particular, Theorem 2.2 implies that every convex monotone functional on C_b admits a dual representation in terms of countably additive measures; cf. Corollary 2.5. However, using the explicit representation of a local base at the origin for the mixed topology allows us to further characterise continuity from below in terms of proximity on compact sets also for nonconvex monotone functionals; see Theorem 2.2.

In Corollary 2.6, we turn our focus to capacities and Choquet integrals. In a first step, we characterise the regularity of general capacities, defined on open sets, in terms of continuity from below of the Choquet integral on the set L_b of all bounded lower semicontinuous functions $\Omega \rightarrow \mathbb{R}$ and of the capacity along sequences of open sets. In a second step, we characterise 2-alternating capacities, for which the related Choquet integral admits a dual representation in terms of countably additive measures on L_b , in terms of continuity from below along sequences of open sets and regularity of the capacity, partially extending the results in Adamski [1].

Corollary 2.7 extends a well-known characterisation of star-shaped risk measures on L^∞ , discussed in Castagnoli et al. [6], to general risk measures on the space C_b . While the proof follows closely that in [6], Corollary 2.5 allows the transition from L^∞ to C_b .

Another question we address in this paper is a characterisation of continuity of convex monotone maps in the mixed topology. Continuity in the mixed topology is of fundamental importance in many situations in robust finance. In the context of superhedging, it has been studied in Cheridito et al. [7]. In the context of dynamic risk measures and semigroups related to stochastic processes under model uncertainty, it

appears in Blessing et al. [3], Goldys et al. [17] and with a different language, it is also used in the analysis of large deviation principles based on max-stable risk measures; cf. Kupper and Zapata [21]. Building on Theorem 2.2, we discuss the equicontinuity of families of convex monotone maps in the mixed topology in Theorem 2.8. There, a characterisation of equicontinuity in terms of continuity from above and uniform equicontinuity on supremum-norm-bounded sets is given.

From a mathematical perspective, the mixed topology has two striking features. On the one hand, unless Ω is compact, it has *no* neighborhood of zero which is bounded with respect to the supremum norm. On the other hand, it has the intrinsic property that for linear operators taking values in an arbitrary locally convex space, continuity is equivalent to continuity on supremum-norm-bounded sets. Corollary 2.11, which is a consequence of Theorem 2.8, extends this intrinsic property by showing that for convex monotone maps taking values in a locally convex vector lattice, $\mu(C_b, \text{ca})$ -continuity is equivalent to $\mu(C_b, \text{ca})$ -continuity on supremum-norm-bounded sets. A particular instance of a locally convex vector lattice is C_b itself, endowed with the mixed topology, which in a financial context corresponds for example to the case of conditional risk measures or conditional nonlinear expectations.

The rest of the paper is organised as follows. In Sect. 2, we state the main results and their corollaries. Section 3 contains all proofs. In Appendix A, we prove an auxiliary result for capacities and Choquet integrals, and in Appendix B, we prove an auxiliary result on locally convex vector lattices.

2 Main results

Throughout, let Ω be a Polish space and $C_b = C_b(\Omega)$ the space of all bounded continuous functions $\Omega \rightarrow \mathbb{R}$. We consider the local base

$$\mathcal{V}^2 := \{ \{g \in C_b : \|g\|_\infty < r\} : r > 0 \}$$

at $0 \in C_b$ for the topology induced by the supremum norm $\|\cdot\|_\infty$, and the local base

$$\mathcal{V}^1 := \left\{ \left\{ g \in C_b : \sup_{x \in C} |g(x)| < r \right\} : r > 0, C \subseteq \Omega \text{ compact} \right\}$$

at $0 \in C_b$ for the vector topology of uniform convergence on compacts. Let \mathcal{V} denote the system consisting of all sets

$$\bigcup_{n \in \mathbb{N}} \sum_{k=1}^n (V_k^1 \cap kV^2) \quad \text{with } (V_k^1)_{k \in \mathbb{N}} \subseteq \mathcal{V}^1 \text{ and } V^2 \in \mathcal{V}^2, \quad (2.1)$$

where $kV^2 := \{kg : g \in V^2\}$ for all $k \in \mathbb{N}$ and

$$\sum_{k=1}^n V_k := \left\{ \sum_{k=1}^n g_k : g_1 \in V_1, \dots, g_n \in V_n \right\}$$

for $n \in \mathbb{N}$ and nonempty subsets V_1, \dots, V_n of C_b . Then \mathcal{V} is a local base at $0 \in C_b$ for a Hausdorff locally convex topology β , which is known as the *mixed topology*. We refer to Wiweger [27] for a detailed discussion on the mixed topology in a more general setting. Clearly, the mixed topology β is finer than the *weak topology* $\sigma(C_b, ca)$ of the dual pair (C_b, ca) , where ca denotes the space of all countably additive signed Borel measures of finite variation. A well-known fact, which we do *not* make use of, is that the mixed topology β coincides with the Mackey topology of the dual pair (C_b, ca) . Moreover, β belongs to the class of *strict topologies*; cf. Wheeler [26]. We also refer to Fremlin et al. [16] and Santilles [24] for additional fine properties of mixed or strict topologies.

We say that a functional $U: C_b \rightarrow \mathbb{R}$ is *monotone* if $U(f) \leq U(g)$ for all $f, g \in C_b$ with $f \leq g$, where for functions $\Omega \rightarrow \mathbb{R}$, the relation \leq and all other order-related objects refer to the pointwise order.

For a sequence $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ and a function $f: \Omega \rightarrow \mathbb{R}$, we write $f_n \nearrow f$ as $n \rightarrow \infty$ if $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in \Omega$. Analogously, we write $f_n \searrow f$ as $n \rightarrow \infty$ if $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$ and $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in \Omega$.

Definition 2.1 a) We say that a monotone functional $U: C_b \rightarrow \mathbb{R}$ is *continuous from below* if $U(f) = \lim_{n \rightarrow \infty} U(f_n)$ for all sequences $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ and $f \in C_b$ with $f_n \nearrow f$ as $n \rightarrow \infty$.

b) We say that a monotone functional $U: C_b \rightarrow \mathbb{R}$ is *continuous from above* if $U(f) = \lim_{n \rightarrow \infty} U(f_n)$ for all sequences $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ and $f \in C_b$ with $f_n \searrow f$ as $n \rightarrow \infty$.

Theorem 2.2 Let $U: C_b \rightarrow \mathbb{R}$ be monotone. Then the following are equivalent:

- (i) U is continuous from below.
- (ii) U is lower semicontinuous in the mixed topology β .
- (iii) U is sequentially lower semicontinuous in the mixed topology β .
- (iv) For all $f \in C_b$, $\varepsilon > 0$ and $r \geq 0$, there exist $\delta > 0$ and a compact $C \subseteq \Omega$ such that

$$U(f) \leq U(f + e) + \varepsilon$$

for all $e \in C_b$ with $\|e\|_\infty \leq r$ and $\sup_{x \in C} |e(x)| \leq \delta$.

For an arbitrary function $U: C_b \rightarrow \mathbb{R}$, its conjugate function $\overline{U}: C_b \rightarrow \mathbb{R}$ is given by $\overline{U}(f) := -U(-f)$ for all $f \in C_b$. Then we obtain the following corollary.

Corollary 2.3 Let $U: C_b \rightarrow \mathbb{R}$ be monotone. Then the following are equivalent:

- (i) U is continuous from above.
- (ii) U is upper semicontinuous in the mixed topology β .
- (iii) U is sequentially upper semicontinuous in the mixed topology β .
- (iv) For all $f \in C_b$, $\varepsilon > 0$ and $r \geq 0$, there exist $\delta > 0$ and a compact $C \subseteq \Omega$ such that

$$U(f + e) \leq U(f) + \varepsilon$$

for all $e \in C_b$ with $\|e\|_\infty \leq r$ and $\sup_{x \in C} |e(x)| \leq \delta$.

A combination of Theorem 2.2 and Corollary 2.3 leads to the following characterisation of continuity in the mixed topology for monotone functionals.

Corollary 2.4 *Let $U : C_b \rightarrow \mathbb{R}$ be monotone. Then the following are equivalent:*

- (i) U is continuous from above and below.
- (ii) U is continuous in the mixed topology β .
- (iii) U is sequentially continuous in the mixed topology β .
- (iv) For all $f \in C_b$, $\varepsilon > 0$ and $r \geq 0$, there exist $\delta > 0$ and a compact $C \subseteq \Omega$ such that

$$|U(f + e) - U(f)| \leq \varepsilon$$

for all $e \in C_b$ with $\|e\|_\infty \leq r$ and $\sup_{x \in C} |e(x)| \leq \delta$.

As a direct consequence of Theorem 2.2, we obtain the main result in Delbaen [9]. We denote by ca_+ the set of all positive elements of ca , i.e., the set of all finite Borel measures.

