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ABSTRACT

A Simple Feasible Alternative Procedure to Estimate Models with High-Dimensional Fixed Effects

In this paper we describe an alternative iterative approach for the estimation of linear regression models with high-dimensional fixed-effects such as large employer-employee data sets. This approach is computationally intensive but imposes minimum memory requirements. We also show that the approach can be extended to non-linear models and potentially to more than two high dimensional fixed effects.

JEL Classification: C01, C81

Keywords: high dimensional fixed effects, linked employer-employee data

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1 Introduction

The increasing availability of large micro-level data sets has spurred interest on methods for estimation of models with high-dimensional fixed-effects. Researchers in several fields such as economics, sociology, and political science among others, find the introduction of fixed effects a particularly appealing way of controlling for unobserved heterogeneity that is shared among groups of observations. In this case it becomes possible to account for all inter-group variability by adding to the set of regressors dummy variables that absorb group specific heterogeneity. This approach has the advantage of allowing for the existence of general patterns of correlation between the unobserved effects and the other regressors. In practice, when estimating a model with a single fixed-effect (i.e. a factor in the analysis of covariance) one is not required to actually add the group dummy variables to the set of regressors. This is particularly convenient when dealing with high-dimensional fixed effects, that is, in a situation when the number of groups (dummy variables) is very large. For several common procedures such as linear regression, Poisson and logit regression the fixed effect can be eliminated from the model making it possible to obtain estimates for the coefficients of the relevant regressors without having to introduce the group dummy variables in the model. For other nonlinear models it is still possible to avoid the explicit introduction of dummy variables to account for the fixed effect by modifying the iterative algorithm used to solve the maximum-likelihood problem [see Greene (2004)]. However, there is no simple solution when there is more than one high-dimensional fixed effect. A notable example are the large employer-employee panel data sets commonly used in the labor economics literature. When studying relations in the labor market researchers often want to simultaneously account for two sources of unobserved heterogeneity, the firm and the worker. Explicit introduction of dummy variables is not an option because the number of units (groups) for either firms or workers is too large. Other well known examples of large data sets with obvious sources of unobserved heterogeneity are panel data sets of patient claims data - here the potential sources of heterogeneity are the patient, the doctor and the hospital; and panel data sets on student performance - the potential sources of heterogeneity being the students, the teachers, and the school.

Abowd et al. (1999) tackled the problem of accounting for two high-dimensional fixed effects in the linear regression problem. In a widely cited paper, the authors proposed several methods that provide approximate solutions to this problem.¹ Later, in an unpublished paper Abowd et al. (2002) presented an iterative algorithm that leads to the exact solution for the least square estimation of the model with two

¹For a discussion on the implementation of these methods in Stata see Andrews et al. (2006).

fixed effects. The user-written command a2reg is a Stata implementation of this algorithm by Amine Ouazad.² In a recent article published in this journal Cornelissen (2008) presented a new user written command, felsdvreg, which consists of a memory saving procedure for estimation of a linear regression model with two high-dimensional fixed effects.

In our own work, Carneiro et al. (2008), we were faced with the problem of estimating a linear regression model with 26 variables and two high-dimensional fixed effects (firm and worker) using a linked employer-employee data set with over 26 million observations. Implementation of the user-written commands discussed above in a computer with 8 Gigabytes of RAM and running Stata MP for Windows failed because of memory limitations. However, using a simple to implement iterative procedure we were able to estimate the regression coefficients and the fixed effects themselves. The approach is computationally intensive but it has the advantage of imposing minimal memory requirements. In this paper we present a detailed discussion of the method proposed in Carneiro et al. (2008) and show how it can be extended to non-linear models and possibly to applications with more than two high-dimensional fixed effects.

2 The Linear Regression Model

2.1 One fixed effect

To start with consider the conventional linear regression model setup:

$$y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_k x_{ki} + \epsilon_i \tag{1}$$

or, more compactly,

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}.\tag{2}$$

Application of the least squares method results in a set of equations (the normal equations) given below:

$$\begin{bmatrix} \frac{\partial SS}{\partial \beta_1} = \sum_i x_{1i}(y_i - \beta_1 x_{1i} - \beta_2 x_{2i} - \dots - \beta_k x_{ki}) = 0\\ \frac{\partial SS}{\partial \beta_2} = \sum_i x_{2i}(y_i - \beta_1 x_{1i} - \beta_2 x_{2i} - \dots - \beta_k x_{ki}) = 0\\ \dots\\ \frac{\partial SS}{\partial \beta_k} = \sum_i x_{ki}(y_i - \beta_1 x_{1i} - \beta_2 x_{2i} - \dots - \beta_k x_{ki}) = 0 \end{bmatrix}$$
(3)

²Apparently a2reg should only be used when there are no nested fixed effects.

