Lan, Hong; Meyer-Gohde, Alexander

Working Paper

Existence and uniqueness of perturbation solutions to DSGE models

SFB 649 Discussion Paper, No. 2012-015

Provided in Cooperation with:
Collaborative Research Center 649: Economic Risk, Humboldt University Berlin

Suggested Citation: Lan, Hong; Meyer-Gohde, Alexander (2012) : Existence and uniqueness of perturbation solutions to DSGE models, SFB 649 Discussion Paper, No. 2012-015, Humboldt University of Berlin, Collaborative Research Center 649 - Economic Risk, Berlin

This Version is available at:
https://hdl.handle.net/10419/56743

SFB 649 Discussion Paper 2012-015

Existence and Uniqueness of Perturbation Solutions to DSGE Models

Hong Lan*
Alexander Meyer-Gohde*

* Humboldt-Universität zu Berlin, Germany

This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

http://sfb649.wiwi.hu-berlin.de
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin
Spandauer Straße 1, D-10178 Berlin

Existence and Uniqueness of Perturbation Solutions to DSGE Models *

Hong Lan † Alexander Meyer-Gohde§

This Version: February 14, 2012

Abstract

We prove that standard regularity and saddle stability assumptions for linear approximations are sufficient to guarantee the existence of a unique solution for all undetermined coefficients of nonlinear perturbations of arbitrary order to discrete time DSGE models. We derive the perturbation using a matrix calculus that preserves linear algebraic structures to arbitrary orders of derivatives, enabling the direct application of theorems from matrix analysis to prove our main result. As a consequence, we provide insight into several invertibility assumptions from linear solution methods, prove that the local solution is independent of terms first order in the perturbation parameter, and relax the assumptions needed for the local existence theorem of perturbation solutions.

JEL classification: C61, C63, E17

Keywords: Perturbation; matrix calculus; DSGE; solution methods; Bézout theorem; Sylvester equations

*We are grateful to Michael Burda and Monique Ebell, as well as participants of research seminars at the HU Berlin for useful comments, suggestions, and discussions. This research was supported by the DFG through the SFB 649 "Economic Risk". Any and all errors are entirely our own.

†Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 1466; Email:lanhong@cms.hu-berlin.de

§Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 5720; Fax: +49-30-2093 5696; E-Mail: alexander.meyer-gohde@wiwi.hu-berlin.de

1 Introduction

Macroeconomists are increasingly using nonlinear methods to analyze dynamic stochastic general equilibrium (DSGE) models. One such method, the perturbation method initiated in macro DSGE modeling by Gaspar and Judd (1997), Judd and Guu (1997), and Judd (1998, ch. 13), has been successfully applied to a variety of applications with a few recent examples including the effects of time varying interest rates in the small open economy in Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011), to multi country real business cycle models in Kollmann, Kim, and Kim (2011), to the yield curve with recursive preferences and long run risks in Rudebusch and Swanson (2012). Intuitively, perturbation rests on the idea that successive differentiation of the equilibrium conditions will generate a set of equations that are sufficient to uniquely recover the coefficeints of the Taylor expansion of the policy function. As emphasized by Judd (1998) and Jin and Judd (2002), this unique recovery rests on solvability conditions that enable the implicit function theorem to guarantee the existence of a unique solution for the undetermined coefficients of higher order terms. Current perturbation analyses proceed under the tenuous assumption that these solvability conditions hold generically, as no general set of conditions has been proven. We fill this gap and provide conditions that guarantee the existence and uniqueness of solutions for DSGE perturbations of an arbitrarily high order. Specifically, our main result shows that assumptions on the linear approximation that are standard in the literature are already sufficient to ensure this existence and uniqueness.

We derive our main result by demonstrating that the set of linear equations in the undetermined coefficients to be solved for each order of approximation can be expressed as a generalized Sylvester equation.[1] Under the assumption of a unique saddle stable solution with respect to the closed unit circle for the homogenous component of the first order perturbation (i.e., (log-)linearization), a fac-

[1] Juillard and Kamenik (2004) and Kamenik (2005) provide a Sylvester representation for many of the unknown coefficients in their perturbation. We formulate the state space, see below, to extend this approach to all coefficients.

torization provided by a corollary of the generalized Bézout theorem relates the set of remaining unstable eigenvalues to a generalized eigenvalue problem with the saddle stable solution as an argument. With this factorization in hand, we relate the spectra of the matrix pencils associated with the leading and trailing coefficients in the generalized Sylvester equation at an arbitrary order to the spectrum of the stable solution and the remaining set of unstable eigenvalues. Due to the separation induced by the unique stable solution, the spectra of the pencils in the generalized Sylvester equation necessarily form a disjoint set (akin to a nonzero determinant in a standard linear equation system), ensuring the existence and uniqueness of solutions to the entire sequence of Sylvester equations.

Our result relies crucially on our ability to provide a closed form representation for the homogenous components of the Sylvester equations. The current standard approach to higher dimensional differentiation resorts to tensor notation,[2] with which Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), and others have shown that the equations to be solved at each order of approximation are linear. Unfortunately, the solvability conditions (that is, invertibility of these linear maps or coefficient matrices) change as the order of approximation changes leading Jin and Judd (2002) to conclude that this invertibility remains an open issue. Our results demonstrate that the choice of tensor notation can obfuscate underlying algebraic relationships:[3] the change in the coefficient matrices leading to the change in the solvability conditions as the analysis proceeds to higher orders of approximation is trivial. We uncover the pattern of the linear map at each order of approximation using the linear-algebraic preserving multidimensional calculus developed in Lan and Meyer-Gohde (2011), enabling the direct application of results from linear algebra described above. At each order, the lone trailing matrix in the Sylvester equation is a Kronecker power of the linear transition matrix of the state space. As the order increases, so too does the Kronecker power; but if the linear transition

[2]See Schmitt-Grohé and Uribe (2004) or Kim, Kim, Schaumburg, and Sims (2008). Gomme and Klein (2011) and Lombardo and Sutherland (2007) provide two exceptions, explicitly avoiding tensor notation in second order calculations. See Lan and Meyer-Gohde (2011) for further discussion.

[3]Gomme and Klein (2011) have argued that deriving perturbation solutions with standard linear algebra increases the transparency of the technique, we extend this idea using our multidimensional mechanical system of differentiation for arbitrarily high orders of approximation and demonstrate that maintaining standard linear algebraic structures enables the derivation of additional analytic results.

matrix is stable with respect to the closed unit circle, so too is an arbitrary Kronecker power of the matrix stable with respect to the closed unit circle. With all other coefficients in the homogenous part of the linear map remaining unchanged at each order, the task of deriving general solvability conditions is greatly reduced.

We construct the Taylor series approximation of the policy function with these uniquely solvable coefficients, proving that the commonly used numerical procedure of successive differentiating the equilibrium conditions of a smooth model uniquely recovers a Taylor approximation. Jin and Judd (2002) provide a local existence theorem for solutions to stochastic nonlinear DSGE models—and hence such Taylor approximations—using an implicit function theorem for Banach spaces, our factorization result of the matrix quadratic equation allows us to eliminate their solvability assumption.[4] Anderson, Levin, and Swanson (2006) show that under the assumption of analyticity of the true policy function, an n'th order perturbation is a global solution in a rigorous sense (inside the Taylor series's domain of convergence). Under their assumption of analyticity, which ensures that the true nonlinear policy function can be uniquely represented by its associated Talyor series within its domain of convergence, our result proves that successive differentiation of the equilibrium conditions is sufficient (in the limit) to recover the policy function.

We proceed to apply our results to several issues in linear and nonlinear perturbations. In numerous studies of linear approximations—from McCallum (1983), to Binder and Pesaran (1997), to Uhlig (1999), to Cho and Moreno (2011), the analyses proceed under the proviso that certain matrices are invertible to deliver a unique solution for the mapping from exogenous to endogenous variables. From our main result, the existence and uniqueness of solutions for these mappings is guaranteed as the existence and uniqueness of a saddle point stable solution for the homogenous component in the endogenous variables is assumed. We show how the factorization provided by the generalized Bézout theorem can be directly applied in their analyses to prove the missing invertibility conditions.

[4]Kim, Kim, Schaumburg, and Sims (2008) show that their assumption of bounded support for exogenous shocks is unnecessary if accuracy in probability instead of an absolute accuracy is sought.

King and Watson (1998) and Klein (2000) exploit the triangularity of their factorizations to prove the existence and uniqueness of their mapping from exogenous to endogenous variables line by line and we relate this scalar approach to the matrix approach with our factorization that allows us to accomplish this task in one step instead of recursively. Nonlinearly, several analyses have pointed out that the first derivative of the policy function with respect to the perturbation parameter ought to be zero. Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004) notably present this result in the context of the first derivative of the policy function with respect to the standard deviation of the shock. Both of these analyses assume the invertibility of the mappings they show to be homogenous, thus enabling our main result to complete their proofs by ensuring this necessary invertibility.

The rest of the paper is organized as follows. In section 2, we lay out a general nonlinear multivariate DSGE model and develop the *n*'th order approximation to its associated policy function by mechanical application of the differentiation rules provided by the linear-algebraic preserving multidimensional calculus and associated Taylor's Theorem. We begin in section 3 with the derivations of the terms associated with the endogenous state space in the first order perturbation, leading to a matrix quadratic problem familiar from the analysis of linear DSGE models. Here we relate the matrix quadratic problem to a generalized eigenvalue problem and introduce the factorization enabled by the generalized Bézout theorem allowing us to place two pencils on different sides of the unit circle. In section 4, we derive the remaining coefficients of the perturbation with a sequence of generalized Sylvester equations and derive our result on the existence and uniqueness of the solutions to these equations, using properties of the solutions to Sylvester equations and our separation of two matrix pencils from the previous section. The existence and uniqueness of these solutions is then linked to the local existence and approximation of the policy function. We apply our results in section 5 to some remaining invertibility assumptions in linear models and address the first order role of the perturbation parameter in nonlinear settings. Finally, section 6 concludes.

2 DSGE Problem Statement and Policy Function

In this section, we introduce the class of models we analyze and the policy function we examine as a solution. Our class of models is a standard system of (nonlinear) second order expectational difference equations compatible with Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot's (2011) Dynare or Anderson, Levin, and Swanson's (2006) PerturbationAIM. We will first present the model class followed by the solution form and then conclude with the Taylor approximation of the solution and the matrix calculus necessary to follow the derivations in subsequent sections.

2.1 Model Class

We analyze a family of discrete-time rational expectations models given by

$$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)] \tag{1}$$

the vector-valued function $f : \mathbb{R}^{ny} \times \mathbb{R}^{ny} \times \mathbb{R}^{ny} \times \mathbb{R}^{ne} \to \mathbb{R}^{ny}$ is assumed C^n, where n is the order of approximation to be introduced subsequently, with respect to all its arguments; $y_t \in \mathbb{R}^{ny}$ the vector of endogenous variables; and $\varepsilon_t \in \mathbb{R}^{ne}$ the vector of exogenous shocks. Note that we assume there are as many equations as endogenous variables.

Additionally, ε_t is assumed independently and identically distributed such that $E[\varepsilon_t] = 0$ and $E\left[{\varepsilon_t}^{\otimes[n]}\right]$ exists and is finite for all n up to and including the order of approximation to be introduced subsequently.[5]

[5]The notation ${\varepsilon_t}^{\otimes[n]}$ represents Kronecker powers, ${\varepsilon_t}^{\otimes[n]}$ is the n'th fold Kronecker product of ε_t with itself: $\underbrace{\varepsilon_t \otimes \varepsilon_t \cdots \otimes \varepsilon_t}_{n \text{ times}}$. For simulations and the like, of course, more specific decisions regarding the distribution of the exogenous processes will have to be made. Kim, Kim, Schaumburg, and Sims (2008, p. 3402) emphasize that distributional assumptions like these are not entirely local assumptions. Dynare (Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot 2011) assumes normality of the underlying shocks. PerturbationAIM (Anderson, Levin, and Swanson 2006) assumes mutual independence of the elements of ε_t.

2.2 Solution Form

Let the policy function be time invariant and ergodic, ruling out explosive and nonfundamental solutions, following Anderson, Levin, and Swanson (2006, p. 3) and let it take

$$z_t = \begin{bmatrix} y_{t-1} \\ \varepsilon_t \end{bmatrix} \in \mathbb{R}^{nz \times 1} \tag{2}$$

as its state vector, where $nz = ny + ne$.

As is usual in perturbation methods, we introduce an auxiliary parameter $\sigma \in [0,\ 1]$ to scale the uncertainty in the model. The "true" stochastic model under study corresponds to $\sigma = 1$ and $\sigma = 0$ represents the deterministic version of the model. Hence, the model has solutions indexed by σ

$$y_t = y(\sigma, z_t), \quad y : \mathbb{R}^+ \times \mathbb{R}^{nz} \to \mathbb{R}^{ny} \tag{3}$$

Time invariance and scaling uncertainty give

$$y_{t+1} = y^+(\sigma, z_{t+1}),\ z_{t+1} = \begin{bmatrix} y_t \\ \sigma\varepsilon_{t+1} \end{bmatrix} \in \mathbb{R}^{nz \times 1},\ y^+ : \mathbb{R}^+ \times \mathbb{R}^{nz} \to \mathbb{R}^{ny} \tag{4}$$

The notation, y and y^+, is adopted so that we can keep track of the source (through y_t and y_{t+1} respectively) of any given partial derivative of the policy function. The necessity of which can be seen by the fact that σ scales the ε_{t+1} in the z_{t+1} argument of y^+, but not that of ε_t in the z_t argument of y, and the the z_{t+1} argument of y^+ is itself a function of y through its dependance on y_t.

