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**Working Paper**

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Jena Economic Research Papers, No. 2008,072

**Provided in Cooperation with:**  
Max Planck Institute of Economics

*Suggested Citation:* Coad, Alex (2008) : The exponential age distribution and the Pareto firm size distribution, Jena Economic Research Papers, No. 2008,072, Friedrich Schiller University Jena and Max Planck Institute of Economics, Jena

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# JENA ECONOMIC RESEARCH PAPERS



# 2008 – 072

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by

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[www.jenecon.de](http://www.jenecon.de)

ISSN 1864-7057

The JENA ECONOMIC RESEARCH PAPERS is a joint publication of the Friedrich Schiller University and the Max Planck Institute of Economics, Jena, Germany. For editorial correspondence please contact [markus.pasche@uni-jena.de](mailto:markus.pasche@uni-jena.de).

Impressum:

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Carl-Zeiss-Str. 3  
D-07743 Jena  
[www.uni-jena.de](http://www.uni-jena.de)

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# The exponential age distribution and the Pareto firm size distribution

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September 19, 2008

## Abstract

Recent work drawing on data for large and small firms has shown a Pareto distribution of firm size. We mix a Gibrat-type growth process among incumbents with an exponential distribution of firm's age, to obtain the empirical Pareto distribution.

**JEL codes:** L20, L25

**Keywords:** Firm size distribution, Firm growth, Gibrat's Law, Pareto distribution, Zipf Law

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## 1 Introduction

Early models of industrial dynamics focused on firms above a certain size threshold, because data on larger firms was easier to obtain. These studies generally observed a lognormal size distribution, that can be explained by a Gibrat process. More recently, work that takes young, small firms into account observes a Zipf distribution of firm size (Axtell (2001), for surveys see de Wit (2005) and Coad (2009)). The explanation suggested by Axtell (2001) consists of a Kesten process in which firm sizes are bounded from below (Kesten (1973)). The combination of a Gibrat random growth model and lower bounds on firm sizes is seen to produce the Zipf distribution (which is a special case of the Pareto distribution). A drawback of this mechanism, however, is that firms are implicitly assumed to all be of the same age, which is of course quite unrealistic. In this model, we begin with a Gibrat-type process, but we relax the restriction that firms are all of the same age. Instead, we posit an exponential distribution of firm age. Mixing these two distributions (Gibrat process for incumbents and an exponential distribution of firm age) yields the observed power law distribution.

The age distribution of firms is considered in Section 2, before an exponential age distribution is combined with a Gibrat growth process to obtain the Pareto firm size distribution (Section 3).

## 2 Age distribution

We can only begin this section by mentioning that the age distribution of a population of firms has barely been investigated in empirical work. However, we consider the age distribution to be an interesting feature of industrial structure. The age distribution may be of direct interest in theoretical models of the firm size distribution (such as the model presented here), as well as providing indirect information on a number of phenomena such as entry rates, survival rates, and possibly even the age of technology used in production.

To the best of our knowledge, the only representations of the age distribution of firms can be found in Coad and Tamvada (2008) and Segarra et al. (2008). Coad and Tamvada (2008) focus on census data covering around 700'000 small scale firms in India, while Segarra et al. (2008) focus on a sample of about 85'000 firms taken from the Spanish Mercantile Register. Figures 1 and 2 show the corresponding age distributions. The shape of the distribution appears to follow an approximately straight line with negative slope over most of the support. Given that the  $y$ -axis is expressed in logarithms, this straight line suggests that an exponential distribution would be a valid approximation of the empirical age distribution.

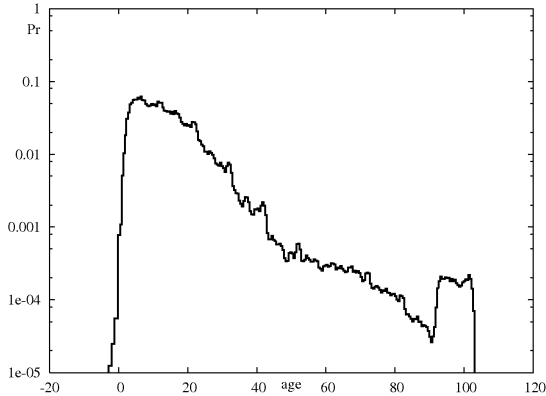


Figure 1: Kernel density of the age distribution of Indian small scale industries. Kernel density computed for equispaced points using an Epanenchnikov kernel (using gbutils 5.2). Taken from Coad and Tamvada (2008).