³

These equations have a closed form solution, the least squares estimator, given by the well-known formula:

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y} \ . \tag{4}$$

However, the above formula is one of several alternatives to find the solution to (3). For instance, we can solve for $\hat{\beta}$ using a partitioned iterative algorithm. An example of such algorithm is shown below:

- Initialize $\beta_1^{(0)},\beta_2^{(0)},...,\beta_k^{(0)}$
- Solve for $\beta_1^{(1)}$ as the solution to $\frac{\partial SS}{\partial \beta_1} = \sum_i x_{1i}(y_i \beta_1 x_{1i} \beta_2^{(0)} x_{2i} \dots \beta_k^{(0)} x_{ki}) = 0$
- Solve for $\beta_2^{(1)}$ as the solution to $\frac{\partial SS}{\partial \beta_2} = \sum_i x_{2i}(y_i \beta_1^{(1)}x_{1i} \beta_2 x_{2i} \dots \beta_k^{(0)}x_{ki}) = 0$
- ...
- Repeat until convergence.

Algorithms such as this are discussed in Smyth (1996). This algorithm is known as the "zigzag" or full Gauss-Seidel algorithm. According to Smyth this algorithm produces a stable but slow iteration depending on the correlation of the parameters. In this particular case use of an iterative algorithm to solve the normal equations is highly inefficient. However, we should note that this implementation has the advantage of not requiring the explicit calculation of the inverse of the $\mathbf{X}'\mathbf{X}$ matrix.

Consider now what happens if we include a set of dummy variables to account for a fixed effect in the regression. In that case,

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \mathbf{D}\boldsymbol{\alpha} + \boldsymbol{\epsilon} \tag{5}$$

where \mathbf{Z} is the matrix of explanatory variables with $N \times k$ dimension and \mathbf{D} is the $N \times G_1$ matrix of "dummy" variables. Now, we can write the normal equations as:

$$\begin{bmatrix} \mathbf{Z'Z} & \mathbf{Z'D} \\ \mathbf{D'Z} & \mathbf{D'D} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{Z'Y} \\ \mathbf{D'Y} \end{bmatrix}$$
(6)

which can be arranged to show

$$\begin{bmatrix} \mathbf{Z}'\mathbf{Z}\beta + \mathbf{Z}'\mathbf{D}\alpha = \mathbf{Z}'\mathbf{Y} \\ \mathbf{D}'\mathbf{Z}\beta + \mathbf{D}'\mathbf{D}\alpha = \mathbf{D}'\mathbf{Y} \end{bmatrix} .$$
(7)

Solving each set of equations independently yields,

$$\begin{bmatrix} \beta = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Y} - \mathbf{D}\alpha) \\ \alpha = (\mathbf{D}'\mathbf{D})^{-1}\mathbf{D}'(\mathbf{Y} - \mathbf{Z}\beta) \end{bmatrix}.$$
(8)

The above partition of the normal equations suggests a convenient iteration strategy. To obtain the exact least squares solution one can simply alternate between estimation of β and estimation of α . But the key thing to note is that we no longer have to worry about the dimensionality of **D**. The expression $(\mathbf{D'D})^{-1}\mathbf{D'}$ used on the estimation of α translates into a simple group average of the residuals of the regression of Y on Z. On the other hand, the expression $\mathbf{D}\alpha$ that shows up on the equation for β is a column vector containing all the elements of α . Estimation of β consists of a simple linear regression of a transformed **Y** on **Z**. In our implementation instead of transforming Y we will keep Y as the dependent variable and add $\mathbf{D}\alpha$ as an additional covariate. When the estimation procedure converges the coefficient on $\mathbf{D}\alpha$ must equal one and the vector $\mathbf{D}\alpha$ will contain all the estimated coefficients for the group dummy variables. With this approach we avoided inversion of a potentially large matrix that would be required if we had simply added **D** to the set of regressors. As an illustration of this approach we use the Stata data set *nlswork* and replicate the coefficient estimates obtained with the fixed effect estimator for the linear regression model as shown in page 400 of the [XT] Stata Manual version 10.

```
webuse nlswork
(National Longitudinal Survey. Young Women 14-26 years of age in 1968)
 generate age2 = age^2
(24 missing values generated)
. generate ttl_exp2 = ttl_exp^2
 generate tenure2 = tenure<sup>2</sup>
(433 missing values generated)
. generate double fe1=0
. local rss1=0
. local dif=1
. local i=0
. while abs('dif')>epsdouble() {
 2. quietly {
 3. regress ln_w age* ttl_exp* tenure* not_smsa south fe1
 4. local rss2='rss1'
 local rss1=e(rss)
 local dif='rss2'-'rss1'
 7. capture drop yhat
 8. predict double yhat
 9. generate double temp1=ln_w-yhat+_b[fe1]*fe1
 10. capture drop fe1
 11. egen double fe1=mean(temp1), by(idcode)
 12. capture drop temp1
 13. local i='i'+1
 14. }
```

15. }						
. di "Total Nu Total Number o	umber of Itera of Iterations	ations> ' > 131	"'i'			
. regress ln_w	w age* ttl_exp	o* tenure* 1	not_smsa	south fe	1	
Source	SS	df	MS		Number of obs	= 28093 = 7007 50
Model Residual	4437.88225 1976.12232	9 493 28083 .070	.098028 0367209		Prob > F R-squared	= 0.0000 = 0.6919
Total	6414.00457	28092 .228	8321393		Adj R-squared Root MSE	= 0.6918 = .26527
ln_wage	Coef.	Std. Err.	t	P> t	[95% Conf.	Interval]
age	.0359987	.0024266	14.83	0.000	.0312424	.0407549
age2	000723	.0000394	-18.34	0.000	0008002	0006457
ttl_exp	.0334668	.0016297	20.53	0.000	.0302724	.0366611
ttl_exp2	.0002163	.0000873	2.48	0.013	.0000452	.0003873
tenure	.0357539	.0013781	25.94	0.000	.0330528	.038455
tenure2	0019701	.0000939	-20.98	0.000	0021541	0017861
not_smsa	0890108	.0036161	-24.62	0.000	0960985	0819231
south	0606309	.0033127	-18.30	0.000	067124	0541379
fe1	1	.0051626	193.70	0.000	.989881	1.010119
_cons	1.037301	.0340432	30.47	0.000	.9705747	1.104027