2.3 Taylor Series Approximation

We seek a Taylor approximation of the solution, (3), expanded around a nonstochastic steady state

Definition 2.1. *Nonstochastic Steady State*

Let $\overline{y} \in \mathbb{R}^{ny}$ be a vector such that

$$0 = f(\overline{y}, \overline{y}, \overline{y}, 0) \tag{5}$$

that is, the function f in (1) with all shocks, set to zero, and the policy function evaluated at the nonstochastic steady state

$$\overline{y} = y(0, \overline{z}) \tag{6}$$

where $\bar{z} = \begin{bmatrix} \overline{y}' & 0' \end{bmatrix}'$*, and all uncertainty regarding the future eliminated* $(\sigma = 0)$.

Note that the nonstochastic steady state need not necessarily be unique as we will admit models that possess unit root solution in the first order approximation.

Following general practice in the perturbation literature, we attempt to pin down the approximation of the unknown policy function (3) by successively differentiating (1) and solving the resulting systems for the unknown coefficients. Notice that, since f is a vector valued function, successive differentiation of f with respect to its vector arguments will generate a hypercube of partial derivatives. We use the method of Lan and Meyer-Gohde (2011) that adapts the structure of matrix derivatives defined in Vetter (1973) to differentiate conformably to the Kronecker product, by deriving partial derivatives from successive differentiation of f as two dimensional matrices. This allows us to avoid tensor notation—mitigating to some extent what Jin and Judd (2002) called a "nontrivial notational challenge"—and use standard linear algebra, operationalizing Gomme and Klein's (2011) goal of two dimensional derivatives to arbitrary orders of differentiation.

Definition 2.2. *Matrix Derivatives*

Let $A(B) : \mathbb{R}^{s\times 1} \to \mathbb{R}^{p\times q}$ *be a matrix-valued function that maps an* $s \times 1$ *vector* B *into an* $p \times q$ *matrix* $A(B)$*, the derivative structure of* $A(B)$ *with respect to* B *is defined as*

$$A_B \equiv \mathscr{D}_{B^T}\{A\} \equiv \begin{bmatrix} \frac{\partial}{\partial b_1} & \cdots & \frac{\partial}{\partial b_s} \end{bmatrix} \otimes A \tag{7}$$

where b_i *denotes* i*'th row of vector* B*,* ^{T} *indicates transposition.*[6] *Structures of* n*'th derivatives are thereby uniquely defined*

$$A_{B^n} \equiv \mathscr{D}_{(B^T)^n}\{A\} \equiv \left(\begin{bmatrix} \frac{\partial}{\partial b_1} & \cdots & \frac{\partial}{\partial b_s} \end{bmatrix}^{\otimes[n]} \right) \otimes A \tag{8}$$

This structure will make the presentation of the solution method more transparent—successive differentiation of f to the desired order of approximation is a mechanical application of the associating calculus

[6]Outside of the derivative structures, we use the apostrophe to indicate transposition.

Theorem 2.3. *A Multidimensional Calculus*

Given the vector $B \in \mathbb{R}^{s\times 1}$ *and the matrix-valued functions* $F : B \to \mathbb{R}^{p\times q}$, $G : B \to \mathbb{R}^{q\times u}$, $H : B \to \mathbb{R}^{u\times v}$ *and given the vector-valued function* $C : B \to \mathbb{R}^{u\times 1}$, $J : C \to \mathbb{R}^{p\times 1}$ *and the matrix-valued function* $A : C \to \mathbb{R}^{p\times q}$, *the following rules of calculus hold*

1. *Matrix Product Rule:* $\mathscr{D}_{B^T}\{FG\} = F_B(I_s \otimes G) + FG_B$, *where* I_s *is an* $s \times s$ *identity matrix*
2. *Matrix Chain Rule:* $\mathscr{D}_{B^T}\{A(C)\} = A_C\left(C_B \otimes I_q\right)$, *where* I_q *is an* $q \times q$ *identity matrix*
3. *Matrix Kronecker Product Rule:* $\mathscr{D}_{B^T}\{F \otimes H\} = F_B \otimes H + (F \otimes H_B)K_{q,vs}\left(I_s \otimes K_{v,q}\right)$, *where* $K_{q,vs}$ *and* $K_{v,q}$ *are* $qvs \times qvs$ *and* $qv \times qv$ *commutation matrices (Magnus and Neudecker 1979).*
4. *Vector Chain Rule:* $\mathscr{D}_{B^T}\{J(C)\} = A_C C_B$

Proof. See Lan and Meyer-Gohde (2011). □

By adapting the notation from Definition 2.2 and writing $y_{z^m\sigma^n}$ as the partial derivative, evaluated at the nonstochastic steady state, of y with respect to σ n times and with respect to z_t m times, we can then write the M-th order Taylor approximation of the policy function (3) using the following

Corollary 2.4. *An M-th order Taylor Approximation of (3) is written as*

$$y_t = \sum_{m=0}^{M} \frac{1}{m!}\left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{z^m\sigma^n}\sigma^n\right](z_t - \bar{z})^{\otimes[m]} \tag{9}$$

Proof. See Appendix. □

Here $\left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{z^m\sigma^n}\sigma^n\right]$ collects all the coefficients associated with the m'th fold Kronecker product of the state vector, z_t. For a given m, the sum over n gathers coefficients in powers of the perturbation parameter σ that correct the coefficients associated with the m'th fold Kronecker product of the state vector, z_t, for uncertainty up to the n-th order. This enables the useful classification of the contributions of uncertainty to the model as corrections to the Taylor series coefficients for uncertainty. That is, moving to a higher order of approximation, M, in (9) comprises two changes:

(i) adding a higher order partial derivative with respect to the state vector z_t and (ii) opening up all existing partial derivatives of current order to a higher order correction for uncertainty.[7] The change in moving from an $M-1$'th to M'th order approximation is

$$\sum_{m=0}^{M} \frac{1}{M!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\frac{1}{(M-m)!} y_{z^m \sigma^{M-m}} \sigma^{M-m} \right] (z_t - \overline{z})^{\otimes [m]} \tag{10}$$

Change (i) adds an M'th order partial derivative with a zeroth order correction for uncertainty (for $m = M$ above, $y_{z^m \sigma^{M-m}} \sigma^{M-m} = y_{z^m \sigma^0} \sigma^0 = y_{z^m}$) and from (ii) comes then additionally a first order uncertainty correction for $M-1$'th order partial derivatives with respect to z_t, a second order uncertainty correction for the $M-2$'th partial derivatives with respect to z_t and so on up to the M'th order correction for uncertainty in the constant. The uncertainty correction at a given order directly depends on the moments of future shocks at each order and so (ii) can be interpreted as successively opening each partial derivatives of current order up to higher moments in the distribution of future shocks, while (i) maintains the deterministic Taylor notion of moving to a higher order polynomial.

2.4 Systems of Equations for the Unknown Coefficients

The procedure can be outlined as follows.[8] Inserting the policy functions for y_t and y_{t+1}—equations, (3) and (4) respectively—into the model (1) yields

$$0 = E_t \left[f \left(y^+ \left(\sigma, \begin{bmatrix} y(\sigma, z_t) \\ \sigma \varepsilon_{t+1} \end{bmatrix} \right), y(\sigma, z_t), z_t \right) \right] \tag{11}$$

a function with arguments σ and z_t. At each order of approximation, we take the collection of derivatives of f from the previous order (for the first-order, we start with the function f itself) and

1. differentiate each of the derivatives of f from the previous order with respect to each of its arguments (i.e., σ and z_t)

2. evaluate the partial derivatives of f and of y at the nonstochastic steady state

3. apply the expectations operator and evaluate using the given moments

[7] We are grateful to Michael Burda for suggesting this interpretation.
[8] See Anderson, Levin, and Swanson (2006) for a similar outline.

4. set the resulting expression to zero and solve for the unknown partial derivatives of y.

The partial derivatives of y, obtained in step (4) at each order, constitute the missing partial derivatives for the Taylor approximation.

3 Solving and Factoring the Matrix Quadratic Equation

In this section, we deal with the only nonlinear equation that needs to be solved, a matrix quadratic equation. The existence and uniqueness of a saddle stable solution—stable with respect to the closed unit circle—for linear approximations is given by the existence and uniqueness of a stable solution the matrix quadratic solution.[9] This is well known, but we will need to make the standard assumptions that guarantee this solution. It has, however, not been appreciated in the DSGE literature that this stable solution can be used to deflate the matrix quadratic equation into a second generalized eigenvalue problem containing the unstable manifold. This factorization, a corollary of the generalized Bézout theorem that relates lambda-matrices, solvents and right division of matrix polynomials, splits the matrix quadratic problem into two disjoint (stable and unstable) components that will be crucial in later sections for ensuring the existence and uniqueness of solutions out to arbitrary orders of approximation.

3.1 Matrix Quadratic Equation

Following Corollary 2.4, the first order Taylor expansion of the policy function (3) around the non-stochastic steady state takes the form

$$y_t = \overline{y} + y_\sigma \sigma + y_z (z_t - \overline{z}) \tag{12}$$

The unknown coefficients are the partial derivatives y_σ and y_z.

Following the method outlined above, we differentiate f in (11) with respect to z_t to generate the

[9] E.g., Uhlig (1999).

equation that determines y_z,

$$\mathscr{D}_{z_t^T}\{f\} = f_{y^+} y_z^+ z_y y_z + f_y y_z + f_z \tag{13}$$

Evaluating this at the nonstochastic steady state and setting its expectation to zero yield

$$E_t\left[\mathscr{D}_{z_t^T}\{f\}\right]\Big|_{\overline{z}} = f_{y^+} y_z z_y y_z + f_y y_z + f_z = 0 \tag{14}$$

Postmultiplying the foregoing with z_y yields

$$f_{y^+}(y_z z_y)^2 + f_y y_z z_y + f_z z_y = 0 \tag{15}$$

This is a matrix quadratic equation in $y_z z_y$. Both Binder and Pesaran (1997) and Uhlig (1999) relate their solutions of linear models explicitly to such quadratic equations, other approaches, such as Blanchard and Kahn (1980) or Klein (2000), work instead directly with model equations by applying matrix factorizations to the model's coefficients. Uhlig (1999) constructs a class of diagonalizable solutions using generalized eigenvalue decomposition. While we dispense with the diagonalizabilty requirements and use a generalized Schur form following Klein (2000), making his assumptions[10] to solve (15), the generalized eigenvalue decomposition will be central for deflation of the quadratic problem given a unique stable solution. We will link our problem in (15), to which we will require a unique stable solution, to the general application of QZ to find the entire set of solutions (or 'solvents') to matrix quadratic problems in Higham and Kim (2000), who note that direct eigenvalue methods may fail to identify solutions to matrix quadratic equations even when they exist.

3.2 The Saddle Stable Solution

We will now construct the stable solution to our matrix quadratic problem (15) using the generalized Schur decomposition. The existence and uniqueness of the stable solution will be guaranteed by three assumptions standard in the literature. The first assumption is a regularity assumption that requires all the equations to be linearly independent, the second is the Blanchard and Kahn (1980) eigenvalue condition in our context requiring exactly as many stable eigenvalues as variables, and,

[10] Though we relax his stability assumption from the open to the closed unit circle, permitting unit-root solutions.

third, a singular version of the Blanchard and Kahn (1980) rank condition—Klein's (2000) translatability assumption—necessary to be able to construct a solution to (15) from the unique set of stable eigenvalues.

In order to construct its solution (or solvent), we need to formalize the definition of our problem as a matrix quadratic equation. Our analysis will proceed initially in the complex plane, but we show—with assumption 3.7—that the results carry over when we restrict solutions to be real valued, see also Klein (2000). We will begin by formalizing the notion of a matrix quadratic problem

Definition 3.1. *Matrix Quadratic Problem*

For f_{y^+}, f_y, and $f_z z_y \in \mathbb{R}^{ny\times ny}$, a matrix quadratic $M(X) : \mathbb{C}^{ny\times ny} \to \mathbb{C}^{ny\times ny}$ in matrix $X \in \mathbb{C}^{ny\times ny}$ is defined as

$$M(X) = f_{y^+}X^2 + f_y X + f_z z_y \tag{16}$$

A solution to the matrix quadratic (16) is called a solvent and is defined as

Definition 3.2. *Solvent of Matrix Quadratic*

A matrix $X \in \mathbb{C}^{ny\times ny}$ is a solvent of the matrix quadratic (16) if and only if $M(X) = 0$

A solvent of the matrix quadratic can be characterized alternatively via the deflating subspace of the associated block companion formulation or linearized pencil of (16), following Higham and Kim (2000)

Lemma 3.3. *Solvent Characterization via Linearization*

A matrix $X \in \mathbb{C}^{ny\times ny}$ is a solvent of the matrix quadratic (16)—i.e., $M(X) = 0$— if and only if

$$D\begin{bmatrix} I_{ny} \\ X \end{bmatrix} X = E\begin{bmatrix} I_{ny} \\ X \end{bmatrix}, \quad D = \begin{bmatrix} 0_{ny\times ny} & I_{ny} \\ f_{y^+} & 0_{ny\times ny} \end{bmatrix}, E = \begin{bmatrix} I_{ny} & 0_{ny\times ny} \\ -f_y & -f_z z_y \end{bmatrix} \tag{17}$$

where I_{ny} is an $ny \times ny$ identity matrix and $0_{ny\times ny}$ is an $ny \times ny$ matrix with all its entries being zero

Proof. See Higham and Kim (2000). □

We will construct solvents of (16) with the generalized Schur decomposition of the matrix pencil $P_{DE}(z) = Dz - E$, where we define a pencil and its spectrum via

Definition 3.4. *Matrix Pencil and Spectrum*

Let $P : \mathbb{C} \to \mathbb{C}^{n\times n}$ be a matrix-valued function of a complex variable; a matrix pencil. Its set of generalized eigenvalues or spectrum $\rho(P)$ is defined via $\rho(P) = \{z \in \mathbb{C} : \det P(z) = 0\}$.

