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Farsighted Stable Sets in Hotelling's Location Games

Junnosuke Shino*

Abstract

We apply farsighted stable set to two versions of Hotelling's location games: one with linear market and another with circular market. It is shown that there always exists a farsighted stable set in both games. In particular, the set of all location profiles that yields equal payoff across all players is shown to be a farsighted stable set. This stable set contains location profiles that reflect minimum differentiation as well as those profiles that reflect local monopoly. These results are in contrast to those obtained by equilibrium analysis. While the stable set uniquely exists when the number of players is 2, the uniqueness is not guaranteed when $n \geq 3$. In particular, we exhibit multiple stable sets in three person location games. We provide possible interpretations of these farsighted stable sets from the viewpoint of players' bargaining power and coalition formation.
(Journal of Economic Literature Classification Number: C71, C72, L13)

Keywords: *Farsighted stable set, Indirect dominance, Hotelling location game, Strategic form game with no-Nash equilibrium, Coalition formation*

1 Introduction

Location model is one of the classical topics in the theory of product differentiation. To describe *"whether sellers will tend to concentrate at one point or to disperse over the area"* (Chamberlin (1933)) as an equilibrium is a fundamental question in this theory. In Hotelling's (1929) duopoly model with linear market, two firms are located at the centre of the market (Minimum Differentiation, hereafter *MD*) in the

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unique equilibrium, rather than differentiating their location point (Local Monopoly, hereafter LM). On the other hand, Eaton and R.G. Lipsey(1975) showed that when the number of firms is more than 2, or when the shape of the market is circle, MD does not necessarily arise in an equilibrium.

Furthermore, in the process of revealing these properties of equilibrium in spatial competition, another fundamental question emerged: As Eaton's paper noted, when the market is linear and the number of firms is three, there exists no pure strategy equilibrium. The existence of pure strategy equilibrium is recovered in a three person location game where after firms' location decisions, consumers has opportunity to relocate (see M. Fujita and J.F. Thisse (1986)). The location game is the special case of games with discontinuous payoff, for which did P. Dasgupta and E. Maskin (1986) show the existence of mixed strategy equilibrium.

In this paper, we employ stable set as a solution concept to analyze these issues in the location games. Specifically, we examine the farsighted stable set, due to Chwe (1994). Stable set was originally proposed by von Neumann and Morgenstern (1953) as a solution to cooperative games. The stable set is a set of outcomes satisfying internal stability and external stability. Internal stability requires that no outcome in the set dominates another. External stability requires that any outcomes outside the set must be dominated by an outcome in the set. In contrast to equilibrium concept, a deviation from an outcome in a stable set is deterred because such deviation induce further deviations, returning to an another outcome in the set by external stability. Internal stability ensures that a final outcome is not preferable for the deviation.

The farsighted stable set is also defined as a set of outcomes satisfying internal and external stabilities. The idea is that given an outcome as a status quo, "far-sighted"players take individual or coalitional moves from it, which in turn may lead to further deviation by another coalition, and so on. Thus, it takes into account a domination of one outcome over an another via chain of such coalitional moves rather than myopic views in the original stable set concept [1] (See Chwe(1994) and Greenberg(1990)).

In this paper, we consider two versions of location game: one with linear market and another with circular market, and we will see that farsighted stable set gives new points of view to the issues discussed above. First, as for the existence of a

[1]For a recent application of farsighted stable set to strategic form games, see Suzuki and Muto (2005). Properties of farsighted stable sets in cooperative game context are investigated in Beal et al. (2008).

solution, it is shown that there always exists a farsighted stable set in both games, no matter what the number of player is. In particular, the set of all location profiles that yields equal payoff across all players is shown to be a farsighted stable set. Therefore, as for the possibility of description of MD or LM, this stable set contains location profiles that reflect minimum differentiation as well as those profiles that reflect local monopoly. These results are in contrast to those obtained in analyses mentioned above. While the stable set uniquely exists when the number of players is 2, the uniqueness is not guaranteed when $n \geq 3$. In particular, we exhibit multiple stable sets in three person location games. We provide possible interpretations of these farsighted stable sets from the viewpoint of players' bargaining power and coalition formation.

The organization of the rest of this paper is as follows. In Section 2, we give formal definition of farsighted stable set and brief comments on its interpretation. In Section 3, we describe two versions of location games as strategic form games. Main results are derived in Section 4. Some concluding remarks are made in Section 5.

2 Farsighted Stable Set

In this section, we give formal definition of farsighted stable set in strategic form games and brief comments on its interpretation.

Let $G = (N, \{Z_i\}_{i \in N}, \{u_i\}_{i \in N})$ be a strategic form game where $N = \{1, 2, ..., n\}$ is the set of players, Z_i is the set of strategies of player i, and u_i is the player i's utility function where $u_i : Z = \times_{i \in N} Z_i \longrightarrow \mathbb{R}$. We call $z = (z_1, ..., z_n) \in Z$ an outcome. When $\sum_{i \in N} u_i(z) = c$ for all $z \in Z$ and for some $c \in \mathbb{R}$, we call that G is constant-sum.

A subset of players $S \subseteq N$ is called a coalition. For any two outcomes $v = (v_1, v_2, ..., v_n) \in Z$ and $w = (w_1, w_2, ..., w_n) \in Z$, we say w is induced from v via coalition $S \subseteq N$, or, S deviates from v to w, denoted $v \longrightarrow_S w$, if $v_i = w_i$ for all $i \in N \backslash S$. $v \longrightarrow_S w$ means that coalition S can move from v to w by itself.

For any two outcomes v and w, we say w indirectly dominates v, denoted $v \ll w$, if there exists a sequence of outcomes $(w^0, w^1, ..., w^K)$ with $v = w^0$ and $w = w^K$, and there exists a sequence of coalitions $(S_0, S_1, ..., S_{K-1})$ such that for all $k \in \{0, 1, ..., K-1\}$, $w^k \longrightarrow_{S_k} w^{k+1}$ and $u_i(w^k) < u_i(w^K)$ for all $i \in S_k$.