Corollary 2.5 *Let $U : C_b \rightarrow \mathbb{R}$ be convex and monotone. Then the following are equivalent:*

- (i) U is continuous from below.
- (ii) U is lower semicontinuous in the mixed topology β .
- (iii) U is lower semicontinuous in the weak topology $\sigma(C_b, \text{ca})$.
- (iv) There exist a nonempty set $\mathcal{M} \subseteq \text{ca}_+$ and a function $\alpha : \mathcal{M} \rightarrow \mathbb{R}$ such that

$$U(f) = \sup_{\mu \in \mathcal{M}} \left(\int_{\Omega} f \, d\mu - \alpha(\mu) \right) \quad \text{for all } f \in C_b. \quad (2.2)$$

We apply Corollary 2.5 to the case of Choquet integrals. In the sequel, let \mathcal{O} denote the family of all open subsets of Ω , i.e., the topology on Ω , and L_b the set of all bounded lower semicontinuous functions $\Omega \rightarrow \mathbb{R}$. A *capacity (on \mathcal{O})* is a map $c : \mathcal{O} \rightarrow [0, \infty)$ with

$$c(\emptyset) = 0 \quad \text{and} \quad c(B_1) \leq c(B_2) \quad \text{for all } B_1, B_2 \in \mathcal{O} \text{ with } B_1 \subseteq B_2.$$

For a capacity $c : \mathcal{O} \rightarrow [0, \infty)$, we define the *Choquet integral* with respect to c as

$$\int_{\Omega} f \, dc := \int_0^{\infty} c(\{f > s\}) \, ds + \int_{-\infty}^0 (c(\{f > s\}) - c(\Omega)) \, ds \quad \text{for all } f \in L_b.$$

By definition, the Choquet integral is *positively homogeneous*, i.e.,

$$\int_{\Omega} (\lambda f) \, dc = \lambda \int_{\Omega} f \, dc \quad \text{for all } f \in L_b \text{ and } \lambda > 0,$$

and *constant additive*, i.e.,

$$\int_{\Omega} (f + m) \, dc = \int_{\Omega} f \, dc + mc(\Omega) \quad \text{for all } f \in L_b \text{ and } m \in \mathbb{R}.$$

A well-known fact is that the Choquet integral is *subadditive*, i.e.,

$$\int_{\Omega} (f_1 + f_2) \, d\mathbf{c} \leq \int_{\Omega} f_1 \, d\mathbf{c} + \int_{\Omega} f_2 \, d\mathbf{c} \quad \text{for all } f_1, f_2 \in L_b,$$

if and only if the capacity c is *2-alternating*, i.e.,

$$c(B_1 \cup B_2) + c(B_1 \cap B_2) \leq c(B_1) + c(B_2) \quad \text{for all } B_1, B_2 \in \mathcal{O}.$$

For the reader's convenience, we provide a proof of this statement in Appendix A.

Another consequence of Theorem 2.2 and Corollary 2.5 is the following result concerning the regularity of general capacities and dual representations of Choquet integrals in terms of countably additive measures for 2-alternating capacities. We point out that the equivalence between (i) and (ii) is a standard result from Choquet theory and can be found for example in the textbook by König [20, Exercise 11.18]. The equivalence between (iii) and (iv) has been discussed in a more general setting in Adamski [1]. The main novelty is the implication (ii) \Rightarrow (iii), which is a consequence of Theorem 2.2 and together with Corollary 2.5 facilitates the proof of the remaining equivalences. For the reader's convenience, we provide a self-contained proof of all equivalences.

Corollary 2.6 *Let $c: \mathcal{O} \rightarrow [0, \infty)$ be a capacity. Then the following are equivalent:*

(i) *For every sequence $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{O}$ with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$,*

$$c\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \lim_{n \rightarrow \infty} c(B_n).$$

(ii) *The Choquet integral is continuous from below on L_b , i.e.,*

$$\int_{\Omega} f \, d\mathbf{c} = \lim_{n \rightarrow \infty} \int_{\Omega} f_n \, d\mathbf{c}$$

for all $f \in L_b$ and any sequence $(f_n)_{n \in \mathbb{N}} \subseteq L_b$ with $f_n \nearrow f$ as $n \rightarrow \infty$.

(iii) *The capacity c is regular, i.e., for all $B \in \mathcal{O}$,*

$$c(B) = \sup_{C \in B} \inf_{\substack{A \in \mathcal{O} \\ A \supseteq C}} c(A), \quad (2.3)$$

where we write $C \in B$ for $C \subseteq B \subseteq \Omega$ with C compact.

If c is 2-alternating, the statements (i)–(iii) are equivalent to the following statement:

(iv) *There exists a nonempty set $\mathcal{M} \subseteq \text{ca}_+$ with $\mu(\Omega) = c(\Omega)$ for all $\mu \in \mathcal{M}$ and*

$$\int_{\Omega} f \, d\mathbf{c} = \sup_{\mu \in \mathcal{M}} \int_{\Omega} f \, d\mu \quad \text{for all } f \in L_b.$$

We now present an application of Corollary 2.5 for star-shaped risk measures on C_b . In the following, we say that a monotone functional $R: C_b \rightarrow \mathbb{R}$ is a *risk*

measure if $R(0) = 0$ and $R(f + m) = R(f) + m$ for all $f \in C_b$ and all constants $m \in \mathbb{R}$. We say that a risk measure $R: C_b \rightarrow \mathbb{R}$ is *star-shaped* if

$$R(\lambda f) \leq \lambda R(f) \quad \text{for all } f \in C_b \text{ and } \lambda \in [0, 1].$$

For a detailed discussion on risk measures, we refer to Föllmer and Schied [15, Chap. 4], and for a survey on star-shaped risk measures to Castagnoli et al. [6]. The following corollary is a variant of [6, Proposition 5] in our setting. The proof heavily uses the insights obtained in the proof of [6, Theorem 2]. We point out that [6, Proposition 5] covers only the case of dominated risk measures, i.e., risk measures on L^∞ , which is discussed in detail in [15, Sect. 4.3], whereas we consider general risk measures restricted to the space of bounded continuous functions.

Corollary 2.7 *Let $R: C_b \rightarrow \mathbb{R}$. Then the following are equivalent:*

- (i) *The map R is a star-shaped risk measure.*
- (ii) *There exist a nonempty set I and a family $(\alpha_i)_{i \in I}$ of functions $\text{ca}_+^1 \rightarrow [0, \infty]$ with $\inf_{\mu \in \text{ca}_+^1} \alpha_i(\mu) = 0$ for all $i \in I$ and*

$$R(f) = \min_{i \in I} \sup_{\mu \in \text{ca}_+^1} \left(\int_{\Omega} f \, d\mu - \alpha_i(\mu) \right) \quad \text{for all } f \in C_b.$$

We conclude this section with various characterisations of continuity in the mixed topology for convex monotone maps. We start with the following theorem which is our second main result. Recall \mathcal{V} from (2.1).

Theorem 2.8 *Let I be a nonempty index set and $(U_i)_{i \in I}$ a family of convex and monotone maps $C_b \rightarrow \mathbb{R}$ with*

$$\sup_{i \in I} (U_i(r) - U_i(0)) < \infty \quad \text{for all constants } r \geq 0. \quad (2.4)$$

Then the following are equivalent:

- (i) *For every sequence $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ with $f_n \searrow 0$ as $n \rightarrow \infty$,*

$$\lim_{n \rightarrow \infty} \sup_{i \in I} (U_i(f_n) - U_i(0)) = 0.$$

- (ii) *For every $r \geq 0$ and every $\varepsilon > 0$, there exists some $V \in \mathcal{V}$ with*

$$\sup_{\|f\|_\infty \leq r} \sup_{i \in I} |U_i(f + e) - U_i(f)| < \varepsilon \quad \text{for all } e \in V.$$

- (iii) *For every $r \geq 0$ and every $\varepsilon > 0$, there exist a compact $C \subseteq \Omega$ and a constant $M \geq 0$ such that*

$$\sup_{i \in I} |U_i(f_1) - U_i(f_2)| \leq M \sup_{x \in C} |f_1(x) - f_2(x)| + \varepsilon$$

for all $f_1, f_2 \in C_b$ with $\max\{\|f_1\|_\infty, \|f_2\|_\infty\} \leq r$.

Theorem 2.8 leads to the following characterisation for continuity in the mixed topology of convex and monotone maps on C_b taking values in a *locally convex vector lattice* (L, τ) , i.e., a vector lattice L together with a locally convex topology τ on L which is generated by a family of lattice seminorms. Recall that for a vector lattice L , a seminorm $p: L \rightarrow [0, \infty)$ is called a *lattice seminorm* if $p(u) \leq p(v)$ for all $u, v \in L$ with $|u| \leq |v|$. We refer to Schaefer and Wolff [23, Sect. V.7] for a detailed study of locally convex vector lattices. For the reader's convenience, we provide an auxiliary result on locally convex vector lattices in Appendix B.

Corollary 2.9 *Let (L, τ) be a locally convex vector lattice and $U: C_b \rightarrow L$ convex and monotone. Then the following are equivalent:*

(i) *U is β - τ -continuous.*

(ii) *For every nonnegative τ -continuous linear functional $\lambda: L \rightarrow \mathbb{R}$ and every sequence $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ with $f_n \searrow 0$ as $n \rightarrow \infty$,*

$$\lim_{n \rightarrow \infty} \lambda(U(f_n)) = \lambda(U(0)).$$

(iii) *For every nonnegative τ -continuous linear functional $\lambda: L \rightarrow \mathbb{R}$, the map*

$$C_b \rightarrow \mathbb{R}, \quad f \mapsto \lambda(U(f))$$

is sequentially upper semicontinuous in the mixed topology β .

(iv) *For every τ -continuous lattice seminorm $p: L \rightarrow [0, \infty)$, every constant $r \geq 0$ and all $\varepsilon > 0$, there exists some $V \in \mathcal{V}$ with*

$$\sup_{\|f\|_\infty \leq r} p(U(f+e) - U(f)) < \varepsilon \quad \text{for all } e \in V.$$

(v) *For every τ -continuous lattice seminorm $p: L \rightarrow [0, \infty)$, every constant $r \geq 0$ and all $\varepsilon > 0$, there exist a compact $C \subseteq \Omega$ and a constant $M \geq 0$ such that*

$$p(U(f_1) - U(f_2)) \leq M \sup_{x \in C} |f_1(x) - f_2(x)| + \varepsilon$$

for all $f_1, f_2 \in C_b$ with $\max\{\|f_1\|_\infty, \|f_2\|_\infty\} \leq r$.

Choosing $L = \mathbb{R}$ with the (usual) topology induced by the absolute value $|\cdot|$, we obtain the following corollary.