As implied earlier, the estimated coefficients are identical to those obtained using [xT] **xtreg** with the **fe** option. Note also that the regression includes an additional variable, **fe1**, with a coefficient of one. This variable was created during estimation and contains the estimates of the fixed effect.

2.2 Two fixed effects

Suppose now that instead of a single high-dimensional fixed-effect we have two highdimensional fixed-effects. That is, we now intend to estimate the following model

$$\mathbf{Y} = \mathbf{Z}\beta + \mathbf{D}_1\alpha + \mathbf{D}_2\gamma + \epsilon , \qquad (9)$$

where \mathbf{D}_1 is $N \times G_1$ and \mathbf{D}_2 is $N \times G_2$ and both G_1 and G_2 have high dimensionality. As discussed earlier, in this particular case estimation of the linear regression model is complicated. However, implementation of the partitioned algorithm discussed above is straightforward. Proceeding as we did before, we can solve the normal equations as

$$\begin{bmatrix} \beta = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Y} - \mathbf{D}_{1}\alpha - \mathbf{D}_{2}\gamma) \\ \alpha = (\mathbf{D}_{1}'\mathbf{D}_{1})^{-1}\mathbf{D}_{1}'(\mathbf{Y} - \mathbf{Z}\beta - \mathbf{D}_{2}\gamma) \\ \gamma = (\mathbf{D}_{2}'\mathbf{D}_{2})^{-1}\mathbf{D}_{2}'(\mathbf{Y} - \mathbf{Z}\beta - \mathbf{D}_{1}\gamma) \end{bmatrix}$$
(10)

Iterating between these sets of equations provides us with the exact least squares solution. All we have to do is compute several linear regressions with k explanatory variables and compute group means of residuals. To illustrate the approach we

modify the above algorithm and apply it to the ancillary data set that accompanies the **felsdvreg** command developed by Thomas Cornelissen. As before, we introduce $\mathbf{D}_1 \alpha$ and $\mathbf{D}_2 \gamma$ as additional regressors instead of subtracting them from the dependent variable.

. use felsdvs:	imul, clear							
. generate dou	uble temp=0							
. generate dou	uble fe1=0							
. generate dou	uble fe2=0							
. local rss1=0)							
. local dif=1								
. local i=0								
<pre>. while abs('c 2. quietly - 3. regress y 4. local rss 6. local dif 7. capture c 8. predict c 9. replace f 10. capture c 11. egen doul 12. replace f 13. capture c 14. egen doul 15. local i= 16. } 17. } . di "Total Number c</pre>	<pre>dif')>epsdoubl { y x1 x2 fe1 fe s2='rss1' s1=e(rss) f='rss2'-'rss1 drop res double res, re temp=res+_b[fe drop fe1 ble fe1=mean(t temp=res+_b[fe lrop fe2 ble fe2=mean(t 'i'+1 umber of Itera of Iterations</pre>	<pre>e() { 2 , s 1]*fe1 emp), 2]*fe2 emp), tions> 69</pre>	<pre>, nop by(i) , nop by(j) by(j)> " 4</pre>	promote promote				
. regress y x2	1 x2 fe1 fe2							
Source	SS	df		MS		Number of obs	=	100
Model Residual	9385.65759 2076.72508	4 95	234 21.	6.4144 860264		F(4, 95) Prob > F R-squared Adi R-squared	= = =	107.34 0.0000 0.8188 0.8112
Total	11462.3827	99	115.	781643		Root MSE	=	4.6755
у	Coef.	Std.	Err.	t	P> t	[95% Conf.	In	terval]
x1	1.029258	.162	224	6.34	0.000	.707203	1	.351314
x2	709482	.1604	068	-4.42	0.000	-1.02793		3910342
fe1	1	.0876	916	11.40	0.000	.8259101		1.17409
ie2	1	.0738	307	13.54 -/ 21	0.000	.8533281 -2 936051	1	.1406/1
	1.3301/9	.4138	501	-4.21	0.000	2.330001	-1	.004000

As hinted above, the estimates for the model coefficients are identical to the least squares results with all dummy variables included that is reported in Cornelissen (2008). We should note that the algorithm took 694 iterations to converge. This is one of the drawbacks of this approach. Fortunately, as discussed below, there is

substantial room for improvement. One obvious simplification is to sweep out one of the fixed effects by subtracting the group mean from all variables. By doing this we only need to deal with one fixed effect. This means that with minor modifications the code shown above can be used to estimate a model with 3 high dimensional fixed effects!