The situation behind the definition of indirect dominance is that any coalitions

can be formed without binding contracts. For example, any members in the coalition S_k can enter another coalition S_{k+1}, which induces further deviation. Moreover, it should be noted that players are assumed to be farsighted in the definition of indirect dominance. All members in S_k compare current outcome w^k with not immediate outcome w^{k+1} but with possible final outcome w^K. Therefore, it is likely that S_k moves from w^k to w^{k+1}, which may not be prefered to w^k, anticipating this deviation induces further deviations.

When $K = 1$, we call w directly dominates v. Direct dominance relation between two outcomes was originally proposed by von Neunmann and Morgenstern(1953) in cooperative game context. Greenberg(1990) also introduced notion of direct dominance into noncooperative games. After that, however, direct dominance is criticized for representing only myopic players, and Harsanyi(1974) and Chwe(1994) proposed indirect dominance relation so as to be consistent with a farsighted player's behavior. As for more detailed discussion, see these articles.

Farsighted stable set is a solution concept based on the indirect dominance. A set of outcomes $V \subseteq Z$ is a farsighted stable set if (1)for all $v, w \in V$, neither $v \ll w$ nor $w \ll v$, and (2)for all $v \in Z \backslash V$, there exists $w \in V$ such that $v \ll w$. Conditions (1) and (2) are called internal stability and external stability, respectively.

A farsighted stable set can be interpreted as a set of outcomes which are supported by a "stable standard of behavior". Now suppose that players engage in open bargaining with moves and counter moves, given an outcome $x \in Z$ as a status quo. If we assume that players behave consistent with indirect dominance, it is natural to think that the set of all possible outcomes is $P(x) = \{x\} \cup \{y \in Z | x \ll z\}$. Furthermore, if players have common understanding about which outcomes are not excluded as a consequence of the open bargaining, we can define a mapping $\phi(\cdot)$ that assigns a subset of $P(x)$ as a set of possible agreement points, to each status quo x. In accordance with Greenberg(1990), we call ϕ a standard of behavior. In principle, ϕ can be any arbitrary mapping, but now consider the following stability condition on ϕ;

$$y \in \phi(x) \Longleftrightarrow y \not\prec_S w \ \textit{for all}\ w \in \phi(z) \ \textit{such that}\ y \longrightarrow_S z^2.$$

where $y \not\prec_S w$ means that there exists $i \in S$ such that $u_i(y) \geq u_i(z)$. Then it can be shown that a farsighted stable set V is identical to the set $\cup_{x \in Z} \phi(x)$ (See

[2]Note that ϕ does not necessarily uniquely exist.

Xue(1998)), that is, a farsighted stable set can be interpreted as a set of outcomes which are supported by a certain "stable standard of behavior [3] ".

3 n-Person Location Games

In this section, we describe two versions of spatial competitions as a strategic form game: one with linear market and another with circular market. We call each game a linear location game and a circular location game, respectively.

In the linear location game, we assume that there is a "linear city,"which is expressed as real number line $[0, 1]$. n firms produce identical physical good, and each of them chooses its location point on the line. A firm's production and location accompany no cost. Consumers are uniformly distributed on the line, and each of them always buys one unit of good from a firm which is nearest to her. A firm's utility is the share to the total market. Note that a firm's utility level for a certain location profile depends not only its own location point but also other ones. If multiple firms locate at the same point, we assume that they equally divide their share.

Denote linear location game by G_l. The set of players is $N = \{1, ..., n\}$, and player i's strategy set is $Z_i = [0, 1]$. To describe i's utility function formally, as well as to make our analysis mathematically concise, we introduce some notations. First, for a outcome $z = (z_1, ..., z_n) \in Z$, let $k(z)$ be the number of distinct values of $(z_1, ..., z_n)$, and denote them, in an increasing order, $L_1(z), L_2(z), ..., L_{k(z)}(z)$. Denote a vector of location points by $L(z) = (L_1(z), L_2(z), ..., L_{k(z)}(z))$. Note that $L(z)$ has finite elements since $k(z) \leq n$. Furthermore, for $L_j(z)$ $(j \in \{1, ..., k(z)\})$, we denote the number of players locating at $L_j(z)$ by $F_j(z)$, and define $F(z) = (F_1(z), F_2(z), ..., F_{k(z)}(z))$. Finally, we denote the distance between z_i and z_j by $\|z_i - z_j\| = |z_i - z_j|$ where $|\cdot|$ denote absolute value.

We give an example in Figure 1 (left). In this example, $z = (3/5, 1/4, 0, 1/4)$, $k(z) = 3$, $L(z) = (0, 1/4, 3/5)$, $F(z) = (1, 2, 1)$, and $\|z_1 - z_2\| = 7/20$, $\|z_2 - z_4\| = 0$, and so forth. i's utility is defined based on our assumption discussed above. For instance, the sum of player 2 and 4's share to the total market is 3/10. Since they divide this equally, $u_2(z) = u_4(z) = 3/20$. We can also check easily that

[3]More precisely, this standard of behavior is defined as optimistic standard of behavior, which was originally proposed in Chwe (1994). He also defined conservative standard of behavior and consistent set which is a set of outcome associated with this standard of behavior.

$u_1(z) = 23/40$ and $u_3(z) = 1/8$. Note that G_l is constant-sum.