Now we can apply Theorem 3 of Higham and Kim (2000) to recover the complete set of solvents of (16).

Theorem 3.5. *The Generalized Schur Decomposition and Solvents*

All solvents of $M(X)$ are given by $X = Z_{21}Z_{11}^{-1} = Q_{11}T_{11}S_{11}^{-1}Q_{11}^{-1}$, where[11]

$$Q^*EZ = T,\ \ Q^*DZ = S \tag{18}$$

is a generalized Schur decomposition with unitary Q and Z and upper triangular S and T, and where Q, Z, S, and T are partitioned as block 2×2 matrices with $ny \times ny$ blocks.

Proof. See Higham and Kim (2000). □

Our interest lies in the unique stable solvent and we will now proceed to the standard assumptions following Klein (2000) and their consequences for the set of solvents. King and Watson's (1998) solvability condition, adapted also as Klein's (2000, p. 1413) Assumption 4.3, requires the matrix pencil $P_{DE}(z) = Dz - E$ to be regular

Assumption 3.6. *Regularity Assumption*

There exists a $z \in \mathbb{C}$ such that $\det(Dz - E) \neq 0$: the matrix pencil $P_{DE}(z) = Dz - E$ is called regular

This assumption rules out a mundane source of singularity which leads to a general nonuniqueness of solvents of the matrix quadratic, (16), merely because the problem is ill specified—e.g., two equations are linearly dependent in the first-order approximation. If this condition were not to hold, the spectrum $\rho(P_{DE})$ would be the entire complex plane—see Golub and Loan (1996, p. 377).

[11] * denotes conjugate transposition.

With this assumption and any generalized Schur decomposition of $P_{DE}(z)$, the spectrum of the pencil $P_{DE}(z)$ is a finite set given by

$$\rho(P_{DE}) = \left\{ \begin{matrix} t_{ii}/s_{ii}, & s_{ii} \neq 0 \\ \infty, & \text{otherwise} \end{matrix} : i = 1, \ldots, 2ny \right\} \tag{19}$$

where s_{ii} and t_{ii} denote the i'th row and i'th column of S and T respectively. With the continuation to infinite generalized eigenvalues,[12] the set of generalized eigenvalues or spectrum has exactly $2ny$ elements.[13]

We will require the solvent to be stable with respect to the closed unit circle. From theorem 3.5, the eigenvalues of a solvent will be equal to the first ny pairs t_{ii}/s_{ii} (suitably extended to infinity as above). Thus, if there exists a unique solvent of the matrix quadratic (16), the Blanchard and Kahn (1980) eigenvalue condition must hold

Assumption 3.7. *Eigenvalue Count*

Of the $2ny$ *generalized eigenvalues of the matrix pencil* $P_{DE}(z) = Dz - E$*, there are exactly* ny *inside or on the unit circle, called stable. Consequently, there are exactly* ny *outside the unit circle, called unstable.*

As the pairs (s_{ii}, t_{ii}) can be arranged in any order, they can be arranged such that ny pairs with $|t_{ii}| \leqslant |s_{ii}|$, or stable eigenvalues, come first. The remaining ny pairs with $|t_{ii}| > |s_{ii}|$, or unstable eigenvalues, follow. As Klein (2000) also notes, with real valued matrices D and E in (17), complex eigenvalues will come in pairs and thus the 2×2 blocks on the diagonals of T and S in the real generalized Schur decomposition[14] would not change the method. Essentially, the possibility of a complex valued solution despite real valued coefficients is ruled out by the separation of the eigenvalues, which come in pairs with equal modulus when complex and are thus both either on one side or the other with an associated real valued solution, see also Uhlig's (1999) discussion. From assumption 3.7, the partitioning of each the four matrices, Q, Z, S and T as (2×2) blocks with

[12] See also Klein (2000, p. 1410).

[13] See J. E. Dennis, Traub, and Weber (1976, p. 835) or Golub and Loan (1996, p. 377), where the regularity in assumption 3.6 rules out the possibility that $s_{ii} = t_{ii} = 0$ for some i.

[14] See Golub and Loan's (1996) Theorem 7.7.2.

$(ny \times ny)$ blocks is conformable with the dimension of the two sets, stable and unstable, generalized eigenvalues. From theorem 3.5, the solvent associated with any generalized Schur decomposition for the matrix quadratic problem is given by $X = Z_{21}Z_{11}^{-1}$ and thus for us to be able to construct a solvent from the combination of stable eigenvalues, we impose following Klein's (2000, p. 1413) Assumption 4.5

Assumption 3.8. *Solvent Constructibility*

The upper right block Z_{11} is nonsingular

As the maximal number of solvents given our regularity assumption is given by the number of different possible combinations of eigenvalues respecting algebraic multiplicities,[15] if a solvent exists for a unique *ny* dimensional set of eigenvalues stable with respect to the closed unit circle then it is the only solvent whose eigenvalues satisfy the stability requirement.

Thus, under assumptions 3.6–3.8, there exists a unique stable solution to (16), which we summarize in the following

Theorem 3.9. *Existence of a Unique Stable Solvent*

There exists a unique solution of (15) with all its eigenvalues inside the closed unit circle (which we will call $y_z z_y$), if the associated linearized pencil is regular (assumption 3.6), has exactly ny stable eigenvalues—inside or on the unit circle (assumption 3.7), and if a generalized Schur decomposition with the ny stable eigenvalues order first admits a solvent (assumption 3.8).

Proof. By construction. □

We will reserve $y_z z_y$ for this unique stable solvent of (15).

[15] See J. E. Dennis, Traub, and Weber (1976), Higham and Kim (2000), or Higham and Kim (2001).

3.3 Factoring the Unstable Solutions

In the previous section, we made three standard[16] assumptions to deliver a unique stable solvent or solution to the matrix quadratic problem at the first order. This solvent is constructed using half (ny out of $2ny$) of the eigenvalues associated with the quadratic problem. We will now apply the generalized Bézout theorem to show that with a solvent (again, in our case the unique stable $y_z z_y$) in hand, the original matrix quadratic problem can be deflated to a generalized eigenvalue problem with all eigenvalues outside the unit circle (i.e., the remaining ny eigenvalues not used in the construction of $y_z z_y$). The generalized eigenvalue problem combines the coefficient matrices of the original quadratic problem with our unique stable $y_z z_y$ into a pencil with unstable eigenvalues, providing us with a means to factor the remaining eigenvalues as pencils involving our stable solution.

From, e.g., J. E. Dennis, Traub, and Weber (1976, p. 835) or Gantmacher's (1959a, p. 228) Theorem 4, the set of eigenvalues of all solvents of (16) are latent roots of the associated lambda-matrix

Definition 3.10. *Lambda-Matrix*

The lambda-matrix $M(\lambda) : \mathbb{C} \to \mathbb{C}^{n\times n}$ *(of degree two) associated with (16) is given by*

$$M(\lambda) \equiv M(\lambda I_n) = f_{y^+}\lambda^2 + f_y\lambda + f_z z_y \tag{20}$$

Its latent roots are values of λ *such that* $\det M(\lambda) = 0$.

The set of latent roots in (20) is identical to the set of eigenvalues of the generalized eigenvalue problem associated with the pencil formed by the matrices in the linearized version (17) of the quadratic problem

Lemma 3.11. *The matrix pencil* $P_{DE}(z) = Dz - E$ *is a linearization of the lambda-matrix (20), hence the latent roots of (20) coincides with the elements of the spectrum* $\rho(P_{DE})$

[16] See Klein (2000).

Proof. See, e.g., Davis (1981), Gohberg, Lancaster, and Rodman (1982, Ch. 7), or Tisseur and Meerbergen (2001). □

Thus, the set of eigenvalues of the pencil $P_{DE}(z)$, $\rho(P_{DE})$, is identical to the set of λ's such that $\det M(\lambda) = 0$.

We are now prepared to link lambda matrices and solvents through the generalized Bézout theorem, repeated in the Appendix, which states that a lambda matrix divided on the right by a binomial in a matrix has as a remainder the matrix polynomial associated with the lambda matrix evaluated at the matrix of the binomial. As noted by Gantmacher (1959a, Ch. 4) and repeated in Lancaster (1966), Davis (1981), Higham and Kim (2000), and Higham and Kim (2001), if this matrix in the binomial is a solvent of the matrix polynomial, the division is without remainder, yielding a factorization of the matrix polynomial. Our matrix polynomial is a matrix quadratic and can thus be factored as follows

Corollary 3.12. *As $y_z z_y$ is a solvent of (16), then (20) has the following factorization*

$$M(\lambda) = (\lambda f_{y^+} + f_{y^+} y_z z_y + f_y)(I_{ny}\lambda - y_z z_y) \tag{21}$$

Proof. Apply theorem A.1 in the Appendix to (16), set $A = y_z z_y$, and note that $M(y_z z_y) = 0$ as $y_z z_y$ is a solvent of $M(X)$. □

Note that the eigenvalues of the pencil $P_{DE}(z)$, $\rho(P_{DE})$, are given by λ's such that

$$\det(\lambda f_{y^+} + f_{y^+} y_z z_y + f_y) \det(I_{ny}\lambda - y_z z_y) = 0 \tag{22}$$

The latter determinant gives the eigenvalues associated with the solvent $y_z z_y$ and the former determinant gives a generalized eigenvalue problem in the coefficients of $M(X)$ and the solvent $y_z z_y$. We can now use assumption 3.7, the Blanchard and Kahn (1980) condition, on the number of eigenvalues to restrict the eigenvalues of the generalized eigenvalue problem $\det(z f_{y^+} + f_{y^+} y_z z_y + f_y) = 0$.

Proposition 3.13. *The eigenvalues of the matrix pencil $P_U(z) \equiv z f_{y^+} + f_{y^+} y_z z_y + f_y$ are contained entirely outside the closed unit circle.*

Proof. From assumption 3.7, there are exactly *ny* eigenvalues of the pencil $P_{DE}(z)$ inside or on the unit circle and exactly *ny* outside the unit circle. From lemma 3.11, then, there are exactly *ny* latent roots of $M(\lambda)$ inside or on the unit circle and exactly *ny* outside the unit circle. The *ny* eigenvalues of the pencil $P_S(z) \equiv I_{ny}z - y_z z_y$ are all inside or on the unit circle by theorem 3.9. Hence, the *ny* eigenvalues of $P_U(z)$ are the *ny* remaining latent roots of $M(\lambda)$, which must be outside the unit circle. □

So the latent roots of $M(\lambda)$ comprise the elements of $\rho(P_S)$—all inside or on the unit circle—and the elements of $\rho(P_U)$—all outside the unit circle. These two spectra are hence disjoint, having no element in common.

Furthermore, the regularity of $P_{DE}(z)$ in assumption 3.6 immediate transfers to both $P_U(z)$ and $P_S(z)$

Lemma 3.14. *The matrix pencils* $P_U(z) = z f_{y^+} + f_{y^+} y_z z_y + f_y$ *and* $P_S(z) = I_{ny}z - y_z z_y$ *are both regular.*

Proof. See Appendix. □

Both the regularity and disjointness of these spectra will be central to the solvability of the undetermined coefficients of perturbations of arbitrary order, to which we will turn in the next section.

Before we proceed, we can now complete the deterministic component of the first order solution. Given our unique stable $y_z z_y$, y_z solves

$$(f_y + f_{y^+} y_z z_y) y_z = -f_z \tag{23}$$

and the existence of its unique solution is summarized in the following

Proposition 3.15. *Under the assumptions of theorem 3.9,* y_z *uniquely solves (23).*

Proof. We need to prove the nonsingularity of the matrix $f_y + f_{y^+} y_z z_y$. This matrix is singular, $\det\left(f_y + f_{y^+} y_z z_y\right) = 0$, if and only if zero is an eigenvalue of the regular pencil $P_U(z) = z f_{y^+} + f_{y^+} y_z z_y + f_y$. From proposition 3.13, the eigenvalues of $P_U(z)$ are outside the unit circle and cannot be zero. □

The factorization provided by the generalized Bézout theorem ensures the nonsingularity of the leading coefficient matrix in (23). Thus, the deterministic component of the first order solution exists and is unique necessarily from the assumptions leading to a unique stable solution to the matrix quadratic equation. We will now extend this result to all the undetermined coefficients of perturbations of arbitrary order.

4 Existence and Uniqueness in Higher Order Perturbations

In this section, we solve for the unknown coefficients of a perturbation with an arbitrarily high order of approximation. A standard result in the literature, noted by Judd (1998, ch. 13), Jin and Judd (2002), Schmitt-Grohé and Uribe (2004) and others, is that the higher order terms of the Taylor expansion are solutions to linear problems taking the coefficients from lower orders as given. Jin and Judd (2002), however, have emphasized that the solvability of these linear systems is not a given and furthermore that the conditions that need to be fulfilled for solvability change with the order of approximation. While they conjecture the generic solvability at all orders, they conclude that this remains an open issue. We will provide conditions for the solvability of perturbation coefficients at all orders using the theorem of Chu (1987) on the existence of unique solutions to generalized Sylvester equations. Surprisingly, we show that the assumptions made in section 3 to guarantee the existence of unique stable transition matrix in the linear approximation are already sufficient to guarantee solvability. As a consequence, it follows that the unknown coefficients of a Taylor expansion of arbitrary order can be uniquely recovered through successive differentiation of the equilibrium conditions if there is a unique stable solvent to the matrix quadratic at first order. Additionally, our solvability results eliminate a key assumption in the local existence proof of Jin and Judd (2002) for stochastic perturbations, leaving only their bounded support assumption as potentially nonstandard.