In Carneiro et al. (2008) we estimated a conventional Mincerian wage equation with two high dimensional fixed effects. Our data source is the *Quadros do Pessoal*, a mandatory employment survey collected yearly by the Portuguese Ministry of Labor and Social Security. The dataset comprised a total of 26,085,101 observations spanning from 1986 to 2005. In our estimation we wanted to account for firm, worker and year fixed effects. With 541,229 firms and 7,155,898 workers employing dummy variables for one of the fixed effects was not an option. But in the following example we show the output result of one specification with 26 covariates and the two high dimensional fixed effects estimated using the approach outlined above.

•	regress ln_1	real_hw schoo	ol* age* te	en* yr* fe1	fe2		
	Source	SS	df	MS		Number of obs F(28,26085072	=26085101)= .
	Model	7623399.13	28	272264.255		Prob > F	= 0.0000
	Residual	606248.16	26085072	.023241192		R-squared	= 0.9263
						Adi R-squared	= 0.9263
	Total	8229647.29	26085100	.315492265		Root MSE	= .15245
	ln_real_hw	Coef.	Std. Err	:. t	P> t	[95% Conf.	Interval]
	school2	.0086473	.0001819	47.54	0.000	.0082908	.0090038
	school3	.0326273	.0001993	163.69	0.000	.0322366	.0330179
	school4	.0482319	.000202	238.71	0.000	.0478359	.0486279
	school5	.1539935	.0002913	528.64	0.000	.1534226	.1545645
	school6	.2373886	.000246	964.89	0.000	.2369064	.2378708
	age	.0257784	.0000212	2 1216.00	0.000	.0257369	.02582
	agesq	0002749	2.77e-07	-993.19	0.000	0002754	0002744
	tenure	.0007567	9.38e-07	806.43	0.000	.0007549	.0007586
	tensq	-1.49e-06	2.59e-09	-573.79	0.000	-1.49e-06	-1.48e-06
	yr1987	.0504275	.0002091	241.22	0.000	.0500178	.0508373
	yr1988	.0659034	.0002072	2 318.08	0.000	.0654973	.0663095
	yr1989	.0800703	.0002049	390.75	0.000	.0796686	.0804719
	yr1991	.1895915	.0002019	938.86	0.000	.1891957	.1899872
	yr1992	.2346536	.0002009	9 1167.74	0.000	.2342597	.2350474
	yr1993	.2418796	.0002008	3 1204.65	0.000	.241486	.2422731
	yr1994	.2669475	.0001995	5 1338.10	0.000	.2665565	.2673386
	yr1995	.2793258	.0001964	4 1422.32	0.000	.2789409	.2797107
	yr1996	.2905584	.0001971	1474.26	0.000	.2901721	.2909447
	yr1997	.3383318	.0001937	1746.45	0.000	.3379521	.3387115
	yr1998	.3827365	.000194	1973.12	0.000	.3823563	.3831166
	yr1999	.4240073	.0001925	2203.00	0.000	.4236301	.4243845
	yr2000	.433117	.0001911	2266.20	0.000	.4327424	.4334916
	yr2002	.450015	.0001921	2342.35	0.000	.4496385	.4503916
	yr2003	.4531523	.0001913	3 2368.54	0.000	.4527773	.4535273
	yr2004	.4784224	.0001903	3 2514.21	0.000	.4780495	.4787954
	yr2005	.4877544	.0001886	3 2586.27	0.000	.4873848	.4881241
	fe1	1	.0001086	9212.19	0.000	.9997872	1.000213

fe2	1	.0001162 8602.81	0.000	.9997722	1.000228
_cons	6465547	.0004291 -1506.62	0.000	6473958	6457136

2.3 Estimation of the Standard Errors

So far the discussion has focused on estimation of the β coefficients. To provide the standard errors associated with the estimator of β we would need to estimate

$$V(\widehat{\beta}) = \sigma^2 \left(\mathbf{X}' \mathbf{X} \right)^{-1} , \qquad (11)$$

which again raises the problem of inverting the $\mathbf{X}'\mathbf{X}$ matrix. An alternative solution to estimate the elements of $V(\hat{\beta})$ is to use the known relation:

$$V(\hat{\beta}_j) = \frac{\sigma^2}{Ns_j^2(1 - R_{j,123...}^2)} .$$
(12)

where s_j^2 is the sample variance associated with the x_j variable and $R_{j,123...}^2$ is the coefficient of determination obtained from a regression of x_j on all other remaining explanatory variables. Estimation of σ^2 is easy because the final regression that provides the estimates of β has the correct sum of squared residuals (SSR). The remaining difficulty is the computation of the number of degrees of freedom associated with SSR. Knowing the dimension of \mathbf{X} is not sufficient because some of the coefficients for the fixed effects may not be identifiable. Abowd et al. (2002) developed an algorithm that can be used for determining the number of identifiable effects. The algorithm is used to identify "mobility groups" but it turns out that the total number of "mobility groups", which we denote by M, is also the number of non-identified coefficients. Hence, the degrees of freedom associated with SSR equal $N-k-G_1-G_2+M$. Computation of $R^2_{i,123...}$ is no longer a problem because we know how to estimate a model with high-dimensional effects. However, this may be time consuming because it would require estimation of a regression with two high dimensional fixed effects for each of the regressors. Fortunately, there is an alternative strategy that will produce faster results. Simply put, the idea consists of estimating the model in two steps. In the first step we expurgate the two fixed effects from all variables in the model. This involves running a linear regression of each individual variable on the high-dimensional effects only and storing the residuals. In the second step we run the regression of interest and the regressions required by (12) using the stored residuals of the variables obtained in the first step instead of the original variables. Because we are not dealing with the high-dimensional fixed effects, the