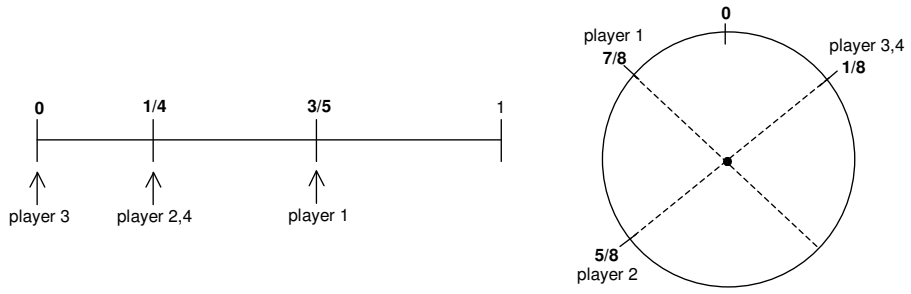

Figure 1: Linear Location Game (Left) and Circular Location Game (Right)

Formally, we define player i's utility function in G_l as follows;

(1)When $k(z) = 1$,

$$u_i(z) = 1/n \quad for\ all\ i \in N$$

(2)When $k(z) = 2$,

$$u_i(z) = \begin{cases} \dfrac{1}{F_1(z)}\left(L_1(z) + \dfrac{L_2(z) - L_1(z)}{2}\right) & if \quad z_i = L_1(z) \\ \dfrac{1}{F_2(z)}\left(1 - L_2(z) + \dfrac{L_2(z) - L_1(z)}{2}\right) & if \quad z_i = L_2(z). \end{cases}$$

(3)When $k(z) \geq 3$,

$$u_i(z) = \begin{cases} \dfrac{1}{F_1(z)}\left(L_1(z) + \dfrac{L_2(z) - L_1(z)}{2}\right) & if \quad z_i = L_1(z) \\ \dfrac{1}{F_m(z)}\left(\dfrac{L_{m+1}(z) - L_{m-1}(z)}{2}\right) & if \quad z_i = L_m(z)\ (m \neq 1,\ m \neq k(z)) \\ \dfrac{1}{F_{k(z)}(z)}\left(1 - L_{k(z)}(z) + \dfrac{L_{k(z)}(z) - L_{k(z)-1}(z)}{2}\right) & if \quad z_i = L_{k(z)}(z). \end{cases}$$

In the circular location game, the only difference is that we assume a "circular city,"which is expressed as a circle with a perimeter 1, instead of a linear city. Consumers are uniformly distributed on the perimeter, and each firm chooses its

location point to maximize the share to the total market. We give a number 0 to an arbitrary point on the perimeter.

Denote the circular location game by G_c. Note that in G_c, player i's strategy set is $Z_i = [0, 1)$. Furthermore, note that $k(z)$, $L(z)$, and $F(z)$ can be defined in the same fashion as those in G_l. We define $\|z_i - z_j\|$ as a shorter length among two arcs formed by z_i an z_j. An example is shown in Figure 1 (right). $z = (7/8, 5/8, 1/8, 1/8)$, $k(z) = 3$, $L(z) = (1/8, 5/8, 7/8)$, $F(z) = (2, 1, 1)$, and $\|z_1 - z_2\| = 1/4$, $\|z_1 - z_3\| = 1/4$, and so on. i's utility is also defined in the same manner as G_l: $u_1(z) = 1/4$, $u_2(z) = 3/8$, and $u_3(z) = u_4(z) = 3/16$. Again, note that G_c is constant-sum.

The formal expression of player i's utility function in G_c can be expressed as follows;

(1)When $k(z) = 1$,

$$u_i(z) = 1/n \quad for\ all\ i \in N$$

(2)When $k(z) = 2$,

$$u_i(z) = \frac{1}{2F_m(z)} \qquad if\ z_i = L_m(z) \quad (m = 1, 2)$$

(3)When $k(z) \geq 3$,

$$u_i(z) = \begin{cases} \dfrac{1}{F_1(z)} \left(\dfrac{1 - L_{k(z)}(z) + L_2(z)}{2} \right) & if \quad z_i = L_1(z) \\ \dfrac{1}{F_m(z)} \left(\dfrac{L_{m+1}(z) - L_{m-1}(z)}{2} \right) & if \quad z_i = L_m(z)\ (m \neq 1,\ m \neq k(z)) \\ \dfrac{1}{F_{k(z)}(z)} \left(\dfrac{1 - L_{k(z)-1}(z) + L_1(z)}{2} \right) & if \quad z_i = L_{k(z)}(z). \end{cases}$$

4 Farsighted Stable Sets in Location Games

In this section, we derive farsighted stable sets in each location game, G_l and G_c. Before doing so, however, it is useful to mention some properties of indirect dominance relations and farsighted stable sets in general strategic form games.

Remark 4.1
In a strategic form game G, for all $x, y, z \in Z$, if $y \ll z$, $x \rightarrow_S y$ and $u_i(x) < u_i(z)$ for all $i \in S$, then $x \ll z$.

Remark 4.1 is straightforward from the definition of the indirect dominance.

Remark 4.2
Let G be a strategic form game and V and $\tilde{V}$ be two different farsighted stable set. Then neither $\tilde{V} \subsetneq V$ nor $V \subsetneq \tilde{V}$ can be true.

Proof.[4] Suppose that $\tilde{V} \subsetneq V$ and pick an outcome x such that $x \in V$ and $x \notin \tilde{V}$. From the external stability of $\tilde{V}$, there exists $\tilde{x} \in \tilde{V}$ such that $x \ll \tilde{x}$. However, since $x \in V$ and $\tilde{x} \in V$, it contradicts the internal stability of V. **Q.E.D**.

Lemma 4.1
Let G be a two-person constant-sum strategic form game and V be a farsighted stable set in G. Then

$$u_i(x) \geq \max_{z_i \in Z_i} \inf_{z_j \in Z_j} u_i(z) \ for\ all\ x \in V\ and\ for\ all\ i \in N.$$

Proof. See Appendix.

Lemma 4.1 says that in any stable sets, every outcome in the set assigns every player equal to or greater than their "maxinf "value when the number of players is 2 and the game is constant. Note that $\min_{z_j \in Z_j} u_i(z)$ is not well-defined since $u_i(z)$ is not continuous. Thus, we consider "maxinf "value instead of "maxmin."

4.1 Linear Location Game

Now we derive farsighted stable sets in the linear and circular location games defined in the previous section. First, we examine farsighted stable sets in G_l.

Theorem 4.1
Let

$$V_e = \{z \in Z | u_i(z) = 1/n \ for\ all\ i \in N\},$$

then V_e is a farsighted stable set in the linear location game G_l.

Proof. The internal stability of V_e is obviously satisfied from the property of V_e. To check the external stability, pick $x^0 \notin V_e$. Note that $k(x^0) \geq 2$ (otherwise $x^0 \in V_e$) and we show this statement by using induction on $k(x^0)$.