Corollary 2.10 *Let $U: C_b \rightarrow \mathbb{R}$ be convex and monotone. Then the following are equivalent:*

(i) *U is continuous from above.*

(ii) *U is sequentially upper semicontinuous in the mixed topology β .*

(iii) *U is continuous in the mixed topology β .*

(iv) *For every $r \geq 0$ and every $\varepsilon > 0$, there exist some $V \in \mathcal{V}$ with*

$$\sup_{\|f\|_\infty \leq r} |U(f+e) - U(f)| < \varepsilon \quad \text{for all } e \in V.$$

(v) For every $r \geq 0$ and every $\varepsilon > 0$, there exist a compact $C \subseteq \Omega$ and a constant $M \geq 0$ such that

$$|U(f_1) - U(f_2)| \leq M \sup_{x \in C} |f_1(x) - f_2(x)| + \varepsilon$$

for all $f_1, f_2 \in C_b$ with $\max\{\|f_1\|_\infty, \|f_2\|_\infty\} \leq r$.

Now let Ω_0 be another Polish space, $C_b(\Omega_0)$ the space of all bounded continuous functions $\Omega_0 \rightarrow \mathbb{R}$ and β_0 the mixed topology on $C_b(\Omega_0)$. To avoid confusion, we write $C_b(\Omega)$ instead of C_b in the following corollary.

Corollary 2.11 *Let $U: C_b(\Omega) \rightarrow C_b(\Omega_0)$ be convex and monotone. Then the following are equivalent:*

- (i) U is β - β_0 -continuous.
- (ii) For every $\omega \in \Omega_0$ and every sequence $(f_n)_{n \in \mathbb{N}} \subseteq C_b(\Omega)$ with $f_n \searrow 0$ as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} (U(f_n))(\omega) = (U(0))(\omega).$$

- (iii) For every compact $K \subseteq \Omega_0$, every constant $r \geq 0$ and all $\varepsilon > 0$, there exist a compact $C \subseteq \Omega$ and a constant $M \geq 0$ such that

$$\sup_{\omega \in K} |(U(f_1))(\omega) - (U(f_2))(\omega)| \leq M \sup_{x \in C} |f_1(x) - f_2(x)| + \varepsilon$$

for all $f_1, f_2 \in C_b(\Omega)$ with $\max\{\|f_1\|_\infty, \|f_2\|_\infty\} \leq r$.

3 Proofs

Before turning our focus on the proof of Theorem 2.2, we collect some well-known facts on the connection between pointwise monotone convergence, uniform convergence on compacts together with uniform boundedness, and convergence in the mixed topology for sequences in C_b .

Remark 3.1 a) Let $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ with $f_n \nearrow f \in C_b$ as $n \rightarrow \infty$. Since $f_n \nearrow f$ as $n \rightarrow \infty$, it follows that $(f_n - f)_{n \in \mathbb{N}}$ is uniformly bounded, i.e.,

$$\sup_{n \in \mathbb{N}} \|f_n - f\|_\infty < \infty, \quad (3.1)$$

and by Dini's lemma, $(f_n - f)_{n \in \mathbb{N}}$ converges uniformly on compacts to $0 \in C_b$, i.e., for all compacts $C \subseteq \Omega$,

$$\sup_{x \in C} |f_n(x) - f(x)| \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

b) Let $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ and $f \in C_b$ with (3.1) and (3.2). Moreover, let $V \in \mathcal{V}$. Then there exist $(V_k^1)_{k \in \mathbb{N}} \subseteq \mathcal{V}^1$ and $V^2 \in \mathcal{V}^2$ such that

$$V = \bigcup_{n \in \mathbb{N}} \sum_{k=1}^n (V_k^1 \cap kV^2).$$

By (3.1), there exists some $k \in \mathbb{N}$ with $f_n - f \in kV^2$ for all $n \in \mathbb{N}$. Moreover, by (3.2), there exists some $n_0 \in \mathbb{N}$ such that $f_n - f \in V_k^1$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Hence $f_n - f \in V_k^1 \cap kV^2 \subseteq V$ for all $n \in \mathbb{N}$ with $n \geq n_0$. This shows that $f_n \rightarrow f$ in the mixed topology β as $n \rightarrow \infty$.

Proof of Theorem 2.2 Let $d: \Omega \times \Omega \rightarrow [0, \infty)$ be a metric consistent with the topology on Ω such that (Ω, d) is a complete separable metric space. Moreover, let $(x_i)_{i \in \mathbb{N}} \subseteq \Omega$ be a sequence such that $D := \{x_i : i \in \mathbb{N}\}$ is dense in Ω .

1) We start with the proof of the implication (i) \Rightarrow (ii). To that end, assume that U is continuous from below. We show that U is lower semicontinuous in the mixed topology β , i.e., for all $f \in C_b$ and $\varepsilon > 0$, there exists some $V \in \mathcal{V}$ such that

$$U(f) \leq U(f + e) + \varepsilon \quad \text{for all } e \in V.$$

In order to do so, fix $f \in C_b$ and $\varepsilon > 0$. Since U is continuous from below, there exists some $\delta > 0$ such that

$$U(f) \leq U(f - \delta) + \frac{\varepsilon}{2}.$$

In a first step, we adapt the main idea from the proof of Ulam's theorem, cf. Dudley [13, proof of Theorem 7.1.4], to our setting and recursively construct families of finite sets $D_k^m \subseteq D$ and continuous functions $\varphi_k^m: \Omega \rightarrow [0, 1]$ which are indexed by $k, m \in \mathbb{N}$ with $k \leq m$ and satisfy the following three properties:

$$D_k^m \subseteq D_k^{m+1} \subseteq \bigcap_{j=k}^m \bigcup_{x \in D_k^j} B\left(x, \frac{1}{j}\right) =: B_k^m \quad \text{for all } k, m \in \mathbb{N} \text{ with } k \leq m; \quad (3.3)$$

$$\varphi_k^m(x) = 0 \quad \text{for all } k, m \in \mathbb{N} \text{ with } k \leq m \text{ and } x \in \Omega \setminus B_k^m; \quad (3.4)$$

$$U(f - \delta) \leq U\left(f - \delta - \frac{m(m+1)}{2} + \sum_{k=1}^m k\varphi_k^m\right) + \frac{\varepsilon}{2} \quad \text{for all } m \in \mathbb{N}. \quad (3.5)$$

In order to simplify notation, let $\ell(0) := 1$ and set

$$f_m := f - \delta - \frac{m(m+1)}{2}, \quad \varphi_m^{m-1} := 1, \quad B_m^{m-1} := \Omega \quad \text{for all } m \in \mathbb{N}.$$

Now let $m \in \mathbb{N}$ and define

$$\psi_k^{\ell, m}(x) := (1 - m \operatorname{dist}(x, \{x_1, \dots, x_\ell\} \cap B_k^{m-1})) \vee 0$$

for all $x \in \Omega$, $k \in \{1, \dots, m\}$ and $\ell \in \mathbb{N}$. Since B_k^{m-1} is open, $\varphi_k^{m-1} \geq 0$ and $\varphi_k^m(x) = 0$ for all $x \in \Omega \setminus B_k^{m-1}$ for all $k \in \{1, \dots, m\}$, it follows that

$$\varphi_k^{m-1} \psi_k^{\ell, m} \nearrow \varphi_k^{m-1} \quad \text{as } \ell \rightarrow \infty$$

for all $k \in \{1, \dots, m\}$. Using the continuity from below of U , there exists $\ell(m) \in \mathbb{N}$ with $\ell(m) \geq \ell(m-1)$ and

$$U\left(f_m + \sum_{k=1}^m k \varphi_k^{m-1}\right) \leq U\left(f_m + \sum_{k=1}^m k \varphi_k^{m-1} \psi_k^{\ell(m), m}\right) + \frac{\varepsilon}{2} 2^{-m},$$

and we define

$$\varphi_k^m := \varphi_k^{m-1} \psi_k^{\ell(m), m}, \quad D_k^m := \{x_1, \dots, x_{\ell(m)}\} \cap B_k^{m-1}. \quad (3.6)$$

We now verify that the sequence constructed in this way satisfies the properties (3.3)–(3.5). By definition, $D_k^m \subseteq B_k^{m-1}$ and since

$$B_k^m = B_k^{m-1} \cap \left(\bigcup_{x \in D_k^m} B\left(x, \frac{1}{m}\right) \right),$$

it follows that $D_k^m \subseteq B_k^m$ for all $m \in \mathbb{N}$. Since $\ell(m) \leq \ell(m+1)$ and $B_k^m \subseteq B_k^{m-1}$, we find that

$$D_k^m = D_k^m \cap B_k^m = \{x_1, \dots, x_{\ell(m)}\} \cap B_k^m \subseteq \{x_1, \dots, x_{\ell(m+1)}\} \cap B_k^m = D_k^{m+1}$$

for all $m \in \mathbb{N}$. By (3.6),

$$\psi_k^{\ell(m), m}(x) = (1 - m \operatorname{dist}(x, D_k^m)) \vee 0$$

for all $x \in \Omega$, $k \in \{1, \dots, m\}$ and $m \in \mathbb{N}$. Hence

$$\varphi_k^m(x) = \prod_{j=k}^m \psi_k^{\ell(j), j}(x) = 0 \quad \text{for all } x \in \Omega \setminus B_k^m, \quad k \in \{1, \dots, m\} \text{ and } m \in \mathbb{N}.$$

Moreover, setting $f_0 := f - \delta$ and using the fact that $\varphi_m^{m-1} = 1$, it follows that

$$U\left(f_{m-1} + \sum_{k=1}^{m-1} k \varphi_k^{m-1}\right) = U\left(f_m + \sum_{k=1}^m k \varphi_k^{m-1}\right) \leq U\left(f_m + \sum_{k=1}^m k \varphi_k^m\right) + \frac{\varepsilon}{2} 2^{-m}$$

for all $m \in \mathbb{N}$. Inductively, we obtain that

$$U(f - \delta) \leq U\left(f - \delta - \frac{m(m+1)}{2} + \sum_{k=1}^m k \varphi_k^m\right) + \frac{\varepsilon}{2} \quad \text{for all } m \in \mathbb{N}.$$

We have therefore verified (3.3)–(3.5) and now define

$$C_k := \bigcap_{j=k}^{\infty} \bigcup_{x \in D_k^j} \overline{B}\left(x, \frac{1}{j}\right) \quad \text{for all } k \in \mathbb{N},$$

where for $x \in \Omega$ and $r > 0$, $\overline{B}(x, r) := \{y \in \Omega : d(x, y) \leq r\}$. Then C_k is a closed and totally bounded subset of a complete metric space and hence compact for all $k \in \mathbb{N}$. For $x \in \Omega$ and $r > 0$, let $B(x, r) := \{y \in \Omega : d(x, y) < r\}$, and observe that

$$\bigcap_{m=k}^{\infty} B_k^m = \bigcap_{m=k}^{\infty} \bigcap_{j=k}^m \bigcup_{x \in D_k^j} B\left(x, \frac{1}{j}\right) = \bigcap_{j=k}^{\infty} \bigcup_{x \in D_k^j} B\left(x, \frac{1}{j}\right) \subseteq C_k \quad \text{for all } k \in \mathbb{N}$$

so that by (3.3), $D_k^m \subseteq C_k$ for all $k, m \in \mathbb{N}$ with $k \leq m$.