regressions in the second step are very easy to implement. One reason why this approach works well is because the calculations in the first step are relatively simple. We can see from (10) that in this case the algorithm involves only the computation of means. In the next example we use again the ancillary data set that accompanies the **felsdvreg** user-written command and show how to compute the (corrected) standard errors in a regression with two high-dimensional fixed effects. In the Stata code shown below we speed up the algorithm by following the common practice of sweeping one of the fixed effects by demeaning the variables.

. use felsdvsi	imul, clear							
. gen double t	cemp=0							
. gen double f	fe2=0							
. gen double 1	Lastfe2=0							
<pre>. gen double 1 . foreach var 2. quietly 4 3. local i=0 4. local dif 5. capture c 6. egen doub 7. replace 1 8. while abs 9. replace 1 10. capture c 11. egen doub 12. capture c 13. egen doub 14. replace 1 15. local dif 16. local i=1 17. } 18. noisily c 19. gen doub 20. } 21. } Total Number c </pre>	<pre>lastfe2=0 of varlist y [of varlist y [] l=1 lrop mean ole mean=mean(var'='var'-me s('dif')>epsfl lastfe2=fe2, n lrop mean ole mean=mean(drop fe2 ole fe2=mean(' temp=sum(reldi t=temp[_N]/_N i'+1 li "Total Numb le 'var'_res=' of Iterations of Iterations of Iterations</pre>	<pre>x1 x2 'var') an oat() opromo fe2), var'+m f(fe2, er of var'-f for y for x1</pre>	<pre>{ , by(te by(i) lastf Itera e2+me > 1 ></pre>	i) by(j) e2)), noj tions fo an 7 19	promote r 'var'	> " 'i'		
regress v re	s v1 res v2 r	es no	cons					
Source	SS X1_105 X2_1	df		MS		Number of obs F(2, 98)	=	100 24,39
Model Residual	1033.49121 2076.72505	2 98	516. 21.1	745607 910719		Prob > F R-squared	=	0.0000
Total	3110.21626	100	31.1	021626		Root MSE	=	4.6034
y_res	Coef.	Std.	Err.	t	P> t	[95% Conf.	In	terval]
x1_res x2_res	1.029258 709482	.1805 .1757	092 232	5.70 -4.04	0.000	.6710437 -1.058199	1 	.387473 3607649

Notice that the iterative procedure was much faster, taking less than 20 iterations for each variable. In this example there are 100 observations, 6 "mobility groups",

 $G_1 = 15$ and $G_2 = 20$ and two regressors, x_1 and x_2 . This means that there are 69 degrees of freedom. We can now proceed to calculate the standard errors associated with the coefficients. We will do it only for the variable x_1 because the calculations are similar for x_2 .

```
. local s2='e(rss)'/69
. quietly sum x1
. local N=r(N)
. local varx1=r(Var)
. local TSSx1='varx1'*('N'-1)
. quietly regress x1_res x2_res
. local RSSx1=e(rss)
. local R2x1=1-'RSSx1'/'TSSx1'
. local sebx1=sqrt('s2'/(('N'-1)*'varx1'*(1-'R2x1')))
. di "The standard error associated with x1 is --> 'sebx1'"
The standard error associated with x1 is --> .2151235245693864
```

The estimates for the coefficients of the fixed effects can be easily recovered by implementing the same iterative procedure to the residuals that are obtained when we subtract the effect of x_1 and x_2 from y.

```
. qui regress y_res x1_res x2_res, nocons
. gen double dum=y-_b[x1_res]*x1-_b[x2_res]*x2
. capture drop mean
. egen double mean=mean(dum), by(i)
. qui replace dum=dum-mean
. local i=0
. local dif=1
. while abs('dif')>epsfloat() {
  2. qui replace lastfe2=fe2, nopromote
 3. capture drop mean
 4. egen double mean=mean(fe2), by(i)
 5. capture drop fe2
 6. egen double fe2=mean(dum+mean), by(j)
 7. qui replace temp=sum(reldif(fe2,lastfe2)), nopromote
 8. local dif=temp[_N]/_N
 9. local i='i'+1
10. }
. di "Total Number of Iterations for fe2 --> " 'i'
Total Number of Iterations for fe2 --> 17
. quietly replace dum=dum-fe2
. egen double fe1=mean(dum), by(i)
. regress y x1 x2 fe1 fe2, nocons
     Source
                     SS
                              df
                                       MS
                                                       Number of obs =
                                                                            100
                                                       F( 4,
                                                                  96) =
                                                                          44.93
                                                       Prob > F
      Model
                 3887.8086
                               4
                                  971.952151
                                                                         0.0000
                2076.72505
                                                                         0.6518
   Residual
                              96
                                  21.6325526
                                                       R-squared
                                                       Adj R-squared =
                                                                         0.6373
       Total
                5964.53365
                             100 59.6453365
                                                       Root MSE
                                                                         4.6511
```

У	Coef.	Std. Err.	t	P> t	[95% Conf	. Interval]
x1	1.029258	.1715101	6.00	0.000	.6888133	1.369703
x2	7094819	.1701882	-4.17	0.000	-1.047303	3716611
fe1	1	.192855	5.19	0.000	.617186	1.382814
fe2	1	.0895005	11.17	0.000	.8223429	1.177657
	1					

We confirm that the estimated coefficients are correct by adding the variables fe1 and fe2 to the linear regression of y on x_1 and x_2 . As expected the estimated β coefficients are the correct ones and the coefficients associated with the variables fe1 and fe2 equal one.