[4]The proof is basically identical as seen in basic game theory textbooks. See M.Osbone and A.Rubinstein(1994), for instance.

First, we show the following;

$$\textit{for every } x^0 \notin V_e \textit{ with } k(x^0) = 2, \textit{ there exists } z \in V_e \textit{ such that } x^0 \ll z.$$

Since $x^0 \notin V_e$ and the game is constant-sum, there exists a location point $L_m(x^0)$ ($m = 1$ or 2) such that $u_i(x^0) < 1/n$ for all i with $x_i^0 = L_m(x^0)$. Let $S_0 = \{i \in N | x_i^0 = L_m(x^0)\}$ and consider $z \in Z$ such that (i) $z_j = x_j^0$ for all $j \in N \setminus S_0$ and (ii) $z_i = z_j$ for all $i \in S_0$. Note that $z \in V_e$ and the fact that $x^0 \longrightarrow_{S_0} z$ is possible and $u_i(x^0) < 1/n = u_i(z)$ for every $i \in S_0$ implies $x^0 \ll z$.

Next, suppose that the following statement is true;

$$\textit{for every } \tilde{x}^0 \notin V_e \textit{ with } k(\tilde{x}^0) = \tilde{k}, \textit{ there exists } z \in V_e \textit{ such that } \tilde{x}^0 \ll z, \quad (1)$$

and we must show;

$$\textit{for every } x^0 \notin V_e \textit{ with } k(x^0) = \tilde{k} + 1, \textit{ there exists } z \in V_e \textit{ such that } x^0 \ll z. \quad (2)$$

Pick $x^0 \notin V_e$ with $k(x^0) = \tilde{k}+1$. Similarly to the previous case, $x^0 \notin V_e$ guarantee the existence of a location point $L_m(x^0)$ ($m \in \{1, ..., \tilde{k}+1\}$) such that $u_i(x^0) < 1/n$ for all i with $x_i^0 = L_m(x^0)$. Let $S_0 = \{i \in N | x_i^0 = L_m(x^0)\}$ and consider S_0's deviation from x^0 to $\hat{x}^0$ such that (i) $\hat{x}_j^0 = x_j^0$ for all $j \in N \setminus S_0$ and (ii) $\hat{x}_i^0 = \hat{x}_j^0$ for all $i \in S_0$. If $\hat{x}^0 \in V_e$, then $u_i(x^0) < 1/n = u_i(\hat{x^0})$ thus $x^0 \ll \hat{x}^0$. Therefore, (2) is true. If $\hat{x}^0 \notin V_e$, then note that $k(\hat{x}^0) = \tilde{k}$ thus there exists $z \in V_e$ satisfying $\hat{x}^0 \ll z$ from (1). Now Remark 4.1 is applied and we conclude that $x^0 \ll z$. **Q.E.D**.

Theorem 4.1 says that a stable set V_e always exists in G_l whatever the number of player is. Note that V_e contains all outcomes such that all players locate at a same point, each of those reflects MD. Moreover, V_e also contains outcomes in which every player locates at different point and each of them enjoys its monopoly power in its own territory. Therefore, we can also see that certain outcomes in V_e describe LM. Obviously, an outcome representing mixture of MD and LM is also in V_e.

Interestingly, the only required property for an outcome to be an element of V_e is that the market share has to be divided equally among all players. One possible interpretation behind this farsighted stable set is that all players' bargaining powers are well-balanced thus they agree with a completely equal allocation. In this sense, V_e can be interpreted as a consequence of grand coalition formation. We will discuss

this point further in the case of $n = 3$ after showing Remark 4.3.

Next, we examine some results about uniqueness.

First, we show that when the number of player is 2, V_e is the unique farsighted stable set in G_l

Theorem 4.2
When $n = 2$, V_e in Theorem 4.1 is the unique farsighted stable set in G_l.

Proof. Consider G_l and suppose $n = 2$. Note that $\max_{z_i \in Z_i} \inf_{z_j \in Z_j} u_i(z) = 1/2$ for all $i \in \{1, 2\}$ since

$$\inf_{z_j \in Z_j} u_i(z_i, z_j) \begin{cases} = \frac{1}{2} & when \quad z_i = \frac{1}{2} \\ < \frac{1}{2} & when \quad z_i \neq \frac{1}{2}. \end{cases}$$

Therefore, from Lemma 4.1 and since the game is constant-sum, if V is a stable set in G_l, $u_i(x) = 1/2$ for all $x \in V$ and for all $i \in \{1, 2\}$. This means that if there exists a stable set $\tilde{V}$ with $\tilde{V} \neq V_e$, it has to satisfy $\tilde{V} \subsetneq V_e$ but it is impossible from Remark 4.2. **Q.E.D**.

When $n \geq 3$, the uniqueness of V_e is not guaranteed. Remark 4.3 shows that when $n = 3$, multiple stable sets exist in G_l

Remark 4.3
For α such that $\dfrac{1}{3} < \alpha < \dfrac{1}{2}$, let $i, j, k \in N$ and

$$V_{ij}(\alpha) \quad = \quad \{z \in Z |\ u_k(z) = \alpha,\ u_i(z) = u_j(z) = \frac{1-\alpha}{2}\}$$

then $V_{ij}(\alpha)$ is a farsighted stable set in G_l when $n = 3$.

Proof. In the following proof, we show $V_{23}(\alpha)$ is a farsighted stable set. The internal stability of $V_{23}(\alpha)$ is obviously satisfied. Before checking $V_{23}(\alpha)$'s external stability, note that:

For $1/3 < \alpha < 1/2$, let z^L and z^R;

$$z^L = (z_1^L, z_2^L, z_3^L) = \left(2\alpha - \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right) \quad z^R = (z_1^R, z_2^R, z_3^R) = \left(\frac{3}{2} - 2\alpha,\ \frac{1}{2},\ \frac{1}{2}\right),$$

then $z^L,\ z^R \in V_{23}(\alpha)$. Note that $1/6 < z_1^L < 1/2$ and $1/2 < z_1^R < 5/6$.