Using the sequence $(C_k)_{k \in \mathbb{N}}$ of compacts, we define

$$V_k^1 := \left\{ e \in C_b : \sup_{x \in C_k} |e(x)| < 2^{-k} \delta \right\} \quad \text{for all } k \in \mathbb{N}$$

and $V^2 := \{g \in C_b(\Omega) : \|g\|_{\infty} < 1\}$. Then

$$V := \bigcup_{n \in \mathbb{N}} \sum_{k=1}^n (V_k^1 \cap kV^2)$$

is a neighborhood of $0 \in C_b$ in the mixed topology. We show that

$$U(f) \leq U(f + e) + \varepsilon \quad \text{for all } e \in V.$$

To that end, let $e \in V$, i.e., there exist $n \in \mathbb{N}$ and $e_k \in V_k^1 \cap kV^2$ for $k \in \{1, \dots, n\}$ such that $e = \sum_{k=1}^n e_k$. Let $\delta_k := \sup_{x \in C_k} |e_k(x)|$. Since $e_k \in V_k^1$, it follows that $\delta_k < 2^{-k} \delta$ for all $k \in \{1, \dots, n\}$. As C_1, \dots, C_n are compact, there exists some $m \in \mathbb{N}$ with $m \geq n$ such that

$$|e_k(y) - e_k(x)| < 2^{-k} \delta - \delta_k$$

for all $k \in \{1, \dots, n\}$, $x \in C_k$ and $y \in \Omega$ with $d(x, y) < \frac{1}{m}$. Since $D_k^m \subseteq C_k$ for all $k \in \{1, \dots, n\}$, this implies that

$$|e_k(y)| \leq 2^{-k} \delta \quad \text{for all } k \in \{1, \dots, n\} \text{ and } y \in \bigcup_{x \in D_k^m} B\left(x, \frac{1}{m}\right).$$

Hence by (3.4), it follows that

$$-k + k\varphi_k^m \leq -k + (k + e_k)\varphi_k^m + 2^{-k} \delta \leq e_k + 2^{-k} \delta \quad \text{for all } k \in \{1, \dots, n\},$$

where the first step uses the fact that $|e_k \varphi_k^m| \leq 2^{-k} \delta$ and the last that $k + e_k \geq 0$ for all $k \in \{1, \dots, n\}$. On the other hand, $-k + k \varphi_k^m \leq 0$ for all $k \in \{n+1, \dots, m\}$. Therefore,

$$f - \delta - \frac{m(m+1)}{2} + \sum_{k=1}^m k \varphi_k^m \leq f - \delta + \sum_{k=1}^n (e_k + 2^{-k} \delta) \leq f + \sum_{k=1}^n e_k = f + e.$$

Using (3.5), it follows that

$$U(f) \leq U(f - \delta) + \frac{\varepsilon}{2} \leq U\left(f - \delta - \frac{m(m+1)}{2} + \sum_{k=1}^m k \varphi_k^m\right) + \varepsilon \leq U(f + e) + \varepsilon.$$

This proves that U is lower semicontinuous with respect to the mixed topology β .

2) Next, we prove that (ii) implies (iv). To that end, let $f \in C_b$ and $\varepsilon > 0$. Then there exist $(V_k^1)_{k \in \mathbb{N}} \subseteq \mathcal{V}^1$ and $V^2 \in \mathcal{V}^2$ such that

$$U(f) \leq U(f + e) + \varepsilon$$

for all $e \in V := \bigcup_{n \in \mathbb{N}} \sum_{k=1}^n (V_k^1 \cap k V^2)$. Let $r \geq 0$. Then there exists some $n \in \mathbb{N}$ such that

$$\{g \in C_b : \|g\|_\infty \leq r\} \subseteq n V^2.$$

Moreover, there exist $\delta > 0$ and a compact $C \subseteq \Omega$ such that

$$\left\{g \in C_b : \sup_{x \in C} |g(x)| \leq \delta\right\} \subseteq V_n^1.$$

Hence for every $e \in C_b$ with $\|e\|_\infty \leq r$ and $\sup_{x \in C} |e(x)| \leq \delta$, it follows that $e \in V$ and therefore $U(f) \leq U(f + e) + \varepsilon$.

3) Clearly, (ii) implies (iii) so that it remains to prove the implications (iii) \Rightarrow (i) and (iv) \Rightarrow (i). To that end, let $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ with $f_n \nearrow f \in C_b$ as $n \rightarrow \infty$. Then due to the monotonicity of U ,

$$\lim_{n \rightarrow \infty} U(f_n) = \sup_{n \in \mathbb{N}} U(f_n) \leq U(f).$$

Since $f_n \nearrow f$ as $n \rightarrow \infty$, it follows by Remark 3.1 a) that the sequence $(f_n - f)_{n \in \mathbb{N}}$ is uniformly bounded and converges uniformly on compacts to 0 $\in C_b$. Moreover, by Remark 3.1 b), it follows that $f_n \rightarrow f$ as $n \rightarrow \infty$ in the mixed topology β . Hence if U satisfies (iii) or (iv), it follows that

$$U(f) \leq \liminf_{n \rightarrow \infty} U(f_n) = \lim_{n \rightarrow \infty} U(f_n).$$

The proof is complete. \square

Proof of Corollary 2.5 The equivalence of (i) and (ii) follows from Theorem 2.2. By standard duality theory in locally convex Hausdorff spaces, cf. Ekeland and

Temam [14, Proposition 3.1], (ii) is equivalent to the fact that U admits a dual representation of the form

$$U(f) = \sup_{\mu \in \mathcal{M}} (\mu f - \alpha(\mu)) \quad \text{for all } f \in C_b$$

with a set \mathcal{M} of β -continuous linear functionals on C_b and a function $\alpha: \mathcal{M} \rightarrow \mathbb{R}$. Let $\mu \in \mathcal{M}$ and $f \in C_b$ with $f \geq 0$. Since U is monotone, it follows that

$$\frac{1}{\lambda} (\mu(-\lambda f) - \alpha(\mu)) \leq \frac{U(-\lambda f)}{\lambda} \leq \frac{U(0)}{\lambda} \quad \text{for all } \lambda > 0.$$

Hence

$$0 = \lim_{\lambda \rightarrow \infty} -\frac{U(0) + \alpha(\mu)}{\lambda} \leq \mu f,$$

which shows that every linear functional in \mathcal{M} is positive. The remaining equivalences and in particular the dual representation (2.2) now follow from the fact that by Theorem 2.2 and the Daniell–Stone theorem, cf. Bogachev [4, Theorem 7.8.1], a positive linear functional $\mu: C_b \rightarrow \mathbb{R}$ is continuous in the mixed topology β if and only if it belongs to ca_+ . \square

Proof of Corollary 2.6 1) We first prove the implication (i) \Rightarrow (ii). To that end, let $(f_n)_{n \in \mathbb{N}} \subseteq L_b$ and $f \in L_b$ with $f_n \nearrow f$ as $n \rightarrow \infty$. Then for all $s \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} c(\{f_n > s\}) = c(\{f > s\}).$$

Using the monotone convergence theorem, it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} f_n \, dc &= \lim_{n \rightarrow \infty} \left(\int_0^{\infty} c(\{f_n > s\}) \, ds + \int_{-\infty}^0 (c(\{f_n > s\}) - c(\Omega)) \, ds \right) \\ &= \int_0^{\infty} c(\{f > s\}) \, ds + \int_{-\infty}^0 (c(\{f > s\}) - c(\Omega)) \, ds \\ &= \int_{\Omega} f \, dc. \end{aligned} \quad (3.7)$$

2) For the implication (ii) \Rightarrow (iii), first observe that

$$c(B) \geq \sup_{C \in B} \inf_{\substack{A \in \mathcal{O} \\ A \supseteq C}} c(A) \quad \text{for all } B \in \mathcal{O}.$$

In order to show the converse inequality, let $B \in \mathcal{O}$ and $\varepsilon > 0$. In a first step, we consider the case $B = \Omega$. Then by Theorem 2.2, there exist $\delta > 0$ and a compact set $C \subseteq \Omega$ such that

$$\int_{\Omega} 1 \, dc \leq \int_{\Omega} g \, dc + \varepsilon$$

for all $g \in C_b$ with $\|g\|_\infty \leq 1$ and $\sup_{x \in C} |g(x) - 1| \leq \delta$. Now let $A \in \mathcal{O}$ with $A \supseteq C$. Since A is open, there exists some $m \in \mathbb{N}$ such that $g: \Omega \rightarrow \mathbb{R}$ given by

$$g(x) := \sup_{y \in C} (1 - md(x, y)) \vee 0 \quad \text{for all } x \in \Omega$$

satisfies $g(x) = 0$ for $x \in \Omega \setminus A$. Since $0 \leq g \leq 1$ and $g(x) = 1$ for all $x \in C$, it follows that

$$c(\Omega) = \int_{\Omega} 1 \, dc \leq \int_{\Omega} g \, dc + \varepsilon \leq \int_{\Omega} 1_A \, dc + \varepsilon = c(A) + \varepsilon.$$

We have therefore shown that

$$c(\Omega) \leq \inf_{\substack{A \in \mathcal{O} \\ A \supseteq C}} c(A) + \varepsilon$$

Taking the supremum over all $C \subseteq \Omega$ and letting $\varepsilon \rightarrow 0$ yields (2.3).

