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MEMORANDUM

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Portfolio Separation Properties of the Skew-Elliptical Distributions



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PORTFOLIO SEPARATION PROPERTIES OF THE SKEW-ELLIPTICAL DISTRIBUTIONS

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Abstract. The two fund separation property of the elliptical distributions is extended to the skew-elliptical and by adding a number of funds equalling the rank of the skewness matrix. Some elements of the generalization to *singular* extended skew-elliptical distributions are covered.

Key words and phrases: Portfolio separation, mutual fund theorem, stochastic dominance, singular extended skew-elliptical distributions.

MSC (2000): 91B28, 60E05, 49K45. *JEL classification:* G11, C61, D81, D53.

0 Introduction

The concept of portfolio separation, a.k.a. the mutual fund theorem, should be well known. Since Tobin [12], numerous works have generalized the result in terms of the preferences which admit separation (like Cass and Stiglitz [2] or even as recently as Schachermayer et al. [11], using a modern approach), or in terms of distributions (Ross [10]). The concept of risk measures falls somewhat in between, see e.g. this author [6] and independently, De Giorgi et al. [7].

This note extends the results of Owen and Rabinovitch [9] and Chamberlain [3], who point out that the elliptical (also frequently referred to as «elliptically contoured») distributions admit two fund separation. It will turn out that a similar result holds for the skew-elliptical class (Branco and Dey [1] and Díaz-García and González-Farías [4]), at the expense of requiring an additional number of funds corresponding to the rank of the skewness matrix. The latter introduce the wider singular extended skew-elliptical (SESE) class, and one of these generalizations will be covered herein. We shall restrict ourselves to the single-period discrete time case. Using this author's refinement [5] of the approach given by Khanna and Kulldorff [8], there will be a continuous-time analogue if the probability law is infinitely divisible (hence the discrete-time setup is more general in terms of probability distributions).

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The result

1 The result

Consider a single period investment in a numéraire (enumerated with a zero) returning Y_0 per monetary unit invested, and another p investment opportunities with returns vector $Y_0\mathbf{1} + \mu + \mathbf{Y}$, so that the return with investments \boldsymbol{u} in the p opportunities and $w - \boldsymbol{u}^{\dagger}\mathbf{1}$ (where wis initial wealth) in the numéraire, will be

$$X = wY_0 + \boldsymbol{u}^{\dagger}(\boldsymbol{\mu} + \boldsymbol{Y}), \tag{1}$$

(where the $\langle * \rangle$ superscript denotes transposition). The market will be assumed free of arbitrage opportunities and of redundant investment opportunities (having removed the latter from the market).

The probability distribution of $\mu + Y$ will be considered conditional on Y_0 – therefore, we can (and will) without loss of generality assume $Y_0 = 0$ (or, for that matter, a risk-free return). μ will be a location parameter, enabling us to assume location at zero in the representation to follow – note however, that we do not assume finite moments of any order.

Recall that an elliptical (a.k.a. elliptically contoured) random variable Z, has characteristic function of the form $e^{-i\theta^{\dagger}\delta}\psi(\theta^{\dagger}M\theta)$, where the matrix M is positive definite. The underlying spherical distribution (i.e. $M^{-1/2}(Z - \delta)$) can then be written as a mixture RS of a positive radial variable R, and S which is independent and uniform on the sphere. A singular elliptical distribution in the sense of [4], is obtained by relaxing the requirement to positive semidefinite M. Therein, it is assumed that R is absolutely continuous, but an approximation argument will allow for general R.