Now we check the external stability of $V_{23}(\alpha)$. Pick $x \notin V_{23}(\alpha)$ and we will show that there exists $z \in V_{23}(\alpha)$ such that $x \ll z$.

First, suppose $k(x) = 1$. (1)when $x_1 = x_2 = x_3 = 1/2$, player 1 deviates from x to $z = z^L$ or $z = z^R$. Note that $z \in V_{23}(\alpha)$. Since $u_1(z) = \alpha > 1/3$, $x \ll z$. (2)when $x_1 = x_2 = x_3 \neq 1/2$, w.l.o.g, suppose that $x_1 = x_2 = x_3 = 1/2 + d$ where $0 < d \leq 1/2$. Assume that player 1 deviates from x to $y = (z_1^L, x_2, x_3)$, then coalition $\{23\}$ deviates from y to $z = z^L$. Note that $u_2(y) = u_3(y) = 1/2\{(1/2 - d) + 1/2(d + 1/2 - 2\alpha + 1/2)\} = 1/2(1 - \alpha) - (1/4)d < u_2(z) = u_3(z)$ and $u_1(x) = 1/3 < u_1(z)$. Therefore, from the definition of indirect dominance relation, we conclude $x \ll z$.

Next, Suppose $k(x) = 2$. We divide this case into the following three cases: (A), (B), and (C).

(A)Suppose $x_1 \neq x_2 = x_3$. (1)When $u_1(x) < \alpha$, consider player 1's deviation from x to $y = (x_2, x_2, x_3)$. Since $k(y) = 1$, there exists $z \in V_{23}(\alpha)$ such that $y \ll z$. Furthermore, since $u_1(x) < \alpha = u_1(z)$, Remark 4.1 implies that $x \ll z$. (2)Suppose $u_1(x) > \alpha$. Since G_l is constant-sum, this is equivalent to $u_2(x) = u_3(x) < (1-\alpha)/2$. Assume that coalition $\{23\}$ deviates from x to $y = (x_1, x_1, x_1)$. Again, since $k(y) = 1$, there exists $z \in V_{23}(\alpha)$ such that $y \ll z$ and $u_2(x) = u_3(x) < u_2(z) = u_3(z)$, thus $x \ll z$. (3)Note that we don't need to consider the case $u_1(x) = \alpha$ because such x is in $V_{23}(\alpha)$ from the assumption $x_1 \neq x_2 = x_3$.

(B)Suppose $x_2 \neq x_1 = x_3$. (1)when $u_2(x) < (1 - \alpha)/2$, consider player 2's deviation from x to $y = (x_1, x_1, x_3)$ where $k(y) = 1$. Same argument holds and it follows that there exists $z \in V_{23}(\alpha)$ such that $x \ll z$. (2)Suppose $u_2(x) \geq (1-\alpha)/2$, which is equivalent to $u_1(x) = u_3(x) \leq (1 + \alpha)/4$. Note that $(1 - \alpha)/2 < (1 + \alpha)/4$ from the assumption on α. If $u_1(x) = u_3(x) < (1-\alpha)/2$, then consider deviation of the coalition $\{13\}$ from x to $y = (x_2, x_2, x_2)$ and note that there exists $z \in V_{23}(\alpha)$ such that $y \ll z$. Since $u_1(x) < (1 - \alpha)/2 < \alpha = u_1(z)$ and $u_3(x) < (1 - \alpha)/2 = u_3(z)$, it follows that $x \ll z$. If $(1 - \alpha)/2 \leq u_1(x) = u_3(x) \leq (1 + \alpha)/4$, note that this is equivalent to $(1 - \alpha)/2 \leq u_2(x) \leq \alpha$. Now consider player 1's deviation from x to $y = (x_2, x_2, x_3)$ and the coalition $\{12\}$'s consecutive deviation from y to $w = (x_3, x_3, x_3)$. Note that there exists $z \in V_{23}(\alpha)$ such that $w \ll z$. Now we need to check whether the sequence of the outcomes $(x, y, w, ..., z)$ satisfies the condition on $x \ll z$. First, $u_1(x) \leq (1 + \alpha)/4 < \alpha = u_1(z)$ since $\alpha > 1/3$. Next, since $L(x) = L(y)$, $u_1(y) = u_2(y) = u_2(x)/2$, which implies that $(1-\alpha)/4 \leq u_1(y) = u_2(y) \leq \alpha/2$. Therefore, $u_1(y) \leq \alpha/2 < \alpha = u_1(z)$ and $u_2(y) \leq \alpha/2 < (1 - \alpha)/2 = u_2(z)$ since $\alpha < 1/2$. Thus, from Remark 4.1., we conclude that $x \ll z$.

(C)When $x_3 \neq x_1 = x_2$, the proof is identical to the case (B).

Finally, suppose that Suppose $k(x) = 3$. Since $x \notin V_{23}(\alpha)$ and G_l is constant-sum, there exists $i \in N$ such that $u_i(x) < u_i(\tilde{z})$ where $\tilde{z} \in V_{23}(\alpha)$. Take i's deviation from x to y satisfying $k(y) = 2$, and note that we can always take such deviation. From the previous part, we know that there exists $z \in V_{23}(\alpha)$ such that $y \ll z$. From Remark 4.1, $x \ll z$. **Q.E.D.**

Now compare V_e in Theorem 4.1 with $V_{ij}(\alpha)$ in Remark 4.3. First of all, note that i's "minsup "value in G_l is $1/4$ when $n = 3$[5]. Minsup value is i's payoff which can be obtained against the other players' worst joint actions for him, provided that i takes a best action to it. Thus, minsup value can be regarded as i's maximum payoff which can be obtained by his own, when the "worst scenario "for him happens. In V_e, every player gets the market share strictly greater than its minsup payoff ($1/4 < 1/3$) and completely equal allocations are attained. Therefore, one possible interpretation of V_e is that players form grand coalition for the purpose to avoid realizing their "worst scenario ".