For general $B \in \mathcal{O}$, the statement now follows from the fact that B , endowed with the subspace topology

$$\mathcal{O}_B := \{A \cap B : A \in \mathcal{O}\} = \{A \in \mathcal{O} : A \subseteq B\} \subseteq \mathcal{O},$$

is again a Polish space, together with the observation that a subset of B is compact in the subspace topology \mathcal{O}_B if and only if it is compact in the original topology \mathcal{O} .

3) To prove that (iii) implies (i), let $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{O}$ with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists some compact $C \subseteq \bigcup_{n \in \mathbb{N}} B_n =: B$ with

$$c(B) \leq \inf_{\substack{A \in \mathcal{O} \\ A \supseteq C}} c(A) + \varepsilon.$$

Since C is compact, $C \subseteq B = \bigcup_{n \in \mathbb{N}} B_n$ and B_n is open with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$, there exists some $n_0 \in \mathbb{N}$ such that $C \subseteq B_{n_0}$. Hence

$$\inf_{\substack{A \in \mathcal{O} \\ A \supseteq C}} c(A) \leq c(B_{n_0}),$$

and it follows that

$$\sup_{n \in \mathbb{N}} c(B_n) \leq c(B) \leq \sup_{n \in \mathbb{N}} c(B_n) + \varepsilon = \lim_{n \rightarrow \infty} c(B_n) + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, it follows that $c(B) = \lim_{n \rightarrow \infty} c(B_n)$.

4) Now we assume that the capacity c is 2-alternating. In order to prove that (iv) implies (i), let $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{O}$ with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} c\left(\bigcup_{n \in \mathbb{N}} B_n\right) &= \sup_{\mu \in \mathcal{M}} \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) \\ &= \sup_{\mu \in \mathcal{M}} \sup_{n \in \mathbb{N}} \mu(B_n) = \sup_{n \in \mathbb{N}} \sup_{\mu \in \mathcal{M}} \mu(B_n) = \lim_{n \rightarrow \infty} c(B_n). \end{aligned}$$

5) In the last step, we prove that (ii) implies (iv). By Corollary 2.5, there exist a set $\mathcal{M} \subseteq \text{ca}_+$ and a function $\alpha: \mathcal{M} \rightarrow \mathbb{R}$ such that

$$\int_{\Omega} f \, dc = \sup_{\mu \in \mathcal{M}} \left(\int_{\Omega} f \, d\mu - \alpha(\mu) \right) \quad \text{for all } f \in C_b.$$

Since the Choquet integral is positively homogeneous, it follows that

$$\int_{\Omega} f \, d\mu - \frac{\alpha(\mu)}{\lambda} = \frac{1}{\lambda} \left(\int_{\Omega} \lambda f \, d\mu - \alpha(\mu) \right) \leq \frac{1}{\lambda} \int_{\Omega} \lambda f \, dc = \int_{\Omega} f \, dc$$

for all $f \in C_b$, $\mu \in \mathcal{M}$ and $\lambda > 0$. Letting $\lambda \rightarrow \infty$, it follows that

$$\sup_{\mu \in \mathcal{M}} \int_{\Omega} f \, d\mu \leq \int_{\Omega} f \, dc \quad \text{for all } f \in C_b.$$

On the other hand,

$$-\alpha(\mu) = \int_{\Omega} 0 \, d\mu - \alpha(\mu) \leq \int_{\Omega} 0 \, dc = 0 \quad \text{for all } \mu \in \mathcal{M}.$$

Hence for all $f \in C_b$,

$$\sup_{\mu \in \mathcal{M}} \int_{\Omega} f \, d\mu \leq \int_{\Omega} f \, dc = \sup_{\mu \in \mathcal{M}} \left(\int_{\Omega} f \, d\mu - \alpha(\mu) \right) \leq \sup_{\mu \in \mathcal{M}} \int_{\Omega} f \, d\mu.$$

In particular,

$$\mu(\Omega) = \int_{\Omega} 1 \, d\mu \leq \int_{\Omega} 1 \, dc = c(\Omega) \quad \text{for all } \mu \in \mathcal{M}.$$

Since the Choquet integral is constant additive, it follows that

$$0 = c(\Omega) + \int_{\Omega} (-1) \, dc \geq c(\Omega) + \int_{\Omega} (-1) \, d\mu = c(\Omega) - \mu(\Omega)$$

for all $\mu \in \mathcal{M}$. We have therefore shown that $\mu(\Omega) = c(\Omega)$ for all $\mu \in \mathcal{M}$. Defining for $f \in L_b$, $x \in \Omega$ and $n \in \mathbb{N}$ the quantity $f_n(x) := \inf_{y \in \Omega} (f(y) + nd(x, y))$ with a metric d consistent with the topology on Ω , there exists for all $f \in L_b$ a sequence $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ with $f_n \nearrow f$ as $n \rightarrow \infty$, and so

$$\begin{aligned} \int_{\Omega} f \, dc &= \sup_{n \in \mathbb{N}} \int_{\Omega} f_n \, dc \\ &= \sup_{n \in \mathbb{N}} \sup_{\mu \in \mathcal{M}} \int_{\Omega} f_n \, d\mu = \sup_{\mu \in \mathcal{M}} \sup_{n \in \mathbb{N}} \int_{\Omega} f_n \, d\mu = \sup_{\mu \in \mathcal{M}} \int_{\Omega} f \, d\mu, \end{aligned}$$

where the first equality uses (3.7) and the last the monotone convergence theorem. \square

Proof of Corollary 2.7 First assume that (ii) is satisfied, i.e., there exist a set $I \neq \emptyset$ and a family $(\alpha_i)_{i \in I}$ of functions $\text{ca}_+^1 \rightarrow [0, \infty]$ with $\inf_{\mu \in \text{ca}_+^1} \alpha_i(\mu) = 0$ for all $i \in I$ and

$$R(f) = \min_{i \in I} \sup_{\mu \in \text{ca}_+^1} \left(\int f \, d\mu - \alpha_i(\mu) \right) \quad \text{for all } f \in C_b.$$

Then one readily verifies that R is monotone with $R(f + m) = R(f) + m$ for all $f \in C_b$ and $m \in \mathbb{R}$. Since $\inf_{\mu \in \text{ca}_+^1} \alpha_i(\mu) = 0$ for all $i \in I$, it follows that

$$R(0) = \min_{i \in I} \sup_{\mu \in \text{ca}_+^1} (-\alpha_i(\mu)) = \min_{i \in I} \left(-\inf_{\mu \in \text{ca}_+^1} \alpha_i(\mu) \right) = 0.$$

We have therefore shown that R is a risk measure. To prove that R is star-shaped, let $f \in C_b$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} R(\lambda f) &= \min_{i \in I} \sup_{\mu \in \text{ca}_+^1} \left(\int_{\Omega} \lambda f \, d\mu - \alpha_i(\mu) \right) \\ &\leq \min_{i \in I} \left(\lambda \sup_{\mu \in \text{ca}_+^1} \left(\int_{\Omega} f \, d\mu - \alpha_i(\mu) \right) \right) = \lambda R(f). \end{aligned}$$

To prove the converse implication (i) \Rightarrow (ii), assume that R is a star-shaped risk measure. Following the proof of Castagnoli et al. [6, Theorem 2], we define

$$\mathcal{A}_\varphi := \{f \in C_b : \exists \lambda \in [0, 1] \text{ with } f \leq \lambda(\varphi - R(\varphi))\} \quad \text{for all } \varphi \in C_b.$$

Then for all $\varphi \in C_b$, the set \mathcal{A}_φ is convex with $g \in \mathcal{A}_\varphi$ for all $g \in C_b$ with $g \leq f$ for some $f \in \mathcal{A}_\varphi$, and $0 \in \mathcal{A}_\varphi$, $m \notin \mathcal{A}_\varphi$ for all $m \in (0, \infty)$. Indeed, for the latter, assume towards a contradiction that there exist $m \in (0, \infty)$ and $\lambda \in (0, 1]$ with $m \leq \lambda(\varphi - R(\varphi))$. Then $\varphi \geq \frac{m}{\lambda} + R(\varphi)$, which contradicts the fact that R is a risk measure. Hence by Föllmer and Schied [15, Proposition 4.7], the functional $R_\varphi: C_b \rightarrow \mathbb{R}$ given by

$$R_\varphi(f) := \inf\{m \in \mathbb{R} : f - m \in \mathcal{A}_\varphi\} \quad \text{for all } f \in C_b$$

defines a convex risk measure on C_b . Let $f \in C_b$ and $m \in \mathbb{R}$ with $m > \inf_{\varphi \in C_b} R_\varphi(f)$. Then there exists some $\varphi \in C_b$ with $m > R_\varphi(f)$, and so

$$f - m \leq \lambda(\varphi - R(\varphi)) \quad \text{for some } \lambda \in [0, 1].$$

Since R is a star-shaped risk measure, it follows that

$$R(f) - m = R(f - m) \leq R(\lambda(\varphi - R(\varphi))) \leq \lambda R(\varphi - R(\varphi)) = 0.$$