4.1 Generalized Sylvester Equations

We generate the linear systems in the unknown coefficients of a perturbation as Sylvester equations for all coefficients at all orders. We construct the linear equations following the method outlined in section 2.4 by mechanical application of the multi-dimensional calculus developed in Lan and Meyer-Gohde (2011) to the equilibrium conditions. The linear algebraic structure reveals a generalized Sylvester equations with leading coefficients containing the unstable and trailing coefficients the stable components of the factorized matrix quadratic equation of the previous section.

The Sylvester form in the higher order perturbation literature is not an innovation, having been identified in previous studies. Aside from the identification of Sylvester equations in a second order context by, e.g., Kim, Kim, Schaumburg, and Sims (2008) or Gomme and Klein (2011), Juillard and Kamenik (2004) and Kamenik (2005) show explicitly that some of the unknown coefficients can be cast as Sylvester equations. To our knowledge, however, this is the first representation that takes this pattern to the limit, showing that all equations of an arbitrary order perturbation can be cast into Sylvester form. While our form is appear wasteful from the numerical perspective of most higher order perturbation analyses, it is precisely this form that enables our proof of the existence and uniqueness of solutions for these equations that numerical studies have taken for granted. Thus, this form is only need for the proof of the validity of the methods and with our results in hand, numerical studies can confidently ignore our form and operate on more efficient compositions.

The first order Taylor expansion that we began in the previous section is incomplete, we still need to determine the stochastic perturbation or first order uncertainty correction, y_σ. We differentiate f in (11) with respect to σ

$$\mathscr{D}_\sigma\{f\} = f_{y^+} y_z^+ z_y y_\sigma + f_{y^+} y_z^+ z_\varepsilon \varepsilon_{t+1} + f_{y^+} y_\sigma^+ + f_y y_\sigma \tag{24}$$

Evaluating the foregoing at the nonstochastic steady state, $\bar{z}$, and setting its expectation to zero yields

$$E_t\left[\mathscr{D}_\sigma\{f\}\right]\Big|_{\bar{z}} = f_{y^+} y_\sigma + (f_y + f_{y^+} y_z z_y) y_\sigma + f_{y^+} y_z z_\varepsilon E_t[\varepsilon_{t+1}] = 0 \tag{25}$$

A generalized Sylvester equation, taking the unique stable solution $y_z z_y$ as given from the previous

section.

For the unknown coefficients of second and higher orders, we successively differentiate (11) with respect to the state vector z_t and the perturbation parameter σ, evaluate the resulting expressions at the nonstochastic steady state and set their expectations equal to zero. This generates a set of generalized Sylvester equations similar to (25). We summarize this in the following

Lemma 4.1. *For all $j,i \in \mathbb{N}^0$ such that $j+i \geqslant 1$ except the case $j=1$ and $i=0$, the undetermined coefficients $y_{z^j\sigma^i}$ solve the following generalized Sylvester equation*

$$f_{y^+} y_{z^j\sigma^i} (z_y y_z)^{\otimes[j]} + \left(f_y + f_{y^+} y_z z_y\right) y_{z^j\sigma^i} + A(j,i) = 0 \tag{26}$$

where $A(j,i)$ is a function of known terms: coefficients from lower orders of approximation and given moments $E\left[{\varepsilon_t}^{\otimes[k]}\right]$, $k \leq i$.

Proof. See the Appendix. □

This representation provides an explicit formulation of the homogenous structure of the equations that the unknown coefficients of each order of approximation must fulfill,[17] which will facilitate the analysis of solvability using linear algebra. At each order, the leading matrix coefficients, f_{y^+} and $f_y + f_{y^+} y_z z_y$, remain unchanged and are formed by the coefficients of unstable factorization P_U of the matrix quadratic as detailed in proposition 3.13. The trailing matrix coefficient, $(z_y y_z)^{\otimes[j]}$, is a Kronecker power of the linear transition matrix of the state space and changes with the order of approximation.

That the trailing matrix changes with j is the source for the problematic dependence of the solvability conditions on the order of approximation identified by Jin and Judd (2002). Specifically, Jin and Judd (2002) first develop a deterministic perturbation, in z_t only, and perturb stochastically, with respect to σ. They point out that the change in the solvability conditions occurs only in a change in the order of approximation in the deterministic perturbation. This is reflected in our

[17] For example, when $(j=0, i=1)$, (26) reduces to (25). In the Appendix, we provide the detailed derivations for the second order Taylor expansion, which yields the three generalized Sylvester equations of (26) with $(j=2, i=0)$, $(j=1, i=1)$ and $(j=0, i=2)$ for the unknown coefficients y_{z^2}, $y_{z\sigma}$ and y_{σ^2} respectively.

Sylvester equations in that the only change occurs with j, the order of the state vector z_t, and that the coefficients are independent of i,[18] the order of the (stochastic) perturbation parameter.

We now proceed to establish conditions under which the solution to (26) exist and are unique. This is crucial for relating assumptions 3.6–3.8 to the characterization of the general solvability condition for the generalized Sylvester equations that follows in the next section.

4.2 Existence and Uniqueness

In this section, we will appeal to Chu's (1987) necessary and sufficient conditions for the existence and uniqueness of solutions to generalized Sylvester equations and prove that they are fulfilled for all our equations in lemma 4.1 as a direct consequence of the existence of the unique stable solution to the matrix quadratic equation (15). Thus, the three standard assumptions—our assumptions 3.6–3.8—from linear analyses to this end are already sufficient to ensure the existence of unique solutions for all unknown coefficients of perturbations of arbitrary order.

The necessary and sufficient conditions proposed by Theorem 1 of Chu (1987) requires the two matrix pencils formed by the leading and trailing matrix coefficients of a generalized Sylvester equation to be regular and have disjoint spectra. We adapt his theorem, adopting his notation temporarily, to our purposes in the following

Proposition 4.2. *There exists a unique solution for $X \in \mathbb{R}^{m\times n}$ in the generalized Sylvester equation*

$$AXB + CXD + E = 0$$

if and only if

1. *$P_{AC}(\lambda) \equiv A\lambda + C$ and $P_{DB}(\lambda) \equiv D\lambda - B$ are regular matrix pencils, and*
2. *$\rho(P_{AC}) \cap \rho(P_{DB}) = \emptyset$*

where $A, C \in \mathbb{R}^{m\times m}$ and $D, B \in \mathbb{R}^{n\times n}$.

[18] $A(j,i)$ is of course dependent on i, reflecting the fact that we can generically expect the value of the solutions associated with different i's to differ. For the solvability conditions to remain unchanged at different i's requires the coefficients of the homogenous portion to remain unchanged.

Proof. See Chu (1987). Notice the rearrangement and redefinition of terms. □

As we have already established the Sylvester form—see lemma 4.1, the existence and uniqueness of solutions for all the coefficients of a perturbation of arbitrarily high order will follow from proposition 4.2 if we can establish the regularity of the following matrix pencils and the disjointness of their spectra

Definition 4.3. *For all $j \in \mathbb{N}^0$, the leading and trailing matrix pencils, respectively, of the generalized Sylvester equation (26) in lemma 4.1 are*

1. $P_U(z) \equiv z f_{y^+} + f_{y^+} y_z z_y + f_y$ *(the $P_U(z)$ in corollary 3.12)*
2. $P_{IS}(z) \equiv z I_{nz^j} - (z_y y_z)^{\otimes[j]}$

Before we examine the regularity and spectral disjointness in the general case, we will highlight the intuition behind proposition 4.2 in the special scalar version of (26), when $f_{y^+}, f_y, y_z z_y$ and $z_y y_z \in \mathbb{R}$ and $A(j,i)$ is a scalar function of known terms.[19] In this case, (26) can be arranged as

$$\left[f_{y^+} (z_y y_z)^j + \left(f_y + f_{y^+} y_z z_y \right) \right] y_{z^j \sigma^i} + A(j,i) = 0 \tag{27}$$

From, e.g., Strang (2009), the foregoing has a unique solution if and only if the leading coefficient is not zero, i.e., $\left[f_{y^+} (z_y y_z)^j + \left(f_y + f_{y^+} y_z z_y \right) \right] \neq 0$. As otherwise there is either no solution (when $A(j,i) \neq 0$) or there exists infinitely many solutions (when $A(j,i) = 0$). The two conditions in proposition 4.2 translate directly into the two ways this leading coefficient can be equal to zero.

The regularity condition in the scalar case translates to both coefficients in either of the pencils being simultaneously equal to zero: either $f_{y^+} = f_y + f_{y^+} y_z z_y = 0$ or $1 = (z_y y_z)^j = 0$. Obviously, both coefficients in the trailing pencil cannot be zero and this general regularity holds in the matrix case as well. The second condition, disjoint spectra, rules out the remaining possibility that the sum of all the coefficients is zero, which can be rearranged as $\frac{f_y + f_{y^+} y_z z_y}{f_{y^+}} \neq (z_y y_z)^j$. Recognize that the

[19] This special case, of course, is not useful practically. Either all shocks or the presence of y_{t-1} has to be shut down, but the mechanisms behind the matrix case are usefully illustrated in this case.

two terms correspond to the eigenvalues of the scalar regular pencils $P_U(z)$ and $P_{IS}(z)$, hence their set of eigenvalues (or spectra) must not contain any identical elements (be disjoint).

Returning to the general matrix case, we will now first establish the regularity of the pencils $P_U(z)$ and $P_{IS}(z)$ and then proceed to prove the disjointness of their spectra. The leading pencil $P_U(z)$ is one of the two pencils in corollary 3.12, its regularity was established in lemma 3.14 and all its eigenvalues were placed outside the closed unit circle in proposition 3.13. The regularity of the trailing pencil is guaranteed by the presence of the identity matrix and we will show that its spectrum is contained inside the closed unit circle by virtue of theorem 3.9.

The regularity of both the pencils is summarized in the following[20]

Lemma 4.4. *For all $j \in \mathbb{N}^0$, the leading and trailing matrix pencils, see definition 4.3, are regular*

Proof. For $P_U(z)$, see lemma 3.14. For $P_{IS}(z)$, this follows from its leading matrix being the identity matrix, see Gantmacher (1959b, pp. 25–27). □

The spectral disjointness follows nearly directly from the factorization of the matrix quadratic in corollary 3.12, with the spectrum of the leading pencil $P_U(z)$ being outside and that of the trailing pencil $P_{IS}(z)$ being inside the closed unit circle. In corollary 3.12, it was the pencil $P_S(z) = I_{ny}z - y_z z_y$ that was the stable pencil, but noting that z_y and z_ε are two constant matrices with all their entries being either unit or zero

$$z_y \equiv \mathscr{D}_{y_{t-1}^T}\{z_t\} = \mathscr{D}_{y_t^T}\{z_{t+1}\} = \begin{bmatrix} I_{ny} \\ 0_{ne\times ny} \end{bmatrix},\ z_\varepsilon \equiv \mathscr{D}_{\varepsilon_t^T}\{z_t\} = \mathscr{D}_{\sigma\varepsilon_{t+1}^T}\{z_{t+1}\} = \begin{bmatrix} 0_{ny\times ne} \\ I_{ne} \end{bmatrix} \tag{28}$$

the matrix $z_y y_z$ in $P_{IS}(z)$ is

$$z_y y_z = \begin{bmatrix} y_z z_y & y_z z_\varepsilon \\ 0_{ne\times ny} & 0_{ne\times ne} \end{bmatrix} \tag{29}$$

and it follows directly that the the eigenvalues of $P_{IS}(z)$ are all stable with respect to the closed unit circle, and thus those of an arbitrary Kronecker power too, if those of $y_z z_y$ are. We summarize the disjointness in the following

[20] The regularity of $P_{IS}(z)$ can also be verified by generalized Schur decomposition. Since the identity matrix is diagonal, it is also upper-triangular, and therefore all s_{ii}'s of $S = Q^* IZ$ are unity. Hence, $s_{ii} = t_{ii} = 0$ is ruled out for all i and $P_{IS}(z)$ is regular.

Lemma 4.5. *For all $j \in \mathbb{N}^0$, the leading and the trailing matrix pencils of definition 4.3 have no eigenvalues in common—their spectra form a disjoint set.*

Proof. See Appendix. □

From lemmata 4.4 and 4.5, proposition 4.2 applies and the existence and uniqueness of solutions to the generalized Sylvester equations (26) in lemma 4.1 follows immediately. In sum,

Proposition 4.6. *Let the assumptions of theorem 3.9 be fulfilled—there exists a unique solution, $y_z z_y$, of the matrix quadratic equation (15) stable with respect to the closed unit circle, then for all $j,i \in \mathbb{N}^0$ such that $j+i \geqslant 1$ except the case $j=1$ and $i=0$, there exist unique $y_{z^j\sigma^i}$ that solve*

$$f_{y^+} y_{z^j\sigma^i} (z_y y_z)^{\otimes[j]} + \left(f_y + f_{y^+} y_z z_y\right) y_{z^j\sigma^i} + A(j,i) = 0$$

the generalized Sylvester equations (26) in lemma 4.1.