Subtracting the influence of the fixed effects from each variable and working only with the residuals has some advantages compared to the process shown earlier entailing direct estimation of the full regression with all the fixed effects added. First, the simple regressions in step 1 are likely to converge at a faster rate. Second, it is possible to test different specifications of the model using only the residuals from the variables without the need to deal with the high dimensional fixed effects. And third, when dealing with very large data sets we can substantially reduce memory requirements because during step 1 we only need to load in memory the variable being handled and the group identifiers for the fixed effects. In fact we could do even better because the solution to the algorithm is performed independently across "mobility groups" meaning that it would be possible to load each "mobility group" in memory separately.

3 Extension to Non-linear Models

In this section we show that the iterative approach outlined earlier for the linear regression model can be extended to non-linear models. Lets first consider a typical Poisson regression model with expected value given by

$$E(y_i) = \lambda_i = \exp(\mathbf{x}'_i\beta) \ . \tag{13}$$

We know that the maximum-likelihood estimators are obtained as the solution to:

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^{i} y_i \mathbf{x}_i - \mathbf{x}_i \exp(\mathbf{x}_i'\beta)$$
$$= \sum_{i=1}^{i} (y_i - \exp(\mathbf{x}_i'\beta)) \mathbf{x}_i = \mathbf{0}$$

If one of the regressors is a dummy variable, say \mathbf{d}_j , then its estimated coefficient, say α_j , has a closed form solution given by:

$$\exp(\alpha_i) = \mathbf{d}'_i \mathbf{y} \times [\mathbf{d}'_i \exp([\mathbf{x}'_i \beta]_{(i)})]^{-1}$$
(14)

where the subscript (j) in the argument of the exponential function is used to note that \mathbf{d}_j is excluded from the argument. The above expression suggests a simple iterative strategy, much like the one that was used for the linear regression. We can alternate between estimation of a Poisson regression with k explanatory variables and calculation of the estimates for all the coefficients of the fixed effects using (14). These estimates can be kept in a single column vector. To show this algorithm at work we replicate the estimates of the Poisson model with fixed effects which appears in page 348 of the [XT] Stata Manual Version 10 as an illustration of the [XT] **xtpoisson** command with the option **fe**. Note that the example shown in the manual includes an exposure variable, **service**, which we incorporated in the algorithm.

```
. webuse ships, clear
. qui poisson acc op_75_79 co_65_69 co_70_74 co_75_79, exposure(service) nolog
. keep if e(sample)
(6 observations deleted)
. bys ship: egen sumy=total(acc)
. gen double off=0
. gen double temp=0
. local dif=1
. local 111=0
. local i=0
. while abs('dif')>epsdouble() {
  2. qui poisson acc op_75_79 co_65_69 co_70_74 co_75_79, off(off) nocon
 3. local 112='111'
 4. local ll1=e(ll)

    capture drop xb
    predict double xb, xb

 7. qui replace temp=xb-off+log(service), nopromote
 8. capture drop sumx
 9. bys ship: egen double sumx=total(exp(temp))
 10. qui replace off=log(sumy/sumx)+log(service), nopromote
 11. local dif='112'-'111'
 12. local i='i'+1
 13.
    }
. di "Total Number of Iterations --> " 'i'
Total Number of Iterations --> 103
. poisson acc op_75_79 co_65_69 co_70_74 co_75_79, nocon off(off) irr nolog
Poisson regression
                                                   Number of obs =
                                                                             34
                                                                           287.40
                                                    Wald chi2(4)
                                                                    =
Log likelihood = -68.280771
                                                   Prob > chi2
                                                                    =
                                                                          0.0000
    accident
                      IRR
                            Std. Err.
                                                 P>|z|
                                                            [95% Conf. Interval]
                                            z
```

1	9
Т	J

3.45

0.001

1.180941

1.826903

op_75_79

1.468831

.1634895

co_65_69	2.008003	.2086755	6.71	0.000	1.637969	2.46163
co_70_74	2.26693	.2814137	6.59	0.000	1.777342	2.89138
co_75_79 off	1.573695 (offset)	.3184952	2.24	0.025	1.058397	2.339875