Hence $R(f) \leq \inf_{\varphi \in C_b} R_\varphi(f)$ for all $f \in C_b$. Moreover, for all $\varphi \in C_b$ and $m \in \mathbb{R}$ with $m < R(\varphi)$, it follows that $\varphi - m > \varphi - R(\varphi)$ so that $R_\varphi(\varphi) = R(\varphi)$ by the

definition of \mathcal{A}_φ . Indeed, if $\varphi - R(\varphi) \leq 0$, it follows that $R(\varphi) = \sup_{x \in \Omega} \varphi(x)$. In this case, the inequality $m < R(\varphi)$ implies that there exists some $x \in \Omega$ with

$$\varphi(x) - m > 0 \geq \lambda(\varphi(x) - R(\varphi)) \quad \text{for all } \lambda \in [0, 1].$$

We have therefore shown that

$$R(f) = \min_{\varphi \in C_b} R_\varphi(f) \quad \text{for all } f \in C_b.$$

In view of Corollary 2.5 and [15, Theorem 4.16], it remains to prove that $R_\varphi: C_b \rightarrow \mathbb{R}$ is continuous from below for all $\varphi \in C_b$. To that end, let $\varphi \in C_b$, $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ with $f_n \nearrow f \in C_b$ as $n \rightarrow \infty$, and $(m_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ with $m_n > \sup_{k \in \mathbb{N}} R_\varphi(f_k)$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} m_n = \sup_{k \in \mathbb{N}} R_\varphi(f_k)$. Then there exists a sequence $(\lambda_n)_{n \in \mathbb{N}} \subseteq [0, 1]$ such that

$$f_n - m_n \leq \lambda_n(\varphi - R(\varphi)) \quad \text{for all } n \in \mathbb{N}.$$

Since $[0, 1]$ is compact, by passing to a subsequence, we may without loss of generality assume that $\lambda_n \rightarrow \lambda \in [0, 1]$ as $n \rightarrow \infty$. Then

$$f - \sup_{k \in \mathbb{N}} R_\varphi(f_k) = \lim_{n \rightarrow \infty} (f_n - m_n) \leq \lim_{n \rightarrow \infty} \lambda_n(\varphi - R(\varphi)) = \lambda(\varphi - R(\varphi)),$$

which proves that $R_\varphi(f) \leq \sup_{k \in \mathbb{N}} R_\varphi(f_k)$. Since $R_\varphi: C_b \rightarrow \mathbb{R}$ is monotone, it follows that $R_\varphi(f) = \lim_{n \rightarrow \infty} R_\varphi(f_n)$. \square

Proof of Theorem 2.8 First observe that by convexity of U_i for all $i \in I$ and (2.4),

$$\begin{aligned} \sup_{i \in I} (U_i(f_1) - U_i(f_2)) &\leq \sup_{i \in I} (U_i(f_1) + U_i(-f_2) - 2U_i(0)) \\ &\leq 2 \sup_{i \in I} (U_i(r) - U_i(0)) < \infty \end{aligned} \quad (3.8)$$

for all $r \geq 0$ and $f_1, f_2 \in C_b$ with $\max\{\|f_1\|_\infty, \|f_2\|_\infty\} \leq r$.

1) The implication (iii) \Rightarrow (i) follows from Remark 3.1 a). To prove that (i) implies (ii), we first show that for every $f \in C_b$ and every $\varepsilon > 0$, there exists some $V \in \mathcal{V}$ with

$$\sup_{i \in I} |U_i(f + e) - U_i(f)| < \varepsilon \quad \text{for all } e \in V.$$

To that end, let $f \in C_b$ and consider the monotone maps $\overline{U}_f, \underline{U}_f: C_b \rightarrow \mathbb{R}$ given by

$$\begin{aligned} \overline{U}_f(g) &:= \sup_{i \in I} (U_i(f + g) - U_i(f)), \\ \underline{U}_f(g) &:= \sup_{i \in I} (U_i(f) - U_i(f - g)) \quad \text{for all } g \in C_b. \end{aligned}$$

Observe that by (3.8), \overline{U}_f and \underline{U}_f are well defined and

$$\underline{U}_f(-g) = \sup_{i \in I} (U_i(f) - U_i(f + g)) \quad \text{for all } g \in C_b.$$

Moreover, for any $V \in \mathcal{V}$, $e \in V$ if and only if $-e \in V$. Hence the auxiliary statement follows from Corollary 2.3 once we have shown that both \overline{U}_f and \underline{U}_f are continuous from above. To that end, let $g \in C_b$ and $(g_n)_{n \in \mathbb{N}} \subseteq C_b$ with $g_n \searrow g$ as $n \rightarrow \infty$. Moreover, let $\varepsilon > 0$, $n \in \mathbb{N}$ and $\lambda \in (0, 1)$. Then using the fact that U_i is convex for all $i \in I$,

$$\begin{aligned} \overline{U}_f(g_n) - \overline{U}_f(g) &\leq \sup_{i \in I} (U_i(f + g_n) - U_i(f + g)) \\ &\leq \lambda \sup_{i \in I} \left(U_i\left(\frac{g_n - g}{\lambda}\right) - U_i(0) \right) + \lambda \sup_{i \in I} (U_i(0) - U_i(f + g)) \\ &\quad + (1 - \lambda) \sup_{i \in I} \left(U_i\left(\frac{f + g}{1 - \lambda}\right) - U_i(f + g) \right) \end{aligned}$$

and

$$\begin{aligned} \underline{U}_f(g_n) - \underline{U}_f(g) &\leq \sup_{i \in I} (U_i(f - g) - U_i(f - g_n)) \\ &\leq \sup_{i \in I} (U_i(f - 2g + g_n) - U_i(f - g)) \\ &\leq \lambda \sup_{i \in I} \left(U_i\left(\frac{g_n - g}{\lambda}\right) - U_i(0) \right) + \lambda \sup_{i \in I} (U_i(0) - U_i(f - g)) \\ &\quad + (1 - \lambda) \sup_{i \in I} \left(U_i\left(\frac{f - g}{1 - \lambda}\right) - U_i(f - g) \right). \end{aligned}$$

Since the maps

$$\mathbb{R} \rightarrow \mathbb{R}, \quad \gamma \mapsto \sup_{i \in I} (U_i(\gamma(f \pm g)) - U_i(f \pm g))$$

are convex and therefore continuous, we obtain

$$\lambda \sup_{i \in I} (U_i(0) - U_i(f \pm g)) + (1 - \lambda) \sup_{i \in I} \left(U_i\left(\frac{f \pm g}{1 - \lambda}\right) - U_i(f \pm g) \right) < \frac{\varepsilon}{2}$$

for $\lambda \in (0, 1)$ sufficiently small. Moreover, by assumption,

$$\lambda \sup_{i \in I} \left(U_i\left(\frac{g_n - g}{\lambda}\right) - U_i(0) \right) < \frac{\varepsilon}{2}$$

for $n \in \mathbb{N}$ sufficiently large since $\frac{g_n - g}{\lambda} \searrow 0$ as $n \rightarrow \infty$. We have therefore shown that

$$0 \leq \overline{U}_f(g_n) - \overline{U}_f(g) < \varepsilon \quad \text{and} \quad 0 \leq \underline{U}_f(g_n) - \underline{U}_f(g) < \varepsilon$$

for $n \in \mathbb{N}$ sufficiently large, and so both \overline{U}_f and \underline{U}_f are continuous from above. We have thus proved the auxiliary statement and are now ready to prove the implication (i) \Rightarrow (ii). For $i \in I$ and $f \in C_b$, let $U_{i,f}: C_b \rightarrow \mathbb{R}$ be given by

$$U_{i,f}(g) := U_i(f + g) - U_i(f) \quad \text{for all } g \in C_b.$$

Then $U_{i,f}$ is convex and monotone with $U_{i,f}(0) = 0$ for all $i \in I$ and $f \in C_b$. Moreover, for all $\lambda \in (0, 1)$, $i \in I$ and $f, g \in C_b$,

$$\begin{aligned} U_{i,f}(g) &\leq \lambda \left(U_i \left(\frac{g}{\lambda} \right) - U_i(f) \right) + (1 - \lambda) \left(U_i \left(\frac{f}{1 - \lambda} \right) - U_i(f) \right) \\ &\leq \lambda \left(U_i \left(\frac{g}{\lambda} \right) - U_i(0) \right) + \lambda (U_i(-f) - U_i(0)) \\ &\quad + (1 - \lambda) \left(U_i \left(\frac{f}{1 - \lambda} \right) - U_i(f) \right), \end{aligned}$$

where the second inequality uses the fact that $U_i(0) - U_i(f) \leq U_i(-f) - U_i(0)$ for all $i \in I$. Let $r \geq 0$, $(g_n)_{n \in \mathbb{N}} \subseteq C_b$ with $g_n \searrow 0$ as $n \rightarrow \infty$ and $\varepsilon > 0$. Then by (3.8), the map

$$\mathbb{R} \rightarrow \mathbb{R}, \quad \gamma \mapsto \sup_{\|f\|_\infty \leq r} \sup_{i \in I} (U_i(\gamma f) - U_i(f))$$

is convex and well defined. Therefore it is continuous and it follows that

$$\lambda (U_i(-f) - U_i(0)) + (1 - \lambda) \left(U_i \left(\frac{f}{1 - \lambda} \right) - U_i(f) \right) < \frac{\varepsilon}{2}$$

for $\lambda \in (0, 1)$ sufficiently small. Hence we get

$$\sup_{\|f\|_\infty \leq r} \sup_{i \in I} U_{i,f}(g_n) \leq \lambda \sup_{i \in I} \left(U_i \left(\frac{g_n}{\lambda} \right) - U_i(0) \right) + \frac{\varepsilon}{2} < \varepsilon$$

for $n \in \mathbb{N}$ sufficiently large, and we have shown that

$$\lim_{n \rightarrow \infty} \sup_{\|f\|_\infty \leq r} \sup_{i \in I} U_{i,f}(g_n) = 0.$$