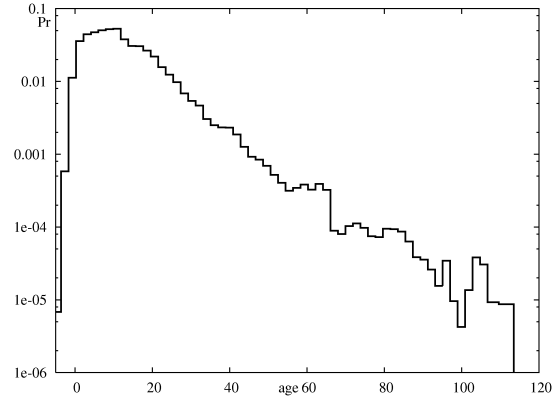


Figure 2: Kernel density of the age distribution of Spanish firms. Kernel density computed for equispaced points using an Epanenchnikov kernel (using gbutils 5.2). Source: author's elaboration of the data in Segarra et al. (2008), page 104.

### 3 Pareto firm size distribution

We now turn to our model of the firm size distribution.

Let  $x_t$  be the size of a firm at time  $t$ , and let  $\varepsilon_t$  be random variable representing an iid idiosyncratic, multiplicative growth shock over the period  $t - 1$  to  $t$ , with mean  $\bar{\varepsilon}$ . We have

$$x_t - x_{t-1} = \varepsilon_t x_{t-1} \tag{1}$$

which can be developed to obtain

$$x_t = (1 + \varepsilon_t)x_{t-1} = x_0(1 + \varepsilon_1)(1 + \varepsilon_2) \dots (1 + \varepsilon_t) \tag{2}$$

It is then possible to take logarithms in order to approximate  $\log(1 + \varepsilon_t)$  by  $\varepsilon_t$  to obtain<sup>1</sup>

$$\log(x_t) \approx \log(x_0) + \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_t = \log(x_0) + \sum_{s=1}^t \varepsilon_s \tag{3}$$

In the limit, as  $t$  becomes large, the  $\log(x_0)$  term will become insignificant, and we obtain:

$$\log(x_t) \approx \sum_{s=1}^t \varepsilon_s \tag{4}$$

Central Limit Theorem implies that  $\log(x_t)$  is normally distributed, which means that firm size (i.e.  $x_t$ ) is lognormally distributed:

<sup>1</sup>This logarithmic approximation is only justified if  $\varepsilon_t$  is 'small' enough (i.e. close to zero), which can be reasonably assumed by taking a short time period (Sutton (1997)).

$$P(x_t) = \frac{1}{x_t \sqrt{2\pi\sigma^2}} e^{-\left(\frac{(\ln x_t - \bar{\varepsilon}t)^2}{2\sigma^2 t}\right)} \quad (5)$$

In this paper, we will no longer assume that  $t$  has the same value for all firms. Instead, we suggest that  $t$  is itself a random variable. In the light of the evidence presented above, it seems reasonable to assume the distribution of firm age to be exponentially distributed. If  $t$  is exponentially distributed, we have:

$$P(t) = \lambda e^{-\lambda t} \quad (6)$$

In order to obtain the mixture of these two distributions, we apply the following rule: if the distribution of a variable  $a$ ,  $p(a, b)$ , depends on a parameter  $b$  which in turn is distributed according to its own distribution  $r(b)$ , then the distribution of  $a$  is given by  $p(a) = \int r(b) \cdot p(a, b) db$  (Adamic and Huberman (1999); Huberman and Adamic (1999)).

This gives us the following:

$$P(x_t) = \int \lambda e^{-\lambda t} \cdot \frac{1}{x_t \sqrt{2\pi\sigma^2}} e^{-\left(\frac{(\ln x_t - \bar{\varepsilon}t)^2}{2\sigma^2 t}\right)} dt \quad (7)$$

and, as in Adamic and Huberman (1999), this can be developed to yield:

$$P(x_t) = C \cdot x_t^{-\beta} \quad (8)$$

where  $C$  is a constant and is given by  $C = \lambda/\sigma(\sqrt{(\bar{\varepsilon}/\sigma)^2 + 2\lambda})$ . The exponent  $\beta$  is in the range  $[1, \infty]$  and is determined by  $\beta = 1 - \frac{\bar{\varepsilon}}{\sigma^2} + \frac{\sqrt{(\bar{\varepsilon}^2 + 2\lambda\sigma^2)}}{\sigma^2}$ . When the mean growth rate is close to 0%,  $\bar{\varepsilon}$  will be close to 1. As a result, if  $\lambda$  is small (implying that the exponential decay is relatively weak, i.e. that it is not uncommon to find firms with an age much greater than one)<sup>2</sup>, and if  $\sigma$  is small (which is not implausible either), then the exponent  $\beta$  will be close to Zipf's value of 1, which has been observed in empirical work (Axtell (2001)).

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<sup>2</sup>This condition is trivial since the duration  $t$  of a Gibrat-type 'shock' can be made arbitrarily short.

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