Extending the algorithm to two fixed effects is straightforward. We use the same data set as before but instead of including the dummy variables for year of construction $(co_65_69, co_70_74 \text{ and } co_75_79)$ we treat the year of construction as a fixed effect, that is, we let the variable yr_con identify a second fixed effect. Now the algorithm is implemented without an exposure variable. For comparability purposes we first run a Poisson regression including dummy variables for both fixed effects.

```
. webuse ships, clear
. local dif=1
. xi: poisson acc op_75_79 i.yr_con i.ship, nolog
                                         (naturally coded; _Iyr_con_1 omitted)
(naturally coded; _Iship_1 omitted)
                   _Iyr_con_1-4
i.vr con
i.ship
                   _Iship_1-5
Poisson regression
                                                      Number of obs
                                                                       =
                                                                                   34
                                                      LR chi2(8)
                                                                        _
                                                                               475.45
                                                      Prob > chi2
                                                                        =
                                                                               0.0000
Log likelihood = -118.47588
                                                                        =
                                                                               0.6674
                                                      Pseudo R2
    accident
                     Coef.
                              Std. Err.
                                                    P>|z|
                                                                [95% Conf. Interval]
                                               z
    op_75_79
                  .2928003
                              .1127466
                                             2.60
                                                    0.009
                                                                 .071821
                                                                             .5137796
                   .5824489
  _Iyr_con_2
                               .1480547
                                                    0.000
                                                                .2922671
                                                                             .8726308
                                             3.93
                                .151248
                                                                .1663437
   _Iyr_con_3
                  .4627844
                                             3.06
                                                    0.002
                                                                             .7592251
  _Iyr_con_4
                 -.1951267
                               .2135749
                                           -0.91
                                                    0.361
                                                              -.6137258
                                                                             .2234724
                   1.79572
                              .1666196
                                           10.78
                                                    0.000
                                                               1.469151
                                                                             2.122288
    _Iship_2
    _Iship_3
                 -1.252763
                              .3273268
                                           -3.83
                                                    0.000
                                                              -1.894312
                                                                            -.6112142
                 -.9044563
                               .2874597
                                                    0.002
                                                              -1.467867
     _Iship_4
                                            -3.15
                                                                            -.3410457
     _Iship_5
                 -.1462833
                               .2351762
                                            -0.62
                                                    0.534
                                                              -.6072202
                                                                             .3146537
       _cons
                  1.308451
                               .1972718
                                            6.63
                                                    0.000
                                                                .9218049
                                                                             1.695096
```

. keep if e(sample)

(6 observations deleted)

. bys ship: egen sumy1=total(acc)

. bys yr_con: egen sumy2=total(acc)

. gen double off=0

- . gen double temp=0
- . gen double temp1=0
- . gen double temp2=0
- . gen double fe1=0
- . gen double fe2=0
- . local 111=0

. local i=0

. while abs('dif')>epsdouble() {

- 2. qui poisson acc op_75_79, nocons offset(off) nolog
- 3. local 112='111'
- 4. local ll1=e(ll)
- 5. capture drop xb

```
6. predict double xb, xb
  7. qui replace temp1=xb-off+fe2, nopromote
  8. capture drop sumx
  9. bys ship: egen double sumx=total(exp(temp1))
 10. qui replace fe1=log(sumy1/sumx), nopromote
 11. qui replace temp2=xb-off+fe1, nopromote
 12. capture drop sumx
 13. bys yr_con: egen double sumx=total(exp(temp2))
 14. qui replace fe2=log(sumy2/sumx), nopromote
 15. qui replace off=fe1+fe2
 16. local dif='112'-'111'
 17. local i='i'+1
 18. }
. di "Total Number of Iterations --> " 'i'
Total Number of Iterations --> 46
. poisson acc op_75_79, nocons offset(off) nolog
Poisson regression
                                                   Number of obs
                                                                   =
                                                                              34
                                                                   =
                                                   Wald chi2(1)
                                                                           18.60
Log likelihood = -118.47588
                                                   Prob > chi2
                                                                   =
                                                                          0.0000
                    Coef.
                                                           [95% Conf. Interval]
    accident
                            Std. Err.
                                                 P>IzI
                                           Z
    op_75_79
                  .2928003
                             .0678844
                                          4.31
                                                 0.000
                                                           .1597493
                                                                        .4258513
         off
                 (offset)
```

Application of the algorithm to Poisson regression was straightforward because we could find a closed form solution for the coefficients associated with the fixed effects. However, in most non-linear regression models the fixed effects do not have a closed-form solution. As shown in Guimarães (2004) models from the multinomial logit family such as logit, multinomial logit and conditional logit can all be estimated using Poisson regression meaning that the above algorithm could be used for these cases. The inexistence of a closed-form solution for the coefficients of the fixed-effects does not invalidate use of the "zigzag" algorithm but it requires the use of a numerical optimization routine to solve for the coefficients of the fixed-effects. This may slow down the algorithm considerably. As an example of this approach we show an application of the "zigzag" algorithm to estimate a negative binomial model with fixed effects.³

. webuse ships	s, clear							
. xi: nbreg ad i.ship	cc op_75_79 co _Iship_1-	_65_69 5) co_7	0_74 co_75 (naturally	5_79 i.sh: 7 coded;	ip, nol _Iship_	log 1 omit	tted)
Negative binor	nial regression	n			Number	of obs	s =	34
					LR chi	2(8)	=	41.45
Dispersion	= mean				Prob >	chi2	=	0.0000
Log likelihood	1 = -88.445258				Pseudo	R2	=	0.1898
accident	Coef.	Std.	Err.	z	P> z	[95%	Conf.	Interval]

³The "fixed effects negative binomial model" ([XT] **xtnbreg** with the **fe** option) is not equivalent to a negative binomial model with dummy variables added for fixed effects (see Guimarães (2008)).