Using the auxiliary statement for the convex monotone functions $U_{i,f}$ with $i \in I$ and $f \in C_b$ with $\|f\|_\infty \leq r$, there exists some $V \in \mathcal{V}$ such that

$$\sup_{\|f\|_\infty \leq r} \sup_{i \in I} |U_i(f + e) - U_i(f)| = \sup_{\|f\|_\infty \leq r} \sup_{i \in I} |U_{i,f}(e)| \leq \varepsilon \quad \text{for all } e \in V.$$

2) It remains to prove the implication (ii) \Rightarrow (iii). Let $r \geq 0$ and $\varepsilon > 0$. Then there exists some $V \in \mathcal{V}$ such that

$$\sup_{\|f\|_\infty \leq r} \sup_{i \in I} |U_i(f + e) - U_i(f)| \leq \varepsilon \quad \text{for all } e \in V.$$

By the definition (2.1) of the local base \mathcal{V} , there exist some compact $C \subseteq \Omega$ and some $\delta > 0$ such that

$$\left\{ e \in C_b : \sup_{x \in C} |e(x)| < \delta \right\} \cap \{e \in C_b : \|e\|_\infty \leq 2r\} \subseteq V.$$

Now let $f_1, f_2 \in C_b$ with $\max\{\|f_1\|_\infty, \|f_2\|_\infty\} \leq r$. Then by the triangle inequality, $\|f_1 - f_2\|_\infty \leq 2r$. If $\sup_{x \in C} |f_1(x) - f_2(x)| < \delta$, then

$$\sup_{i \in I} |U_i(f_1) - U_i(f_2)| \leq \varepsilon.$$

On the other hand, if $\sup_{x \in C} |f_1(x) - f_2(x)| \geq \delta$, then by (3.8), it follows that

$$\sup_{i \in I} |U_i(f_1) - U_i(f_2)| \leq M \sup_{x \in C} |f_1(x) - f_2(x)|$$

with $M := \frac{2}{\delta} \sup_{i \in I} (U_i(r) - U_i(0))$. The proof is complete. \square

Proof of Corollary 2.9 Since (L, τ) is a locally convex vector lattice, (iv) implies (i). The implication (i) \Rightarrow (iii) is trivial, and by Corollary 2.3, (ii) and (iii) are equivalent. By Theorem 2.8 and Lemma B.1, (ii) \Rightarrow (iv) and (ii) \Rightarrow (v), since (ii) together with Dini's lemma implies that for every convex and weak* compact set K of nonnegative τ -continuous linear functionals, the map

$$U_K: C_b \rightarrow \mathbb{R}, \quad f \mapsto \sup_{\mu \in K} \mu(U(f) - U(0))$$

is convex with $\lim_{n \rightarrow \infty} U_K(f_n) = 0$ for every sequence $(f_n)_{n \in \mathbb{N}} \subseteq C_b$ with $f_n \searrow 0$ as $n \rightarrow \infty$. Since for every nonnegative τ -continuous linear functional $\lambda: L \rightarrow \mathbb{R}$, there exists a τ -continuous lattice seminorm $p: L \rightarrow [0, \infty)$ with $|\lambda u| \leq p(u)$ for all $u \in L$, (v) implies (ii) by Dini's lemma. \square

Appendix A: Capacities and Choquet integrals

The setup and notation in this section follow that of the main part. The following lemma is a sort of folklore result; cf. König [20, Property 11.8 and Theorem 11.11]. For the reader's convenience, we nevertheless provide a short proof.

Lemma A.1 *Let $c: \mathcal{O} \rightarrow [0, \infty)$ be a capacity. Then the following are equivalent:*

(i) *For all $B_1, B_2 \in \mathcal{O}$,*

$$c(B_1 \cup B_2) + c(B_1 \cap B_2) \leq c(B_1) + c(B_2).$$

(ii) *For all $f_1, f_2 \in L_b$,*

$$\int_{\Omega} (f_1 + f_2) \, dc \leq \int_{\Omega} f_1 \, dc + \int_{\Omega} f_2 \, dc.$$

Proof We first prove the implication (ii) \Rightarrow (i). To that end, let $B_1, B_2 \in \mathcal{O}$. Then

$$\begin{aligned} c(B_1 \cup B_2) + c(B_1 \cap B_2) &= \int_{\Omega} (1_{B_1} + 1_{B_2}) \, dc \\ &\leq \int_{\Omega} 1_{B_1} \, dc + \int_{\Omega} 1_{B_2} \, dc = c(B_1) + c(B_2). \end{aligned}$$

We proceed with the proof of the implication (i) \Rightarrow (ii). In the first step, we prove by induction over $n \in \mathbb{N}$ that

$$\int_{\Omega} \sum_{i=1}^n 1_{B_i} \, dc \leq \sum_{i=1}^n c(B_i) \quad \text{for all } B_1, \dots, B_n \in \mathcal{O}. \quad (\text{A.1})$$

For $n = 1$, the statement is trivial. Assume that (A.1) is proved for some $n \in \mathbb{N}$ and let $B_1, \dots, B_{n+1} \in \mathcal{O}$. Then

$$\sum_{k=1}^{n+1} 1_{B_k} = 1_{\bigcup_{i=1}^{n+1} B_i} + \sum_{k=1}^n 1_{(\bigcup_{i=1}^k B_i) \cap B_{k+1}}.$$

Using (A.1) and (i), we obtain that

$$\begin{aligned} \int_{\Omega} \sum_{i=1}^{n+1} 1_{B_i} \, dc &= c\left(\bigcup_{i=1}^{n+1} B_i\right) + \int_{\Omega} \sum_{k=1}^n 1_{(\bigcup_{i=1}^k B_i) \cap B_{k+1}} \, dc \\ &\leq c\left(\bigcup_{i=1}^{n+1} B_i\right) + \sum_{k=1}^n c\left(\left(\bigcup_{i=1}^k B_i\right) \cap B_{k+1}\right) \leq \sum_{i=1}^{n+1} c(B_i). \end{aligned}$$

Now let $f_1, f_2 \in L_b$. Since the Choquet integral is constant additive, we may without loss of generality assume that $f_1 \geq 0$ and $f_2 \geq 0$. Let $r := \max\{\|f_1\|_{\infty}, \|f_2\|_{\infty}\}$. For $i = 1, 2, n \in \mathbb{N}$ and $k = 1, \dots, 2^n$, define $B_{i,n}^k := \{f_k > k2^{-n}r\}$. Then for $i = 1, 2$,

$$\left\| f_i - \sum_{k=1}^{2^n} 2^{-n} r 1_{B_{i,n}^k} \right\|_{\infty} \leq 2^{-n} r \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But positive homogeneity of the Choquet integral and (A.1) give

$$\begin{aligned} \int_{\Omega} \sum_{k=1}^{2^n} 2^{-n} r (1_{B_{1,n}^k} + 1_{B_{2,n}^k}) \, dc &= 2^{-n} r \int_{\Omega} \sum_{k=1}^{2^n} (1_{B_{1,n}^k} + 1_{B_{2,n}^k}) \, dc \\ &\leq 2^{-n} r \sum_{k=1}^{2^n} (c(B_{1,n}^k) + c(B_{2,n}^k)) \\ &= \int_{\Omega} \sum_{k=1}^{2^n} 2^{-n} r 1_{B_{1,n}^k} \, dc + \int_{\Omega} \sum_{k=1}^{2^n} 2^{-n} r 1_{B_{2,n}^k} \, dc, \end{aligned}$$

and so it follows that

$$\int_{\Omega} (f_1 + f_2) \, dc \leq \int_{\Omega} f_1 \, dc + \int_{\Omega} f_2 \, dc. \quad \square$$

Appendix B: Locally convex vector lattices

Throughout this section, let (L, τ) be a locally convex vector lattice, i.e., a vector lattice L together with a locally convex topology τ on L which is generated by a family of lattice seminorms.

Let $L_+ := \{u \in L : u \geq 0\}$ and L' be the topological dual space of L , i.e., the space of all continuous linear functionals $L \rightarrow \mathbb{R}$. Moreover, let

$$L'_+ := \{\lambda \in L' : \lambda u \geq 0, \forall u \in L_+\}$$

be the set of all positive continuous linear functionals on L . For $u \in L$, we use the standard notation $u_+ := u \vee 0$ and $u_- := -(u \wedge 0)$. Then $u = u_+ - u_-$ and $|u| := u_+ + u_-$ for all $u \in L$. The following lemma can be deduced from Aliprantis and Border [2, Theorem 8.24 and Corollary 8.25] together with the fact that every linear functional that is bounded by a lattice seminorm is order bounded. For the sake of a self-contained exposition, we provide a short proof.