Proof. From lemmata 4.4 and 4.5, the two conditions of proposition 4.2 are fulfilled and the result is immediate. □

Thus, given the unique stable solution of the matrix quadratic equation (15), all coefficients of in a perturbation of arbitrary order exist and are unique. We will now proceed to examine the consequences of this result for the policy function or exact solution $y(\sigma, z_t)$ and its Taylor approximation.

4.3 Discussion and Consequences for Nonlinear Perturbation Methods

In this section, we will examine the conditions for the local existence of a solution to our model (1) and then construct a Taylor approximation using the solutions to the generalized Sylvester equations (26) along with the first order term y_z from the previous section. We then highlight the insight of Anderson, Levin, and Swanson (2006) that this local solution can take on global facets as the order of approximation is increased. As we have shown that the solutions for the coefficients that we will use to construct our Taylor approximation exist and are unique, our analysis proves that

the method of successively differentiating the equilibrium conditions of a smooth model as used by many numerical algorithms necessarily leads to a unique recovery of this Taylor series.

Jin and Judd (2002) provide a local existence theorem for the solution to stochastic models and note the importance of checking whether a particular model fulfills these necessary conditions. We eliminate their solvability assumption, as their assumption of a unique locally asymptotically stable solution implies our theorem 3.9 holds, enabling us to apply our factorization from the generalized Bézout theorem of section 3 and confirm that their solvability assumption is necessarily fulfilled, analogously to our proposition 4.6.

Theorem 4.7. *Simplified Local Existence Theorem of Jin and Judd (2002)*
If (i) the function f in (1) exists and is analytic for all ε_t in some neighborhood of $\bar{z}$ defined in (5), (ii) there exists a unique deterministic solution $y(0,z_t)$ locally analytic in z_t and locally asymptotically stable, (iii) $E[\varepsilon_t]=0$, and (iv) ε_t has bounded support, then there is an $r>0$ such that for all (z_t,σ) in a ball with radius r centered at $(0,\bar{z})$ there exists a unique solution $y(\sigma,z_t)$ to (11). Furthermore, all derivatives of $y(\sigma,z_t)$ exist in a neighborhood of $(0,\bar{z})$ and can be solved by implicit differentiation.

Proof. See Jin and Judd's (2002) Theorem 6, where we have adapted notation to our exposition. Note that their assumption (iii) has been eliminated. See the Appendix. □

This simplification is potentially important, as it eliminates the only prohibitive assumption that has not been addressed elsewhere for the extension of local existence from a deterministic to a stochastic setting. Kim, Kim, Schaumburg, and Sims (2008) have discussed the assumption of bounded support and argue that if an accuracy in probability perspective is taken, then this assumption is not needed for finite time simulations and estimations. All told, what is needed for the local existence of a solution to a stochastic problem is sufficient differentiability of the equilibrium conditions, the existence of a solution to the deterministic variant of the model and restrictions on the moments of the stochastic processes that ensure the model remains well defined.

Taking now the existence, at least locally, of a solution for granted, we will gather the solutions to the generalized Sylvester equations (26) along with the first order term y_z as the unknown coefficients in a Taylor approximation of the policy function. Recalling the assumed differentiability of the equilibrium conditions and the existence of the nonstochastic steady state, we apply our results thus far and conclude that successive differentiation of the equilibrium conditions (11) is sufficient to uniquely recover a Taylor approximation of arbitrary order

Theorem 4.8. *Let the assumptions of theorem 3.9 be fulfilled—there exists a unique solution, $y_z z_y$, of the matrix quadratic equation (15) stable with respect to the closed unit circle, a Taylor series approximation at a nonstochastic steady state of the policy function $y_t = y(\sigma, z_t)$*

$$y_t = \sum_{m=0}^{M} \frac{1}{m!} \left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{z^m \sigma^n} \sigma^n \right] (z_t - \overline{z})^{\otimes [m]}$$

exists and is unique for all M and can be uniquely recovered by successive (implicit) differentiation of the equilibrium conditions (11).

Proof. The existence and uniqueness of the coefficients in the Taylor approximation follows from proposition 3.15 for y_z and from proposition 4.6 for all remaining coefficients. Recalling the assumed differentiability of (1), successive differentiation of (11) is then well defined. □

This result ensures that a Taylor approximation of the policy function can be unambiguously recovered by the obvious method of successive differentiation of the equilibrium conditions and solving the resulting linear system of equations for the unknown coefficients. This method is, of course, the basis of the numerous numerical algorithms for calculating perturbation solutions to DSGE models and this result proves that their users can be assured that perturbation applied to sufficiently smooth problems at a nonstochastic steady state must deliver a solution and that it must be unique under standard saddle stability conditions on the linear approximation.

Perturbation methods generate local approximations of the policy function—the Taylor expansion around the nonstochastic steady state at which the solution of the model is (assumed) known.

As Jin and Judd (2002) state, these methods do well for small but nontrivial neighborhoods of the point around which we approximate and, as Anderson, Levin, and Swanson (2006) point out, this nontrivial neighborhood is potentially very large if the policy function along with the function of the equilibrium conditions are analytic. As a consequence, the validity of perturbation methods can be extended past the local level to the entire domain of convergence of the Taylor expansion of the true policy function. If we assume that the policy function is analytic, the Taylor expansion converges to the policy function as the order of approximation becomes infinite. As corollary 4.8 ensures that perturbation methods can uniquely recover a Taylor expansion that satisfies the model's equilibrium conditions out to the order of approximation, then this uniquely recovered Taylor expansion must be a valid solution everywhere within its domain of convergence. We formalize this in the following

Corollary 4.9. *Under the assumptions that the policy function y in $y_t = y(\sigma, z_t)$ is analytic and asymptotically stable at the point $(0, \bar{z})$, the function f in the model statement (1) in analytic at the point $(\bar{y}, \bar{y}, \bar{y}, 0)$ and the Taylor series (9) of the policy function expanded around the point $(0, \bar{z})$ converges in any (compact) subset of the domain of the policy function, $\mathbb{R}^+ \times \mathbb{R}^{nz}$, successive differentiation of the equilibrium conditions (11) is sufficient to recover the policy function in this subset.*

Proof. If the policy function is analytic at the point around which we expand the Taylor series, then the Taylor expansion converges to the policy function as the order of approximation becomes infinite. Theorem 4.6 ensures the unique recovery of such an asymptotic expansion. □

Our Sylvester characterization of the equations to be solved at each order of approximation along with the factorization provided by the generalized Bézout theorem leads to a linear algebraic characterization of the solvability conditions for a perturbation of arbitrarily high order. While we confirm Jin and Judd's (2002) assessment that these conditions change with the order of approximation, the change is minimal comprising only Kronecker powers of the linear transition matrix of the state space. Our same factorization enabled us to weaken the requirements for the local existence proof of a solution, which provides the theoretical foundation for the Taylor expansion that we have proven

is necessarily uniquely recoverable by successive differentiation of the equilibrium conditions given a unique stable solution at the first order of approximation.

5 Applications

Here we will address to specific components of our arbitrary order perturbation: the linear mapping from exogenous (in our case ε_t) to endogenous (here y_t) variables and the first order independence of the policy function on the perturbation parameter σ. Many studies on linear solution methods have paid the existence and uniqueness of the first mapping little attention, directing focus towards the endogenous mapping associated (in our formulation) with the matrix quadratic equation. Our factorization from the generalized Bézout theorem can be applied directly in the context of such linear studies—we center our analysis around Uhlig (1999)—to prove the existence and uniqueness of this mapping under saddle stability conditions. In a nonlinear result, Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004) have conjectured the independence of the policy function from first order effects of the perturbation parameter ($y_{z^j\sigma^i} = 0$ for $i = 1$), as the equations that these coefficients solve are homogenous. Our analysis adds the missing link, showing not only that zero is a solution (as follows from the homogeneity), but that it is the only solution.

5.1 Uhlig's (1999) Q or the Linear Mapping from Exogenous Variables

The literature on linear DSGE models is well established, but the matrix factorization provided by the generalized Bézout theorem can also be applied to the solvability of the mapping from exogenous to endogenous variables in existing linear solution methods. Specifically, we show how the techniques of the previous two sections can be applied to this mapping in several linear solution methods spanning the last three decades. The result that this mapping can be uniquely resolved is not new— the procedure of King and Watson (2002, pp. 73–74) and Klein (2000, p. 1416) is a recursive scalar alternative to our direct matrix approach. However, the main focus of most research on linear solutions concentrates on the quadratic equation—the mapping of endogenous variables through

time. McCallum (1983), Binder and Pesaran (1997), Uhlig (1999), and Cho and Moreno (2011) are a few papers in this expansive literature that leave the existence and uniqueness of the mapping from exogenous variables to endogenous variables unresolved. While this solvability is guaranteed by our proposition 3.15, it is instructive to apply the underlying linear algebra—proposition 3.13 and theorem 4.2—directly to this well known literature. We will focus in detail on Uhlig (1999), adopting his notation for this section, and then relate the solvability of his exogenous to endogenous mapping to that of McCallum (1983), Binder and Pesaran (1997), and Cho and Moreno (2011).

Uhlig (1999) solves a linear model by the method of undetermined coefficients, with the following problem statement[21]

$$0 = E_t \left[F x_{t+1} + G x_t + H x_{t-1} + L z_{t+1} + M z_t \right], \; z_t = N z_{t-1} + \varepsilon_t \tag{30}$$

where x_t is a vector ($nx \times 1$) of endogenous variables, z_t is an exogenous vector ($nz \times 1$) autoregressive process, and ε_t a vector of serially uncorrelated innovations to z_t.[22]

Proceeding with the method of undetermined coefficients using the postulated solution

$$x_t = P x_{t-1} + Q z_t \tag{31}$$

the matrix P solves a matrix quadratic equation

$$F P^2 + G P + H = 0 \tag{32}$$

Uhlig (1999) constructs a solvent with a set of *nx* eigenvalues and eigenvectors associated with the linearization of (32). Assuming there is a unique solution stable with respect to the closed unit circle, we can apply the generalized Bézout theorem for right division by a solvent—corollary 3.12— and combine with Uhlig's (1999) assumption that N has only stable eigenvalues to yield

Lemma 5.1. *The matrix pencils $P_{FPG}(\lambda) = FP + G + \lambda F$ and $P_N(\lambda) = I_{nz}\lambda - N$ are regular. The spectrum of $P_{FPG}(\lambda)$ is wholly outside the closed unit circle and that of $P_N(\lambda)$ wholly inside.*

Proof. For $P_{FPG}(\lambda)$, see proposition 3.13 and lemma 3.14; $P_N(\lambda)$ is by assumption. □

[21] This is his "brute force" formulation. The same logic applies to his "with sensitivity" approach and the results carry over to that formulation too. We choose this formulation to conserve space.

[22] Note that our problem statement (1) would put Uhlig's (1999) x_t and z_t in our y_t

This leaves the mapping from z_t to x_t, the matrix Q, to be determined. Uhlig (1999) shows that Q solves a generalized Sylvester equation

$$FQN + (FP + G)Q + M + LN = 0 \tag{33}$$

He applies the vec operation on (33) to solve for Q, yielding $V \times vec(Q) = -vec(M + LN)$ and states that "if [V] is invertible, then [the equation in Q has] a unique solution for Q." If there is a unique solution to P with all eigenvalues inside or on the unit circle, however, this proviso is not needed, lemma 5.1 enables a direct application of Chu (1987), repeated here as theorem 4.2) to (33,

Proposition 5.2. *If there is a unique solution P stable with respect to the closed unit circle, then there exists a unique solution for Q.*

Proof. From lemma 5.1, the pencils $P_{FPG}(\lambda)$ and $P_N(\lambda)$ are regular and their eigenvalues form a disjoint set. Thus, following Chu's (1987) Theorem 1, there exists a unique solution to (33). □

Again, the uniqueness of a stable solution to the matrix quadratic equation—here (32)—guarantees the existence of a unique solution to all remaining coefficients—here Q as a solution to (33).

Our matrix factorization can be applied directly to numerous other linear methods from the past thirty years. Beginning with McCallum (1983, p. 163),[23] who sets $H, L = 0$ and states "Q will be unique for almost all values of F and N." Proposition 5.2 applies here directly with $P = 0$ always being a stable solution to (32) and assuming its uniqueness, lemma 5.1 necessarily applies. Binder and Pesaran (1997) examine the special case of $G = -I_{nx}$ and add the proviso of "if $I_{nx} - FP$ is invertible" to their solution method, stating that "[a]lthough it is in general difficult to establish strong analytic results regarding the existence and multiplicity of solutions [...] we so far have not encountered any well-specified economic model for which $I_{nx} - FP$ would have been singular." Indeed, under the assumption that P has a unique stable solution, the invertibility of $I_{nx} - FP$ is guaranteed by lemma 5.1. Recently, Cho and Moreno (2011) have explored the forward solution as

[23] McCallum (1983, p. 164) then extends his analysis to allow $H \neq 0$, apparently claiming unique solvability in this more general case. No indication is provided as to why his reservations in the more restricted case are eliminated when he loosens his assumptions.

a refinement mechanism, likewise under the assumption of $G = -I_{nx}$ and like Binder and Pesaran (1997), their results hold "provided that $det(I_{nx} - FP) \neq 0$." Again, if P is the unique stable solution to the matrix quadratic equation, this condition necessarily holds. The generalized Bézout theorem and the solvability of Sylvester equations ensure in the context of linear models with a unique stable solution that Q is unique and that $G + FP$ is indeed invertible.