This paper does only to a limited extent use singular properties covered by [4], but will utilize their multivariate generalization of the case treated in [1]. Following their notation, one takes as starting point a singular elliptical vector $\boldsymbol{E} = (\boldsymbol{E}_1^{\dagger}, \boldsymbol{E}_2^{\dagger})^{\dagger}$ located at $\boldsymbol{\delta} = \boldsymbol{0}$ and with associated matrix $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Delta} \end{pmatrix}$, and where the marginals \boldsymbol{E}_1 and \boldsymbol{E}_2 (*p*-vector and *q*-vector, respectively) have associated positive semidefinite matrices $\boldsymbol{\Sigma} \in \mathbf{R}^{p \times p}$ and $\boldsymbol{\Delta} \in \mathbf{R}^{q \times q}$ – observe that each \boldsymbol{E}_i is allowed intra-dependent components. Now for arbitrary non-random $\boldsymbol{\mu} \in \mathbf{R}^p$, $\boldsymbol{\nu} \in \mathbf{R}^q, \boldsymbol{D} \in \mathbf{R}^{q \times p}$, then

 $|\boldsymbol{\mu} + \boldsymbol{E}_1| \boldsymbol{D} \boldsymbol{E}_1 + \boldsymbol{E}_2 - \boldsymbol{\nu} \ge \mathbf{0}$ (component-wise inequality, i.e. positive orthant)

has the singular vector-variate skew-elliptical distribution. In [4], this is parametrized as $\text{SESE}_{r}^{(p)}(q, k_1, \mu, \Sigma, k, D, \nu, \Delta, h_r^{(p)})$ where r, k and k_1 are the ranks of Σ , Δ and $\Delta + D\Sigma D^{\dagger}$, respectively, and $h_r^{(p)}$ denotes the density generating function with respect to some appropriate Hausdorff measure (which is not unique – however, the results won't depend on the choice). We remark that integrability assumptions are not needed, despite the literature's common use of terms like e.g. covariance matrix.

We shall assume $\mu + Y$ to have such a distribution. Then Y belongs to the same class, except with location μ replaced by null. In order to ensure absence of arbitrage and of redundant investment opportunities, we shall assume Σ positive definite (so that in particular, r = p); the only «singular» property left then is a possible rank-deficiency of Δ . We can adapt the following special case from [4, Theorem 5.1]:

References

LEMMA. Suppose that Y is absolutely continuous and distributed

$$\boldsymbol{Y} \sim \text{SESE}_p^{(p)}(q, k_1, \boldsymbol{0}, \boldsymbol{\Sigma}, k, \boldsymbol{D}, \boldsymbol{\nu}, \boldsymbol{\Delta}, h_p^{(p)}),$$
(2)

where Σ is positive definite and $h_p^{(p)}$ is the density generating function with respect to pdimensional Lebesgue measure. Then, for any non-random non-null p-vector u:

$$\boldsymbol{u}^{\dagger}\boldsymbol{Y} \sim \text{SESE}_{1}^{(1)}(\boldsymbol{q}, \boldsymbol{k}_{1}, \boldsymbol{0}, \boldsymbol{u}^{\dagger}\boldsymbol{\Sigma}\boldsymbol{u}, \text{rank}(\boldsymbol{\Delta}_{\boldsymbol{u}}), \boldsymbol{D}_{\boldsymbol{u}}, \boldsymbol{\nu}, \boldsymbol{\Delta}_{\boldsymbol{u}}, \boldsymbol{h})$$
 (3)

where $h = h_1^{(1)}$ is a univariate density-generating function, and

$$D_{u} = \frac{1}{u^{\dagger} \Sigma u} D \Sigma u, \qquad \Delta_{u} = \Delta + D \Sigma D^{\dagger} - D_{u} (u^{\dagger} \Sigma u) D_{u}^{\dagger}.$$
(4)

Recalling that non-absolutely continuous components in the underlying radial distribution can be recovered by approximation, we then have the following:

THEOREM. Assume the market (1) with the returns distributed according to (2), where Σ is positive definite. Suppose the agents rank portfolios according to first-order stochastic dominance of the return. Then we have $2 + \operatorname{rank}(D)$ fund separation. Furthermore, under the additional constraint of $\mathbf{u}^{\dagger}\mathbf{1} = w$ (i.e. the absence of opportunity to invest in the («safe») numéraire), we have $1 + \operatorname{rank}(D^{\dagger}, \mathbf{1}^{\dagger}\Sigma^{-1})$ fund separation.