Lemma B.1 *Let $p: L \rightarrow [0, \infty)$ be a continuous lattice seminorm.*

a) *For every $\lambda \in L'$ with $|\lambda u| \leq p(u)$ for all $u \in L$, there exist $\lambda_+, \lambda_- \in L'_+$ with*

$$\lambda u = \lambda_+ u - \lambda_- u \quad \text{and} \quad \max\{\lambda_+ |u|, \lambda_- |u|\} \leq p(u) \quad \text{for all } u \in L. \quad (\text{B.1})$$

b) *There exists a convex and weak* compact set $K \subseteq L'_+$ with*

$$\max_{\mu \in K} |\mu u| \leq \max_{\mu \in K} \mu |u| = p(u) \leq 2 \max_{\mu \in K} |\mu u| \quad \text{for all } u \in L.$$

Proof a) Since p is a lattice seminorm, it follows that $p(u) = p(v)$ for all $u, v \in L$ with $|u| = |v|$. In particular,

$$p(u) = p(|u|) \quad \text{for all } u \in L. \quad (\text{B.2})$$

Let $\lambda \in L'$ with $|\lambda u| \leq p(u)$ for all $u \in L$, and define

$$\lambda_+ u := \sup\{\lambda v : v \in L_+, v \leq u\} \quad \text{for all } u \in L_+.$$

Since p is a lattice seminorm, $0 \leq \lambda_+ u \leq p(u)$ and $\lambda_+(\alpha u) = \alpha \lambda_+ u$ for all $u \in L_+$ and $\alpha \geq 0$. In order to prove that λ_+ is additive, let $u_1, u_2 \in L_+$. Then for $v_1, v_2 \in L_+$ with $v_1 \leq u_1$ and $v_2 \leq u_2$,

$$\lambda v_1 + \lambda v_2 \leq \lambda(v_1 + v_2) \leq \lambda_+(u_1 + u_2).$$

Hence $\lambda_+ u_1 + \lambda_+ u_2 \leq \lambda_+(u_1 + u_2)$. On the other hand, for $v \in L_+$ with $v \leq u_1 + u_2$, let

$$v_1 := (v - u_2)_+ \geq 0, \quad v_2 := v - v_1 = v + (u_2 - v) \wedge 0 = v \wedge u_2 \leq u_2.$$

Moreover, $v_1 \leq u_1$ since $u_1 \geq 0$, and $v_2 = v \wedge u_2 \geq 0$ since $v \geq 0$ and $u_2 \geq 0$. Hence

$$\lambda v = \lambda v_1 + \lambda v_2 \leq \lambda_+ u_1 + \lambda_+ u_2.$$

We have therefore shown that $\lambda_+(u_1 + u_2) = \lambda_+u_1 + \lambda_+u_2$. For $u \in L$, define

$$\lambda_+u := \lambda_+u_+ - \lambda_+u_-.$$

Let $u, v \in L$. Then

$$(u + v)_+ - (u + v)_- = u + v = u_+ - u_- + v_+ - v_-,$$

and so

$$\begin{aligned} \lambda_+(u + v)_+ + \lambda_+u_- + \lambda_+v_- &= \lambda_+((u + v)_+ + u_- + v_-) \\ &= \lambda_+((u + v)_- + u_+ + v_+) \\ &= \lambda_+(u + v)_- + \lambda_+u_+ + \lambda_+v_+, \end{aligned}$$

which implies that $\lambda_+(u + v) = \lambda_+u + \lambda_+v$. Moreover, for all $u \in L$ and $\alpha > 0$,

$$\lambda_+(\alpha u) = \lambda_+(\alpha u)_+ - \lambda_+(\alpha u)_- = \lambda_+(\alpha u_+) - \lambda_+(\alpha u_-) = \alpha \lambda_+u.$$

Since by definition $(-u)_+ = -u_-$ and $(-u)_- = -u_+$ for all $u \in L$, it follows that $\lambda_+ : L \rightarrow \mathbb{R}$ is linear, and by (B.2),

$$|\lambda_+u| \leq \lambda_+|u| \leq p(|u|) = p(u),$$

which implies that $\lambda_+ : L \rightarrow \mathbb{R}$ is continuous. Now, defining $\lambda_-u := \lambda_+u - \lambda u$ for all $u \in L$, we find that

$$\lambda_-u = \sup\{\lambda(v - u) : v \in L_+, v \leq u\} = \sup\{\lambda(-w) : w \in L_+, w \leq u\} = (-\lambda)_+u$$

for all $u \in L_+$. Using the fact that λ_- is linear and replacing λ by $-\lambda$, it follows that

$$|\lambda_-u| \leq \lambda_-|u| = (-\lambda)_+|u| \leq p(u).$$

In particular, λ_- is continuous, and the proof of part a) is complete.

b) Let $V := \{u \in L : p(u) \leq 1\}$. Then by the Banach–Alaoglu theorem, the set $V^\circ := \{\lambda \in L' : \lambda u \leq 1, \forall u \in V\}$ is convex and weak* compact. By (B.2) and the Hahn–Banach theorem,

$$p(u) = \max_{\lambda \in V^\circ} |\lambda u| \quad \text{for all } u \in L.$$

Since the set $L'_+ := \bigcap_{u \in L_+} \{\lambda \in L' : \lambda u \geq 0\}$ is convex and weak* closed, it follows that $K := V^\circ \cap L'_+$ is convex and weak* compact. By (B.2),

$$\max_{\mu \in K} |\mu u| \leq \max_{\mu \in K} \mu |u| \leq p(|u|) = p(u) \quad \text{for all } u \in L.$$

By (B.1), for all $\lambda \in V^\circ$ and $u \in L$,

$$-\lambda_-|u| \leq \lambda|u| \leq \lambda_+|u| \quad \text{and} \quad \max\{\lambda_+|u|, \lambda_-|u|\} \leq p(u).$$

This shows that $\lambda_+, \lambda_- \in K$ and $|\lambda|u| \leq \max\{\lambda_+|u|, \lambda_-|u|\}$ for all $\lambda \in V^\circ$. Hence by (B.2),

$$p(u) = p(|u|) = \max_{\lambda \in V^\circ} |\lambda|u| \leq \sup_{\lambda \in V^\circ} \max\{\lambda_+|u|, \lambda_-|u|\} \leq \max_{\mu \in K} \mu|u|.$$

We have therefore shown that $p(u) = \max_{\mu \in K} \mu|u|$ for all $u \in L$. On the other hand,

$$p(u) = \max_{\lambda \in V^\circ} |\lambda u| \leq \max_{\lambda \in V^\circ} (|\lambda_+ u| + |\lambda_- u|) \leq 2 \max_{\mu \in K} |\mu u|. \quad \square$$

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Declarations

Competing Interests The author declares no competing interests.

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References

1. Adamski, W.: Capacitylike set functions and upper envelopes of measures. *Math. Ann.* **229**, 237–244 (1977)
2. Aliprantis, C.D., Border, K.C.: *Infinite-Dimensional Analysis. A Hitchhiker's Guide*, 2nd edn. Springer, Berlin (1999)
3. Blessing, J., Denk, R., Kupper, M., Nendel, M.: Convex monotone semigroups and their generators with respect to Γ -convergence (2022). Preprint, Available online at <https://arxiv.org/abs/2202.08653>
4. Bogachev, V.I.: *Measure Theory. Volume II*. Springer, Berlin (2007)
5. Burzoni, M., Maggis, M.: Arbitrage-free modeling under Knightian uncertainty. *Math. Financ. Econ.* **14**, 635–659 (2020)
6. Castagnoli, E., Cattelan, G., Maccheroni, F., Tebaldi, C., Wang, R.: Star-shaped risk measures. *Oper. Res.* **70**, 2637–2654 (2022)
7. Cheridito, P., Kiiski, M., Prömel, D.J., Soner, H.M.: Martingale optimal transport duality. *Math. Ann.* **379**, 1685–1712 (2021)
8. Cheridito, P., Kupper, M., Tangpi, L.: Representation of increasing convex functionals with countably additive measures. *Stud. Math.* **260**, 121–140 (2021)
9. Delbaen, F.: Convex increasing functionals on $C_b(X)$ spaces. *Stud. Math.* **271**, 107–120 (2023)
10. Delbaen, F.: Monetary utility functions on $C_b(X)$ spaces. *Int. J. Theor. Appl. Finance* **27**, 2350033 (2024)
11. Denis, L., Hu, M., Peng, S.: Function spaces and capacity related to a sublinear expectation: application to G -Brownian motion paths. *Potential Anal.* **34**, 139–161 (2011)

12. Denk, R., Kupper, M., Nendel, M.: Kolmogorov-type and general extension results for nonlinear expectations. *Banach J. Math. Anal.* **12**, 515–540 (2018)
13. Dudley, R.M.: *Real Analysis and Probability*. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove (1989)
14. Ekeland, I., Temam, R.: *Convex Analysis and Variational Problems*. North-Holland, Amsterdam (1976)
15. Föllmer, H., Schied, A.: *Stochastic Finance. An Introduction in Discrete Time*, Fourth revised and extended edn. de Gruyter, Berlin (2016)
16. Fremlin, D.H., Garling, D.J.H., Haydon, R.G.: Bounded measures on topological spaces. *Proc. Lond. Math. Soc.* **s3-25**, 115–136 (1972)
17. Goldys, B., Nendel, M., Röckner, M.: Operator semigroups in the mixed topology and the infinitesimal description of Markov processes. *J. Differ. Equ.* **412**, 23–86 (2024)
18. Herdegen, M., Khan, N.: Mean- ρ portfolio selection and ρ -arbitrage for coherent risk measures. *Math. Finance* **32**, 226–272 (2022)
19. Jouini, E., Schachermayer, W., Touzi, N.: Law invariant risk measures have the Fatou property. In: Kusuoka, S., Yamazaki, A. (eds.) *Advances in Mathematical Economics*, vol. 9, pp. 49–71. Springer, Tokyo (2006)
20. König, H.: *Measure and Integration. An Advanced Course in Basic Procedures and Applications*. Springer, Berlin (1997)
21. Kupper, M., Zapata, J.M.: Large deviations built on max-stability. *Bernoulli* **27**, 1001–1027 (2021)
22. Peng, S.: *Nonlinear Expectations and Stochastic Calculus Under Uncertainty with Robust CLT and G-Brownian*. Springer, Berlin (2019)
23. Schaefer, H.H., Wolff, M.P.: *Topological Vector Spaces*, 2nd edn. Springer, New York (1999)
24. Sentes, F.D.: Bounded continuous functions on a completely regular space. *Trans. Am. Math. Soc.* **168**, 311–336 (1972)
25. Wang, S.S., Young, V.R., Panjer, H.H.: Axiomatic characterization of insurance prices. *Insur. Math. Econ.* **21**, 173–183 (1997)
26. Wheeler, R.F.: A survey of Baire measures and strict topologies. *Expo. Math.* **1**, 97–190 (1983)
27. Wiweger, A.: Linear spaces with mixed topology. *Stud. Math.* **20**, 47–68 (1961)

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