Of course, the uniqueness of Q has been addressed in other analyses. Klein (2000) and King and Watson (2002) both provide a recursive procedure that proceeds element by element through the combined vector Qz_t.[24] The recursivity follows from the triangularization provided by Schur decomposition (see our theorem 3.5). In particular, Klein (2000) highlights that the method will fail if $s_{ii} = t_{ii} = 0$ (the notation for s and t aligns with our section 3), which is ruled out by the regularity assumption 3.6, and moves through the unstable triangular block, inverting the matrix $s_{ii}N - t_{ii}I_{nz}$. If an eigenvalue t_{ii}/s_{ii} were to coincide with an eigenvalue in the exogenous transition matrix N, Klein's (2000) inversion would not succeed. But he is moving through the unstable block and thus the eigenvalues of N and the unstable t_{ii}/s_{ii} form a disjoint set, guaranteeing the necessary invertibility. This is the same mechanism as we present above. The only difference being that our approach uses matrix techniques to solve the problem in one step, whereas Klein (2000) and King and Watson (2002) move element by element through the unstable set of eigenvalues.

5.2 First Order Independence from σ

This section confirms the conjecture of both Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004) that the policy function is independent of the perturbation parameter σ. This follows intuitively, we argue, as the first moment of the exogenous shocks is assumed to be zero, thus eliminating its impact at all orders. Some studies, e.g., Kim, Kim, Schaumburg, and Sims (2008), deriving their second or higher order Taylor expansions assuming without proof that these coefficients are all zero.

[24] Klein (2000) also provides a matrix formulation in terms of a Sylvester equation as above, but does not address the solvability of the equation. While he advocates the recursive method for computation reasons, its formulation enables the solvability to be directly verified in his analysis from his assumptions.

The root of the difficulty lies in the solvability of these systems: Schmitt-Grohé and Uribe (2004) to second order and Jin and Judd (2002) to arbitrary order prove that the unknown coefficients involving the perturbation parameter solve homogeneous equations. Of course, the zero solution solves these equations, but the claim that the solution is uniquely zero requires solvability in addition to homogeneity—see, e.g., Strang (2009). Our main result confirms the conjecture by providing the necessary solvability so as to add uniqueness to their existence of the zero solution.

With the first moment of exogenous shocks and all $y_{z^k\sigma}$ for $k < j$ zero, the generalized Sylvester equations in $y_{z^j\sigma}$ are homogenous

$$f_{y^+} y_{z^j\sigma} \left(z_y y_z\right)^{\otimes[j]} + \left(f_y + f_{y^+} y_z z_y\right) y_{z^j\sigma} = 0 \tag{34}$$

As the zero matrix is always a solution to (34) and the solution must be unique following theorem 4.6, $y_{z^j\sigma} = 0$ is the unique solution for all j. We formalize this in the following

Proposition 5.3. *For all $j \in \mathbb{N}^0$, $y_{z^j\sigma}$ is zero.*

Proof. See the Appendix. □

The intuition behind this is simple: the unknown coefficient $y_{z^j\sigma}$ is the comparative static matrix measuring the impact of the first order moment of exogenous shocks on the policy function y (and its derivatives with respect to the state vector z_t). As the first order moment is assumed to be zero, the first order moment of exogenous shocks has no impact at all.

6 Conclusion

We have proven the existence and uniqueness of solutions for the undetermined coefficients in perturbations of an arbitrarily high order. For users of numerical perturbation algorithms, such as Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot's (2011) Dynare or Anderson, Levin, and Swanson's (2006) PerturbationAIM, we have answered two questions. First, given a nonlinear perturbation solution from a numerical algorithm, is this solution the only solution? Second, should a numerical algorithm fail to deliver a solution: does a solution not exist at all or did the

numerical algorithm simply fail to find it? Given a unique stable solution at first order, our results provide a definitive assurance that a solution must exist and that is must be unique. In essence, we show that successive differentiation of the equilibrium condition will generate set of equations that are sufficient to uniquely recover the coefficients of the Taylor expansion of the policy function.

Our method exploits the analytic factorization provided by the generalized Bézout theorem of the matrix quadratic equation from the linear (or first order) problem, taking a unique stable solution at that order as given. The factorization separates the original matrix quadratic problem into two regular pencils with disjoint sets of eigenvalues. These two pencils form the basis of the pencils of the leading and trailing coefficient matrices in the generalized Sylvester equations that govern the undetermined coefficients at all higher orders of approximation. Our results make extensive use of the multidimensional calculus of Lan and Meyer-Gohde (2011) that preserves linear algebraic structures, enabling us to provide this explicit representation of the homogenous components of these linear equations. The existence and uniqueness of the solutions for the undetermined coefficients is then a straightforward application of Chu's (1987) theorem on solutions to generalized Sylvester equations and follows from the regularity and disjointness of the sets of eigenvalues of the pencils of the leading and trailing coefficient matrices.

With the recent proliferation of interest in nonlinear methods and general familiarity of economists with the simplest perturbation—i.e., the first order or (log-)linearization, our results should provide researchers applying perturbation methods numerically with the confidence that a perturbation of arbitrary order is guaranteed to provide a unique solution if the linear approximation has a unique stable solution.

References

ADJEMIAN, S., H. BASTANI, M. JUILLARD, F. MIHOUBI, G. PERENDIA, M. RATTO, AND S. VILLEMOT (2011): "Dynare: Reference Manual, Version 4," Dynare Working Papers 1, CEPREMAP.

ANDERSON, G. S., A. LEVIN, AND E. SWANSON (2006): "Higher-Order Pertubation Solutions to Dynamic Discrete-Time Rational Expectations Models," Federal Reserve Bank of San Francisco Working Paper Series 2006-01.

BINDER, M., AND M. H. PESARAN (1997): "Multivariate Linear Rational Expectations Models: Characterization of the Nature of the Solutions and Their Fully Recursive Computation," *Econometric Theory*, 13(6), pp. 877–888.

BLANCHARD, O. J., AND C. M. KAHN (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(5), 1305–1311.

CHO, S., AND A. MORENO (2011): "The Forward Method as a Solution Refinement in Rational Expectations Models," *Journal of Economic Dynamics and Control*, 35(3), 257–272.

CHU, K.-W. E. (1987): "The Solution of the Matrix Equations $AXB - CXD = E$ and $(YA - DZ, YC - BZ) = (E, F)$," *Linear Algebra and its Applications*, 93, 93–105.

DAVIS, G. J. (1981): "Numerical Solution of a Quadratic Matrix Equation," *SIAM Journal on Scientific and Statistical Computing*, 2(2), 164–175.

FERNÁNDEZ-VILLAVERDE, J., P. A. GUERRÓN-QUINTANA, J. RUBIO-RAMÍREZ, AND M. URIBE (2011): "Risk Matters: The Real Effects of Volatility Shocks," *American Economic Review*, 101(6), 2530–61.

GANTMACHER, F. R. (1959a): *The Theory of Matrices*, vol. I. Chelsea Publishing Company, New York, NY.

——— (1959b): *The Theory of Matrices*, vol. II. Chelsea Publishing Company, New York, NY.

GASPAR, J., AND K. L. JUDD (1997): "Solving Large-Scale Rational-Expectations Models," *Macroeconomic Dynamics*, 1(01), 45–75.

GOHBERG, I., P. LANCASTER, AND L. RODMAN (1982): *Matrix Polynomials*. Academic Press, Inc., New York, NY.

GOLUB, G. H., AND C. F. V. LOAN (1996): *Matrix Computations*. The Johns Hopkins University Press, Baltimore, MD, and London, UK, 3 edn.

GOMME, P., AND P. KLEIN (2011): "Second-Order Approximation of Dynamic Models without the Use of Tensors," *Journal of Economic Dynamics and Control*, 35(4), 604–615.

HIGHAM, N. J., AND H.-M. KIM (2000): "Numerical Analysis of a Quadratic Matrix Equation," *IMA Journal of Numerical Analysis*, 20, 499–519.

——— (2001): "Solving a Quadratic Matrix Equation by Newton's Method with Exact Line Searches," *SIAM Journal on Matrix Analysis and Applications*, 23(2), 499–519.

J. E. DENNIS, J., J. F. TRAUB, AND R. P. WEBER (1976): "The Algebraic Theory of Matrix Polynomials," *SIAM Journal on Numerical Analysis*, 13(6), 831–845.

JIN, H.-H., AND K. L. JUDD (2002): "Pertubation Methods for General Dynamic Stochastic Models," Mimeo April.

JUDD, K. L. (1998): *Numerical Methods in Economics*. MIT Press, Cambridge, MA.

JUDD, K. L., AND S.-M. GUU (1997): "Asymptotic Methods for Aggregate Growth Models," *Journal of Economic Dynamics and Control*, 21(6), 1025–1042.

JUILLARD, M., AND O. KAMENIK (2004): "Solving Stochastic Dynamic Equilibrium Models: a k-Order Perturbation Approach," Mimeo Autumn.

KAMENIK, O. (2005): "Solving SDGE Models: A New Algorithm for the Sylvester Equation," *Computational Economics*, 25(1), 167–187.

KIM, J., S. KIM, E. SCHAUMBURG, AND C. A. SIMS (2008): "Calculating and Using Second-Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models," *Journal of Economic Dynamics and Control*, 32(11), 3397–3414.

KING, R. G., AND M. W. WATSON (1998): "The Solution of Singular Linear Difference Systems under Rational Expectations," *International Economic Review*, 39(4), 1015–26.

KING, R. G., AND M. W. WATSON (2002): "System Reduction and Solution Algorithms for Singular Linear Difference Systems under Rational Expectations," *Computational Economics*, 20, 57–86.

KLEIN, P. (2000): "Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model," *Journal of Economic Dynamics and Control*, 24(10), 1405–1423.

KOLLMANN, R., J. KIM, AND S. H. KIM (2011): "Solving the Multi-Country Real Business Cycle Model Using a Perturbation Method," *Journal of Economic Dynamics and Control*, 35(2), 203–206.

LAN, H., AND A. MEYER-GOHDE (2011): "Solving DSGE Models with a Nonlinear Moving Average," SFB 649 Discussion Papers 2011-087.

LANCASTER, P. (1966): *Lambda-Matrices and Vibrating Systems*. Pergamon Press, Oxford, UK.

LOMBARDO, G., AND A. SUTHERLAND (2007): "Computing Second-Order-Accurate Solutions for Rational Expectation Models Using Linear Solution Methods," *Journal of Economic Dynamics and Control*, 31(2), 515–530.

MAGNUS, J. R., AND H. NEUDECKER (1979): "The Commutation Matrix: Some Properties and Applications," *The Annals of Statistics*, 7(9), 383–394.

——— (2007): *Matrix Differential Calculus with Applications in Statistics and Econometrics*. John Wiley & Sons, West Sussex, UK, 3d edn.

MCCALLUM, B. T. (1983): "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective," *Journal of Monetary Economics*, 11(2), 139–168.

RUDEBUSCH, G. D., AND E. T. SWANSON (2012): "The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks," *AEJ: Macroeconomics*, 4(1), 105–43–61.

SCHMITT-GROHÉ, S., AND M. URIBE (2004): "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function," *Journal of Economic Dynamics and Control*, 28(4), 755–775.

STRANG, G. (2009): *Introduction to Linear Algebra*. Wellesley-Cambridge Press and SIAM, Wellesley, MA, 4 edn.

TISSEUR, F., AND K. MEERBERGEN (2001): "The Quadratic Eigenvalue Problem," *SIAM Review*, 43(2), 235–286.

UHLIG, H. (1999): "A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily," in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon, and A. Scott, chap. 3, pp. 30–61. Oxford University Press.

VETTER, W. J. (1973): "Matrix Calculus Operations and Taylor Expansions," *SIAM Review*, 15(2), 352–369.

A Appendices

A.1 Proof of corollary 2.4

From Vetter (1973, pp. 358–363), a multidimensional Taylor expansion using the structure of derivatives (evaluated at $\bar{B}$) in Lan and Meyer-Gohde (2011) is given by

$$\underset{(p\times 1)}{M}\left(\underset{(s\times 1)}{B}\right) = M(\bar{B}) + \sum_{n=1}^{N} \frac{1}{n!} \mathscr{D}^{n}_{B^{T^n}} M(\bar{B}) \left(B-\bar{B}\right)^{\otimes[n]} + R_{N+1}\left(\bar{B},B\right) \tag{A-1}$$

$$\text{where } R_{N+1}\left(\bar{B},B\right) = \frac{1}{N!}\int_{\xi=\bar{B}}^{B} \mathscr{D}^{N+1}_{B^{T^{N+1}}} M(\xi) \left(I_s \otimes \left(B-\xi\right)^{\otimes[N]}\right) d\xi \tag{A-2}$$

Differentiating (3) with respect to all its arguments M times and noting permutations of the order of differentiation, a Taylor approximation about the nonstochastic steady state $\bar{z}$ is

$$\begin{aligned}
y_t = {} & \frac{1}{0!}\left(\frac{1}{0!}\bar{y} + \frac{1}{1!}y_{\sigma}\sigma + \frac{1}{2!}y_{\sigma^2}\sigma^2 + \ldots + \frac{1}{M!}y_{\sigma^M}\sigma^M\right) \\
& + \frac{1}{1!}\left(\frac{1}{0!}y_z + \frac{1}{1!}y_{z\sigma}\sigma + \frac{1}{2!}y_{z\sigma^2}\sigma^2 + \ldots + \frac{1}{(M-1)!}y_{z\sigma^{M-1}}\sigma^{M-1}\right)\left(z_t-\bar{z}\right) \\
& + \frac{1}{2!}\left(\frac{1}{0!}y_{z^2} + \frac{1}{1!}y_{z^2\sigma}\sigma + \frac{1}{2!}y_{z^2\sigma^2}\sigma^2 + \ldots + \frac{1}{(M-2)!}y_{z^2\sigma^{M-2}}\sigma^{M-2}\right)\left(z_t-\bar{z}\right)^{\otimes[2]} \\
& \vdots \\
& + \frac{1}{M!}\frac{1}{0!}y_{z^M}\left(z_t-\bar{z}\right)^{\otimes[M]}
\end{aligned}$$

Writing the foregoing more compactly yields (9) in the text.