Proof. We observe from (4) that the distribution (3) depends on \boldsymbol{u} only through $\sqrt{\boldsymbol{u}^{\dagger}\boldsymbol{\Sigma}\boldsymbol{u}} \in \mathbf{R}_{+}$ and $\boldsymbol{D}\boldsymbol{\Sigma}\boldsymbol{u} \in \mathbf{R}^{q}$. For given values Q > 0 and $Q\boldsymbol{q} \in \mathbf{R}^{q}$ of these, the agent will

$$\max_{\boldsymbol{u}} \boldsymbol{\mu}^{\dagger} \boldsymbol{u} \quad \text{subject to} \quad \boldsymbol{u}^{\dagger} \boldsymbol{\Sigma} \boldsymbol{u} = Q^{2}, \quad \boldsymbol{D} \boldsymbol{\Sigma} \boldsymbol{u} = Q \boldsymbol{q}$$

or equivalently, putting $\boldsymbol{v} = \boldsymbol{\Sigma} \boldsymbol{u}, \, \boldsymbol{a} = \boldsymbol{\mu} \boldsymbol{\Sigma}^{-1}$

$$\max_{\boldsymbol{a}} \boldsymbol{a}^{\dagger} \boldsymbol{v} \qquad \text{subject to} \quad \boldsymbol{v}^{\dagger} \boldsymbol{\Sigma}^{-1} \boldsymbol{v} = Q^{2}, \quad \boldsymbol{D} \boldsymbol{v} = Q \boldsymbol{q}$$

where for the case without safe investment opportunity, augment with the additional constraint $\mathbf{1}^{\dagger} \boldsymbol{u} = (\mathbf{1}^{\dagger} \boldsymbol{\Sigma}^{-1}) \boldsymbol{v} = \boldsymbol{w}$. Now the constraints $\boldsymbol{D} \boldsymbol{v} = Q \boldsymbol{q}$ form rank (\boldsymbol{D}) linear equations in \boldsymbol{v} . Rewriting these constraints – including $\mathbf{1}^{\dagger} \boldsymbol{\Sigma}^{-1} \boldsymbol{v} = \boldsymbol{w}$ if appropriate – into $\boldsymbol{D} \boldsymbol{v} = \boldsymbol{q}$ where \boldsymbol{D} has full rank, the proof is now a standard procedure: The associated Lagrangian becomes

$$oldsymbol{a}^{\dagger}oldsymbol{v} - oldsymbol{\lambda}^{\dagger}oldsymbol{\check{D}}oldsymbol{v} - \Lambdaoldsymbol{v}^{\dagger}oldsymbol{\Sigma}^{-1}oldsymbol{v},$$

which is stationary when $\boldsymbol{a} - \boldsymbol{\lambda}^{\dagger} \boldsymbol{\check{D}} = 2\Lambda \boldsymbol{\Sigma}^{-1} \boldsymbol{v} = 2\Lambda \boldsymbol{u}$. To complete the proof, we merely need to address degeneracies: First, if the constraint qualification fails (where the ellipsoid $\boldsymbol{v}^{\dagger} \boldsymbol{\Sigma}^{-1} \boldsymbol{v} = Q^2$ is tangent to one of the hyperplanes), the solution is obtained as a limiting case, and spanned by the rows of $\boldsymbol{\check{D}}$. Finally, the case $\Lambda = 0$ is only possible when \boldsymbol{a} is spanned by the rows of $\boldsymbol{\check{D}}$, and the one fund saved this way will be replaced by an additional orthogonal vector in order to achieve the desired dispersion Q^2 (since no risk aversion is assumed).

Observe that the result reduces to three-fund separation for the setup of Branco and Dey [1] (who restrict their analysis to D being a vector), and that by putting D = 0 we recover the Owen and Rabinovich [9] two-fund separation property as a corollary.

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