On the other hand, note that in $V_{ij}(\alpha)$, $1/3 < u_k(z) < 1/2$, $1/4 < u_i(z) < 1/3$ and $1/4 < u_j(z) < 1/3$ for every $z \in V_{ij}(\alpha)$. Therefore, in $V_{ij}(\alpha)$, all players get market shares strictly greater than their minsup value. This is a common property we can observe both in V_e and $V_{ij}(\alpha)$. However, at the same time, note that player k succeeds to get the share strictly greater than that in V_e, while i and j's share is strictly smaller than those in V_e. One possible interpretation of $V_{ij}(\alpha)$ is that these uneven allocations reflect each player's bargaining power. Namely, in $V_{ij}(\alpha)$, player k has "strong"power relative to i and j, while in V_e, each player's power is well-balanced. Coalition $\{ij\}$ convinces player k to accept $V_{ij}(\alpha)$ by giving strictly greater payoff than what k would get in V_e, and both of the "weaker"players in the coalition succeed to avoid falling into their minsup values.

[5]This can be checked as follows: (1)for any pairs (x_j, x_k) satisfying $x_j = x_k$, player i can get at least nearly to $1/2$ by locating close to $x_j = x_k$ but in the longer halfline formed by $x_j = x_k$ and the edge of the market. Then w.l.o.g, suppose $x_j < x_k$. (2)when $\|x_j - x_k\| > 1/2$, i locates in the open set (x_j, x_k) and gets strictly more than $1/4$. (3)when $\|x_j - x_k\| \leq 1/2$ and $[x_j > 1/4$ *or* $x_k < 3/4]$, i gets at least $1/4$ by locating close to x_j (or x_k) but in the halfline between 0 (or 1) and x_j (or x_k). (4)Finally, suppose $\|x_j - x_k\| \leq 1/2$ and $[x_j \leq 1/4$ *and* $x_k \geq 3/4]$. In this case, the only possibility is $(x_j, x_k) = (1/4, 3/4)$. For this pair, i's best response is locating in the closed set $[1/4, 3/4]$ and gets $1/4$.

4.2 Circular Location Game

Next, we examine farsighted stable sets in G_c.

Theorem 4.3
Let

$$V_e = \{z \in Z | u_i(z) = 1/n \ for \ all \ i \in N\},$$

then V_e is a farsighted stable set in the circular location game G_c.

Proof. The proof is identical to the case of linear location game.

As stated in Theorem 4.1, Theorem 4.3 also shows that V_e always exists in G_c, and V_e includes outcomes representing both MD and LM. It should be noted that the characteristics of V_e does not depend on the shapes of the markets, which is contrast to results obtained by equilibrium analysis (See Eaton and Lipsey (1975)). Obviously, we can adopt same interpretation of V_e as discussed after Theorem 4.1 in terms of grand coalition formation.

Then, we show that when the number of player is 2, V_e is the unique farsighted stable set in G_c.

Theorem 4.4
When $n = 2$, V_e in Theorem 4.3 is the unique farsighted stable set in G_c.

Proof. Consider G_c and suppose $n = 2$. Note that, in this case, $V_e = Z$. Since V_e is a farsighted stable set from Theorem 4.3, Remark 4.2 directly implies that V_e is the unique farsighted stable set. **Q.E.D**.

Similarly to G_l, the uniqueness of V_e is not guaranteed when $n \geq 3$. We exhibit multiple stable sets in the case of $n = 3$ in Remark 4.4.

Remark 4.4
For α such that $0 < \alpha < \frac{1}{4}$, let $i, j, k \in N$ and

$$V_{ij}(\alpha) \quad = \quad \{z \in Z | \ u_i(z) = \frac{1}{4} + \alpha, \ u_j(z) = \frac{1}{2} - \alpha, \ u_k(z) = \frac{1}{4}\}$$

then $V_{ij}(\alpha)$ is a farsighted stable set in G_c when $n = 3$.

Proof. In the following proof, we show that $V_{23}(\alpha)$ is a farsighted stable set. The internal stability of $V_{23}(\alpha)$ is obviously satisfied. Before checking $V_{23}(\alpha)$'s external stability, for $0 < \alpha < 1/4$ and for $(x_1, x_2, x_3) \in Z$, define a set $\tilde{Z}(\alpha, x_1)$ such that

$$\tilde{Z}(\alpha, x_1) = \{\tilde{z} \in Z | \tilde{z}_1 = x_1, \|\tilde{z}_1 - \tilde{z}_2\| = 2\alpha, \|\tilde{z}_2 - \tilde{z}_3\| = 1/2\}.$$

Note that $\tilde{Z}(\alpha, x_1)$ contains two elements and note that if $\tilde{z} \in \tilde{Z}(\alpha, x_1)$, then $\tilde{z} \in V_{23}(\alpha)$. Now we check the external stability of $V_{23}(\alpha)$. Pick $x \notin V_{23}(\alpha)$ and we will show that there exists $z \in V_{23}(\alpha)$ such that $x \ll z$.

First, pick $x \notin V_{23}(\alpha)$ such that $\underline{k(x) = 1}$. Pick $\tilde{z} \in \tilde{Z}(\alpha, x_1)$ and note that from the assumption on α, either $u_2(\tilde{z}) > 1/3$ or $u_3(\tilde{z}) > 1/3$ is satisfied. (1)Suppose $u_2(\tilde{z}) > 1/3$. Then take player 2's deviation from x to $y = (x_1, \tilde{z}_2, x_3)$ and player 3's consecutive deviation from y to $z = \tilde{z}$. Now we need to check whether the sequence of the outcomes (x, y, z) satisfies the condition on $x \ll z$. First, $u_2(x) = 1/3 < u_2(z)$. Next, $u_3(y) = 1/4 < 1/2 - \alpha = u_3(z)$ since $0 < \alpha < 1/4$. Therefore, $x \ll z$ holds. (2)Suppose $u_3(\tilde{z}) > 1/3$. In this case, the sequence of deviations $x \longrightarrow_{\{3\}} y \longrightarrow_{\{2\}} z$ such that $y = (x_1, x_2, \tilde{z}_3)$ and $z = \tilde{z}$ satisfies the condition on $x \ll z$.