A.2 The Generalized Bézout Theorem

Theorem A.1. *The Generalized Bézout Theorem*

The arbitrary lambda-matrix

$$M(\lambda) = M_0\lambda^m + M_1\lambda^{m-1} + \ldots M_m, \text{ where } M_0 \neq \underset{(n\times n)}{0}$$

when divided on the right by the binomial

$$I_n\lambda - A$$

yields

$$M(\lambda) = Q(\lambda)\left(I_n\lambda - A\right) + M(A)$$

where

$$Q(\lambda) = M_0\lambda^{m-1} + (M_0A + M_1)\,\lambda^{m-2} + \ldots M_0A^{m-1}M_1A^{m-2} + \ldots M_m$$

Proof. See Gantmacher (1959a). □

A.3 Proof of Lemma 3.14

The regularity of the matrix pencil $P_{DE}(z)$ in assumption 3.6 means

$$\det\left(f_{y^+}z^2 + f_y z + f_z z_y\right) \neq 0 \tag{A-3}$$

Following corollary 3.12, the matrix polynomial inside the foregoing admits the following factorization

$$\det\left((z f_{y^+} + f_{y^+}y_z z_y + f_y)(I_{ny}z - y_z z_y)\right) \neq 0 \tag{A-4}$$

and using the product rule of matrix determinants of Strang (2009, ch. 5), the foregoing rewrites

$$\det(z f_{y^+} + f_{y^+}y_z z_y + f_y)\det(I_{ny}z - y_z z_y) \neq 0 \tag{A-5}$$

This means neither of the two determinants is zero, or equivalently, matrix pencils $P_U(z) = z f_{y^+} + f_{y^+}y_z z_y + f_y$ and $P_S(z) = I_{ny}z - y_z z_y$ are both regular.

A.4 Proof of Lemma 4.1

We will first show that for all $j, i \in \mathbb{N}^0$ such that $j + i \geqslant 1$ except the case $j = 1$ and $i = 0$, successive differentiation of the function f with respect to its arguments, z_t and σ, yields

$$\mathscr{D}_{z_t^{T\,j}\sigma^i}\{f\} = f_{y^+}y_{z^j\sigma^i}\,(z_y y_z)^{\otimes[j]} + \left(f_y + f_{y^+}y_z z_y\right)y_{z^j\sigma^i} + B(j,i) \tag{A-6}$$

where the function $B(j,i)$ is (i) linear in ε_{t+1} up to and including i-th Kronecker power and contains (ii) products involving derivatives of y and y^+ with respect to z_t $j+i$ or less times and σ i or less times except for the unknown $y_{z^j\sigma^i}$ under consideration

$$B(j,i) = B\left(y^+_{z^l\sigma^k}, y_{z^l\sigma^k}, \varepsilon^{\otimes[k]}_{t+1}\right) \tag{A-7}$$

$$\text{where } l = 0,1,2,\ldots,j+i;\ k = 0,1,2,\ldots,i;\ l+k \leq j+i;\ \text{but not } l = j \text{ and } k = i \tag{A-8}$$

The index rule (A-8) ensures that all the terms in the $B(j,i)$ function are given by previous calculations as the unknown under consideration, $y_{z^j\sigma^i}$, has been excluded by $l=j$ and $k=i$ simultaneously having been disallowed.

We will proceed inductively by differentiating (A-6) with respect to z_t and σ respectively and confirming that the two resulting expressions take the form of (A-6). First, differentiating (A-6) with respect to z_t yields

$$\begin{aligned}\mathscr{D}_{z_t^T j+1\sigma^i}\{f\} = & f_{y^+}y_{z^{j+1}\sigma^i}\left(z_y y_z\right)^{\otimes[j+1]} + \left(f_y + f_{y^+}y_z z_y\right)y_{z^{j+1}\sigma^i} \\ & + \mathscr{D}_{z_t^T}\{f_{y^+}\}\left(I_{ne}\otimes\left[y^+_{z^j\sigma^i}\left(z_y y_z\right)^{\otimes[j]}\right]\right) + f_{y^+}y^+_{z^j\sigma^i}\mathscr{D}_{z_t^T}\left\{\left(z_y y_z\right)^{\otimes[j]}\right\} \\ & + \mathscr{D}_{z_t^T}\{f_y\}\left(I_{ne}\otimes y_{z^j\sigma^i}\right) + \mathscr{D}_{z_t^T}\{f^+_y y_z z_y\}\left(I_{ne}\otimes y_{z^j\sigma^i}\right) \\ & + \mathscr{D}_{z_t^T}\left\{B\left(y^+_{z^l\sigma^k}, y_{z^l\sigma^k}, \varepsilon_{t+1}^{\otimes[k]}\right)\right\}\end{aligned} \tag{A-9}$$

The terms in the second and third lines of the foregoing contain products involving the derivatives of y and y^+ with respect to z_t $j+i$ or less times and σ i or less times, all known from previous calculations.

The terms in the last line contain products involving the derivatives of y and y^+ with respect to z_t $j+i+1$ or less times and σ i or less times. This can be shown by differentiating through $B(j,i)$ in the last line with respect to z_t in which

$$\begin{aligned}&\mathscr{D}_{z_t^T}\left\{y^+_{z^l\sigma^k}\right\} = y^+_{z^{l+1}\sigma^k}[(z_y y_z)\otimes I_{z^l}],\ \mathscr{D}_{z_t^T}\left\{y_{z^l\sigma^k}\right\} = y_{z^{l+1}\sigma^k}[(z_y y_z)\otimes I_{z^l}] \\ &\text{where } l=0,1,2,\ldots,j+i;\ k=0,1,2,\ldots,i;\ l+k\leq j+i;\ \text{but not } l=j \text{ and } k=i\end{aligned} \tag{A-10}$$

Importantly, the unknown under consideration upon differentiation, $y_{z^{j+1}\sigma^i}$, is excluded by advancing the exclusion in the index rule: with no $y_{z^j\sigma^i}$ in $B(j,i)$, there can be no $y_{z^{j+1}\sigma^i}$ in $B(j+1,i)$. Second the terms constitute a linear function in ε_{t+1} up to and including i-th Kronecker power as differentiating $\varepsilon_{t+1}^{\otimes[k]}$ in the last line does not advance the index i. Hence (A-9) can be rewritten

$$\mathscr{D}_{z_t^T j+1\sigma^i}\{f\} = f_{y^+}y_{z^{j+1}\sigma^i}\left(z_y y_z\right)^{\otimes[j+1]} + \left(f_y + f_{y^+}y_z z_y\right)y_{z^{j+1}\sigma^i} + B(j+1,i) \tag{A-11}$$

Differentiating (A-6) with respect to σ yields

$$\begin{aligned}\mathscr{D}_{z_t^T j_{\sigma^{i+1}}}\{f\} = & f_{y^+} y_{z^j\sigma^{i+1}} (z_y y_z)^{\otimes[j]} + \left(f_y + f_{y^+} y_z z_y\right) y_{z^j\sigma^{i+1}} \\ & + \mathscr{D}_\sigma\{f_{y^+}\}\left[y^+_{z^j\sigma^i}(z_y y_z)^{\otimes[j]}\right] + f_{y^+} y^+_{z^{j+1}\sigma^i}(z_y y_z)^{\otimes[j+1]} + f_{y^+} y^+_{z^{j+1}\sigma^i} z_\varepsilon \varepsilon_{t+1}(z_y y_z)^{\otimes[j]} \\ & + f_{y^+} y^+_{z^j\sigma^i} \mathscr{D}_\sigma\left\{(z_y y_z)^{\otimes[j]}\right\} + \mathscr{D}_\sigma\{f_y\} y_{z^j\sigma^i} + \mathscr{D}_\sigma\{f_y^+ y_z z_y\} y_{z^j\sigma^i} \\ & + \mathscr{D}_\sigma\left\{B\left(y^+_{z^l\sigma^k}, y_{z^l\sigma^k}, \varepsilon^{\otimes[k]}_{t+1}\right)\right\} \end{aligned} \tag{A-12}$$

The terms in the second and third lines of the foregoing contain products involving the derivatives of y and y^+ with respect to z_t $j+i+1$ or less times and σ i or less times, all known from previous calculations. Note again that the unknown, here $y_{z^j\sigma^{i+1}}$, only appears in the first line.

The last line contains products involving the derivatives of y and y^+ with respect to z_t $j+i+1$ or less times and σ $i+1$ or less times. This can be shown by differentiating through $B(j,i)$ in the last line with respect to σ in which

$$\mathscr{D}_\sigma\left\{y^+_{z^l\sigma^k}\right\} = y^+_{z^{l+1}\sigma^k}(z_y y_\sigma + z_\varepsilon \varepsilon_{t+1}) + y^+_{z^l\sigma^{k+1}} \tag{A-13}$$

$$\mathscr{D}_\sigma\left\{y_{z^l\sigma^k}\right\} = y_{z^l\sigma^{k+1}} \tag{A-14}$$

$$\text{where } l = 0,1,2,\ldots,j+i;\ k = 0,1,2,\ldots,i;\ l+k \le j+i; \text{ but not } l=j \text{ and } k=i$$

Importantly, the unknown $y_{z^j\sigma^{i+1}}$ is again not present here either, as when $k=i$ or equivalently, $k+1=i+1$, $l=j$ is not allowed by the index rule: with no $y_{z^j\sigma^i}$ in $B(j,i)$, there can be no $y_{z^j\sigma^{i+1}}$ in $B(j,i+1)$. Notice that an additional ε_{t+1} is included in (A-13). The possibility that this term multiplies with the existing $\varepsilon^{\otimes[k]}_{t+1}$ necessitates the advancement of the index associated with Kronecker powers of ε_{t+1} for $B(j,i+1)$ to remain linear in the set of $\varepsilon^{\otimes[k+1]}_{t+1}$.

All terms in the last three lines of (A-12) can thus be collected in $B\left(y^+_{z^l\sigma^{k+1}}, y_{z^l\sigma^{k+1}}, (\varepsilon_{t+1})^{\otimes[k+1]}\right) = B(j,i+1)$ and (A-12) can be rewritten

$$\mathscr{D}_{z_t^T j_{\sigma^{i+1}}}\{f\} = f_{y^+} y_{z^j\sigma^{i+1}} (z_y y_z)^{\otimes[j]} + \left(f_y + f_{y^+} y_z z_y\right) y_{z^j\sigma^{i+1}} + B(j,i+1) \tag{A-15}$$

The second step is to evaluate (A-6), having been verified by induction above, with the given moments of ε_{t+1} and at the nonstochastic steady state. Setting the resulting expression equal to zero

and letting $A(j,i) \equiv E_t\left[B(j,i)\right]\Big|_{\overline{z}}$ yields (26) in the text.

All that remains is to address the cases that where excluding by the indexing and initialize the induction. The two cases that were excluded are (i) $(j=0, i=0)$ corresponding to the nonstochastic steady state value of y which was assumed given in the text; (ii) $(j=1, i=0)$ for y_z, which was solved separately as (23) in the text. Noting that the case $(j=0, i=1)$ for y_σ in (25) also conforms to the pattern, we can start the induction with the three second order terms $(j=2, i=0)$, y_{z^2}; $(j=1, i=1)$, $y_{z\sigma}$; and $(j=0, i=2)$ y_{σ^2} which are provided in the next section separately and thus complete the proof.

A.5 Generalized Sylvester Equations for Second Order Terms

Following corollary 2.4, the second order Taylor expansion of the policy function (3) takes the form

$$y_t = \overline{y} + y_\sigma \sigma + \frac{1}{2} y_{\sigma^2}\sigma^2 + \left(y_z + y_{z\sigma}\sigma\right)\left(z_t - \overline{z}\right) + \frac{1}{2} y_{z^2}\left(z_t - \overline{z}\right)^{\otimes[2]} \tag{A-16}$$

Given coefficients from the first order, there are three unknowns, y_{z^2}, $y_{z\sigma}$ and y_{σ^2}, to be solved.

To find y_{z^2}, we differentiate (13) with respect to z_t

$$\begin{aligned} \mathscr{D}_{z_t^T z_t^T}\{f\} =& \mathscr{D}_{z_t^T}\left\{f_{y^+}\right\}\left(I_{nz} \otimes y_z^+ z_y y_z\right) + f_{y^+} y_{z^2}^+ \left(z_y y_z\right)^{\otimes 2} + f_{y^+} y_z^+ z_y y_{z^2} \\ &+ \mathscr{D}_{z_t^T}\left\{f_y\right\}\left(I_{nz} \otimes y_z\right) + f_y y_{z^2} + \mathscr{D}_{z_t^T}\left\{f_z\right\} \end{aligned} \tag{A-17}$$

$$\begin{aligned} \text{where } \mathscr{D}_{z_t^T}\left\{f_{y^+}\right\} &= f_{y^{+2}}\left[\left(y_z^+ z_y y_z\right) \otimes I_{ny}\right] + f_{yy^+}\left(y_z \otimes I_{ny}\right) + f_{zy^+} \\ \mathscr{D}_{z_t^T}\left\{f_y\right\} &= f_{y^+ y}\left[\left(y_z^+ z_y y_z\right) \otimes I_{ny}\right] + f_{y^2}\left(y_z \otimes I_{ny}\right) + f_{zy} \end{aligned}$$

Evaluating the foregoing at the nonstochastic steady state, taking its expectation, and setting the resulting expression equal to zero yields

$$\begin{aligned} E_t\left[\mathscr{D}_{z_t^T z_t^T}\{f\}\right]\Big|_{\overline{z}} =& f_{y^+ y_{z^2}}\left(z_y y_z\right)^{\otimes 2} + \left(f_{y^+ y_z} z_y + f_y\right) y_{z^2} \\ &+ E_t\left[\mathscr{D}_{z_t^T}\left\{f_{y^+}\right\}\left(I_{nz} \otimes y_z^+ z_y y_z\right) + \mathscr{D}_{z_t^T}\left\{f_y\right\}\left(I_{nz} \otimes y_z\right) + \mathscr{D}_{z_t^T}\left\{f_z\right\}\right]\Big|_{\overline{z}} \\ =& 0 \end{aligned} \tag{A-18}$$

This is the generalized Sylvester equation (26) with $j=2$ and $i=0$, and under the expectation are known terms from previous orders.