	I					
op_75_79	.3324104	.328116	1.01	0.311	3106852	.975506
co_65_69	.8380919	.4378077	1.91	0.056	0199955	1.696179
co_70_74	1.658684	.4850461	3.42	0.001	.708011	2.609357
co_75_79	.8604224	.5955773	1.44	0.149	3068876	2.027732
_Iship_2	2.35359	.4701847	5.01	0.000	1.432045	3.275135
_Iship_3	-1.104561	.5214874	-2.12	0.034	-2.126657	082464
_Iship_4	9606946	.4905212	-1.96	0.050	-1.922098	.0007092
_Iship_5	077889	.4780747	-0.16	0.871	-1.014898	.8591201
_cons	.4230202	.5218569	0.81	0.418	5998006	1.445841
/lnalpha	7372302	.3814595			-1.484877	.0104166
alpha	.4784372	.1825044			.2265302	1.010471

Likelihood-ratio test of alpha=0: chibar2(01) = 60.06 Prob>=chibar2 = 0.000 . keep if e(sample) (6 observations deleted)

. egen id=group(ship)

. qui sum id

. local maxg=r(max)

. local dif=1

. local 111=0

. local i=0

. gen double off1=0

. gen double off2=0

. while abs('dif')>epsdouble() {

4. constraint define 1 [lnalpha]_cons='lna'

```
2. qui nbreg acc op_75_79 co_65_69 co_70_74 co_75_79, nocons offset(off1) nol > og
```

> og
3. local lna=log(e(alpha))

```
5. local 112='111'
  6. local ll1=e(ll)
  7. capture drop xb
  8. predict double xb, xb
  9. qui replace off2=xb-off1, nopromote
 10. forval j=1/'maxg' {
11. qui nbreg acc if id=='j', offset(off2) constraint(1) nolog
12. qui replace off1=_b[_cons] if e(sample), nopromote
 13. }
 14. local dif='112'-'111'
 15. local i='i'+1
 16. }
. di "Total Number of Iterations -->" 'i'
Total Number of Iterations -->122
. nbreg acc op_75_79 co_65_69 co_70_74 co_75_79, offset(off1) nocons nolog
Negative binomial regression
                                                           Number of obs =
                                                                                         34
                                                                                       76.36
```

Dispersion = mean	Wald chi2(4) =	= 76.36
Log likelihood = -88.445258	Prob > chi2 =	• 0.0000

accident	Coef.	Std. Err.	z	P> z	[95% Conf	. Interval]
op_75_79	.3324104	.3022232	1.10	0.271	2599362	.924757
co_65_69	.838092	.3036241	2.76	0.006	.2429996	1.433184
co_70_74	1.658684	.3187183	5.20	0.000	1.034008	2.283361
co_75_79	.8604225	.4997525	1.72	0.085	1190744	1.839919

off1	(offset)			
/lnalpha	7372302	.3659848	-1.454547	0199132
alpha	. 4784373	.1751007	.2335061	.9802837

Likelihood-ratio test of alpha=0: chibar2(01) = 88.87 Prob>=chibar2 = 0.000

Following an approach similar to the one used for the negative binomial model it should be possible to extend the algorithm to other non-linear models. The algorithm should work well with models that have globally concave log-likelihood functions such as the ones discussed here. Finally, we should note that with the "zigzag" algorithm we obtain the correct value of the log-likelihood function. This means that we can very easily implement statistical tests based upon likelihood ratios.

4 Conclusion

In this paper we successfully explored the implementation of the full Gauss-Seidel algorithm to estimate regression models with high-dimensional fixed-effects. The main advantage of this procedure is the ability to estimate a very large number of twoway fixed effects under minimal memory requirements. Generalizing the procedure to non-linear regression models is straightforward, in particular if there is a closedform solution for the fixed-effect. Extending the procedure to three-way fixed effects (for example: worker, firm, and match-specific effects) is also feasible, and is likely to prove very fruitful.

We do not claim, however, that our procedure is a superior estimation strategy. Quite to the contrary, the "zigzag" algorithm can be very slow and researchers should use more efficient estimation techniques whenever available. We know that the linear regression model with two high-dimensional fixed effects is estimated much more efficiently with the user-written command felsdvreg the same way that xtpoisson is the better approach to estimate a Poisson model with a single high dimensional fixedeffect. Nevertheless, there are circumstances when the "zigzag" algorithm may prove useful, namely when existing approaches do not work because of hardware (memory) limitations or when there are no other known ways of estimating the model. As we mentioned earlier the estimation strategy outlined in this paper is time consuming but it does have the advantage of imposing minimum memory requirements and being simple to implement. There are many ways to improve the speed of the algorithms discussed above and research is needed to figure out how to improve them. Some obvious things to try is to write more efficient Stata code (possibly Mata), to work on obtaining good starting values and to use some acceleration technique for

the algorithm. This should not be hard to accomplish because the estimates of fixed effects tend to converge monotonically and it should be possible to use the information from the last iterations to adjust the trajectory of the fixed effect estimates and thus obtain faster convergence. Anyway, such enterprise is beyond the scope of this paper.

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