Next, suppose $\underline{k(x) = 2}$. We divide this case into the following three cases: (A), (B), and (C).

(A)Suppose $x_1 \neq x_2 = x_3$. Consider a deviation of the coalition $\{23\}$ from x to z where $z \in \tilde{Z}(\alpha, x_1)$. Since $u_2(x) = 1/4 < 1/4 + \alpha = u_2(z)$ and $u_3(x) = 1/4 < 1/2 - \alpha = u_3(z)$, $x \ll z$.

(B)Suppose $x_2 \neq x_1 = x_3$. Take player 3's deviation from x to $y = (x_1, x_2, x_2)$ and coalition $\{23\}$'s further deviation from y to z where $z \in \tilde{Z}(\alpha, x_1)$. Since $u_3(x) = 1/4 < u_3(z)$, $u_2(y) = 1/4 < u_2(z)$, and $u_3(y) = 1/4 < u_3(z)$, we conclude $x \ll z$.

(C)When $x_3 \neq x_1 = x_2$, take the sequence of deviations $x \longrightarrow_{\{2\}} y \longrightarrow_{\{23\}} z$ such that $y = (x_1, x_3, x_3)$ and $z \in \tilde{Z}(\alpha, x_1)$. Then as discussed in (B), $x \ll z$ holds.

Finally, suppose that $\underline{\text{Suppose } k(x) = 3}$. Since $x \notin V_{23}(\alpha)$ and G_c is constant-sum, there exists $i \in N$ such that $u_i(x) < u_i(\hat{z})$ where $\hat{z} \in V_{23}(\alpha)$. Take i's deviation from x to y where $k(y) = 2$, and note that we can always take such deviation. From the previous part, we know that there exists $z \in V_{23}(\alpha)$ such that $y \ll z$. Thus from Remark 4.1, $x \ll z$. **Q.E.D.**

Now compare V_e in Theorem 4.3 with $V_{ij}(\alpha)$ in Remark 4.4. First, in G_c, minsup value is 1/4 when $n = 3$ [6], which is identical to that in G_l. Therefore, V_e gives every player the market share strictly greater than its minsup payoff, and completely equal allocations are attained. Thus, we can interpret V_e as a result of grand coalition formation, as discussed in linear market case.

On the other hand, note that in $V_{ij}(\alpha)$, $1/4 < u_i(\cdot) < 1/2$ and $1/4 < u_j(\cdot) < 1/2$. The common property of $V_{ij}(\alpha)$s in G_l and G_c is that every player gets more than his minsup value. One interesting point uniquely lies in G_c comes from the fact that at least one of $u_i(\cdot)$ and $u_j(\cdot)$ is always strictly greater than 1/3. Keep this in mind, we can interpret $V_{ij}(\alpha)$ in G_c as follows. Player i possessing relatively "strong"bargaining power proposes to form coalition $\{ij\}$ to player j, guaranteeing to give j strictly greater share than j's minsup value. Which α is realized depends on relative strength of j's bargaining power over i. For example, when j's power is not so much weaker than that of i, i and j are likely to reach an agreement that both of them get share more than 1/3 ($1/12 < \alpha < 1/6$ would be realized). On the other hand, if j's bargaining power over i is weak, it seems to be natural to predict that j would be forced to agree with a share greater than minsup value but less than 1/3 ($\alpha > 1/6$), which would have been attained if their power were well balanced.

Recall that when considering $V_{ij}(\alpha)$ in G_l, i and j form coalition and both of them are assumed to have "weaker"bargaining power. In G_c, contrastingly, we can interpret that at least one of the members of the coalition can be assumed to have "strong"bargaining power in $V_{ij}(\alpha)$.

5 Concluding Remarks

In this paper we studied farsighted stable sets in location games when firms act with farsightedness. We showed that there always exists a farsighted stable set, which depicts both of two seemingly contrast phenomena; MD and LM. These phenomena were integrated under the same cause; completely equal allocations between all players. Then we showed that the stable set exists uniquely when

[6]Check i's minsup value. First, if $x_j = x_k$ i's best response is to locate any points different from $x_j = x_k$ and get 1/2. Next, if $\|x_j - x_k\| = 1/2$, i gets 1/4 no matter which point he chooses. Finally, if $x_j \neq x_k$ and $\|x_j - x_k\| < 1/2$, i can get the share strictly greater than 1/4 by locating in the wider arc of the two formed by x_j and x_k. Thus note that in this case, "minmax "value is well defined, instead of "minsup"one. But here we use the term "minsup "to keep consistency of our usage of the term.

$n = 2$, while other stable sets exist when $n = 3$. In the case of $n = 3$, we also provided a possible interpretation that all of these farsighted stable sets can be seen as a consequence of players' coalition formation.

We conclude this paper by pointing out several questions for further research. First, our approach in this paper reminds us of another notion of farsighted stability proposed by Chwe (1994), which is called consistent set. As discussed in his paper, consistent set can be regarded as representation of players' conservative stable standard of behavior, while farsighted stable set can be considered as optimistic one. By applying consistent set to location games, we might obtain some further insights.

Next, in this paper we did not find all farsighted stable sets in location games in the case of $n = 3$. Moreover, when $n > 3$, we did not show any statements about the uniqueness of V_e. Finding other stable sets in these cases could be an interesting topic.

Finally, one interesting question is which and how a certain farsighted stable set is realized corresponding each player's bargaining power. For example, as discussed in section 4, we showed that V_e and $V_{ij}(\alpha)$ reflect players' underling bargaining powers in each case. Moreover, in a specific $V_{ij}(\alpha)$, a realized α also depends on their bargaining powers. To answer which α is realized, which coalition formation are plausible for a certain stable set, and more generally, which stable standard of behavior would be formed among all players, we need to formulate the dynamic process of communication or bargaining with farsighted players.