To determine $y_{z\sigma}$, we differentiate (24) with respect to z_t

$$\begin{aligned}\mathscr{D}^2_{z_t^T\sigma}\{f\} =& \mathscr{D}_{z_t^T}\left\{f_{y^+}\right\}\left(I_{nz}\otimes\left(f_{y^+}\left[y_z^+\left(z_y y_\sigma + z_\varepsilon \varepsilon_{t+1}\right)+y_\sigma^+\right]\right)\right)\\ &+ f_{y^+}\mathscr{D}_{z_t^T}\left\{y_z^+\right\}\left[I_{nz}\otimes\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)\right] + f_{y^+}y_z^+ z_y y_{z\sigma}\\ &+ f_{y^+}y_{z\sigma}^+ z_y y_z + \mathscr{D}_{z_t^T}\left\{f_y\right\}\left(I_{nz}\otimes y_\sigma\right) + f_y y_{z\sigma}\end{aligned} \tag{A-19}$$

$$\text{where}\quad \mathscr{D}_{z_t^T}\left\{y_z^+\right\} = y_{z^2}^+\left(z_y y_z\right)^{\otimes 2} + y_z^+ z_y y_{z^2}$$

Setting the expectation of the foregoing evaluated at the nonstochastic steady state to zero yields

$$\begin{aligned}E_t\left[\mathscr{D}_{z_t^T\sigma}\{f\}\right]\Big|_{\overline{z}} =& f_{y^+}y_{z\sigma}\left(z_y y_z\right) + \left(f_{y^+}y_z z_y + f_y\right)y_{z\sigma}\\ &+ E_t\Big[\mathscr{D}_{z_t^T}\left\{f_{y^+}\right\}\left(I_{nz}\otimes\left(f_{y^+}\left[y_z^+\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)+y_\sigma^+\right]\right)\right)\\ &+ f_{y^+}\mathscr{D}_{z_t^T}\left\{y_z^+\right\}\left[I_{nz}\otimes\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)\right] + \mathscr{D}_{z_t^T}\left\{f_y\right\}\left(I_{nz}\otimes y_\sigma\right)\Big]\Big|_{\overline{z}}\\ =& 0\end{aligned} \tag{A-20}$$

This is (26) with $j = 1$ and $i = 1$.

To determine y_{σ^2}, we differentiate (24) with respect to σ

$$\begin{aligned}\mathscr{D}^2_{\sigma^2}\{f\} =& \mathscr{D}_\sigma\left\{f_{y^+}\right\}\left(I_{nz}\otimes\left(f_{y^+}\left[y_z^+\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)+y_\sigma^+\right]\right)\right)\\ &+ f_{y^+}\mathscr{D}_\sigma\left\{y_z^+\right\}\left(I_{nz}\otimes\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)\right) + f_{y^+}y_z^+ z_y y_{\sigma^2} + f_{y^+}y_{\sigma^2}^+\\ &+ \mathscr{D}_\sigma\left\{f_y\right\}\left(I_{nz}\otimes y_\sigma\right) + f_y y_{\sigma^2}\end{aligned} \tag{A-21}$$

$$\begin{aligned}\text{where}\quad \mathscr{D}_\sigma\left\{f_{y^+}\right\} &= f_{y^{+2}}\left[\left(y_z^+\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)+y_\sigma^+\right)\otimes I_{ny}\right] + f_{yy^+}\left(y_\sigma\otimes I_{ny}\right)\\ \mathscr{D}_\sigma\left\{y_z^+\right\} &= y_{z^2}^+\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)+y_{\sigma z}^+\\ \mathscr{D}_\sigma\left\{f_y\right\} &= f_{y+y}\left[\left(y_z^+\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)+y_\sigma^+\right)\otimes I_{ny}\right] + f_{y^2}\left(y_\sigma\otimes I_{ny}\right)\end{aligned}$$

Evaluating the foregoing at the nonstochastic steady state, taking its expectation, and setting the resulting expression equal to zero yields

$$\begin{aligned}E_t\left[\mathscr{D}^2_{\sigma^2}\{f\}\right]\Big|_{\overline{z}} =& f_{y^+}y_{\sigma^2} + \left(f_{y^+}y_z z_y + f_y\right)y_{\sigma^2}\\ &+ E_t\Big[\mathscr{D}_\sigma\left\{f_{y^+}\right\}\left(I_{nz}\otimes\left(f_{y^+}\left[y_z^+\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)+y_\sigma^+\right]\right)\right)\\ &+ f_{y^+}\mathscr{D}_\sigma\left\{y_z^+\right\}\left(I_{nz}\otimes\left(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}\right)\right) + \mathscr{D}_\sigma\left\{f_y\right\}\left(I_{nz}\otimes y_\sigma\right)\Big]\Big|_{\overline{z}}\end{aligned}$$

(A-22) $=0$

This is (26) with $j=0$ and $i=2$.[25]

A.6 Proof of Lemma 4.5

From (29), it follows that the eigenvalues of $z_y y_z$ are those of $y_z z_y$ plus a zero eigenvalue with algebraic multiplicity *ne* and are thus, following theorem 3.9, all inside the closed unit circle. From Theorem 1 of Magnus and Neudecker (2007, ch .2), the eigenvalues of the Kronecker product of two matrices are equal to the products of the eigenvalues of the two matrices and hence it follows immediately that all the eigenvalues of $(z_y y_z)^{\otimes[j]}$ for all $j \in \mathbb{N}^0$, and hence the trailing pencil of definition 4.3, are also inside the closed unit circle. The eigenvalues of the leading pencil of definition 4.3 are all outside the closed unit circle from proposition 3.13. The two pencils in question have thusly no eigenvalue in common as their spectra are separated by the unit circle.

A.7 Proof of Theorem 4.7

Under our problem statement (1), the derivative of Jin and Judd's (2002) operator $\mathcal{N}(y,\sigma)$ has a leading coefficient matrix given by $f_y + f_{y^+} y_z z_y$ at the steady state. From proposition 3.13, this matrix is necessarily invertible. Hence, we conclude that Jin and Judd's (2002) assumption (ii), from which our theorem 3.9 follows, ensures that their assumption (iii), the invertibility of $\mathcal{N}_y(y,0)$, is necessarily fulfilled.

A.8 Proof of Proposition 5.3

Following the proof of lemma 4.1 in section A.4, we can write the set of equations governing $y_{z^j\sigma}$, for $j \geq 0$, as

(A-23) $$f_{y^+} y_{z^j\sigma} (z_y y_z)^{\otimes[j]} + \left(f_y + f_{y^+} y_z z_y\right) y_{z^j\sigma} + A(j,1) = 0$$

[25] The second moment of future shocks in (A-22) can be identified by multiplying out the terms under the expectation operator. Terms of the form, i.e., $(z_\varepsilon \varepsilon_{t+1}) \otimes (z_\varepsilon \varepsilon_{t+1})$ can be rewritten as $\left(z_\varepsilon^{\otimes[2]}\right)\left(\varepsilon_{t+1}^{\otimes[2]}\right)$ using the mixed Kronecker product rule.

where $A(j,1) = E_t[B(j,1)]$. We will proceed inductively over the terms in $B(j,1)$ where the homogeneity of the set of equations governing $y_{z^j\sigma}$ will follow inductively from the solvability proven in proposition 4.6.

To begin, assume that for some $j \geq 0$, $B(j,1)$ is a set of terms involving a product of at least one of $y_{z^k\sigma}$, $k < j$, or ε_{t+1}, but at most one of the latter. As differentiating

$$f_{y^+} y_{z^j\sigma} (z_y y_z)^{\otimes[j]} + \left(f_y + f_{y^+} y_z z_y\right) y_{z^j\sigma} + B(j,1) = 0 \tag{A-24}$$

with respect to z_t only advances the index j, see section A.4, it follows that

$$\mathscr{D}_{z_t^T\sigma}\{B(j,1)\} = B(j+1,1) \tag{A-25}$$

with $B(j+1,1)$ being a set of terms involving a product of at least one of $y_{z^k\sigma}$, $k < j+1$, or ε_{t+1}, but at most one of the latter. To start the induction, note from (24) that

$$B(0,1) = f_{y^+} y_z^+ z_\varepsilon \varepsilon_{t+1} \tag{A-26}$$

thus, confirming the composition of $B(j,1)$ as a set of terms involving a product of at least one of $y_{z^k\sigma}$, $k < j$, or ε_{t+1}, but at most one of the latter.[26]

Taking expectations

$$A(j,1) = E_t[B(j,1)] \tag{A-27}$$

and as the first moment of ε_t was assumed zero, all terms except those involving only products of $y_{z^k\sigma}$, $k < j$ are eliminated. Thus, if all $y_{z^k\sigma}$, $k < j$ are zero, then $A(j,1)$ and the equation in $y_{z^j\sigma}$ is homogenous. From proposition 4.6 it then follows that $y_{z^j\sigma}$ must also be zero, as a unique solution exists and zero is always a solution of a homogenous equation. Hence by induction, starting from the homogenous equation for y_σ, all $y_{z^j\sigma} = 0$, for $j \geq 0$.

[26] As $k < j$ permits only negative powers of k in $y_{z^k\sigma}$ in $B(0,1)$, it is perhaps useful to examine $B(1,1)$ as well. Eaxmining (A-19) for the second order case, which gives

$$B(1,1) = \mathscr{D}_{z_t^T}\left\{f_{y^+}\right\}\left(I_{nz} \otimes (f_{y^+}[y_z^+(z_y y_\sigma + z_\varepsilon \varepsilon_{t+1}) + y_\sigma^+])\right) + f_{y^+}\mathscr{D}_{z_t^T}\left\{y_z^+\right\}\left[I_{nz} \otimes (z_y y_\sigma + z_\varepsilon \varepsilon_{t+1})\right] + \mathscr{D}_{z_t^T}\left\{f_y\right\}(I_{nz} \otimes y_\sigma)$$
$$\text{where } \mathscr{D}_{z_t^T}\left\{y_z^+\right\} = y_{z^2}^+ (z_y y_z)^{\otimes 2} + y_z^+ z_y y_{z^2}$$

notice that all terms involve a product of at least one of y_σ, or ε_{t+1}, but at most one of the latter.

SFB 649 Discussion Paper Series 2012

For a complete list of Discussion Papers published by the SFB 649, please visit http://sfb649.wiwi.hu-berlin.de.

001 "HMM in dynamic HAC models" by Wolfgang Karl Härdle, Ostap Okhrin and Weining Wang, January 2012.
002 "Dynamic Activity Analysis Model Based Win-Win Development Forecasting Under the Environmental Regulation in China" by Shiyi Chen and Wolfgang Karl Härdle, January 2012.
003 "A Donsker Theorem for Lévy Measures" by Richard Nickl and Markus Reiß, January 2012.
004 "Computational Statistics (Journal)" by Wolfgang Karl Härdle, Yuichi Mori and Jürgen Symanzik, January 2012.
005 "Implementing quotas in university admissions: An experimental analysis" by Sebastian Braun, Nadja Dwenger, Dorothea Kübler and Alexander Westkamp, January 2012.
006 "Quantile Regression in Risk Calibration" by Shih-Kang Chao, Wolfgang Karl Härdle and Weining Wang, January 2012.
007 "Total Work and Gender: Facts and Possible Explanations" by Michael Burda, Daniel S. Hamermesh and Philippe Weil, February 2012.
008 "Does Basel II Pillar 3 Risk Exposure Data help to Identify Risky Banks?" by Ralf Sabiwalsky, February 2012.
009 "Comparability Effects of Mandatory IFRS Adoption" by Stefano Cascino and Joachim Gassen, February 2012.
010 "Fair Value Reclassifications of Financial Assets during the Financial Crisis" by Jannis Bischof, Ulf Brüggemann and Holger Daske, February 2012.
011 "Intended and unintended consequences of mandatory IFRS adoption: A review of extant evidence and suggestions for future research" by Ulf Brüggemann, Jörg-Markus Hitz and Thorsten Sellhorn, February 2012.
012 "Confidence sets in nonparametric calibration of exponential Lévy models" by Jakob Söhl, February 2012.
013 "The Polarization of Employment in German Local Labor Markets" by Charlotte Senftleben and Hanna Wielandt, February 2012.
014 "On the Dark Side of the Market: Identifying and Analyzing Hidden Order Placements" by Nikolaus Hautsch and Ruihong Huang, February 2012.
015 "Existence and Uniqueness of Perturbation Solutions to DSGE Models" by Hong Lan and Alexander Meyer-Gohde, February 2012.

SFB 649, Spandauer Straße 1, D-10178 Berlin
http://sfb649.wiwi.hu-berlin.de

This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".