Appendix

Proof of Lemma 4.1. Let G be a two person constant-sum strategic form game and V a stable set in G. Assume that, in negation, there exists $\tilde{z} \in V$ and there exists $i \in N$ such that

$$u_i(\tilde{z}) < \max_{z_i \in Z_i} \inf_{z_j \in Z_j} u_i(z),$$

and we derive contradiction. Let z_i^* be i's "maxinf "strategy, that is, $\inf_{z_j \in Z_j} u_i(z_i^*, z_j) = \max_{z_i \in Z_j} \inf_{z_j \in Z_j} u_i(z_i, z_j)$. Then, from the assumption,

$$u_i(\tilde{z}_i, \tilde{z}_j) < \max_{z_i \in Z_i} \inf_{z_j \in Z_j} u_i(z) = \inf_{z_j \in Z_j} u_i(z_i^*, z_j),$$

and since $\inf_{z_j \in Z_j} u_i(z_i^*, z_j) \leq u_i(z_i^*, \tilde{z}_j)$,

$$u_i(\tilde{z}_i, \tilde{z}_j) < u_i(z_i^*, \tilde{z}_j). \tag{3}$$

Therefore, from the definition of indirect dominance relation, $(\tilde{z}_i, \tilde{z}_j) \ll (z_i^*, \tilde{z}_j)$. Since $(\tilde{z}_i, \tilde{z}_j) \in V$, $(z_i^*, \tilde{z}_j) \notin V$ must be true to preserve the internal stability of V. From the external stability, it follows that there has to exist $(\tilde{w}_i, \tilde{w}_j) \in V$ such that $(z_i^*, \tilde{z}_j) \ll (\tilde{w}_i, \tilde{w}_j)$. From the definition of indirect dominance, the following has to be satisfied;

there exists a sequence of outcomes $(w^0, w^1, ..., w^K)$ *with* $w^0 = (z_i^*, \tilde{z}_j)$ *and* $w^K = (\tilde{w}_i, \tilde{w}_j)$*, and there exists a sequence of coalitions* $(S_0, S_1, ..., S_{K-1})$ *such that for all* $k \in \{0, 1, ..., K-1\}$*, (A)*$w^k \longrightarrow_{S_k} w^{k+1}$ *and (B)*$u_i(w^k) < u_i(w^K)$ *for all* $i \in S_k$*.* (4)

Now consider a coalition S_0 in (4) above.

First, suppose $\underline{i \in S_0}$. Since $(z_i^*, \tilde{z}_j) \ll (\tilde{w}_i, \tilde{w}_j)$, $u_i(z_i^*, \tilde{z}_j) < u_i(\tilde{w}_i, \tilde{w}_j)$. From the inequality (3) above, it follows that $u_i(\tilde{z}_i, \tilde{z}_j) < u_i(\tilde{w}_i, \tilde{w}_j)$. Since $(\tilde{z}_i, \tilde{z}_j) \longrightarrow_{\{i\}} (z_i^*, \tilde{z}_j)$ is possible, from Remark 4.1, $(\tilde{z}_i, \tilde{z}_j) \ll (\tilde{w}_i, \tilde{w}_j)$. However, since $(\tilde{z}_i, \tilde{z}_j) \in V$ and $(\tilde{w}_i, \tilde{w}_j) \in V$, it contradicts the internal stability of V.

Next, suppose $\underline{S_0 = \{j\}}$. (A) if $S_1 = \cdots = S_{K-1} = \{j\}$ in the sequence of coalition of (4), note that $(\tilde{w}_i, \tilde{w}_j) = (z_i^*, \tilde{w}_j)$. In this case, $u_i(\tilde{z}_i, \tilde{z}_j) < \inf_{z_j \in Z_j} u_i(z_i^*, z_j) \leq u_i(z_i^*, \tilde{w}_j) = u_i(\tilde{w}_i, \tilde{w}_j)$. Therefore, since $u_i(\tilde{z}_i, \tilde{z}_j) < u_i(\tilde{w}_i, \tilde{w}_j)$, $(z_i^*, \tilde{z}_j) \ll (\tilde{w}_i, \tilde{w}_j)$ and $(\tilde{z}_i, \tilde{z}_j) \longrightarrow_{\{i\}} (z_i^*, \tilde{z}_j)$ is possible, $(\tilde{z}_i, \tilde{z}_j) \ll (\tilde{w}_i, \tilde{w}_j)$, from Remark 4.1. Again, this violates the internal stability of V. (B)Suppose there exists a coalition S_p ($p \in \{1, ..., K-1\}$) such that $i \in S_p$. Let $S_{\tilde{p}}$ be the first coalition including i in the sequence of coalitions in (4), that is, if $\tilde{k} < \tilde{p}$, then $S_{\tilde{k}} = \{j\}$. Now consider $(w_i^{\tilde{p}}, w_j^{\tilde{p}}) \longrightarrow_{S_{\tilde{p}}} (w_i^{\tilde{p}+1}, w_j^{\tilde{p}+1})$ in (4). Since $z_i^* = w_i^0 = w_i^1 = ... = w_i^{\tilde{p}}$, this is equal to $(z_i^*, w_j^{\tilde{p}}) \longrightarrow_{S_{\tilde{p}}} (w_i^{\tilde{p}+1}, w_j^{\tilde{p}+1})$. Recall that $i \in S_{\tilde{p}}$ thus from the definition of indirect dominance, $u_i(z_i^*, w_j^{\tilde{p}}) < u_i(\tilde{w}_i, \tilde{w}_j)$. Furthermore, $u_i(\tilde{z}_i, \tilde{z}_j) < \inf_{z_j \in Z_j} u_i(z_i^*, z_j) \leq u_i(z_i^*, \tilde{w}_j^{\tilde{p}})$. Therefore, $u_i(\tilde{z}_i, \tilde{z}_j) < u_i(\tilde{w}_i, \tilde{w}_j)$. Now recall that $(\tilde{z}_i, \tilde{z}_j) \longrightarrow_{\{i\}} (z_i^*, \tilde{z}_j)$ is possible and $(z_i^*, \tilde{z}_j) \ll (\tilde{w}_i, \tilde{w}_j)$. Thus, Remark 4.1 implies $(\tilde{z}_i, \tilde{z}_j) \ll (\tilde{w}_i, \tilde{w}_j)$, which violates the internal stability of V. **Q.E.D.**

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